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Cubic complexes and finite type invariants

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Abstract Cubic complexes appear in the theory of finite type invariants so often that one can ascribe them to basic notions of the theory. In this paper we begin the exposition of finite type invariants from the 'cubic' point of view. Finite type invariants of knots and homology 3-spheres fit perfectly into this conception. In particular, we get a natural explanation why they behave like polynomials.

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1 Introduction

Polynomial functions play a fundamental role in mathematics. While they are usually defined on Euclidean spaces, linear and even quadratic maps are commonly considered for more general spaces, for example for abelian groups. This observation leads to a natural question: on which spaces can one define polynomial functions, and which structure is required for that? Certain hints pointing to a possible answer can be extracted from the theory of difference schemes on cubic lattices. For example, a continuous function is linear, if its forward second difference derivative at any point x_0 vanishes, i.e. $f(x_0+x_1+x_2)-f(x_0+x_1)-f(x_0+x_2)+f(x_0)=0$ for any x_1 and x_2. Similarly, quadratic functions are characterized by the identity $f(x_0+x_1+x_2+x_3)-f(x_0+x_1+x_2)-f(x_0+x_1+x_3)-f(x_0+x_2+x_3)+f(x_0+x_1)+f(x_0+x_2)+f(x_0+x_3)-f(x_0)\equiv 0$. It should be clear now how to generalize this to higher degrees:

Theorem 1.1 *A continuous function $f:\mathbb{R}^d\to\mathbb{R}$ is polynomial of degree less than n if and only if $\sum_\sigma(-1)^{|\sigma|}f(x_\sigma)=0$ for any $x_0,x_1,\dots,x_n\in\mathbb{R}^d$. Here the summation is over all $\sigma=(\sigma_1,\dots\sigma_{n+1})\in\{0,1\}^n$, $|\sigma|=\sum_i\sigma_i$ and $x_\sigma=x_0+\sum_i\sigma_ix_i$.*

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Note that σ are nothing more than the vertices of the standard n-cube $[0,1]^n$ and x_σ are their images under an affine map $\varphi : [0,1]^n \to \mathbb{R}^d$. Alternatively, if we replace the cube $[0,1]^n$ by $[-1,1]^n$, we get the following characterization of polynomial functions:

$$\sum_\sigma \sigma_1 \dots \sigma_n f(x_\sigma) \equiv 0,$$

where $\sigma = (\sigma_1, \dots \sigma_n) \in \{-1,1\}^n$ and $x_\sigma = x_0 + \sum_i \sigma_i x_i$. This formula corresponds to vanishing of n-th central difference derivatives of f at x_0.

Both formulas have the same meaning: the function is polynomial of degree less than n, if the alternating sum of its values on the vertices of every affine (possibly degenerate) n-cube in $\mathbb{R}^d$ vanishes. Therefore, one may expect that given a set X_n of "n-cubes" in a space W, we may define a notion of a polynomial function $W \to \mathbb{R}$. What are such n-cubes and how should they be related for different n? An appropriate object is well-known in topology under the name of a cubic complex.

While cubic complexes were used in topology for decades, their relation to polynomials became apparent only recently in the framework of finite type invariants. It turns out that cubic complexes underlie so many properties of finite type invariants, that one may ascribe them to basic notions of the theory. Probably M. Goussarov was one of the first to notice this relation and realize its importance in full generality; the second author learned this idea from him in 1996. This relation was also noticed and discussed in an interesting unpublished preprint [5]. In this paper we begin the exposition of the theory of finite type invariants from the "cubic" point of view. Finite type invariants of knots and homology 3-spheres fit perfectly into this conception.

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2 Cubic complexes

2.1 Semicubic complexes

Simplicial complexes, i.e., unions of simplices in $\mathbb{R}^d$, are widely used in topology. While somewhat less intuitively clear, a notion of a semisimplicial complex (see [14]) is used in situations when the number of simplices is infinite, especially locally. Semicubic complexes are similar to semisimplicial ones. The only difference is that instead of simplices we take cubes.

Definition 2.1 A *semicubic complex* $\bar{X}$ is a sequence of arbitrary sets and maps $\ldots \Rightarrow X_n \Rightarrow X_{n-1} \Rightarrow \ldots \Rightarrow X_0$. Here each arrow $X_n \Rightarrow X_{n-1}$ stands for $2n$ maps $\partial_i^\varepsilon \colon X_n \to X_{n-1}$, $1 \le i \le n$, $\varepsilon = \pm$, called the *boundary operators.* The boundary operators are required to commute after reordering: if $i > j$, then $\partial_j^{\varepsilon_1}\partial_i^{\varepsilon_2} = \partial_{i-1}^{\varepsilon_2}\partial_j^{\varepsilon_1}$. Elements of X_n are called *n-dimensional cubes*, the maps $\partial_i^\pm$ are *boundary operators.* A pair $\partial_i^-(x)$, $\partial_i^+(x) \in X_{n-1}$ form the i-th pair of the *opposite faces* of $x \in X_n$. One can consider also a semicubic complex $\bar{X}$ as a *semicubic structure* on the set X_0.

The above relations between maps mimic the usual identities for the standard cube $I^n = \{(x_1,\ldots,x_n) \in \mathbb{R}^n\colon -1 \le x_i \le 1\}$ with vertices $(\pm1,\pm1,\ldots,\pm1)$. Each time when we take an $(n-1)$-face, we identify it with the standard cube of dimension $n-1$ by renumbering the coordinate axes monotonicaly. See Fig. 1.

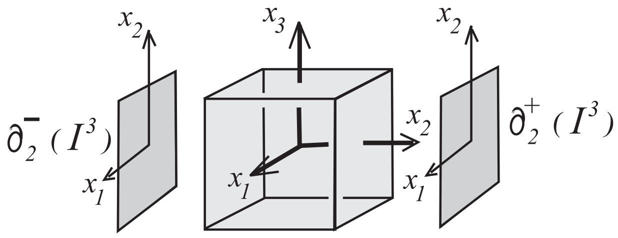

Figure 1: Boundary operators

It follows from the commutation relations that any superposition of n boundary operators taking X_n to X_0 coincides with a monotone superposition $\partial_1^{\varepsilon_1}\partial_2^{\varepsilon_2}\ldots\partial_n^{\varepsilon_n}$ (or with a superposition $\partial_1^{\varepsilon_n}\partial_1^{\varepsilon_{n-1}}\ldots\partial_1^{\varepsilon_1}$, whichever you like). Therefore, any cube $x \in X_n$ has 2^n 0-dimensional vertices, if we count them with multiplicities. The set of all vertices is naturally partitioned into two groups: we set a vertex $\partial_1^{\varepsilon_1}\partial_2^{\varepsilon_2}\ldots\partial_n^{\varepsilon_n}$ to be *positive* if $\varepsilon_1\varepsilon_2\ldots\varepsilon_n$ is $+$, and *negative* otherwise.

By a map of a semicubic complex X to a semicubic complex Y we mean a sequence $\bar{\psi} = \{\psi_n\}$ of maps $\psi_n\colon X_n \to Y_n, 0 \le n < \infty$, such that they commute with the boundary operators, i.e., $\partial_i^\varepsilon\psi_n = \psi_{n-1}\partial_i^\varepsilon$ for all $n \ge i \ge 1$ and $\varepsilon = \pm$.

Evidently, semicubic complexes and maps between them form a category.

2.2 Incidence complexes

To exclude the ordering of boundary operators (which is intrinsic to semicubic complexes but often is inessential), we define cubic complexes in more general terms of *incidence relations.*

Let A, B be an ordered pair of arbitrary sets. By an *incidence relation* between A and B we mean any subset R of $A \times B$. If $(a, b) \in R$ for some $a \in A, b \in B$, then we write $a \succ b$. The same notation $A \succ B$ will be used for indicating that A, B are equipped with a fixed incidence relation.

Definition 2.2 An *incidence complex* $\bar{X}$ is a sequence $\cdots \succ X_n \succ X_{n-1} \succ \ldots \succ X_0$ of arbitrary sets and incidence relations between neighboring sets. Elements of X_n are called *n-dimensional cells.* A cell $c_m \in X_m$ is a *face* of a cell $c_n \in X_n, 0 \leq m \leq n$, if there exist cells $c_{n-1}, \ldots, c_{m+1}$ such that $c_n \succ c_{n-1} \succ \ldots \succ c_m$.

By a map $\bar{\psi}\colon \bar{X} \to \bar{X}'$ between two incidence complexes we mean a sequence of maps $\psi_n\colon X_n \to X'_n$ such that $c_m \in X_m$ is a face of $c_n \in X_n$ implies $\psi_m(c_m) \in X'_m$ is a face of $\psi_n(c_n) \in X'_n$.

Example 2.3 The standard cube I^n and all its faces form an incidence complex $\bar{I}^n$ in an evident manner. Obviously, each face I^m of I^n is a cube such that the inclusion $I^m \subset I^n$ induces an inclusion of the corresponding incidence complexes.

Definition 2.4 Let $\bar{X}$ be an incidence complex. Then any map $\bar{\varphi}\colon \bar{I}^n \to \bar{X}$ (considered as a map between incidence complexes) is called a *cubic chart* for $\bar{X}$.

Evidently, for any face I^m of $I^n, 0 \leq m \leq n$, the restriction $\bar{\varphi}|_{\bar{I}^m}$ of any cubic chart $\bar{\varphi}\colon \bar{I}^n \to \bar{X}$ is also a cubic chart.

Definition 2.5 An incidence complex $\bar{X}$ equipped with a set Φ of cubic charts for $\bar{X}$ is called *cubic* if the following holds:

(1) For any cube $x \in X_n$ there is at least one chart $\bar{\varphi}\colon \bar{I}^n \to \bar{X}$ in Φ such that $\varphi_n(I^n) = x$. We will say that *φ covers x.*
(2) The restriction of any cubic chart $\bar{\varphi}\colon \bar{I}^n \to \bar{X}$ in Φ onto any face subcomplex $\bar{I}^m$ of $\bar{I}^n$ belongs to Φ.
(3) For any two charts $\bar{\varphi}_1\colon \bar{I}^n \to \bar{X}$ in Φ, $\bar{\varphi}_2\colon \bar{I}^n \to \bar{X}$ which cover the same cube $x \in X_n$ there exists a combinatorial isomorphism (called *a transient map*) $\bar{\psi}\colon \bar{I}^n \to \bar{I}^n$ such that $\bar{\varphi}_1 = \bar{\varphi}_2\bar{\psi}$.

2.3 Oriented cubic complexes

Of course, oriented cubic complexes are composed from oriented cubes, but our definition of orientation of a cube drastically differs from the usual one.

Definition 2.6 An *orientation* of a cube is an orientation of all its edges such that all the parallel edges of the cube are oriented coherently. It means that if two edges are related by a parallel translation, then so are their orientations.

To any vertex v of an oriented n-dimensional cube one can assign a sign $+$ or $-$, depending on the number of edges outgoing of v: $+$ if it is even, and $-$ if odd. Also, every pair of opposite $(n-1)$-dimensional faces of the cube consists of a *negative* and a *positive* face. We distinguish them by the behaviour of orthogonal edges: an $(n-1)$-dimensional face is negative if all the orthogonal edges go out of its vertices. If all the orthogonal edges are incoming, then the face is positive.

Note that the standard cube I^n is equipped with the canonical orientation induced by the orientations of the coordinate axes. See Fig. 2 for the signs of its vertices.

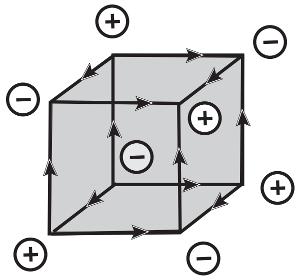

Figure 2: The signs of vertices. They will be used for taking alternative sums.

Remark 2.7 Obviously, any orientation-preserving isomorphism $I^n \to I^n$ preserves the signs of vertices and $(n-1)$-dimensional faces, and keeps fixed the *source vertex* (having only outgoing edges) as well as the *sink vertex* (having only incoming ones).

Definition 2.8 A cubic incidence complex $\bar{X}$ is *oriented*, if every two its charts covering the same n-cube are related by an orientation-preserving transient map $\bar{I}^n \to \bar{I}^n$.

It follows from Remark 2.7 that if a cubic incidence complex $\bar{X}$ is oriented, then all vertices and $(n-1)$-dimensional faces of any cube $x \in X_n$ have correctly defined signs. Of course, if x has less than 2^n vertices, then some of them have several signs. Source and sink vertices of x are also defined, as well as its positive and negative faces.

We say that a map $\bar{\psi}\colon \bar{X} \to \bar{X}'$ between two oriented cubic incidence complexes is orientation-preserving, if for any cube $x \in X_n$ of $\bar{X}$ there exist cubic charts $\bar{\varphi}\colon \bar{I}^n \to \bar{X}$ of $\bar{X}$ and $\bar{\varphi}'\colon \bar{I}^n \to \bar{X}'$ of $\bar{X}'$ and an orientation-preserving map $\bar{\psi}'\colon \bar{I}^n \to \bar{I}^n$ such that $\bar{\varphi}$ covers x and $\bar{\psi}\bar{\varphi} = \bar{\varphi}'\bar{\psi}'$. Of course, oriented cubic complexes and orientation-preserving maps form a category.

Remark 2.9 Any semicubic complex $\bar{Y}$ determines an oriented cubic complex $\bar{X}$: we simply forget about ordering of the boundary operators, preserving the information on positive and negative $(n-1)$-dimensional faces. Vice versa, any oriented cubic complex $\bar{X}$ determines a semicubic complex $\bar{Y}$ as follows. The set Y_n consists of all cubic maps $\bar{I}^n \to \bar{X}$ which are related to cubic charts of $\bar{X}$ by orientation-preserving transient maps. The boundary operators $\partial_i^{\pm}$ are defined by taking restrictions onto positive and negative i-th faces of I^n. Both constructions are functorial.

Example 2.10 Let W be a topological space. Then *singular cubes* in W, i.e., continuous maps $f\colon [-1,1]^n \to W$ of standard cubes into W, can be organized into a semicubic complex as well as into an oriented cubic complex $\bar{X} = \bar{X}(W)$ in an evident way: X_n is the set of all singular cubes of dimension n, and the boundary operators, respectively, incidence relations are given by taking restrictions onto the faces.

Other examples are discussed in Section 4. As we have seen in Remark 2.9, semicubic complexes and oriented cubic incidence complexes are related very closely. Further on we will use the semicubic complexes, but occasionally return to oriented ones. For brevity, in both cases we will call them "cubic complexes".

2.4 Cubes vs simplices

Cubes enjoy all good properties of simplices and have the following advantages:

(1) Each face of a cube has the opposite face;

(2) Two cubes with a common face can be glued together into a new cube (well, parallelepiped, but it does not matter);

(3) The direct product of two cubes of dimensions m and n is a cube of dimension $m+n$.

The above properties of cubes may be included as axioms. We will say that a cubic complex is *good*, if

(1) For each n and $i, 1 \leq i \leq n$, there is an involution $J_i : X_n \to X_n$, such that $\partial_j^\varepsilon J_i = J_{i-1}\partial_j^\varepsilon$ for $j < i$, $\partial_j^\varepsilon J_i = J_i \partial_j^\varepsilon$ for $j > i$, and $\partial_i^\varepsilon J_i = \partial_i^{-\varepsilon}$;

(2) For each $x, y \in X_n$ with $\partial_i^+(x) = \partial_i^-(y)$ there is $x \circ y \in X_n$ such that $\partial_i^+(x \circ y) = \partial_i^+(y)$ and $\partial_i^-(x \circ y) = \partial_i^-(x)$;

(3) For each $x \in X_n$, $y \in X_m$ with $\partial_1^- \dots \partial_n^-(x) = \partial_1^- \dots \partial_m^-(y)$ there is $xy \in X_{n+m}$ such that $\partial_1^- \dots \partial_n^-(xy) = y$ and $\partial_{n+1}^- \dots \partial_{n+m}^-(xy) = x$.

These axioms guarantee a rich algebraic structure (duality, composition, and product) on good cubic complexes. One can easily show that any cubic complex can be embedded into a good cubic complex. However, in all interesting examples which we presently know the cubic complexes are good. Note that axioms 2 and 3 descend to the level of oriented cubic complexes.

Another important advantage of cubes over simplices, as we will see below, is that they turn out to be extremely useful for a study of polynomial functions.

3 Finite type functions and n-equivalence

3.1 Functions on vertices

Just as for semisimplicial complexes, for a semicubic complex $\bar{X}$ one may consider *n-chains* $C_n(\bar{X})$, i.e., linear combinations of n-cubes with, say, rational coefficients. Any function f on X_n extends to a function on n-chains $C_n(\bar{X})$ by linearity. The boundary operators $\partial_i^\varepsilon : X_n \to X_{n-1}$, picked with an appropriate signs, may be combined into a differential $\sum_i (-1)^i (\partial_i^+ - \partial_i^-)$ on chains. This differential brings us to homology groups.

Having in mind a study of polynomial functions we, however, choose the signs differently. Namely, we define the operator $\partial \colon C_n(\bar{X}) \to C_{n-1}(\bar{X})$ by

$$\partial = \frac{1}{n} \sum_i (\partial_i^+ - \partial_i^-).$$

This operator does not satisfy $\partial^2 = 0$; of a special interest for us will be 0-chains $\partial^n(x)$, for any $x \in X_n$. The chain $\partial^n(x)$ contains all 2^n vertices of the

n-cube x with signs shown in Fig. 2:

$$\partial^n(x) = \sum_{\varepsilon_1,\dots,\varepsilon_n} \varepsilon_1 \dots \varepsilon_n \partial_1^{\varepsilon_1} \partial_2^{\varepsilon_2} \dots \partial_n^{\varepsilon_n}(x).$$

Remark 3.1 Note that the sum $\frac{1}{n}\sum_i(\partial_i^+ - \partial_i^-)$ does not depend on the order of the summands. It means that ∂ is determined for any oriented cubic complex.

3.2 Polynomials in $\mathbb{R}^d$

Let us start with the following simple example.

Let X_n consist of all affine maps $\varphi\colon I^n \to \mathbb{R}^d$ and the boundary operators $\partial_i^\pm$ assign to each φ its restrictions to the faces $\{x_i = \pm 1\}$. Then X_0 can be identified with R^d. We would like to interpret polynomiality of functions $f : \mathbb{R}^d \to \mathbb{R}$ in terms of their values on $\partial^n(x)$, $x \in X_n$. In these terms Theorem 1.1 may be restated as follows:

Theorem 3.2 *A continuous function $f : \mathbb{R}^d \to \mathbb{R}$ is polynomial of degree less than n, if and only if $f(\partial^n(x)) = 0$ for all $x \in X_n$.*

Recall that the cubes may be highly degenerate: to obtain a polynomial dependence on i-th coordinate, we consider affine maps with the image of I^n contained in a line parallel to the i-th coordinate axis.

3.3 Finite type functions

The example above motivates the following definition, which works for both semicubic and oriented cubic complexes.

Definition 3.3 Let $\bar{X}$ be a cubic complex, and A an abelian group. A function $f : X_0 \to A$ is of *finite type of degree less than* n, if for all $x \in X_n$ we have $f(\partial^n(x)) = 0$.

Remark 3.4 This definition looks more familiar in terms of the dual *cochain complex* $C^*(\bar{X})$ of linear functions on $C_*(\bar{X})$ with the coboundary operator $df(x) = f(\partial x)$ dual to ∂: a function $f \in C^0(\bar{X})$ is of degree less than n if $d^n(f) = 0$.

Note that the cubic complex $\bar{U}$ whose n-cubes are affine functions $f\colon I^n \to R$ has the following interesting property: any finite type function on U_0 is a polynomial. This observation explains why in many respects finite type functions behave analogously to polynomials [2].

Given a function $f : X_n \to A$ on n-cubes, sometimes one may extend it to all the cubes of dimension $< n$, including vertices, by the following descending method (first described in different terms by Vassiliev [20] for a cubic complex of knots). By a *jump* through an n-cube $x \in X_n$ we mean the transition from an $(n-1)$-face of x to the opposite one. The value $c = f(x)$ is the *price* of the jump. We add c, if we jump from a negative face to the opposite one, and subtract c, if the jump is from a positive face to the opposite negative one. See Fig. 3.

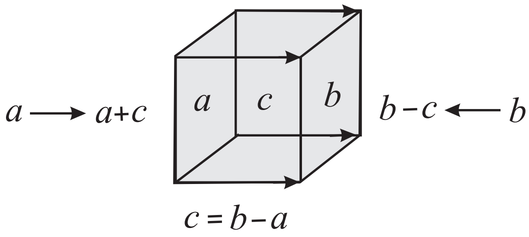

Figure 3: Jumping in positive direction we gain c, jumping back we loose c.

Theorem 3.5 *A function $f\colon X_n \to A$ extends to X_{n-1} if and only if the algebraic sum of the prices of any cyclic sequence of jumps is zero.*

Proof Call two $(n-1)$-cubes *parallel*, if one can pass from one to the other by jumps. In each equivalence class we choose a representative r, assign a variable, say, y to it, and set $f(r) = y$. Then we calculate the value of f for any other cube from the same equivalence class by paying $\pm f(x)$ for jumping across any cube $x \in X_n$. It is clear that we get a correctly defined function on X_{n-1} if and only if the above cyclic condition holds, see Fig. 4. □

One can look at the descending process as follows. To construct a function of degree n, we start with the zero function $X_{n+1} \to A$ and try to descend it successively to functions on $X_n, X_{n-1}, \ldots, X_0$. At each step we create a lot of new variables, and at each next step subject them to some linear homogeneous restrictions. If the system has a nonempty solution space, then we can descend further.

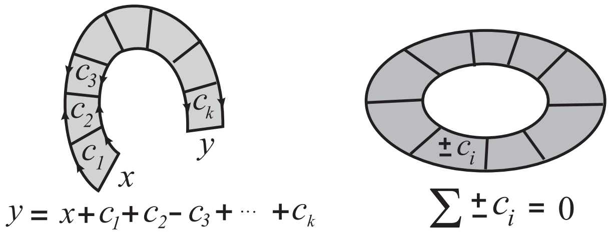

Figure 4: Prices of jumps and the cyclic condition

Remark 3.6 The number of equations at each step can be infinite. Sometimes it can be made finite by the following two tricks. First, it suffices to consider only basic cycles, which generate all cyclic sequences of n-cubes. Second, if one cyclic sequences of n-cubes consists of some faces of a cyclic sequence of $(n+1)$-cubes, then the sequence of the opposite faces is also cyclic and has the same sum of jumps. Therefore, it suffices to consider only one of these two chains.

The question when the solution space is always nonempty (i.e., when the descending process gives us nontrivial invariants) is usually very hard. For the case of real-valued finite type invariants of knots (see the next section), when the number of variables and equations at each step can be made finite, the affirmative answer follows from the Kontsevich theorem [16]. For the case of finite type invariants of homology spheres the number of variables is infinite, which makes this case especially difficult. See [7], where the authors managed to get rid off all but finitely many variables by borrowing additional relations from the lower levels.

3.4 N-equivalence and chord diagrams

For any cubic complex one may define a useful notion of n-equivalence:

Definition 3.7 Let $\bar{X}$ be a cubic complex. Elements $x, y \in X_k$ are *n-equivalent*, if there exists an $(n+k+1)$-chain $z \in C_{n+k+1}(\bar{X})$, such that $\partial^{n+1}(z) = y - x$. We denote $x \underset{n}{\sim} y$.

Remark 3.8 Our definition is somewhat different from the one used by M. Goussarov for links and 3-manifolds. He defines the notion of n-equivalence only on X_0 and uses a certain additional geometrical structure present in these cubic complexes. Roughly speaking, his relation is generated by $(n+1)$-cubes

z with $\partial^{n+1}(z) = (-1)^{n+1}(y - x)$, but only of a special type, namely such that $\partial_1^+ \dots \partial_{n+1}^+(z) = y$ and $\partial_1^{\varepsilon_1} \dots \partial_{n+1}^{\varepsilon_{n+1}}(z) = x$ otherwise. In other words, all the vertices of z should coincide with an 0-cube x, except the unique sink vertex y. One may show that x and y are n-equivalent in a sense of Goussarov, if and only if there exist two $(n+1)$-cubes z_x and z_y all vertices of which coincide, except for the sink vertex, which is x for z_x and y for z_y.

While *a priori* Goussarov's definition is finer, these definitions are equivalent for knots; also, if the theorems announced in [10, 11] are taken in the account, these definitions should coincide for string links and homology cylinders.

It is easy to see that:

Lemma 3.9 *n-equivalence is an equivalence relation.*

Remark 3.10 Given a product on the set X_0, often the classes of n-equivalence form a group. See [9, 10, 11] for groups of knots, string links, and homology spheres.

If a cubic complex has more than one class of 0-equivalence, one may also consider a restriction of the theory of finite type functions to some fixed class of 0-equivalence. There exists also a more general theory of partially defined finite type invariants, see [10].

The following simple theorem shows that functions of degree $\leq n$ are constant on classes of n-equivalence:

Theorem 3.11 *Let $\bar{X}$ be a cubic complex , A an abelian group, and let $f : X_0 \to A$ be a function of degree $\leq n$. Then for any $x, y \in X_0$ such that $x \underset{n}{\sim} y$ we have $f(x) = f(y)$.*

Proof By the definition of n-equivalence, there exists an $(n+1)$-chain z such that $\partial^{n+1}(z) = y - x$. But f is of degree $\leq n$, hence $f(\partial^{n+1}(z)) = 0$. It remains to notice that $f(x) - f(y) = f(x - y) = f(\partial^{n+1}(z))$. □

Remark 3.12 The opposite is not true: in general functions of finite type do not distinguish classes of n-equivalence, i.e., the equality $f(x) = f(y)$ for any f of degree $\leq n$ does not imply that $x \underset{n}{\sim} y$ (see [10] for the case of the link cubic complex).

In some important cases, however, e.g., for the case of the knot cubic complex , functions of finite type do distinguish classes of n-equivalence, see [19, 10, 11].

Goussarov [10, 11] also announced similar results for the cubic complexes of string links and homology cylinders, but we do not know what were his ideas on the subject and no proofs seem to be known.

A study of conditions under which functions of finite type distinguish classes of n-equivalence present an important problem.

It is also interesting to consider the quotients of n-equivalence classes by the relation of $(n+1)$-equivalence.

A closely related space $H_n(\bar{X})$ of *chord diagrams* of a cubic complex $\bar{X}$ is defined as $H_n(\bar{X}) = C_n/\partial(C_{n+1})$, where C_n are n-chains in $\bar{X}$. The *weight system* $\{f(h)|h \in H_n\}$ of a function f of degree n is defined by setting $f(h) = f(\partial^n z)$ for any representative $z \in h$. For any other representative $z + \partial z' \in h$, $z' \in X_{n+1}$ we have $f(\partial^n(z + \partial z')) = f(\partial^n(z)) + f(\partial^{n+1}(\partial^{n+1}(z')) = f(\partial^n(z))$, since f is of degree n.

4 Examples of cubic complexes

The examples in this section mimic the following definition of an n-dimensional cube x in $\mathbb{R}^d$ with sides parallel to the axes. Let x be a sequence of d symbols, $d-n$ of which are real numbers and n are $*$, together with n pairs (s_i^-, s_i^+) of real numbers. An $(n-1)$-dimensional face $\partial_i^{\pm} x$ is obtained by plugging $s_i^{\pm}$ in x instead of i-th $*$. To obtain a cube with sides which are not parallel to the axes, instead of i-th $*$ coordinate (varying from s_i^- to s_i^+ along i-th side), one may consider several $*$'s united in i-th group.

4.1 Cubic structure on a group

Let G be a group. We define a cubic structure on G in the following way. An n-cube $x \in X_n$ is a word $g_0(a_1 b_1)g_2 \dots (a_n b_n)g_n$ which contains n bracketed pairs $(a_i, b_i) \in G \times G$ separated by elements $g_0, \dots, g_n$ of G. Elements of X_0 are identified with G. The boundary operator ∂_i^- changes the i-th bracket (a_i, b_i) into the product $a_i b_i$, while the boundary operator ∂_i^+ changes the i-th bracket into $b_i a_i$. Since each ∂_i^+ differs from ∂_i^- by the transposition of a pair of elements a_i and b_i, it is easy to see that 0-equivalence classes coincide with the abelinization $G/[G,G]$ of G. More generally, n-equivalence classes coincide with the double cosets $G_n \backslash G / G_n$ of G by the $(n+1)$-th lower central subgroup G_n. Here the lower central subgroups G_n of a group G are generated by $[G, G_{n-1}]$, with $G_0 = G$.

Another, closely related, but somewhat more general cubic structure on G may be defined as follows. Set X_n to be a free product of $n+1$ copies G^0, G^1, ..., G^n of G (with $X_0 = G^0$ identified with G). The boundary operator ∂_i^- is the projection of G^i to 1. The boundary operator ∂_i^+ is the map identifying i-th copy G^i of G with G^0 (and renumbering the other copies of G by $j \to j-1$ for $j > i$). It is easy to see that linear functions $f : G \to A$ such that $f(1) = 0$ are exactly homomorphisms of G into A (since vanishing of f on a 2-cube gh with $g \in G^1$, $h \in G^2$ implies $f(ab) - f(a) - f(b) + f(1) = 0$).

Theorem 4.1 *Classes of n-equivalence of the above cubic structure on G coincide with the cosets $G_n \backslash G / G_n$ of G by the n-th lower central subgroup G_n.*

Proof Indeed, from any element of the form $y = x \cdot ghg^{-1}h^{-1}$ we may construct a 2-cube as follows. Write $xghg^{-1}h^{-1}$ as an element z of the free product of three copies G^0, G^1 and G^2 of G, with $x \in G^0$, $g, g^{-1} \in G^1$, and $h, h^{-1} \in G^2$. An application of ∂_1^- removes g and g^{-1}, leaving $xhh^{-1} = x$; similarly, an application of ∂_2^- removes h and h^{-1}, leaving $xgg^{-1} = x$. Thus $\partial_1^+\partial_2^+(xghg^{-1}h^{-1}) = y$ and $\partial_1^{\varepsilon_1}\partial_2^{\varepsilon_2}(xghg^{-1}h^{-1}) = x$ otherwise, so the 2-cube z satisfies $\partial^2(z) = y - x$. Hence x and y are 1-equivalent.

In a similar way, if $y = x \cdot c$ differs from x by an n-th commutator c, then we may write it as an element z in a free product of $n+2$ copies of G, such that again $\partial_i^-(z) = x$ for any $1 \le i \le n+1$, and hence $\partial^{n+1}(z) = y - x$ and $x \underset{n}{\sim} y$. The opposite direction is rather similar. □

The algebra H_n of chord diagrams is in this case a free product of n copies of $G/[G, G]$.

4.2 Cubic structures on trees, operads, and graphs

Let P be an operad. A cubic structure on P may be introduced by plugging some fixed $s_i^\pm$ in some $x \in P$. We will illustrate this idea on an example of the rooted tree operad.

Define $x \in X_n$ as a collection $(T, T_i^\pm, j)$, where T, $T_i^\pm$, $i = 1, \ldots, n$ are trees and $j = j_1, \ldots, j_n$ is a set of n leaves of T. The face $\partial_i^\pm x$ is obtained by attaching (grafting) the root of $T_i^\pm$ to the j_i-th leaf of T. See Figure 5.

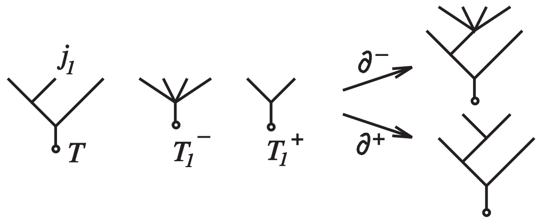

Figure 5: Grafting of trees

It would be interesting to study the theory of finite type functions on this cubic complex . The example below shows that the most basic functions of trees fit nicely in the theory of finite type functions.

Theorem 4.2 *The number of edges and the number of vertices of a given valence (as well as any of their functions) are degree one functions on the cubic complex of trees. The number of n-leaved trees of some fixed combinatorial type is a function of degree $\leq n/2$.*

A similar cubic structure may be defined on graphs (using insertions of some subgraphs $G_i^{\pm}$ in n vertices). In particular, using subgraphs which contain just one edge, we obtain the following cubic structure. Define an n-cube $x \in X_n$ to be a graph G with n marked vertices $v_1, \dots, v_n$, together with two fixed partitions s_i^+, s_i^- of edges incident to v_i. The boundary operator $\partial_i^- x$ (resp. $\partial_i^+ x$ acts by inserting in v_i a new edge, splitting it into two vertices in accordance with the partition s_i^- (resp. s_i^+) of the edges. Here are some simple examples of finite type functions on the cubic complex of graphs.

Theorem 4.3 *The number of edges, the number of vertices, and the number of loops (as well as any of their functions) are degree zero functions. The number of vertices of some fixed valence is a degree one function. The number of edges with the endpoints being vertices of some fixed valences is a degree two function. The number of n-vertices subgraphs of some fixed combinatorial type is a function of degree $\leq n/2$.*

One of the relations in the algebra of chord diagrams for this cubic complex is Stasheff's pentagon relation. We do not know whether there are any other relations. It may be also interesting to investigate the relation of this cubic complex to the graph cohomology.

4.3 Vassiliev knot complex

Let X_n consist of singular knots in R^3 having n ordered transversal double points. The boundary operators $\partial_i^{\pm}$ act by a positive, respectively, negative resolution of i-th double point, by shifting one string of the knot from the other. See Fig. 6. Here the resolution is positive, if the orientation of the fixed string, the orientation of the moving string, and the direction of the shift determine the positive orientation of R^3. The vertices of an n-cube thus may be identified with 2^n knots, obtained from an n-singular knot by all the resolutions of its double points.

Finite type functions for this complex are known as finite type invariants of knots (also known as Vassiliev or Vassiliev-Goussarov invariants), see [1] for an elementary introduction to the theory of Vassiliev invariants.

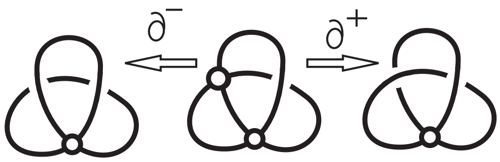

Figure 6: A 2-singular knot and resolutions of a double point

4.4 More general knot complex

Let an element of X_n be a knot together with a set of its n fixed modifications. More precisely, fix a set $H_1, \dots , H_n$ of disjoint handlebodies in $\mathbb{R}^3$. An n-cube is a tangle T in $\mathbb{R}^3 \smallsetminus \cup H_i$, together with a set of $2n$ tangles $T_i^{\pm} \subset H_i$, such that for any choice of signs $\varepsilon_1, \dots , \varepsilon_n$ the glued tangle $K_{\varepsilon_1,\dots,\varepsilon_n} = T \cup T_1^{\varepsilon_1} \cup \dots \cup T_n^{\varepsilon_n}$ is a knot. Here by a tangle in a manifold M with boundary we mean a 1-dimensional manifold, properly embedded in M. The boundary operators ∂_i^{ε} act by forgetting H_i and gluing T_i^{ε} to T. See Figure 7. The vertices may be thus identified with 2^n knots $K_{\varepsilon_1,\dots,\varepsilon_n}$.

This cubic structure on knots was introduced by Goussarov [12] under the name of "interdependent knot modifications". From the construction (restricting the modifications to crossing changes) it is clear that any finite type function in this theory is a Vassiliev knot invariant. As shown in [12], the opposite is also true, so the finite type functions for this cubic complex are exactly Vassiliev knot invariants (with a shifted grading); see [12, 3].

It would be interesting to construct similar cubic complexes for virtual knots and plane curves with cusps.

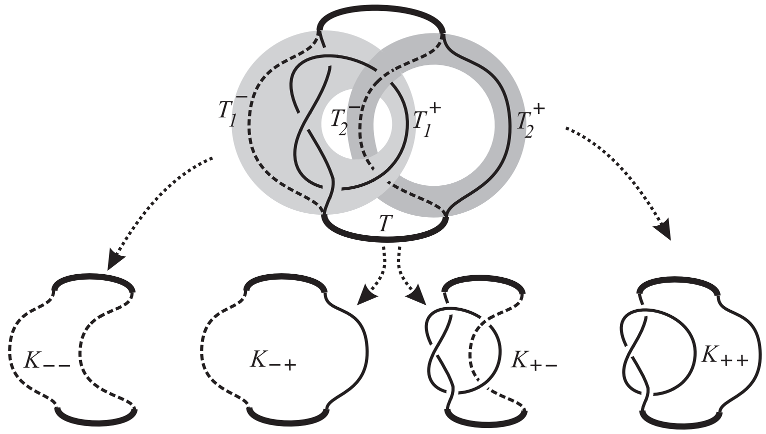

Figure 7: A tangle and its modifications

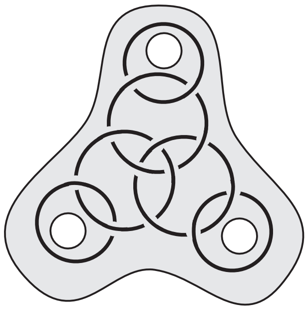

Figure 8: A Y-clasper

4.5 Borromean surgery in 3-manifolds

Let H be a standard genus 3 handlebody presented as a 3-ball with three index one handles attached to it. Consider a 6-component link $L \subset H$ consisting of the Borromean link B in the ball and three circles which run along the handles and are linked with the corresponding components of B, see Fig. 8. We equip L with the zero framing.

Definition 4.4 An n-component Y-*clasper* (or a Y-graph) in a 3-manifold M is a collection of n embeddings $h_i : H \to M$, $1 \leq i \leq n$, such that the images $h_i(H)$ are disjoint.

Let us construct a cubic complex as follows. X_n consists of all pairs $(M, \mathcal{Y})$, where M is a 3-manifold and $\mathcal{Y}$ an n-component Y-clasper in M. The pairs are considered up to homeomorphisms of pairs. The boundary operator ∂_i^- acts by forgetting the i-th component h_i of $\mathcal{Y}$ (the manifold M remains the same). The operator ∂_i^+ also removes h_i, but, in contrast to ∂_i^-, M is replaced by the new manifold obtained from M by the surgery along $h_i(L)$. Such a surgery is called *Borromean* (see [17]). It is known [17] that one 3-manifold may be obtained from another by Borromean surgeries (so belong to the same 0-equivalence class) if and only if they have the same homology and the linking pairing in the homology. In particular, M is a homology 3-sphere if and only if it can be obtained from S^3 by Borromean surgeries. It is easy to see that the sets X_n together with operators $\partial_i^{\pm}$ form a cubic complex .

Its finite type invariants are invariants of 3-manifolds in the sense of [11, 13]. One may also restrict it to homology 3-spheres.

4.6 Whitehead surgery in 3-manifolds

There are several other approaches to the finite type invariants of homology spheres. They are based on surgery on algebraically split links [18], boundary links [6], blinks [8], and so on. All of them fit into the conception of cubic complexes and turn out to be equivalent, see [7].

Here is a new approach, based on Whitehead surgery.

Definition 4.5 An n-component Y-clasper $h_i : H \to M$ in a 3-manifold M is a *Whitehead clasper*, if for each i one of the handles of $h_i(H)$ bounds a disc in $M \smallsetminus \cup_j h_j(H)$ and the framing of this handle is ± 1.

A surgery along a Whitehead clasper is called *Whitehead surgery*; it was introduced in [17] in different terms. From the results of [17] it follows that:

Theorem 4.6 *M is a homology 3-sphere if and only if it can be obtained from S^3 by surgery on a Whitehead clasper.*

We obtain the Whitehead cubic complex of homology 3-spheres by considering only Whitehead Y-claspers in homology spheres in the definition of the Borromean cubic complex above. It is easy to see that the sets X_n together with operators $\partial_i^{\pm}$ form a cubic complex.

From the construction it is clear that all finite type invariants of homology spheres of degree $< n$ in the sense of Borromean theory above are also finite type invariants of degree $< n$ in the sense of Whitehead surgery. We expect the opposite to be also true (probably up to a degree shift). Considering this theory for arbitrary 3-manifolds, we get, however, a theory which is finer than the theory based on the Borromean surgery. The reason is that the Whitehead surgery preserves the triple cup product in the homology, while the Borromean surgery in general does not. The study of this theory and its comparison with the theory introduced in [4] seem to be promising.

4.7 More on polynomiality

In many examples (see above) there exist several different cubic structures on the same space X_0. However, in all presently known non-trivial examples the set of finite type functions remains the same, up to a shift of grading. See [12] for the case of knots, and [7] for homology 3-spheres.

It would be quite interesting to understand better this "robustness" of finite type functions, and to formulate conditions which would imply such a uniqueness.

In conclusion we note that finite type invariants of knots and homology spheres are obtained by the same schema as polynomials (see Section 3.2). This observation explains once again their polynomial nature [2]. It is also worth noting a curious "secondary" polynomiality of finite type invariants: any finite type invariant is a polynomial in primitive finite type invariants.

Finally, let us remark that an oriented cubic complex (with n-cubes being certain commutative diagrams of vector spaces) appear in the construction of Khovanov's homology [15] for the Jones polynomial. It would be interesting to investigate Khovanov's construction from this point of view.

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