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Matrix-tree theorems and the Alexander-Conway polynomial

Gregor Masbaum

Abstract This talk is a report on joint work with A. Vaintrob [12, 13]. It is organised as follows. We begin by recalling how the classical Matrix-Tree Theorem relates two different expressions for the lowest degree coefficient of the Alexander-Conway polynomial of a link. We then state our formula for the lowest degree coefficient of an algebraically split link in terms of Milnor's triple linking numbers. We explain how this formula can be deduced from a determinantal expression due to Traldi and Levine by means of our Pfaffian Matrix-Tree Theorem [12]. We also discuss the approach via finite type invariants, which allowed us in [13] to obtain the same result directly from some properties of the Alexander-Conway weight system. This approach also gives similar results if all Milnor numbers up to a given order vanish.

AMS Classification 57M27; 17B10

Keywords Alexander-Conway polynomial, Milnor numbers, finite type invariants, Matrix-tree theorem, spanning trees, Pfaffian-tree polynomial

1 The Alexander-Conway polynomial and its lowest order coefficient

Let L be an oriented link in S^3 with m (numbered) components. Its Alexander-Conway polynomial

$$\nabla_L(z) = \sum_{i\geq 0} c_i(L) z^i \in \mathbb{Z}[z]$$

is one of the most thoroughly studied classical isotopy invariants of links. It can be defined in various ways. For example, if V is a Seifert matrix for L, then

$$\nabla_L(z) = \det(tV - t^{-1}V^T) \tag{1}$$

where $z = t - t^{-1}$. Another definition is via the *skein relation*

$$\nabla_{L_+} - \nabla_{L_-} = z\nabla_{L_0} \ , \tag{2}$$

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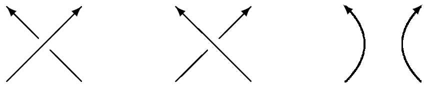


Figure 1: A skein triple

where (L_+, L_-, L_0) is any skein triple (see Figure 1).

Indeed, the Alexander-Conway polynomial is uniquely determined by the skein relation (2) and the initial conditions

$$\nabla_{U_m} = \begin{cases} 1 & \text{if } m = 1 \\ 0 & \text{if } m \geq 2, \end{cases} \tag{3}$$

where U_m is the trivial link with m components.

Hosokawa [7], Hartley [6, (4.7)], and Hoste [8] showed that the coefficients $c_i(L)$ of ∇_L for an m-component link L vanish when $i \leq m-2$ and that the coefficient $c_{m-1}(L)$ depends only on the linking numbers $\ell_{ij}(L)$ between the ith and jth components of L. Namely,

$$c_{m-1}(L) = \det \Lambda^{(p)}, \tag{4}$$

where $\Lambda = (\lambda_{ij})$ is the matrix formed by linking numbers

$$\lambda_{ij} = \begin{cases} -\ell_{ij}(L), & \text{if } i \neq j \\ \sum_{k \neq i} \ell_{ik}(L), & \text{if } i = j \end{cases} \tag{5}$$

and $\Lambda^{(p)}$ denotes the matrix obtained by removing from Λ the pth row and column (it is easy to see that $\det \Lambda^{(p)}$ does not depend on p).

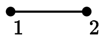
Formula (4) can be proved using the Seifert matrix definition (1) of ∇_L. We will not give the proof here, but let us indicate how linking numbers come in from this point of view. Let Σ be a Seifert surface for L. The key point is that the Seifert form restricted to $H_1(\partial\Sigma; \mathbb{Z}) \subset H_1(\Sigma; \mathbb{Z})$ is just given by the linking numbers ℓ_{ij}. In particular, for an appropriate choice of basis for $H_1(\Sigma; \mathbb{Z})$, the Seifert matrix V contains the matrix $\Lambda^{(p)}$ as a submatrix, which then leads to Formula (4).

Hartley and Hoste also gave a second expression for $c_{m-1}(L)$ as a sum over trees:

$$c_{m-1}(L) = \sum_T \prod_{\{i,j\} \in edges(T)} \ell_{ij}(L) \ , \tag{6}$$

where T runs through the spanning trees in the complete graph K_m. (The complete graph K_m has vertices $\{1, 2, \dots, m\}$, and one and only one edge for every unordered pair $\{i, j\}$ of distinct vertices.)

For example, if $m = 2$ then $c_1(L) = \ell_{12}(L)$, corresponding to the only spanning tree in

$K_2 =$ 

If $m = 3$, then

$$c_2(L) = \ell_{12}(L)\ell_{23}(L) + \ell_{23}(L)\ell_{13}(L) + \ell_{13}(L)\ell_{12}(L) \ ,$$

corresponding to the three spanning trees of K_3 (see Figure 2).

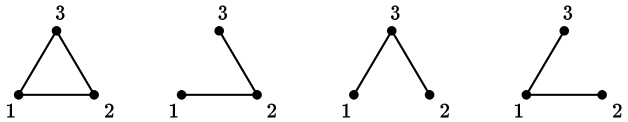


Figure 2: The complete graph K_3 and its three spanning trees

2 The classical Matrix-Tree Theorem

It is a pleasant exercise to check by hand that Formulas (4) and (6) give the same answer for $m = 2$ and $m = 3$. For general m this equality can be deduced from the classical Matrix-Tree Theorem applied to the complete graph K_m.

The statement of this theorem is as follows. Consider a finite graph G with vertex set V and set of edges E. If we label each edge $e \in E$ by a variable x_e, then a subgraph of G given as a collection of edges $S \subset E$ corresponds to the monomial

$$x_S = \prod_{e \in S} x_e.$$

Form a symmetric matrix $\Lambda(G) = (\lambda_{ij})$, whose rows and columns are indexed by the vertices of the graph and entries given by

$$\lambda_{ij} = - \sum_{\substack{e \in E, \\ v(e) = \{i,j\}}} x_e, \quad \text{if } i \neq j, \quad \text{and} \quad \lambda_{ii} = \sum_{\substack{e \in E, \\ i \in v(e)}} x_e.$$

Here, we denote by $v(e) \subset V$ the set of endpoints of the edge e. Since the entries in each row of $\Lambda(G)$ add up to zero, the determinant of this matrix vanishes and the determinant of the submatrix $\Lambda(G)^{(p)}$ obtained by deleting the pth row and column of $\Lambda(G)$ is independent of p. This gives a polynomial

$$\mathcal{D}_G = \det \Lambda(G)^{(p)} \tag{7}$$

in variables x_e which is called the Kirchhoff polynomial of G. The *Matrix-Tree Theorem* [16, Theorem VI.29][2, Theorem II.12] states that this polynomial is the generating function of spanning subtrees of the graph G (*i.e.* connected acyclic subgraphs of G with vertex set V). In other words, one has

$$\mathcal{D}_G = \sum_T x_T \ , \tag{8}$$

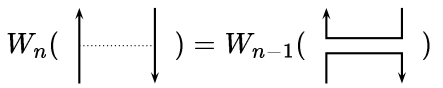
where the sum is taken over all the spanning subtrees in G.

In the case of the complete graph K_m, let us denote its Kirchhoff polynomial by $\mathcal{D}_m$. If we write $x_{ij} = x_{ji}$ for the indeterminate x_e corresponding to the edge $e = \{i,j\}$, then $\Lambda(K_m)$ becomes identified with the matrix Λ defined in (5). Formula (4) says that the coefficient $c_{m-1}(L)$ is the Kirchhoff polynomial $\mathcal{D}_m$ evaluated at $x_{ij} = \ell_{ij}(L)$, while Formula (6) says that $c_{m-1}(L)$ is the generating function of spanning trees of K_m, again evaluated at $x_{ij} = \ell_{ij}(L)$. Thus, the Matrix-Tree Theorem (8) applied to the complete graph K_m shows that these two formulas for $c_{m-1}(L)$ are equivalent.

3 An interpretation via finite type invariants

Formula (6) can also be proved directly by induction on the number of components of L [6, 8]. This proof can be formulated nicely in the language of finite type (Vassiliev) invariants, as follows. (See [13] for more details.)

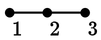
Recall that the coefficient c_n of the Alexander-Conway polynomial is a finite type invariant of degree n. Let us denote its weight system by W_n. It can be computed recursively by the formula


(9)

which follows immediately from the skein relation (2). For a chord diagram D on m circles with n chords, let D' be the result of smoothing of all chords by means of (9). If D' consists of just one circle, then

$$W_n(D) = W_0(D') = 1 \ .$$

Otherwise, one has $W_n(D) = W_0(D') = 0$.

To see how this relates to formula (6), note that smoothing of a chord cannot reduce the number of circles by more than one. Thus, for $W_n(D)$ to be non-zero we need at least $m-1$ chords. Moreover, the diagrams D with exactly $m-1$ chords satisfying $W_{m-1}(D) \neq 0$ must have the property that if each circle of D is shrinked to a point, the resulting graph formed by the chords is a tree. See Figure 3 for an example of a chord diagram D whose associated graph is the tree .

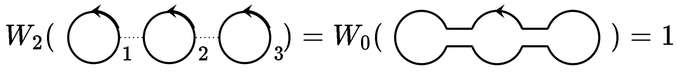


Figure 3: A degree 2 chord diagram D with $W_2(D) = 1$

In other words, the weight system W_{m-1} takes the value 1 on precisely those chord diagrams whose associated graph is a spanning tree on the complete graph K_m, and W_{m-1} is zero on all other chord diagrams.

This simple observation implies Formula (6), as follows. The linking number ℓ_{ij} is a finite type invariant of order 1 whose weight system is the linear form dual to the chord diagram having just one chord connecting the ith and jth circle. It follows that the right hand side of (6) (which is the spanning tree generating function of K_m evaluated in the ℓ_{ij}'s) is a finite type invariant of order $m-1$ whose weight system is equal to W_{m-1}. This proves Formula (6) on the level of weight systems. The proof can be completed using the fact that the Alexander-Conway polynomial is (almost) a canonical invariant [1] (see [13]).

4 Algebraically split links and Levine's formula

If the link L is *algebraically split*, *i.e.* all linking numbers ℓ_{ij} vanish, then not only $c_{m-1}(L) = 0$, but, as was proved by Traldi [14, 15] and Levine [10], the next $m-2$ coefficients of ∇_L also vanish

$$c_{m-1}(L) = c_m(L) = \ldots = c_{2m-3}(L) = 0.$$

For algebraically split oriented links, there exist well-defined integer-valued isotopy invariants $\mu_{ijk}(L)$ called the *Milnor triple linking numbers.* These invariants generalize ordinary linking numbers, but unlike ℓ_{ij}, the triple linking numbers are antisymmetric with respect to their indices, $\mu_{ijk}(L) = -\mu_{jik}(L) = \mu_{jki}(L)$. Thus, for an algebraically split link L with m components, we have $\binom{m}{3}$ triple linking numbers $\mu_{ijk}(L)$ corresponding to the different 3-component sublinks of L.

Levine [10] (see also Traldi [15, Theorem 8.2]) found an expression for the coefficient $c_{2m-2}(L)$ of ∇_L for an algebraically split m-component link in terms of triple Milnor numbers

$$c_{2m-2}(L) = \det \Lambda^{(p)}, \tag{10}$$

where $\Lambda = (\lambda_{ij})$ is an $m \times m$ skew-symmetric matrix with entries

$$\lambda_{ij} = \sum_k \mu_{ijk}(L), \tag{11}$$

and $\Lambda^{(p)}$, as before, is the result of removing the pth row and column.

For example, if $m = 3$, we have

$$\Lambda = \begin{pmatrix} 0 & \mu_{123}(L) & \mu_{132}(L) \\ \mu_{213}(L) & 0 & \mu_{231}(L) \\ \mu_{312}(L) & \mu_{321}(L) & 0 \end{pmatrix}$$

and

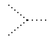
$$c_4(L) = \det \Lambda^{(3)} = -\mu_{123}(L)\mu_{213}(L) = \mu_{123}(L)^2 \ . \tag{12}$$

This formula (in the $m = 3$ case) goes back to Cochran [3, Theorem 5.1].

Similar to Formula (4) for $c_{m-1}(L)$, Levine's proof of Formula (10) uses the Seifert matrix definition (1) of ∇_L.

5 The Pfaffian-tree polynomial $\mathcal{P}_m$

Formula (10) is similar to the first determinantal expression (4). One of the main results of [12, 13] is that there is an analog of the tree sum formula (6) for algebraically split links. To state this result, we need to introduce another tree-generating polynomial analogous to the Kirchhoff polynomial.

Namely, instead of usual graphs whose edges can be thought of as segments joining pairs of points, we consider 3-*graphs* whose edges have three (distinct) vertices and can be visualized as triangles or Y-shaped objects  with the three vertices at their endpoints.

The notion of spanning trees on a 3-graph is defined in the natural way. A sub-3-graph T of a 3-graph G is *spanning* if its vertex set equals that of G, and it is a tree if its topological realization (*i.e.* the 1-complex obtained by gluing together Y-shaped objects  corresponding to the edges of T) is a tree (*i.e.* it is connected and simply connected). See Figure 4 for an example.

Similarly to the variables x_{ij} of $\mathcal{D}_m$, for each triple of distinct numbers $i, j, k \in \{1, 2, \dots, m\}$ we introduce variables y_{ijk} antisymmetric in i, j, k

$$y_{ijk} = -y_{jik} = y_{jki}, \text{ and } y_{iij} = 0 \ .$$

These variables correspond to edges $\{i, j, k\}$ of the *complete* 3*-graph* Γ_m with vertex set $\{1, \dots, m\}$.

As in the case of ordinary graphs, the correspondence

$$\text{variable } \ y_{ijk} \quad \mapsto \quad \text{edge } \{i, j, k\} \text{ of } \ \Gamma_m$$

assigns to each monomial in y_{ijk} a sub-3-graph of Γ_m.

The generating function of spanning trees in the complete 3-graph Γ_m is called the *Pfaffian-tree polynomial* $\mathcal{P}_m$ in [12, 13]. It is

$$\mathcal{P}_m = \sum_T y_T$$

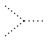
where the sum is over all spanning trees T of Γ_m, and y_T is, up to sign, just the product of the variables y_{ijk} over the edges of T. Because of the antisymmetry of the y_{ijk}'s, signs cannot be avoided here. In fact, the correspondence between monomials and sub-3-graphs of Γ_m is not one-to-one and a sub-3-graph determines a monomial only up to sign. But these signs can be fixed unambiguously, although we won't explain this here (see [12, 13]).

If m is even, then one has $\mathcal{P}_m = 0$, because there are no spanning trees in 3-graphs with even number of vertices. If m is odd, then $\mathcal{P}_m$ is a homogeneous polynomial of degree $(m-1)/2$ in the y_{ijk}'s. For example, one has

$$\mathcal{P}_3 = y_{123}$$

(the 3-graph Γ_3 with three vertices and one edge is itself a tree). If $m = 5$, we have

$$\mathcal{P}_5 = y_{123}\, y_{145} - y_{124}\, y_{135} + y_{125}\, y_{134} \ \pm \ \dots \ , \tag{13}$$

where the right-hand side is a sum of 15 similar terms corresponding to the 15 spanning trees of Γ_5. If we visualize the edges of Γ_m as Y-shaped objects  , then the spanning tree corresponding to the first term of (13) will look like on Figure 4.

We can now state one of the main results of [12, 13].

Theorem 5.1 [12, 13] *Let L be an algebraically split oriented link with m components. Then*

$$c_{2m-2}(L) = \big(\mathcal{P}_m(\mu_{ijk}(L))\big)^2 \ , \tag{14}$$

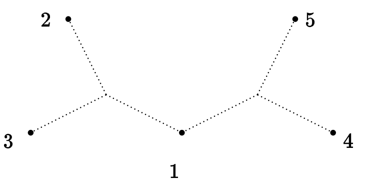


Figure 4: A spanning tree in the complete 3-graph Γ_5. It has two edges, $\{1,2,3\}$ and $\{1,4,5\}$, and contributes the term $y_{123}\,y_{145}$ to $\mathcal{P}_5$.

where $\mathcal{P}_m(\mu_{ijk}(L))$ means the result of evaluating the polynomial $\mathcal{P}_m$ at $y_{ijk} = \mu_{ijk}(L)$.

For $m=3$, we find again Cochran's formula (12), but for $m \geq 5$ our formula is new. For example, when $m=5$, we obtain that the first non-vanishing coefficient of $\nabla_L(z)$ for algebraically split links with 5 components is equal to

$$\begin{aligned} c_8(L) &= \mathcal{P}_5(\mu_{ijk}(L))^2 \\ &= \left(\mu_{123}(L)\mu_{145}(L) - \mu_{124}(L)\mu_{135}(L) + \mu_{125}(L)\mu_{134}(L) \ \pm\ \ldots\right)^2, \end{aligned}$$

where $\mathcal{P}_5(\mu_{ijk}(L))$ consists of 15 terms corresponding to the spanning trees of Γ_5.

6 A proof via the Pfaffian Matrix-Tree Theorem of [12]

The first proof of Theorem 5.1 was given in [12]. One of the main results of that paper is a *Pfaffian Matrix-Tree Theorem* which is the analog for 3-graphs of the classical Matrix-Tree Theorem (see Section 2). It expresses the generating function of spanning trees on a 3-graph G as the Pfaffian of a matrix $\Lambda(G)^{(p)}$ associated to G.

If G is the complete 3-graph Γ_m, this theorem says the following.

Theorem 6.1 [12] *The generating function of spanning trees on the complete 3-graph Γ_m is given by*

$$\mathcal{P}_m = (-1)^{p-1}\,\mathrm{Pf}(\Lambda(\Gamma_m)^{(p)}) \ ,$$

where $\Lambda(\Gamma_m)$ is the $m \times m$ skew-symmetric matrix with entries $\Lambda(\Gamma_m)_{ij} = \sum_k y_{ijk}$, and Pf *denotes the Pfaffian.*

Recall that the Pfaffian of a skew-symmetric matrix A is a polynomial in the coefficients of A such that

$$(\mathrm{Pf}\, A)^2 = \det A\ .$$

The matrix Λ defined in (11) is obtained from $\Lambda(\Gamma_m)$ by substituting the triple Milnor number $\mu_{ijk}(L)$ for the indeterminate y_{ijk}. Hence, Theorem 6.1 implies Theorem 5.1, since we know from Formula (10) that

$$c_{2m-2}(L) = \det \Lambda^{(p)} = (\mathrm{Pf}\, \Lambda^{(p)})^2\ .$$

For a definition of the matrix $\Lambda(G)$ and a statement of the Pfaffian Matrix-Tree Theorem in the case of general 3-graphs G, as well as for a proof, see [12].

7 A proof via finite type invariants [13]

As explained in Section 3, the appearance of spanning trees in Formula (6) for the coefficient c_{m-1} is very natural from the point of view of finite type invariants. A similar approach also leads to a proof of Theorem 5.1 via finite type invariants. This argument naturally generalizes to higher Milnor numbers. Let us briefly describe this approach (see [13] for details).

The connection between the Alexander-Conway polynomial and the Milnor numbers is established by studying their weight systems and then using the Kontsevich integral. In the dual language of the space of chord diagrams, the Milnor numbers correspond to the tree diagrams (see [5]) and the Alexander-Conway polynomial can be described in terms of certain trees and wheel diagrams (see [9] and [17]). However, for first non-vanishing terms, only tree diagrams matter, as the following Vanishing Lemma shows.

Proposition 7.1 (Vanishing Lemma [13]) *Let D be a degree-d diagram on $m \geq 2$ solid circles, such that D has no tree components of degree $\leq n-1$. Let W_d be the Alexander-Conway weight system. If $d \leq n(m-1)+1$, then $W_d(D) = 0$ unless D has exactly $m-1$ components, each of which is a tree of degree $\geq n$.*

This result is the generalization of the fact, shown in Section 3, that the Alexander-Conway weight system W_d for m-component links is always zero in degrees $d < m-1$. Indeed, this fact is the $n=1$ case of the Vanishing Lemma. However, the proof in the general case is more complicated. It uses properties of the Alexander-Conway weight system from [4] which are based on the connection between ∇ and the Lie superalgebra $gl(1|1)$.

In view of the relationship between Milnor numbers and tree diagrams studied in [5], the Vanishing Lemma implies in a rather straightforward way the following result, which was first proved by Traldi [15] and Levine [11] using quite different methods.

Proposition 7.2 [15, 11] *Let L be an oriented link such that all Milnor invariants of L of degree*[1] *$\leq n-1$ vanish. Then for the coefficients $c_i(L)$ of the Alexander-Conway polynomial $\nabla_L(z) = \sum_{i\geq 0} c_i(L) z^i$ we have*

(i) $c_i(L) = 0$ *for* $i < n(m-1)$,

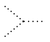
(ii) $c_{n(m-1)}(L)$ *is a homogeneous polynomial $F_m^{(n)}$ of degree $m-1$ in the Milnor numbers of L of degree n.*

Using the approach via Seifert surfaces, Levine [11] (see also Traldi [14, 15]) gives a formula for the polynomial $F_m^{(n)}$ as a determinant in the degree n Milnor numbers of L. For $n=1$ and $n=2$, this formula specializes to Formulas (4) and (10), respectively.

From the point of view of the Alexander-Conway weight system, however, one is lead to an expression for the polynomial $F_m^{(n)}$ in terms of the spanning tree polynomials $\mathcal{D}_m$ and $\mathcal{P}_m$. Indeed, as explained in Section 3, for $n=1$ the polynomial $F_m^{(1)}$ is easily recognized to be the spanning tree polynomial $\mathcal{D}_m$ in the linking numbers ℓ_{ij}. How does this generalize to higher n?

Consider for example the case $n=2$, that is, the case of algebraically split links. Proposition 7.2(ii) tells us that $c_{2m-2}(L)$ is a homogeneous polynomial $F_m^{(2)}$ of degree $m-1$ in triple Milnor numbers $\mu_{ijk}(L)$.

Theorem 7.3 [13] *The polynomial $F_m^{(2)}$ is equal to $\mathcal{P}_m^2$, the square of the Pfaffian-tree polynomial $\mathcal{P}_m$.*

Here is a sketch of the proof. Triple Milnor numbers are dual to Y-shaped diagrams  , and the coefficients of the polynomial $F_m^{(2)}$ can be computed from the Alexander-Conway weight system. For example, the coefficient of the monomial

$$\mu_{123}\,\mu_{145}\,\mu_{235}\,\mu_{345}$$

in $F_5^{(2)}$ is equal to the value of the Alexander-Conway weight system on the diagram in Figure 5.

[1]Here, the *degree* of a Milnor invariant is its Vassiliev degree, which is one less than its length (the number of its indices). For example, linking numbers have degree one, and triple linking numbers have degree two.

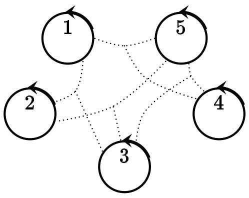


Figure 5: A diagram contributing to $F_5^{(2)}$

The coefficients of $F_m^{(2)}$ can be computed recursively by the relation in Figure 6, which follows from identities proved in [4].

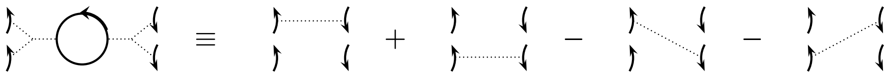


Figure 6: An identity modulo the Alexander-Conway relations

Indeed, the relation in Figure 6 together with the smoothing relation (9) allows one to reduce a diagram consisting of $m-1$ Y's on m solid circles to a linear combination of diagrams consisting of $m-3$ Y's on $m-2$ solid circles. (We are leaving out some details here.) This gives recursion formulas expressing $F_m^{(2)}$ in terms of $F_{m-2}^{(2)}$.

Let us state an example of such a recursion formula. It is convenient to write the antisymmetric triple Milnor number formally as an exterior product

$$\mu_{ijk} = v_i \wedge v_j \wedge v_k$$

and to consider $F_m^{(2)}$ as an expression in the indeterminates v_i:

$$F_m^{(2)} = F_m^{(2)}(v_1, v_2, \dots, v_m) \ .$$

Then the relation in Figure 6 implies for example that $F_m^{(2)}$ satisfies the recursion relation

$$\begin{aligned}\left[\frac{\partial^2 F_m^{(2)}}{\partial\mu_{123}\,\partial\mu_{145}}\right]_{v_1=0} &= F_{m-2}^{(2)}(v_3+v_4, v_2, v_5, \dots) + F_{m-2}^{(2)}(v_2+v_5, v_3, v_4, \dots) \\ &\quad - F_{m-2}^{(2)}(v_2+v_4, v_3, v_5, \dots) - F_{m-2}^{(2)}(v_3+v_5, v_2, v_4, \dots) \ .\end{aligned}$$

It turns out that this and similar recursion relations are enough to determine the polynomial $F_m^{(2)}$ for all m, once one knows it for $m=2$ and $m=3$. But

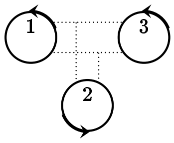


Figure 7: The only diagram contributing to $F_3^{(2)}$

it is easy to see that $F_2^{(2)} = 0$, while $F_3^{(2)} = \mu_{123}^2$, the only non-zero diagram contributing to $F_3^{(2)}$ being the diagram in Figure 7.

We now claim that this implies that $F_m^{(2)}$ is equal to $\mathcal{P}_m^2$, the square of the Pfaffian-tree polynomial $\mathcal{P}_m$. Clearly this is true for $m = 2$ and $m = 3$, and the proof consists of showing that $\mathcal{P}_m^2$ satisfies the same recursion relations as $F_m^{(2)}$. This uses the following two relations (15) and (16) satisfied by $\mathcal{P}_m$ itself, which follow more or less directly from the definition of $\mathcal{P}_m$ as the spanning tree generating function of the complete 3-graph Γ_m (see [12]).

The first is a *contraction-deletion* relation

$$\mathcal{P}_m = y_{123}\mathcal{P}_{m-2}(v_1 + v_2 + v_3, v_4, \dots, v_m) + [\mathcal{P}_m]_{y_{123}=0} \ . \tag{15}$$

Here, we have again written the indeterminate y_{ijk} as an exterior product $v_i \wedge v_j \wedge v_k$. The first term on the right hand side corresponds to the spanning trees on Γ_m containing the edge $\{1, 2, 3\}$, and the second term to those that do not. Note that a similar contraction-deletion relation exists for the classical spanning tree generating function for usual graphs.

The second relation is called *Three-term relation* in [12]. It states that

$$\mathcal{P}_m(v_2 + v_3, v_4, \dots) + \mathcal{P}_m(v_3 + v_4, v_2, \dots) + \mathcal{P}_m(v_2 + v_4, v_3, \dots) = 0 \tag{16}$$

where the dots stand for $v_5, v_6, \dots, v_{m+2}$.

The contraction-deletion relation and the three-term relation imply, by some algebraic manipulation, that $\mathcal{P}_m^2$ satisfies the same recursion relations as $F_m^{(2)}$.

Thus, although the recognition of the polynomial $F_m^{(2)}$ as being equal to the squared spanning tree polynomial $\mathcal{P}_m^2$ is not quite as immediate from the Alexander-Conway weight system as the identification of $F_m^{(1)}$ with the spanning tree polynomial $\mathcal{D}_m$ in Section 3, it is still quite natural. Indeed, it is based on the fact that the recursion relations have two natural interpretations, one coming from the weight system relations in Figure 6, and one coming from the contraction-deletion relation and the three-term relation for the Pfaffian-tree polynomial $\mathcal{P}_m$.

8 Some generalizations to higher Milnor numbers

The polynomial $F_m^{(n)}$ can be determined explicitly for higher values of n also. The answer can be expressed in terms of the spanning tree polynonmials $\mathcal{D}_m$ or $\mathcal{P}_m$. One obtains the following result for links with vanishing Milnor numbers up to a given degree.

Theorem 8.1 [13] *Let L be an oriented m-component link with vanishing Milnor numbers of degree $p < n$ and let $\nabla_L(z) = \sum_{i\geq 0} c_i(L)z^i$ be its Alexander-Conway polynomial. Then $c_i = 0$ for $i < n(m-1)$ and*

$$c_{n(m-1)}(L) = \begin{cases} \mathcal{D}_m(\ell_{ij}^{(n)}) , & \text{if } n \text{ is odd} \\ (\mathcal{P}_m(\mu_{ijk}^{(n)}))^2 , & \text{if } n \text{ is even,} \end{cases}$$

where $\ell_{ij}^{(n)}$ and $\mu_{ijk}^{(n)}$ are certain linear combinations of the Milnor numbers of L of degree n.

Note that if m is even then $\mathcal{P}_m = 0$ and so if n is also even, then the coefficient $c_{n(m-1)}(L)$ is always zero. In this case the Vanishing Lemma 7.1 leads to an expression for the next coefficient $c_{n(m-1)+1}(L)$ in terms of a certain polynomial $G_m^{(n)}$. (See Figure 8 for an example of a diagram contributing to $G_4^{(2)}$.)

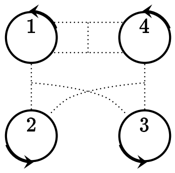


Figure 8: A diagram contributing to $G_4^{(2)}$

This polynomial can again be expressed via spanning trees (see [13]).

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Institut de Mathématiques de Jussieu, Université Paris VII
Case 7012, 75251 Paris Cedex 05, France

Email: `masbaum@math.jussieu.fr`

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