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A surgery formula for the 2-loop piece of the LMO invariant of a pair

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Abstract Let $\Theta(M,K)$ denote the 2-loop piece of (the logarithm of) the LMO invariant of a knot K in M, a $\mathbb{Z}HS^3$. Forgetting the knot (by which we mean setting diagrams with legs to zero) specialises $\Theta(M,K)$ to $\lambda(M)$, Casson's invariant. This note describes an extension of Casson's surgery formula for his invariant to $\Theta(M,K)$. To be precise, we describe the effect on $\Theta(M,K)$ of a surgery on a knot which together with K forms a boundary link in M. Whilst the presented formula does not characterise $\Theta(M,K)$, it does allow some insight into the underlying topology.

AMS Classification 57M27; 57M25

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0 Introduction

The simplest characterisation of λ, Casson's invariant of integral homology three-spheres, is the following. Let K be an f-framed knot, where f is plus or minus 1, in M, a $\mathbb{Z}HS^3$; and let M_K denote the result of surgery on K. Then,

$$\lambda(M_K) = \lambda(M) + f\, a_2(M,K). \tag{1}$$

In the equation above $a_2(M,K)$ denotes the coefficient of k^2 in the power series $A_{(M,K)}(e^k)$, where $A_{(M,K)}(t)$ is the symmetric Alexander polynomial of a knot K in M, a $\mathbb{Z}HS^3$, (by symmetric is meant the representative satisfying $A_{(M,K)}(t^{-1}) = A_{(M,K)}(t)$ and $A_{(M,K)}(1) = 1$).

What actually happened, of course, is that this formula was used to "discover" Casson's invariant in the image of the LMO invariant. But let us reverse this history and ask how a hypothetical student of finite-type invariants, ignorant of Casson's formula, might "derive" it from the LMO invariant (we recall the identification in detail in Section 2).

To begin, (considering S^3_K for simplicity), Casson's invariant is the co-efficient of the theta graph in the image of the LMO invariant:

$$\mathrm{Z}^{LMO}(S^3_K) = 1 + \frac{\lambda(S^3_K)}{2} \quad + \text{terms of higher degree.} \tag{2}$$

The Alexander polynomial, on the other hand, is to be found in the image of the Kontsevich invariant, amongst the coefficients of the "wheel" diagrams (see Theorem 5):

$$\hat{Z}(K) = \hat{Z}(U)\left(1 - \frac{1}{2}a_2(S^3,K) \quad + \text{terms of higher degree}\right). \tag{3}$$

According to the LMO surgery formula, $Z^{LMO}(S^3_K)$ is obtained from the Kontsevich integral of K by "gluing chords" into its legs, in a certain way (see Equation 12). Casson's formula then follows from the obvservation that the only term in the surgery formula which leads to a theta diagram is when a single chord is glued into the legs of the wheel with 2 legs:

$$\left\langle -\frac{1}{2}f \quad , -\frac{1}{2}a_2(S^3,K) \quad \right\rangle = f a_2(S^3,K)\, \frac{1}{2} \quad . \tag{4}$$

The simple aim of this article is to tell this familiar story, and then to repeat it, but in a more general context. This more general story concerns $\Theta(M,K)$, the 2-loop piece of the LMO invariant of a pair. We may think of it as a Casson's invariant of M equipped with a knot $K \subset M$. Indeed, if we set all diagrams with legs to zero we recover $\frac{\lambda(M)}{2}\theta$.

More formally: in this setting diagrams may have their edges labelled by power series in a single variable, $\quad a_0 + a_1k_1 + a_2k^2 + \ldots$. This describes a series of Jacobi diagrams: add a leg for every power of k (see Notation 2). In these terms, $\Theta(M,K)$ is defined to be that part of the LMO invariant arising from marked thetas (supressing labels below):

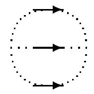
$$\begin{aligned} \mathrm{Z}^{LMO}(M,K) &= (\text{wheels}) \sqcup \\ &\quad \exp_{\sqcup}\left(\sum \quad + \text{conn. diags. with} \geq 3 \text{ loops} \right). \end{aligned}$$

At first glance, $\Theta(M,K)$ may not seem an interesting invariant. It appears to be a rather arbitrary collection of diagrams from the image of the LMO invariant. But here is the thing:

Theorem 1 ([10], Rozansky's conjecture) *$\Theta(M,K)$ is rational. That is, it is expressible as a finite combination of labelled theta diagrams, where each label is a quotient of the form $\frac{p(e^k)}{A_{(M,K)}(e^k)}$, for $p(e^k)$ a polynomial in e^k.*

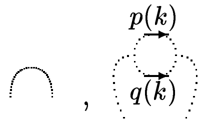
Our main theorem (Theorem 8) concerns the effect on $\Theta(M,K)$ of the following move on K: ± 1-framed surgery on a knot K' which together with K forms a boundary link. (See [7] for a theory of the rationality of the Kontsevich integral of a knot or a boundary link.) It observes a generalisation of Casson's formula (Equation 1) of the following general form. The supressed labels are of the form $p(e^k)/A_{(M,K)}(e^k)$, $p(e^k)$ a polynomial, and the sum is finite.

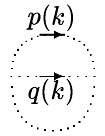
$$\Theta((M,(K,K'))_{K'}) = \Theta(M,K) + \sum \qquad (5)$$



The contributing terms arise from the LMO surgery formula in a familiar way (see Section 3):

$$\left\langle -\frac{f}{2} \quad , \quad \right\rangle = -f \quad . \qquad (6)$$





Section 1, "The 1-loop piece of $\mathrm{Z}^{LMO}(M,L)$ for a boundary link L", introduces the background for the second term above, i.e., the wheel with 2 legs with marked edges. This arises from a certain non-commutative generalisation of the Alexander polynomial of a knot to a 2-component boundary link.

Section 2, "Casson's invariant", recalls the identification of the Casson invariant with the 2-loop piece of the LMO invariant of a $\mathbb{Z}HS^3$. The proof of our generalised formula will be an adaptation of this proof. Then, Section 3 recalls some elements of the recent theory of the *rational expansion of the Kontsevich invariant.* This will describe the origin and character of $\Theta(M,K)$.

Our extension of Equation 1 to $\Theta(M,K)$ (see Theorem 8) is the subject of Section 4, "Surgery on a sublink of a boundary link". Whilst this is our advertised goal, our ulterior motive is to use this discussion to highlight, in the **simplest possible terms**, one or two results from the recent theory of the rationality of the Kontsevich invariant, and expose some techniques from these (sometimes dense) papers.

We illustrate this technique in Section 5, where a step is taken towards finding a Seifert surface based formula for $\Theta(M,K)$.

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1 The 1-loop piece of $Z^{LMO}(M,L)$ for a boundary link L

The diagram-valued determinant Consider Equation 3: actually, one can interpret the entire 1-loop piece of the Kontsevich invariant of a knot (that is, the projection to the subspace generated by the *wheels*), as a certain representation of the Alexander polynomial of a knot. To do this, write down the usual definition, but replace the determinant you have written with the *diagram-valued* determinant, to follow.

Consider, then, the (usual) determinant $|M|$ of a square matrix M of power series in a single variable k which augments to a matrix M_ε which is invertible over $\mathbb{Q}$ (one *augments* M by setting k to zero). Such a determinant satisfies the following equation (which, observe, is well-defined, $(1-MM_\varepsilon^{-1})$ being small in the k-adic topology):

$$|M| = |M_\varepsilon| \exp\left(-\mathrm{Tr}\left(\sum_{l=1}^{\infty}\frac{(1-MM_\varepsilon^{-1})^l}{l}\right)\right). \qquad (7)$$

This equality follows (for example) from a few straightforward **determinant-like properties** of the right-hand side $\Psi'(M)$, which we take this opportunity to collect, below. (In detail: DLP's 1 and 2 imply that $\Psi'(M)$ is unaffected by elementary row operations. Having assumed that M augments to an invertible matrix, elementary row operations may be used to transform M to upper-triangular form, whence the result follows from DLP's 3 and 4.) All matrices mentioned below augment to invertible matrices.

(1) $\Psi'(M) = |M|$ if $M = M_\varepsilon$.

(2) $\Psi'(M_1M_2) = \Psi'(M_1)\Psi'(M_2)$.

(3) If M is of the form $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$, then $\Psi'(M) = \Psi'(A)\Psi'(C)$.

(4) $\Psi'([x]) = x, x \in \mathbb{Q}[[k]]$.

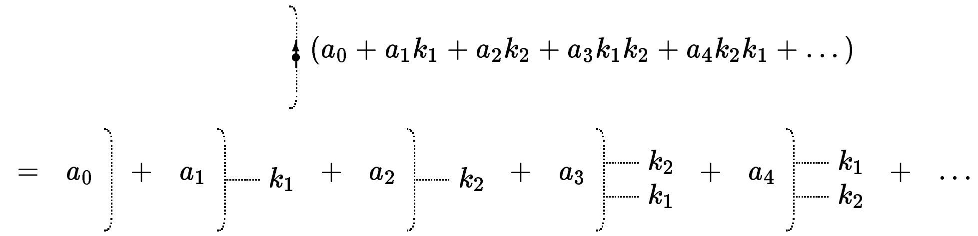
For example, property (2) follows from the BCH formula together with the cyclic invariance of the trace (which kills commutators).

The diagram-valued determinant, to be introduced presently, is naturally motivated by the problem: Define a "determinant" $\Psi(M)$ of a square matrix M of elements of $\mathbb{Q}\langle\langle k_1,\dots,k_\mu\rangle\rangle$ (where M augments to an invertible matrix). The notation $\mathbb{Q}\langle\langle k_1,\dots,k_\mu\rangle\rangle$ denotes the ring of power series in μ non-commuting variables. To be "precise": Define a "useful" function of such matrices of non-commutative power series satisfying DLP's one through four.

We might try to use the right-hand side of Equation 7 to *define* such a determinant. As it stands, however, the expression is inadequate: $Tr(AB)$ is no longer equal to $Tr(BA)$ if the entries of A and B do not commute, and as a consequence multiplicativity (property (2)) fails.

We can, however, restore the cyclic invariance of the trace by replacing the trace operation with a certain *wheel-valued* trace. To recall this, a notation for certain series of Jacobi diagrams helps. By Jacobi diagram we mean the familiar thing (unitrivalent diagrams modulo AS, STU, IHX relations). This note, for example, will frequently use a space $\mathcal{A}(\star_{\{k,k'\}})$, the diagrams of which have univalent vertices (no skeleton) labelled by either k or k'. The object we require we will call a *generating diagram*: a trivalent graph whose edges may have some extra oriented (i.e. the pair of incident edges ordered) bivalent vertices each of which has an element of $\mathbb{Q}\langle\langle k_1, \dots, k_\mu\rangle\rangle$ affixed to it. Such a diagram denotes an element of $\mathcal{A}(\star_{\{k_1,\dots,k_\mu\}})$, which we will say *corresponds to* (or is *generated by*) the generating diagram. For example:

Notation 2



Definition 3

(1) The *wheel-valued trace*, $\mathrm{Tr}^{\circlearrowleft}$, takes an element of $\mathcal{M}(\mathbb{Q}\langle\langle k_1, \dots, k_\mu\rangle\rangle)$, the set of square matrices of elements of $\mathbb{Q}\langle\langle k_1, \dots, k_\mu\rangle\rangle$, to

$$\sum_i M_{ii} \in \mathcal{A}(\star_{\{k_1,\dots,k_\mu\}}).$$

(2) Let $\mathcal{M}(\mathbb{Q}\langle\langle k_1, \dots, k_\mu\rangle\rangle)^\varepsilon \subset \mathcal{M}(\mathbb{Q}\langle\langle k_1, \dots, k_\mu\rangle\rangle)$, denote the subset of matrices which augment to invertible matrices. The *diagram-valued determinant* $\tilde{\Psi}(M) : \mathcal{M}(\mathbb{Q}\langle\langle k_1, \dots, k_\mu\rangle\rangle)^\varepsilon \to \mathcal{A}(\star_{\{k_1,\dots,k_\mu\}})$ is defined by

$$M \mapsto |M_\varepsilon| \exp_\sqcup \left(-\mathrm{Tr}^{\circlearrowleft} \left(\sum_{l=1}^{\infty} \frac{(1 - MM_\varepsilon^{-1})^l}{l} \right) \right) \tag{8}$$

Lemma 4 *$\tilde{\Psi}$ satsifies determinant-like properties (1) through (4).*

The discussion below is in terms of the normalisation $\Psi(M) = \frac{1}{|M_\varepsilon|}\tilde{\Psi}(M)$.

The 1-loop piece (Note on terminology: by "1-loop piece" we mean the 1-loop piece of the logarithm of $Z^{LMO}(M,K)$.)

The story which leads to the following theorem begins with a certain influential conjecture and proof – the conjecture due to Melvin and Morton [15], and the proof, one of the first successes of the theory of finite-type invariants, due to Bar-Natan and Garoufalidis [2].

Our theorem below connects the Kontsevich integral of a boundary link to a certain abelian invariant of boundary links. By abelian we mean arising from classical techniques, depending ultimately on certain linking numbers. By boundary link we mean a link for the components of which we can find a a set of disjoint Seifert surfaces.

So, let L be a boundary link of μ 0-framed components in M, a $\mathbb{Z}HS^3$. Take a set of disjoint Seifert surfaces $\Sigma = \{\Sigma_1, \ldots, \Sigma_\mu\}$ for the components of L. Say Σ_j has genus g_j. On each surface Σ_j choose a system $\{c^j_1, \ldots, c^j_{2g_j}\}$ of oriented curves which present a basis for the first homology of Σ_j. Let S denote the Seifert matrix corresponding to these choices; that is, S is a square matrix with rows and columns indexed by the ordered list $\{c^1_1, \ldots, c^1_{2g_1}, \ldots, c^\mu_1, \ldots, c^\mu_{2g_\mu}\}$. Observe that the Seifert matrices of the individual surfaces appear in blocks along the diagonal, and that the off-diagonal blocks measures linking between curves from different surfaces. We require, furthermore, the following matrix of power series of $\mathbb{Q}\langle\langle k_1, \ldots, k_\mu\rangle\rangle$, which we sometimes write $T(k_1, \ldots, k_\mu)$:

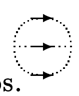
$$T = \mathrm{Diag}(\underbrace{e^{k_1}, \ldots, e^{k_1}}_{2g_1}, \ldots, \underbrace{e^{k_\mu}, \ldots, e^{k_\mu}}_{2g_\mu}).$$

Observe that the matrix $ST^{-1/2} - S^* T^{1/2}$ (where M^* denotes the transpose of a matrix M) augments to an invertible matrix. We call it the *Alexander matrix* corresponding to the Seifert matrix S, and denote it by Λ_S.

Theorem 5 [7] *For L a 0-framed boundary link in M, a $\mathbb{Z}HS^3$, and S a Seifert matrix for L, the value of $Z^{LMO}(M,L)$, an element of the space $\mathcal{A}(\circledast_{\{k_1,\ldots,k_\mu\}})$, is equal to*

$$(\nu(k_1) \sqcup \ldots \sqcup \nu(k_\mu)) \sqcup \Psi(\Lambda_S)^{-\frac{1}{2}} \sqcup (1 + \ldots), \tag{9}$$

where the error in the bracket above is a series of diagrams in $\mathcal{A}(\star_{\{k_1,\ldots,k_\mu\}})$ corresponding to a series of generating diagrams each connected component of which has at least two loops.

Remarks (1) That is, the error is a series of terms like  (supressing labels), i.e. from closed trivalent graphs with at least 2 loops.

(2) Let us recall how and where this is proved in the literature. Observe that given a collection of Seifert surfaces for L, the factor $\Psi(\Lambda_S)$ does not depend on the choice of a basis of curves. It suffices, then, to prove the theorem for some special choice of curves. The special choice of curves the proof uses is made by putting the surface in some standard disc-band form, and taking the associated standard collection of curves. Now, the relevant piece of the LMO invariant is expressed in terms of Ψ of the equivariant linking matrix of a ('nice') surgery presentation. Here, one chooses a certain surgery presentation that is canonically associated to a disc-band decomposed Seifert surface: the so-called Y-view of boundary links (see e.g. [6]). It turns out that Ψ of the equivariant linking matrix of this surgery presentation can then be manipulated to produce the above function of the Seifert matrix corresponding to the chosen basis of curves. (Elements of this strategy were introduced in [11].)

(3) $\Psi(\Lambda_S)$ *does* depend on the isotopy class of the collection. Its image in the space $\mathcal{A}(\circledast_{\{k_1,\dots,k_\mu\}})$, however, does not. These issues, and the above proof, are discussed in full detail in [7]. We remark that, in the form presented, this theorem does not depend on the two pieces of heavy machinery employed by [7] – it depends on neither the adapted Kirby-Fenn-Rourke theorem nor the [5] calculation of the Kontsevich integral of the unknot.

(4) The notation $\nu(k)$ denotes the Kontsevich integral of the unknot in $\mathcal{A}(\star_k)$ (see the Appendix). The space $\mathcal{A}(\circledast_{\{k_1,\dots,k_\mu\}})$ is the quotient of $\mathcal{A}(\star_{\{k_1,\dots,k_\mu\}})$ obtained by imposing the notorious *link relations* (see [3]). Note that, for starters, the $\sqcup$-product is not well-defined in the presence of such relations. The meaning of the above equation, then, is the following: the factors lie in $\mathcal{A}(\star_{\{k_1,\dots,k_\mu\}})$, by definition, and they are to be multiplied in that space and then the result is to be projected to $\mathcal{A}(\circledast_{\{k_1,\dots,k_\mu\}})$.

2 Casson's invariant

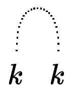
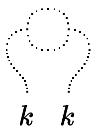
We now recall the identification of Casson's formula with the degree 1 piece of the LMO invariant. We are going to do this in a way that is far from the easiest, as a warm-up and to provide context for the proof of our main theorem.

So, we wish to calculate the effect on the degree 1 piece of the LMO invariant of a surgery on a knot K in an integral homology sphere M. In this case, the determinant-like properties give

$$\Psi(ST^{-1/2} - S^*T^{1/2}) = \Psi(A_K(e^k)).$$

and we may rewrite equation 9:

Corollary 6 *For a pair of an f-framed knot K in M, $\nu(k)\#Z^{LMO}(M,K)$ is equal to*

$exp_{\sqcup}\left(\frac{f}{2}\right.$  $\left.\right)\sqcup\left(1+\frac{\lambda(M)}{2}\right.$  $+\left(2\kappa-\frac{1}{2}a_2(K)\right)$  $\left.+\ \dots\right),$ (10)

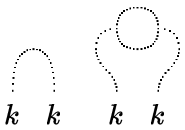
where the error inside the bracket is a series of diagrams each of which has more than 2 trivalent vertices, and κ denotes the coefficient of the wheel with 2 legs in $\nu(k)$.

Two multiplicative corrections were required to obtain this formula from equation 9: First, we had to multiply in a copy $\nu(k)$ to allow use in the LMO surgery formula (equation 12). Then, we had to adjust the framing of K from 0 to f. See the Appendix for some details on an elegant way to perform such corrections (using the beautiful wheeling isomorphism).

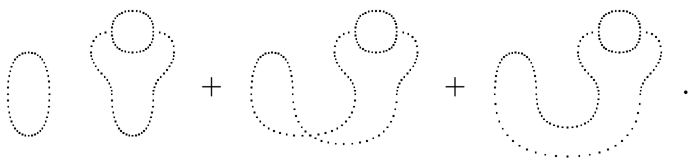
How to put this formula to use? Let us briefly recall how the degree $\leq n$ part of $Z^{LMO}(M_K)$ is determined from the series $Z^{LMO}(M,K)$. This uses the map

$$\iota_n : \mathcal{A}(\star_k) \to \mathcal{A}(\phi).$$

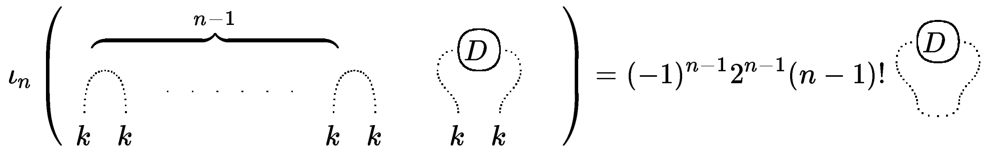
We recall this by calculating:

$\iota_2\left(\right.$  $\left.\right).$

The map ι_n sends a diagram with other than $2n$ legs to zero. Otherwise, (our example, for example) there are 3 steps. First, glue n chords into the legs, in all possible ways:



Then, replace any vertex-free loops that have arisen with multiplicative factors of $-2n$, and, finally, set any resulting diagrams with more than $2n$ vertices to zero. Thus, our example yields -2θ. We will presently need the following result, which is fun to prove [13] (D is any diagram of degree less than or equal to n with exactly 2 legs):



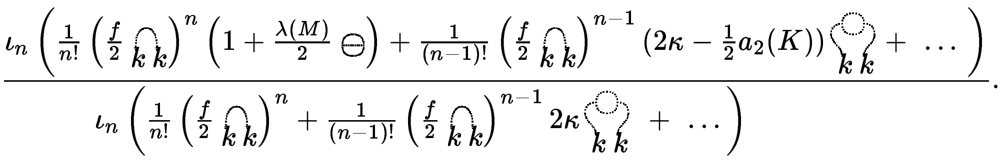
(11)

The degree $\leq n$ part of $Z^{LMO}(M_K)$ is given by the degree $\leq n$ part of the expression [14, 12]:

$$\frac{\iota_n\left(Z^{LMO}(M,K)\#\nu(k)\right)}{\iota_n\left(Z^{LMO}(S^3,U_f)\#\nu(k)\right)} \in \mathcal{A}(\phi), \tag{12}$$

where U_f is an unknot with the framing of K, and we have to take the degree less than or equal to n part of the result.

We can now observe the effect of surgery on Casson's invariant. Setting n to 1 leads to a quick calculation, but we will learn more (on the way to our main theorem) if we do this for arbitrary (positive) n. So, substituting equation 10 into this expression, we find that $1 + \frac{\lambda(M_K)}{2}\theta + \ldots$ equals



Here is what is happening: terms which do not have precisely $2n$ legs are killed by ι_n. Assuming ι_n is to be operated on an expression of the form $\exp_{\sqcup}(\text{chords}) \sqcup X$, for X some series of diagrams, then to each term in X we multiply as many chords as we need to get the right number of legs, and that is what we see above. The error terms above arise from diagrams in X with more than 2 trivalent vertices: acting with ι_n on such a diagram (producted with an appropriate number of chords) inevitably yields a diagram with more than 2 loops. Evaluating returns the expected $1 + (\lambda(M) + fa_2(M,K))\,\frac{1}{2}\theta +$ diagrams with > 2 loops .

3 The loop expansion and rationality

So, the 1-loop part of $\mathrm{Z}^{LMO}(M,K)$ is a polynomial, more or less. This and general principles let us write $\mathrm{Z}^{LMO}(M,K)$ as

$$\underbrace{\nu(k) \sqcup \left(\Psi(A_K(e^k))\right)^{-\frac{1}{2}}}_{\text{The wheels}} \sqcup \exp_{\sqcup}(\underbrace{R_2 + R_3 + R_4 + \dots}_{>1 \text{ loops}}),$$

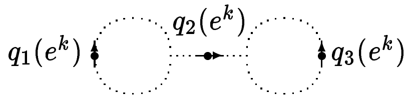
where R_l is a (unique modulo IHX and AS) series of connected diagrams each with l loops. Lev Rozansky's studies of the Melvin-Morton expansion of the coloured Jones polynomial [19], combined with results in the theory of finite-type invariants (for example the proof of Melvin-Morton [2]), led him to conjecture that each piece R_l might be generated from "finite" data as well: from a finite collection of 1-variable polynomials [18].

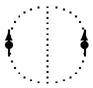
Theorem 7 [10] *There exists a finite $\mathbb{Q}$-linear combination of generating diagrams with l loops, each of whose edges is labelled by exactly one power series of the form*

$$\frac{\text{Laurent polynomial in } e^k}{A_K(e^k)},$$

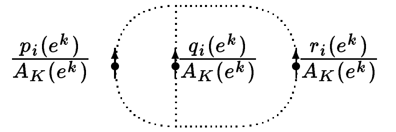
which generates the series R_l.

The 2-loop piece Let us focus now on the 2-loop piece, R_2. According to this theorem, and the following exercise (for rational functions $\{q_i(t)\}$ non-singular at 1),



$$= \quad q_2(1) \times (q_1(e^k) - q_1(e^{-k}))$$  $$(q_3(e^k) - q_3(e^{-k})) \ ,$$

there exists a finite set of 4-tuples $\{\lambda_i, p_i(t), q_i(t), r_i(t)\}$ of a rational λ_i together with three polynomials, such that

$$R_2 = \textstyle\sum_i \lambda_i$$ 

. (13)

We digress for 2 paragraphs on the the question of how to present this result. It is possible, at this point, to pass to a space of trivalent graphs with edges labelled by rational functions of t (that is, with no mention of power series). For example, in [7] the authors defined a space, $\mathcal{A}(\Lambda_{loc})$, which is generated by such diagrams, and which has certain relations which allow one to "Push" around the labels.

Keeping in mind our aim of presenting this work "in the simplest possible terms" we will not discuss these issues here (see [7] for a full discussion). That is, we will continue to present results at the level of declaring that some element in question is an element of $\mathcal{A}(\star_{\{k\}})$ generated by some finite combination of trivalent diagrams with edges labelled by power series in k generated by rational functions in e^k. Suffice it to say that, at least for the 2 loop piece, such an expression uniquely determines some polynomials (up to some symmetries)[1].

Returning to the main discussion: Let us write R_2, the 2-loop invariant, as $\Theta(M,K) \in \mathcal{A}(\star_k)$. This invariant generalises Casson's invariant in the sense that, obviously,

$$\big(\Theta(M,K)\big)\big|_{k=0} = \frac{\lambda(M)}{2} \quad (14)$$

in other words, when diagrams with legs are set to zero. That this generalisation is natural is well illustrated by our main theorem:

4 Surgery on a sublink of a boundary link

We come, at last, to our generalisation to $\Theta(M,K)$ of Casson's formula (equation 1). Ideally, we would like to describe the effect on $\Theta(M,K)$ of a surgery on *any* ± 1-framed knot K' in $M - K$ (this, for example, would let us change crossings). Here, however, we can only give a formula describing the effect of a surgery on a ± 1-framed knot K' with the property that (K,K') forms a *boundary link* in M. The pair that results we denote $(M,(K,K'))_{K'}$.

There are several, related, reasons why this is a natural class of surgeries to consider. Firstly, to anticipate the proof: if (K,K') forms a boundary link, then the invariant $Z^{LMO}(M,(K,K'))$ can be written *without trees*, which lets us control the contributions to $\Theta(M,K)$, the 2-loop piece, when we evaluate the

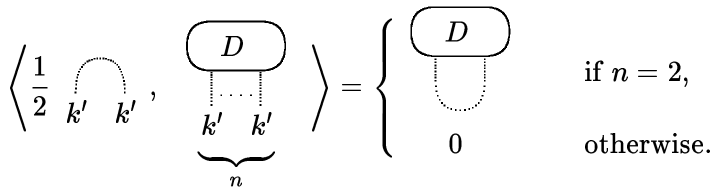
[1] Note added: this is not true for a general number of loops. The question of injectivity has recently been resolved [17] (these spaces do not inject into $\mathcal{A}(\star_{\{k\}})$).

LMO surgery formula. Secondly, recall that $\Theta(M,K)$ appears in $Z^{LMO}(M,K)$ as

$$\nu(k) \sqcup \left(\Psi(A_K(e^k))\right)^{-\frac{1}{2}} \sqcup (1 + \Theta(M,K) + \ldots).$$

A general principle of quantum topology then suggests that it should be possible to compare $\Theta(M_1,K_1)$ and $\Theta(M_2,K_2)$ when their Alexander polynomials coincide. As it happens, surgery on a ± 1-framed component K' of a boundary link (K,K') does not affect the Alexander polynomial of K (this is because the lift of K' bounds in the universal cyclic cover of K, for example).

The statement uses the following notation, where D is a diagram in $\mathcal{A}(\star_{\{k,k'\}})$ (possibly with some number of k-labelled legs, not shown).



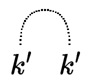
(15)

Instead of taking the coefficient of k^2 in a series associated to the Alexander polynomial (as appears in Casson's formula), we use this operation to "take the coefficient of $(k')^2$" in the diagram-valued determinant associated to the boundary link (K,K'). To be precise:

Theorem 8 (The Main Theorem)

Let $(M,(K,K'))$ be a $(0,f)$-framed boundary link (K,K') in M, a $\mathbb{Z}HS^3$, (where f is plus or minus 1,) and let S be a Seifert matrix corresponding to some choice of a disjoint pair of Seifert surfaces (Σ,Σ') for (K,K'). Then,

$$\Theta\left((M,(K,K'))_{K'}\right) = \Theta(M,K) - f\left\langle \frac{1}{2}\right.$$



$$\left. , \ \Psi(W)^{-1/2} \right\rangle, \qquad (16)$$

where

$$W = \Lambda_S(k,k')\Lambda_S(k,0)^{-1}.$$

Recall that $\Lambda_S(k,k')$ denotes the Alexander matrix corresponding to S:

$$\Lambda_S(k,k') = ST(k,k')^{-1/2} - S^*T(k,k')^{1/2}.$$

Three remarks:

(1) Let S_K be the Seifert matrix of K by itself. Observe that it follows from the multiplicativity of the determinant that the exponential inside the pairing can be written

$$\left(\frac{\Psi\left(\Lambda_S\right)}{\Psi\left(\Lambda_{S_K}\right)}\right)^{-\frac{1}{2}}.$$

(2) If we are to believe this formula, then it must, at least, give us a "rational" $\Theta((M,(K,K'))_{K'})$ (see Theorem 7). To see that this is the case, write Ψ as $\exp\operatorname{tr}\log$, and consider the argument of the exponential:

$$\frac{1}{2}\sum_{p=1}^{\infty}\operatorname{Tr}^{\circlearrowleft}\left(\frac{\left((\Lambda_S(k,0)-\Lambda_S(k,k'))\Lambda_S(k,0)^{-1}\right)^p}{p}\right). \tag{17}$$

First, we perform some cancellations to write this in terms of integral powers of e^k. Let $\hat{\Lambda}_S=\Lambda_S T(-k/2,0)$, that is, $\hat{\Lambda}_S=ST(-k,-k'/2)-S^*T(0,k'/2)$. Observe that the expression (17) may be written:

$$\frac{1}{2}\sum_{p=1}^{\infty}\operatorname{Tr}^{\circlearrowleft}\left(\frac{\left((\hat{\Lambda}_S(k,0)-\hat{\Lambda}_S(k,k'))\hat{\Lambda}_S(k,0)^{-1}\right)^p}{p}\right).$$

Now, note that the factor $(\hat{\Lambda}_S(k,0)-\hat{\Lambda}_S(k,k'))$ is independent of k and is divided by k'. In fact, for M_a and M_b matrices of integers:

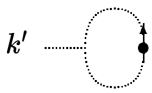
$$(\hat{\Lambda}_S(k,0)-\hat{\Lambda}_S(k,k'))=k'M_a+{k'}^2M_b+(\text{terms with}\geq 3\text{ factors }k').$$

Consider now the second factor, $\hat{\Lambda}_S(k,0)^{-1}$. Observe that this factor has the form (for B and C some matrices of integers, $|C|=1$, and $\hat{\Lambda}_{S_K}(k)$ denoting $(S_Ke^{-k}-S_K^*)$):

$$\begin{bmatrix}\hat{\Lambda}_{S_K}(k) & 0\\ (e^{-k}-1)B & C\end{bmatrix}^{-1}=\frac{1}{A_K(e^k)}M_c(e^k),$$

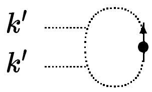
where $M_c(e^k)$ is some matrix of Laurent polynomials in e^k.

Now, only terms with less than 2 legs labelled k' will contribute to formula (16). Thus, the contribution of the $p=1$ term is precisely the sum of:



$\sum_{i,j}(M_a)_{ij}\frac{1}{A_K(e^k)}(M_c(e^k))_{ji}$ (18)

and



$\sum_{i,j}(M_b)_{ij}\frac{1}{A_K(e^k)}(M_c(e^k))_{ji}$ (19)

For the purposes of the discussion, this is a good point to make precise some terminology:

Definition 9 A *fragment of a generating diagram* is a unitrivalent diagram with univalent vertices labelled by k', with edges possibly labelled by power series in k, and such that each connected component has at least 1 trivalent vertex. A *fragment of a rational generating diagram* satisfies, in addition, that the edge labels are rational functions in e^k.

Diagrams (18) and (19) are clearly such things – each a fragment of a rational generating diagram.

The pairing $<,>$ will map a product of such diagrams to a rational generating diagram. It is easy to see that the $p=2$ term is a sum of such fragments, and that terms for $p \geq 3$ will all have at least 3 legs labelled k', and hence will be killed by the pairing $<,>$.

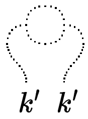
(3) Our theorem suggests studying the following "finite-type" filtration. Let $\mathbf{MK}$ be the Abelian group freely generated by pairs (M,K) of a knot K in a $\mathbb{Z}HS^3$ M. Let $\mathbf{MK}_n^{\partial}$ denote the subgroup generated by elements of $\mathbf{MK}$ corresponding to pairs (M,L) of a boundary link L with a distinguished component K in M, a $\mathbb{Z}HS^3$: the correspondence is to take the obvious alternating sum $\sum_{L'\subset L-K}(-1)^{\epsilon}(M,(L',K))_{L'}$. This filtration is clearly as steep as the "loop" filtration (see [9]).

Proof of the Main Theorem We now turn to the proof of the main theorem. This will be a step-by-step copy of the proof of Casson's formula. First, in analogy with Equation 10, we need an expression for $\mathrm{Z}^{LMO}(M,(K.K'))$ that identifies the contributing terms.

We start with Theorem 5. Let K_0' be K' with the zero framing. Theorem 5, together with the fact that

$$\mathrm{Z}^{LMO}(M,(K,K'))|_{k'=0} = \mathrm{Z}^{LMO}(M,K),$$

tells us that $\mathrm{Z}^{LMO}(M,(K,K_0'))$ is equal to:

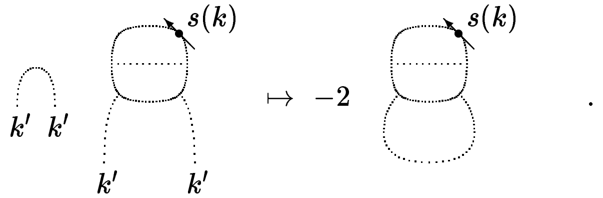
$$\nu(k) \sqcup (\Psi(\Lambda_S))^{-\frac{1}{2}} \sqcup \left(1 + \Theta(M,K) + \kappa\right.$$


$$\left. + \ \ldots \right),$$

where the error term is a series of fragments of generating diagrams (see Defintion 9) with more than 2 trivalent vertices[2]. Note that the maps $\{\iota_n\}$ send

[2]We remark on a possible source of confusion here. This sentence refers to the number of trivalent vertices in the generating diagram, not to the number of trivalent vertices in the diagrams it generates. For example: the fragment in diagram 20 has 4 trivalent vertices while the series it generates contains diagrams with an arbitrarily high number of trivalent vertices (and k-labelled legs).

products of such terms with chords to generating diagrams with *at least* 3 loops. Let us examine an example. Under ι_2, the following product of a fragment of a generating diagram and a chord is mapped:



$$. \qquad (20)$$

It is to be understood that the error terms in the equations that follow are also of this form, and consequently do not contribute to the 2-loop piece, for the same reason.

Following the Appendix, we can adjust the framing of K_0' to f, and $\#$-multiply by a copy of $\nu(k')$ to find that $\mathrm{Z}^{LMO}(M,(K,K'))\#_{k'}(\nu(k'))$ is equal to

$$\exp_{\sqcup}\left(\frac{f}{2}\underset{k'\ k'}{\frown}\right)\sqcup\nu(k)\sqcup\left(\Psi(\Lambda_S)\right)^{-\frac{1}{2}}\sqcup\left(1+\Theta(M,K)+2\kappa\underset{k'\ k'}{\bigcirc}+\ldots\right).$$

This can be re-organised to isolate the factors with no legs labelled k' and with no more than 2 loops:

$$\begin{aligned}&=\left(\nu(k)\sqcup\Psi(\Lambda_{S_K})^{-\frac{1}{2}}\sqcup(1+\Theta(M,K))\right)\\&\qquad\sqcup\exp_{\sqcup}\left(\frac{f}{2}\underset{k'\ k'}{\frown}\right)\sqcup\left(\frac{\Psi(\Lambda_S)}{\Psi(\Lambda_{S_K})}\right)^{-\frac{1}{2}}\sqcup\left(1+2\kappa\underset{k'\ k'}{\bigcirc}+\ldots\right).\end{aligned}\qquad(21)$$

We will presently substitute this expression into the following formula, which calculates, for some n, the degree $\leq n$ part of $\mathrm{Z}^{LMO}((M,(K,K'))_{K'})$ (see [14, 12, 16]):

$$\frac{\iota_n\left(\mathrm{Z}^{LMO}(M,(K,K'))\#_{k'}\nu(k')\right)}{\iota_n\left(\mathrm{Z}^{LMO}(S^3,U_f)\#_k\nu(k)\right)}\in\mathcal{A}(\star_k),\qquad(22)$$

where the ι_n in the numerator acts on the labels k'. We will observe that the 2-loop part of the result, as n varies, is the degree $\leq n$ part of some fixed series, which we can conclude calculates the 2-loop part of $\mathrm{Z}^{LMO}((M,(K,K'))_{K'})$.

Let us fix some n, then, and consider the numerator. Denote the leading bracket of expression (21) by α. Let ψ denote the piece of $(\Psi(\Lambda_S)/\Psi(\Lambda_{S_K}))^{-\frac{1}{2}}$ with precisely 2 legs labelled by k'. The numerator, then, is clearly the degree $\leq n$ piece of:

$$\iota_n\left(\alpha\sqcup\left(\frac{1}{n!}\left(\frac{f}{2}\underset{k'\ k'}{\frown}\right)^n+\frac{1}{(n-1)!}\left(\frac{f}{2}\underset{k'\ k'}{\frown}\right)^{n-1}\sqcup\left(\psi+2\kappa\underset{k'\ k'}{\bigcirc}\right)+\ \ldots\right)\right),$$

which evaluates (see equation 11) to the degree $\leq n$ piece of

$$(-1)^n f^n \alpha \sqcup \left(1 - f^{-1}\left\langle \frac{1}{2}\ {}_{k'}\!\frown_{k'}\ ,\ \psi \right\rangle - 2f^{-1}\kappa\ \ominus + \ldots\right).$$

The denominator, on the other hand, may be similarly calculated to be the degree $\leq n$ part of:

$$(-1)^n f^n \left(1 - 2f^{-1}\kappa\ \ominus + \ldots\right).$$

The quotient of these (which we then take the degree $\leq n$ part of) is as required:

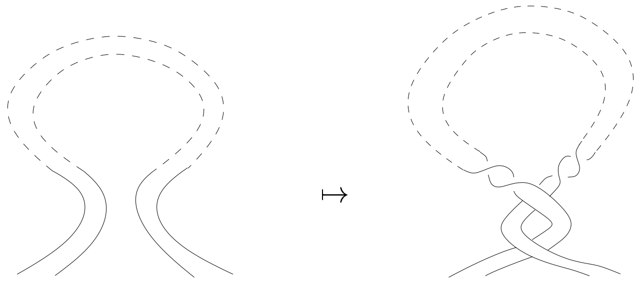
$$\nu(k) \sqcup \Psi(\Lambda_K) \sqcup \left(1 + \Theta(M,K) - f\left\langle \frac{1}{2}\ {}_{k'}\!\frown_{k'}\ ,\ \psi \right\rangle + \ldots\right). \qquad \square$$

5 Changing band self-crossings

It seems important to discover a full topological formula for $\Theta(M,K)$ (noting that Lev Rozansky can compute the invariant in all cases [20]). Perhaps some enlargement of Turaev's "multiplace generalisation of the Seifert matrix" [21] is the key. Let Σ be a Seifert surface for K in M and let $\{c_1, \ldots, c_{2g}\}$ be a system of curves presenting a basis for the first homology of Σ. Then:

Problem: Express $\Theta(M,K)$ in terms of the finite type invariants of degree ≤ 3 of the links obtained by pushing the curves $\{c_i\}$ off Σ.

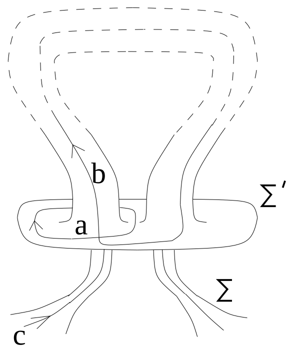
We wrap this article up with a step in this direction. Let Σ be a Seifert surface of genus $2g$ for a knot (M,K). Focus on a band of this surface (the dashed part, below, follows some arbitrarily knotty path around the other bands in M), and consider another knot $(M,K^{\bowtie})$ obtained from (M,K) by a move of the following sort (which we will call a *band twist*):



(23)

We can now apply Theorem 8 to calculate $\Theta(M,K^{\bowtie}) - \Theta(M,K)$ for the reason that these two knots are related by a surgery on a knot K' which together with

K forms a boundary link in M. K' can be taken to be the -1-framing of the boundary of Σ' (see diagram 24 below).



(24)

First, we choose a convenient system of curves presenting a basis for the first homology of (Σ, Σ'). Let c_1 be some curve in Σ which traverses the part of the band shown in the diagram exactly once (shown as c). Complete the choice of the $\{c_i\}$ in any fashion, so long as the chosen curves lie in the complement in Σ of the displayed part of the band. Let c_1' be a and let c_2' be b (this choice is fixed by the diagram if we demand that $\mathrm{lk}(b^+, b) = 0$, where b^+ is the push-off of b). The corresponding $(2g+2) \times (2g+2)$ Seifert matrix can be written:

$$S = \left[\begin{array}{c|cc} & -1 & \\ & 0 & \\ S_K & . & \mathbf{b}^* \\ & . & \\ & 0 & \\ \hline -1 \;\; 0 \;\; . \;\; . \;\; 0 & 0 & -1 \\ \mathbf{b} & 0 & 0 \end{array}\right],$$

where the vector $\mathbf{b}$ collects the linking numbers of the curve b with the system of curves $\{c_i\}$ on Σ:

$$\mathbf{b} = \{\mathrm{lk}(b, c_1), \mathrm{lk}(b, c_2), \dots, \mathrm{lk}(b, c_{2g})\}.$$

Substituting this Seifert matrix into the surgery formula (Theorem 8), one arrives (via some involved matrix algebra) at the following statement. Let $\mathbf{p}$ denote the projector $\overbrace{(1, 0, \dots, 0)}^{2g}$, and let $\langle k \rangle$ (resp. $\langle k' \rangle$) denote $e^{k/2} - e^{-k/2}$ (resp. $e^{k'/2} - e^{-k'/2}$).

Theorem 10 *Letting $f = -1$ for a positive twist (as shown earlier), and $f = +1$ for a negative twist:*

$$\Theta(M, K^{\bowtie}) = \Theta(M, K) - f\left\langle \frac{1}{2}\ \overset{\frown}{k'\ \ k'}\ ,\ \Psi\left(Z\right)^{-1/2} \right\rangle,$$

where Z is the matrix

$$I \ -\ \Lambda_{S_K}^{-1}\,\langle k'\rangle \left(e^{k'/2}\mathbf{b}^*\mathbf{p} - e^{-k'/2}\mathbf{p}^*\mathbf{b}\right)\langle k\rangle\,.$$

This is technical, yes, but straightforward to compute. Observe that it has the expected rationality structure, with the Alexander polynomial of $K^{\bowtie}$ appearing in the denominators of the labels.

There is an obvious extension to the case of a twist of $2\pi p$ radians.

Appendix: Multiplicative corrections

Frequently we encounter something like the following problem: we are presented an element of $\mathcal{A}(\circledast_{\{k,k'\}})$ as (the projection from $\mathcal{A}(\star_{\{k,k'\}})$ of) an exponential (w.r.t. the "disjoint-union product") of some series of connected "symmetrised" diagrams. For example:

$$\exp_{\sqcup}(W + S_1 + S_2 + \dots) \in \mathcal{A}(\circledast_{\{k,k'\}}),$$

where W is a series of wheels all of whose legs are labelled k and S_i is a series of connected fragments of generating diagrams, each with exactly i trivalent vertices (see Definition 9). We are then asked to multiply it by some given factor using the "connect-sum multiplication", and then re-express the result in some convenient "symmetrised" form. We will examine, as a guiding example, the following expression (note that the error term on the LHS below is *different* to the error term on the RHS):

Lemma 11

$$\begin{aligned} &exp_{\#}\left(\frac{f}{2}\ \xrightarrow{\ \frown\ }\right)\#_{\{k'\}}exp_{\sqcup}(W + S_1 + S_2 + \dots) \\ =\ \ &exp_{\sqcup}\left(\frac{f}{2}\ \overset{\frown}{k'\ \ k'} + W + S_1 + S_2 + \dots\right). \end{aligned}$$

Proof There are brute force ways to do this, but an elegant approach is available: use the *wheeling isomorphism.* This theory was conjectured by Bar-Natan, Garoufalidis, Rozansky and Thurston, conjectures which have since been proved, including a Kontsevich integral based theory due to Bar-Natan, Le and Thurston [5].

All we need to know about this beautiful game here is the following: there exists an element $\nu(k) \in \mathcal{A}(\star_k)$, which is of the form $\exp_\sqcup(W)$, for W a series of connected *wheels* (1-loop diagrams), with the property that if $X \in \mathcal{A}(\star_x)$, and if $Y \in \mathcal{A}(\circledast_{x,y})$ then

$$\widehat{\nu(x)}(X) \#_{\{x\}} \widehat{\nu(x)}(Y) = \widehat{\nu(x)}(X \sqcup Y), \tag{25}$$

where $\widehat{\nu(x)}(X)$, for example, is the operation of joining, in all ways, *all* legs of some diagram appearing in $\nu(x)$ to *some* of the x-labelled legs of some diagram appearing in X, bilinearly extended to all diagrams in those series. (At the end of this Appendix we remind why the BLT theory applies in this generality.)

To apply this here, note that (recalling that κ denotes the coefficient of the wheel with two legs in $\nu(x)$):

$$\frac{f}{2} \ \overset{\frown\, x}{\longrightarrow} \ = \widehat{\nu(x)} \left(\frac{f}{2} \underset{x\ \ x}{\frown} - f\kappa \ominus \right), \tag{26}$$

and also that

$$\exp_\sqcup(W + S_1 + S_2 + \dots) = \widehat{\nu(k')} \left(\exp_\sqcup(W + S_1 + S_2 + \dots) \right), \tag{27}$$

because any way of gluing a wheel to the legs of a fragment of a generating diagram will result in a fragment of a generating diagram with at least three trivalent vertices. So, taking the $\#$-exponential of (26), and then $\#$-multiplying it into (27) gives:

$$\begin{aligned} &\widehat{\nu(k')} \left(\exp_\sqcup \left(\frac{f}{2} \underset{k'\ k'}{\frown} - f\kappa \ominus + W + S_1 + S_2 + \dots \right) \right) \\ = &\left(\exp_\sqcup \left(\frac{f}{2} \underset{k'\ k'}{\frown} - f\kappa \ominus + f\kappa \ominus + W + S_1 + S_2 + \dots \right) \right), \end{aligned}$$

as required. $\square$

We finish with a remark reminding why the BLT theory includes equation 25. Following [5]:

$$\hat{Z} \left(\overset{|x}{\circlearrowleft} k \right) = \exp_\sqcup \left(\frac{f}{2} \underset{x\ \ k}{\frown} \right) \sqcup \nu(k) \in \mathcal{A}(\star_{\{x\}} \circledast_{\{k\}}).$$

We may "double" the component k in 2 different ways: by taking its parallel or by vertically composing 2 copies of the long Hopf link ("1+1=2"). The functorial properties of $\hat{Z}$ lead to a corresponding algebraic equation in $\mathcal{A}(\star_{\{x\}} \circledast_{\{k_1,k_2\}})$. Pairing with $X(k_1) \sqcup Y(k_2, y)$ sends this equation to equation

25. The unusual point that must be observed is that this pairing is still well-defined (despite the y-labelled legs) because after pairing a “link relation” in k_2 may “sweep” across Y to become a link relation in y, which we are modding out by.

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