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Power sums and Homfly skein theory

Hugh R. Morton

Abstract The Murphy operators in the Hecke algebra H_n of type A are explicit commuting elements, whose symmetric functions are central in H_n. In [7] I defined geometrically a homomorphism from the Homfly skein $\mathcal{C}$ of the annulus to the centre of each algebra H_n, and found an element P_m in $\mathcal{C}$, independent of n, whose image, up to an explicit linear combination with the identity of H_n, is the mth power sum of the Murphy operators. The aim of this paper is to give simple geometric representatives for the elements P_m, and to discuss their role in a similar construction for central elements of an extended family of algebras $H_{n,p}$.

AMS Classification 57M25; 20C08

Keywords Homfly skein theory, Murphy operators, power sums, supersymmetric polynomials, annulus, Hecke algebras

Introduction

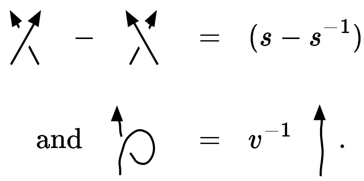
This article is a combination of material presented at a conference in Siegen in January 2001 and at the Kyoto low-dimensional topology workshop in September 2001. I am grateful to the organisers of both these meetings for the opportunity to discuss this work. I must also thank the Kyoto organisers and RIMS for their hospitality during the production of this account.

Since much of the material in the talks is already contained in [7] I refer readers there for many of the details, while giving a brief review here. In this paper I concentrate on some of the related results and developments which did not appear in [7]. In particular I include a direct skein theory proof in theorem 3 of a result originally due to Aiston [1] giving an unexpectedly simple geometric representation for a sequence of elements $\{P_m\}$ in the skein of the annulus as sums of a small number of closed braids. The elements P_m were interpreted in [1] as the result of applying Adams operations to the core curve of the annulus, when the skein of the annulus is viewed as the representation ring of $sl(N)$ for large N. The same elements P_m are shown in [7] to give rise to the power sums of the Murphy operators in the centre of the Hecke algebras H_n.

I extend the ideas to algebras $H_{n,p}$ based on tangles with strings allowed in both directions at top and bottom, where a similar uniform description for much, in fact probably all, of the centre, can be given. This suggests analogues of the Murphy operators in these cases, which now come in two sets $\{T(j)\}, j = 1, \ldots, n$ and $\{U(k)\}, k = 1, \ldots, p$, of commuting elements, and provide central elements as the supersymmetric functions of these sets of elements in the algebra $H_{n,p}$.

1 Framed Homfly skeins

The skein theory setting described more fully in [7] is based on the framed Homfly skein relations,



The framed Homfly skein $\mathcal{S}(F)$ of a planar surface F, with some designated input and output boundary points, is defined as linear combinations of oriented tangles in F, modulo Reidemeister moves II and III and these two local skein relations. The coefficient ring can be taken as $\Lambda = \mathbf{Z}[v^{\pm 1}, s^{\pm 1}]$ with powers of $s^k - s^{-k}$ in the denominators.

When F is a rectangle with n inputs at the bottom and n outputs at the top the skein $R_n^n(v,s) = \mathcal{S}(F)$ of n-tangles is an algebra under composition of tangles, isomorphic to the Hecke algebra $H_n(z)$, with $z = s - s^{-1}$ and coefficients extended to Λ. This algebra has a presentation with generators $\{\sigma_i\}, i = 1, \ldots, n-1$ corresponding to Artin's elementary braids, which satisfy the braid relations and the quadratic relations $\sigma_i^2 = z\sigma_i + 1$. The braids $T(j), j = 1, \ldots, n$, shown in figure 1 make up a set of commuting elements in H_n. These are shown by Ram [8] to be the Murphy operators of Dipper and James [2], up to linear combination with the identity of H_n.

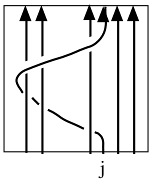
$T(j) \;=$ 

Figure 1

Their properties are discussed further in [7], where the element $T^{(n)}$ in figure 2 is shown to represent their sum, again up to linear combination with the identity.

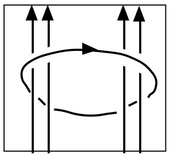
$T^{(n)} \;=$ 

Figure 2

In the same paper the power sums $\sum T(j)^m$ are presented as skein elements which clearly belong to the centre of H_n, in terms of an element P_m in the skein $\mathcal{C}$ of the annulus. This leads to skein theory presentations for any symmetric function of the Murphy operators as central elements of H_n, and gives a pictorial view of the result of Dipper and James which identifies the centre of H_n for generic parameter with these symmetric functions.

Before giving the skein presentation for P_m as a sum of m closed braids in the annulus I discuss briefly the construction of central elements in H_n, and in some extended variants of these algebras.

Definition Write $H_{n,p}$ for the skein $\mathcal{S}(F)$ where F is the rectangle with n outputs and p inputs at the top, and matching inputs and outputs at the bottom as in figure 3.

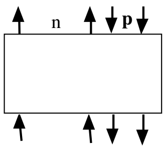
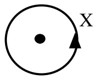
$F \;=$ 

Figure 3

There is a natural algebra structure on $H_{n,p}$ induced by placing oriented tangles one above the other. When $p = 0$ we have the Hecke algebra $H_n = H_{n,0}$. The resulting algebra $H_{n,p}$ has been studied by Kosuda and Murakami, [4], in the context of $sl(N)_q$ endomorphisms of the module $V^{\otimes n} \otimes \overline{V}^{\otimes p}$, where V is the fundamental N-dimensional module. Hadji [3] has also described an explicit skein-theoretic basis for it; while there is a linear isomorphism of $H_{n,p}$ with $H_{(n+p)}$ this is not in general an algebra isomorphism.

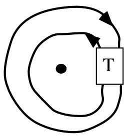
2 The annulus

The skeins $H_{n,p}$ are closely related to the skein $\mathcal{C} = \mathcal{S}(F)$ where F is the annulus. An element $X \in \mathcal{C}$, which is simply a linear combination of diagrams in the annulus, modulo the skein relations, is indicated schematically as



.

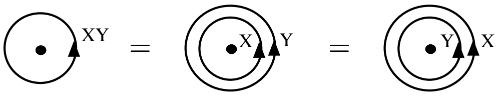
The closure map $H_{n,p} \to \mathcal{C}$, induced by taking a tangle T to its closure $\hat{T}$ in the annulus, is defined by

$\hat{T} =$ 

.

This is a Λ-linear map, whose image we call $\mathcal{C}_{n,p}$. Every diagram in the annulus lies in some $\mathcal{C}_{n,p}$.

The skein $\mathcal{C}$ has a product induced by placing one annulus outside another,

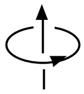


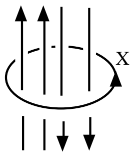
.

Under this product $\mathcal{C}$ becomes a commutative algebra, since diagrams in the inner annulus can be moved across the outer annulus using Reidemeister moves II and III.

The evaluation map $\langle\ \rangle : \mathcal{C} \to \Lambda$ is defined by setting $\langle X \rangle$ to be the framed Homfly polynomial of X, regarded as a linear combination of diagrams in the plane. When the Homfly polynomial is normalised to take the value 1 on the empty diagram, and hence the value $\delta = \dfrac{v^{-1} - v}{s - s^{-1}}$ on the zero-framed unknot, the map $\langle\ \rangle$ is a multiplicative homomorphism.

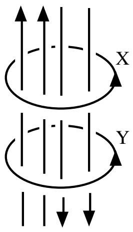
3 Central elements of $H_{n,p}$

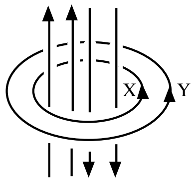
There is an easily defined algebra homomorphism $\psi_{n,p}$ from $\mathcal{C}$ to the centre of each algebra $H_{n,p}$, induced from $D =$  by placing $X \in \mathcal{C}$ around the circle and the identity of $H_{n,p}$ on the arc, to get

$\psi_{n,p}(X) =$ 

$\in H_{n,p}.$

It is clear that

$\psi_{n,p}(XY) =$ 

$=$ 

$= \psi_{n,p}(X)\psi_{n,p}(Y),$

and that the elements $\psi_{n,p}(X)$ all lie in the centre of $H_{n,p}$. It is shown in [7] that the image of $\psi_{n,0}$, called ψ_n there, consists of all symmetric polynomials in the Murphy operators in H_n and so, by [2], makes up the whole of the centre of H_n in the generic case.

It is natural to suspect that the same is true for $H_{n,p}$.

Conjecture *The image of $\psi_{n,p}$ is the whole centre of $H_{n,p}$.*

Remark I don't have an immediate skein theory argument showing that central elements can always be written as $\psi(X)$, even in the case of H_n. I suspect that the best way would be to show that the restriction of $\psi_{n,p}$ to $\mathcal{C}_{n,p}$ followed by closure is an isomorphism, and couple this with an upper bound on the dimension of the centre, although I don't know of an algebraic determination of the centre in the general case $n, p > 0$.

In [7] the element $P_m \in \mathcal{C}$ is constructed, by manipulating generating functions, as a polynomial in elements $\{h_i\}$, while each h_i is itself initially a linear combination of $i!$ closed braids. It satisfies

$$\psi_{n,0}(P_m - \langle P_m\rangle) = (s^m - s^{-m})v^{-m}\sum_{j=1}^{n} T(j)^m,$$

for every n. This result can be generalised to $H_{n,p}$ as follows.

Theorem 1 *The central element $\psi_{n,p}(P_m)$ of $H_{n,p}$ can be written, up to linear combination with the identity, as the power sum difference*

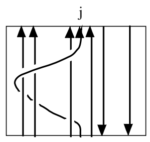
$$v^{-m}\sum_{j=1}^{n} T(j)^m - v^m\sum_{k=1}^{p} U(k)^m,$$

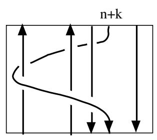
for some commuting elements $\{T(j)\}\cup\{U(k)\}$.

Proof Apply the arguments of [7] using the skein $\mathcal{A}$ of the annulus with a boundary input and output to show that

$$\psi_{n,p}(P_m) - \langle P_m\rangle \mathrm{Id} = (s^m - s^{-m})\left(v^{-m}\sum_{j=1}^{n} T(j)^m - v^m\sum_{k=1}^{p} U(k)^m\right),$$

where

$T(j) =$ 

, $U(k) =$ 

.

□

Remark Every central element of the form $\psi_{n,p}(X)$ is a polynomial in the elements $\psi_{n,p}(P_m)$. Stembridge [9] defines a supersymmetric polynomial in two sets of commuting variables $\{x_i\}$ and $\{y_j\}$ to be a polynomial which is symmetric in $\{x_i\}$ and $\{y_j\}$ separately, and which is independent of t after making the substitution $x_1 = y_1 = t$. He then characterises them as polynomials in the power sum differences $\sum x_i^m - \sum y_j^m$. The central elements $\psi_{n,p}(X)$ can thus be described as supersymmetric polynomials in the sets of commuting elements $\{v^{-1}T(j)\}$ and $\{vU(k)\}$.

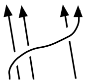
These elements might be considered as a set of Murphy operators in the general setting. There is certainly some measure of choice for the set of elements; conjugating all of them by a fixed element will not alter their supersymmetric polynomials, while further choices are also possible.

4 Geometric representatives of P_m

The main result of this paper is to give a geometrically simple representative for P_m as a sum of m closed braids.

Write $A_m \in \mathcal{C}$ for the closure of the m-braid

$$\sigma_{m-1}\cdots\sigma_2\sigma_1 \;=$$



.

The central element $T^{(n)} \in H_n$ can be written as $T^{(n)} = \psi_{n,0}(A_1)$, so we can take P_1 to be the core curve A_1 of the annulus.

The construction of the elements P_m in [7] makes use of the sequence of elements $h_i \in \mathcal{C}_{i,0}$ which arise as the closure of one of the two most basic idempotents in H_i. The properties of these elements which will be used here are their relations to the elements A_m. These are expressed compactly in terms of formal power series with coefficients in the ring $\mathcal{C}$ in theorem 2.

Theorem 2 (Morton)

$$A(t) = \frac{H(st)}{H(s^{-1}t)},$$

where

$$H(t) = 1 + \sum_{n=1}^{\infty} h_n t^n$$

and

$$A(t) = 1 + (s - s^{-1}) \sum_{m=1}^{\infty} A_m t^m.$$

Proof This is given in [7], using simple skein properties of the elements h_n and the skein $\mathcal{A}$ of the annulus with a single input on one boundary curve and a matching output on the other. □

The elements P_m are defined in [7] by the formula

$$\sum_{m=1}^{\infty} \frac{P_m}{m} t^m = \ln(H(t)).$$

Now every skein admits a *mirror map*, $\overline{} : \mathcal{S}(F) \to \mathcal{S}(F)$ induced by switching all crossings in a tangle, coupled with inverting v and s in Λ. When this is applied to the series $A(t)$ the result is the series

$$\overline{A}(t) = 1 - z \sum_{m=1}^{\infty} \overline{A}_m t^m,$$

where $\overline{A}_m$ is the mirror image of the closed braid A_m, and $z = s - s^{-1}$.

The series $H(t)$ is invariant under the mirror map, as shown in [7], and hence so is P_m, although I will not need these facts here. The simple geometric representative for P_m in theorem 3 can be thought of as 'almost' A_m, but averaged in a way that ensures mirror symmetry.

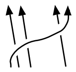
Definition Write $A_{i,j}$ for the closure of the braid

$$\sigma_{i+j}^{-1}\sigma_{i+j-1}^{-1}\dots\sigma_{i+1}^{-1}\sigma_i\dots\sigma_2\sigma_1 =$$ 

on $i+j+1$ strings.

Then $A_{i,j}$ is the closed braid with i positive and j negative crossings which results from switching the last j crossings of A_{i+j+1}, with $A_m = A_{m-1,0}$ and $\overline{A}_m = A_{0,m-1}$.

Theorem 3 (Aiston) *The element P_m is a multiple of the sum of all m of the closed braids $A_{i,j}$ which have m strings. Explicitly,*

$$[m]P_m = \sum_{\substack{i=0\\ i+j=m}}^{m-1} A_{i,j-1} = \overline{A}_m + \dots + A_m =$$  $+\cdots+$  $+\cdots+$  ,

where $[m] = \dfrac{s^m - s^{-m}}{s - s^{-1}}$.

While Aiston's proof required a diversion through results about $sl(N)_q$ representations, the proof here uses only the skein-based result of theorem 2 and one further simple skein theory lemma.

Lemma 4 *For all $i,j > 0$ we have*

$$A_{i,j-1} - A_{i-1,j} = zA_{i-1,0}A_{0,j-1}\ (= zA_i\overline{A}_j).$$

Proof Apply the skein relation to the ith positive crossing in $A_{i,j-1}$. Switching this crossing gives $A_{i-1,j}$ while smoothing it gives the product $A_i\overline{A}_j$. $\square$

Corollary 5

$$A(t)\overline{A}(t) = 1.$$

Proof We have

$$z\sum_{\substack{i=1\\ i+j=m}}^{m-1} A_i\overline{A}_j = A_{m-1,0} - A_{0,m-1} = A_m - \overline{A}_m, m>1,$$

by lemma 4. The coefficient of t^m in $A(t)\overline{A}(t)$ is

$$zA_m - z\overline{A}_m - z^2\sum_{i+j=m} A_i\overline{A}_j = 0, m>0.$$

□

Remark This result appears in [7]; the proof here does not need theorem 2 or properties of the mirror map.

Proof of theorem 3 Write

$$\Pi_m = \sum_{\substack{i=0\\ i+j=m}}^{m-1} A_{i,j-1}.$$

Then

$$\sum_{\substack{i=1\\ i+j=m}}^{m-1} i(A_{i,j-1} - A_{i-1,j}) + \Pi_m = mA_m,$$

giving

$$\Pi_m = mA_m - z\sum_{\substack{i=1\\ i+j=m}}^{m-1} iA_i\overline{A}_j.$$

This is the coefficient of t^m in $\left(\sum_{i=1}^{\infty} iA_it^i\right)\overline{A}(t)$. Now

$$z\sum_{i=1}^{\infty} iA_it^i = t\frac{d}{dt}(A(t)).$$

We then have

$$\begin{aligned} z\sum_{m=1}^{\infty}\Pi_m t^{m-1} &= \frac{d}{dt}(A(t))\overline{A}(t) \\ &= \frac{d}{dt}\ln(A(t)), \qquad (1) \end{aligned}$$

since $\overline{A}(t) = A(t)^{-1}$ by corollary 5. Now

$$\begin{aligned} \ln A(t) &= \ln H(st) - \ln H(s^{-1}t), \\ &= \sum_{m=1}^{\infty}(s^m - s^{-m})\frac{P_m}{m}t^m, \qquad (2) \end{aligned}$$

by theorem 2 and the definition of P_m. Comparing the coefficients of t^{m-1} in (1), using(2), gives

$$(s-s^{-1})\Pi_m = (s^m - s^{-m})P_m,$$

and hence the result. $\square$

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Department of Mathematical Sciences, University of Liverpool
Peach St, Liverpool, L69 7ZL, UK

Email: morton@liv.ac.uk

URL: http://www.liv.ac.uk/\char'176su14

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