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Generalisations of Rozansky-Witten invariants

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Abstract We survey briefly the definition of the Rozansky-Witten invariants, and review their relevance to the study of compact hyperkähler manifolds. We consider how various generalisations of the invariants might prove useful for the study of non-compact hyperkähler manifolds, of quaternionic-Kähler manifolds, and of relations between hyperkähler manifolds and Lie algebras. The paper concludes with a list of additional problems.

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1 Introduction

In 1996 Rozansky and Witten [24] discovered that a *hyperkähler manifold* X^{4k} gives rise to a *topological quantum field theory* Z_X in dimension $(2+1)$. In particular, to each closed oriented 3-manifold M is associated a *partition function* $Z_X(M)$, a complex-valued topological invariant of M, which is defined by means of a path integral over the space of maps $M \to X$.

This path-integral definition is not mathematically rigorous, but Rozansky and Witten were able to give a rigorous formulation of the (exact) *perturbative expansion* of $Z_X(M)$. This is an expansion in terms of Feynman diagrams, which in this theory are simply trivalent graphs Γ:

$$Z_X(M) = \sum_{\Gamma} b_{\Gamma}(X) I_{\Gamma}(M). \tag{1}$$

The expansion decouples the partition function into quantities $b_\Gamma(X)$ depending only on X, and quantities $I_\Gamma(M)$ depending only on M. Habegger and Thompson [10] showed that when $b_1(M) \geq 1$, the numbers $I_\Gamma(M)$ are essentially coefficients of the *Le-Murakami-Ohtsuki invariant* [19] of M; this is believed to hold also when $b_1(M)$ is zero. Therefore the new ingredient in

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Rozansky and Witten's construction is the *weight system* $b_\Gamma(X)$, a complex-valued function on trivalent graphs, constructed from the geometry of X.

In this paper we will investigate the weight systems primarily *as geometers*, with a view to using them to study the manifold X. Although it is also very interesting to study the invariants from the point of view of Vassiliev theory [22], or of TQFT [23], we will not pursue these ideas here.

Let us review the definition of the Rozansky-Witten weight systems, assuming for the moment that X is *compact*. Recall that a *hyperkähler manifold* X^{4k} is a Riemannian manifold admitting three orthogonal parallel complex structures I, J, K which satisfy the quaternion identities. Its holonomy group is contained in $\mathrm{Sp}(k)$, and therefore the frame bundle can be reduced to a principal $\mathrm{Sp}(k)$-bundle. The complexification of the tangent bundle therefore splits as a tensor product

$$TX \otimes_{\mathbb{R}} \mathbb{C} = E \otimes_{\mathbb{C}} H \tag{2}$$

where E is the bundle induced by the standard representation of $\mathrm{Sp}(k)$ on $\mathbb{H}^k$ ($\mathbb{H}$ denoting the quaternions) and H is a trivialisable complex rank-two bundle. (This notation is consistent with Salamon's [25] but differs from that of Rozansky and Witten [24].)

The Riemann curvature tensor reduces to a section Ψ of $\mathrm{Sym}^4 E$, and the metric induces skew-symmetric two-forms on E and H. Suppose we are given a trivalent graph Γ with $2k$ vertices. We place a copy of Ψ at each vertex, labelling the outgoing edges by the indices of Ψ (with one extra index at each vertex). Then we tensor these copies of Ψ together, contract indices along edges using the skew two-form on E, and project the result from $\otimes^{2k}E$ to $\Lambda^{2k}E$. The latter bundle has a trivialisation given by the kth power of the skew two-form, and hence we obtain a well-defined scalar field which we can integrate over X. Up to normalization (including orientation conventions to handle the signs) this gives $b_\Gamma(X)$, an invariant of the hyperkähler structure on X which we will refer to as a *Rozansky-Witten invariant* of X.

Rozansky and Witten showed that $b_\Gamma(X)$ depends on the graph Γ only up to the *anti-symmetry* and *IHX relations*, and hence determines a weight system on the *graph homology* space of rational linear combinations of equivalence classes of graphs — that is, the space of Jacobi diagrams typically denoted elsewhere in these proceedings by $\mathcal{A}(\emptyset)$. In particular, this implies that $Z_X(M)$ really is a topological invariant of M, for the terms $I_\Gamma(M)$ actually depend on a choice of metric on M and vary as the metric is deformed, but the variations in the sum (1) cancel out.

If we choose a complex structure on X which is compatible with the hyperkähler metric, then the resulting complex manifold has a holomorphic symplectic form $\omega \in \mathrm{H}^0(X, \Lambda^2 T^*)$, where T^* is the holomorphic cotangent bundle. Kontsevich [15] gave a reinterpretation of the weights $b_\Gamma(X)$ using characteristic classes of the resulting complex-valued symplectic foliation. At the same time Kapranov [13] recognised that from this point of view, the curvature tensor can be viewed as a Dolbeault representative of the *Atiyah class* α_T of the holomorphic tangent bundle. The Atiyah class α_F of a holomorphic bundle F is the extension class of the short exact sequence

$$0 \to T^* \otimes F \to \mathrm{J}^1(F) \to F \to 0$$

where $\mathrm{J}^1(F)$ is the first jet bundle of F (see [2]). Thus the Atiyah class α_F is an element of $\mathrm{Ext}^1(F, T^* \otimes F) = \mathrm{H}^1(X, T^* \otimes \mathrm{End}(F))$; it vanishes if and only if the sequence splits, and hence α_F is the obstruction to the existence of a global holomorphic connection on F. We can define the weights $b_\Gamma(X)$ using the Atiyah class instead of the curvature tensor, with the advantage that we no longer need to know the metric on X (usually we know only of the *existence* of a hyperkähler metric, by Yau's Theorem). This approach required us to choose a complex structure on X, but we already know from the original definition that $b_\Gamma(X)$ is independent of this choice, which amounts to a choice of holomorphic trivialisation of the bundle H, and gives an identification of E with the holomorphic tangent bundle T. Consequences of this approach are that Rozansky-Witten invariants are invariant under *deformation* of the hyperkähler metric on X, and in fact can be defined more generally for *holomorphic symplectic manifolds*, which are hyperkähler if and only if they are Kähler.

The first extensive calculations of Rozansky-Witten invariants were carried out by the second author in [26]. Let Θ^k be the (disconnected) graph given by k copies of the *theta graph* Θ. If X is an *irreducible* hyperkähler manifold (i.e. no covering space of X splits into a product) then $b_{\Theta^k}(X)$ can be written

$$\frac{2^k k^k}{k!}\frac{(\int_X c_2[\omega\bar{\omega}]^{k-1})^k}{(\int_X[\omega\bar{\omega}]^k)^{k-1}} = \frac{k!}{(4\pi^2 k)^k}\frac{\|R\|^{2k}}{(\mathrm{vol}(X))^{k-1}} \tag{3}$$

where c_2 is the second Chern class of T, $[\omega\bar{\omega}]$ is the cohomology class represented by $\omega\bar{\omega}$, $\|R\|$ is the L^2-norm of the curvature, and $\mathrm{vol}(X)$ is the volume. (This is equation (10) in [11].) Examples of irreducible hyperkähler manifolds of dimension $4k$ are the *Hilbert scheme* of k points on a K3 surface and the generalized Kummer varieties (see Beauville [3]).

The *Chern numbers* of X can also be expressed (see [26]) as Rozansky-Witten invariants $b_\Gamma(X)$, where we choose Γ to be the *closure of a disjoint union of* *wheel graphs*. Not all graphs can be expressed as a linear combination of such closures, and indeed Rozansky-Witten invariants are strictly more general than characteristic numbers (see [27], page 360). However, it is a surprising fact that Θ^k can be expressed in such a way: this is a simple corollary of the Wheeling Theorem of Bar-Natan, Le, and Thurston [31]; a more direct proof was recently given by Britze and Nieper [5]. We find that

$$b_{\Theta^k}(X) = 48^k k! \hat{A}^{1/2}[X] \tag{4}$$

where $\hat{A}^{1/2}$ is the square root of the $\hat{A}$-polynomial. Combining this with the expression (3) for $b_{\Theta^k}(X)$ gives us a formula (the main result of [11]) for the L^2-norm of the curvature in terms of characteristic numbers and the volume of X. This formula was an important ingredient in Huybrechts' proof [12] of a finiteness result for the number of diffeomorphism types of compact hyperkähler manifolds.

There is a useful generalisation [13, 27, 30] of the basic construction. A *holomorphic bundle* (or coherent sheaf) F on X allows us to define numbers $b_\Gamma(X,F)$ associated to any trivalent graph Γ which has a preferred oriented circle. This is the procedure by which one obtains genuine Vassiliev weight systems, defined on the usual algebra $\mathcal{A}$ of Jacobi diagrams.

2 Non-compact hyperkähler manifolds

There are several families of *non-compact* hyperkähler manifolds X whose Rozansky-Witten invariants are interesting for physical reasons.

(a) Monopole spaces The story of the invariants actually began with a dimensionally reduced version of Seiberg-Witten theory with gauge group SU(2). There is a topologically twisted version of this theory which has as its partition function the *Casson invariant*, by which we mean the generalisation due to Walker and Lescop [20] which works for arbitrary 3-manifolds, and not just for homology spheres. In the low energy limit, the theory becomes a sigma-model with target space the *Atiyah-Hitchin manifold* X_{AH}, which is the reduced moduli space of *two-monopoles* on $\mathbb{R}^3$, and is a smooth non-compact hyperkähler manifold of dimension four. Rozansky and Witten observed that in fact their model was still defined with X_{AH} replaced by any (compact or asymptotically flat) hyperkähler manifold.

There should be versions of the Casson invariant for gauge groups other than SU(2). For a general group G we should replace X_{AH} by the moduli space of vacua of the 3-dimensional SUSY gauge theory with gauge group G; for example, when G is $\mathrm{SU}(N)$ we should choose the target space X to be the reduced moduli space of N*-monopoles.* The resulting partition function would be a finite-type invariant of Ohtsuki order $3(N-1)$, and it would be particularly interesting to compute it in the case of $SU(3)$ (perhaps by a surgery formula - see Thompson [30], for example), and to investigate whether there is any connection with the generalized Casson invariant of Boden and Herald [4].

(b) Hilbert schemes The (reduced) Hilbert scheme of points on $\mathbb{C}^2$ are hyperkähler manifolds whose Rozansky-Witten theory is supposed to arise when reducing M-theory on manifolds with G_2 holonomy to associative submanifolds. Consequently, it would be useful to calculate $b_\Gamma(X)$ for these manifolds.

(c) Gravitational instantons (Kronheimer's *ALE spaces* [17]) These spaces are resolutions of quotient singularities $\mathbb{C}^2/G$, where G is a finite subgroup of $\mathrm{SU}(2)$. The McKay correspondence says that such subgroups correspond to Dynkin diagrams of type A, D, and E. For example, the cyclic group of order N corresponds to A_{N-1}. Kronheimer gave a construction of these spaces as hyperkähler quotients.

When X is non-compact we encounter a couple of basic problems. Most obviously, the integral in the definition of $b_\Gamma(X)$ may not converge; but even if it does, we won't necessarily get a weight system on *graph homology*, as it may be necessary to add a boundary term to the IHX relation.

One can always work with weight systems taking graph homology to the *Dolbeault cohomology* of the manifold, but this might itself be very complicated, and therefore useless. So let us consider some possible approaches to the convergence problem for the quantities $b_\Gamma(X)$. The first is a direct differential geometric approach: there are examples of non-compact hyperkähler manifolds for which the metric is *explicitly known*, unlike in the compact case. However, the calculations involved are not simple; see Sethi, Stern, and Zaslow [29] for this direct calculation when X is the Atiyah-Hitchin manifold. A second approach is to try to mimic Kapranov's approach, but using *compactly supported* (or perhaps L^2) cohomology. A third approach is to *compactify* the manifold; we won't get a compact *hyperkähler* manifold but we can at least try to define Rozansky-Witten invariants of the manifold so obtained.

This third approach has been investigated by Goto [8]. He generalised holomorphic symplectic manifolds by defining *log symplectic manifolds* as (compact) complex manifolds having a meromorphic symplectic form with logarithmic poles along a given divisor D (which must be the anti-canonical divisor). Under certain hypotheses, Rozansky-Witten invariants can be defined for this class of manifolds. If X is non-compact, we can try to compactify by adding a divisor at infinity. In the case of the reduced monopole moduli spaces this results in a log symplectic manifold. We don't yet know whether the (log symplectic) Rozansky-Witten invariants of this compactification agree with the Rozansky-Witten invariants of the non-compact space, as defined in terms of the metric, but this approach seems highly promising.

The second approach can be illustrated in the case of gravitational instanton spaces. First note that in four dimensions there is essentially just one Rozansky-Witten invariant

$$b_\Theta(X) = \frac{1}{(2\pi)^2}\int_X \mathrm{Tr}(R^2)$$

which can be calculated using the Gauss-Bonnet formula. We find

$$b_\Theta(X) = 2\chi(X) + B(X)$$

where χ is the Euler characteristic and B is a boundary term. For example, when X is $\mathbb{C}^2$ we have $b_\Theta(\mathbb{C}^2) = 0$ and $\chi(\mathbb{C}^2) = 1$, and therefore the boundary term $B(\mathbb{C}^2)$ must equal -2. Now gravitational instanton spaces are *asymptotically locally Euclidean* (ALE), which means their metric is asymptotic to the metric on $\mathbb{C}^2/G$ at infinity. In particular, this means they have boundary term

$$B(X) = \frac{B(\mathbb{C}^2)}{|G|},$$

where $|G|$ is the order of the group G, which acts freely on $\mathbb{C}^2$ at infinity. The homology group $\mathrm{H}_2(X,\mathbb{Z})$ is generated by the irreducible curves C_i which make up the exceptional divisor E, and these correspond to the nodes of the Dynkin diagram; apart from $\mathrm{H}_0(X,\mathbb{Z}) \cong \mathbb{Z}$, all other homology groups are zero. Therefore

$$\chi(X) = 1 + \mathrm{rank}(G).$$

Combining the above, we get

$$b_\Theta(X) = 2 + 2\,\mathrm{rank}(G) - \frac{2}{|G|}.$$

In particular, for the A-series we get $2N - \frac{2}{N}$. This happens to be twice the value of the Casimir in the fundamental representation of $SU(N)$, but we don't know a general Lie-theoretic interpretation of the formula.

Now recall that the Atiyah class α_T is the obstruction to the existence of a global holomorphic connection. If we choose a cover $\{U_i\}$ of X and local holomorphic connections ∇_i on each open set U_i, then the differences

$$\nabla_i - \nabla_j \in \mathrm{H}^0(U_i \cap U_j, T^* \otimes \mathrm{End}(T))$$

form a 1-cocycle representing α_T in Čech cohomology. Using the ALE property again, we can choose a holomorphic (in fact flat) connection on the complement U_0 of a compact neighbourhood of the exceptional divisor E. This gives an Atiyah class which is *compactly supported* in a neighbourhood of E, and this Atiyah class can be used to calculate $b_\Theta(X)$. In the case when X is the A-series of gravitational instantons, this calculation has been carried out by the second author and the answer agrees with the one given above.

In general, we don't expect there to be a compactly supported representative of the Atiyah class on an arbitrary non-compact hyperkähler manifold X. However, if X is asymptotically flat then there ought to be a representative with some kind of nice asymptotic behaviour (perhaps lying in L^2-cohomology, for instance). We are not yet aware of any such examples.

3 Quaternionic-Kähler manifolds

Hyperkähler manifolds can be regarded as particular examples of *quaternionic-Kähler* manifolds. Whereas the former have holonomy contained in $\mathrm{Sp}(k)$, quaternionic-Kähler manifolds are characterized by having holonomy contained in

$$\mathrm{Sp}(k).\mathrm{Sp}(1) = \mathrm{Sp}(k) \times \mathrm{Sp}(1)/\mathbb{Z}_2$$

where $\mathbb{Z}_2$ is generated by $(-1, -1)$. The scalar curvature s of a quaternionic-Kähler manifold X vanishes if and only if it is locally hyperkähler (in particular, if X is simply connected then s vanishes if and only if X is globally hyperkähler). However, if s is non-zero then X is not in general Kähler and does not have a holomorphic symplectic form (indeed it may not even admit an almost complex structure). Thus at first sight it may appear impossible to define anything resembling Rozansky-Witten invariants for quaternionic-Kähler manifolds. Let us assume once again that all our manifolds are compact; our results on quaternionic-Kähler manifolds will be drawn from Salamon's survey article [25].

For a quaternionic-Kähler manifold X there is an obstruction $\epsilon \in \mathrm{H}^2(X, \mathbb{Z}_2)$ to the lifting of the structure group $\mathrm{Sp}(k).\mathrm{Sp}(1)$ of X to $\mathrm{Sp}(k) \times \mathrm{Sp}(1)$. If ϵ vanishes then the decomposition (2) of the complexified tangent bundle is still valid, but with H now a *non-trivial* rank-two complex vector bundle. Indeed H is the bundle induced by the standard representation of $\mathrm{Sp}(1)$ on $\mathbb{H}$, just as E is induced by the standard representation of $\mathrm{Sp}(k)$ on $\mathbb{H}^k$. More generally, over open sets where ϵ vanishes, there is a *local* decomposition of this form; for simplicity we assume ϵ vanishes globally.

Let us return to Rozansky and Witten's original definition of the invariants using Ψ, the section of $\mathrm{Sym}^4 E$. In the quaternionic-Kähler case the curvature tensor of X equals $R_Q + s\rho_1$ where R_Q is a section of $\mathrm{Sym}^4 E$ and ρ_1 is a certain invariant element (see [25], Corollary 3.4). We can then proceed as before using R_Q instead of Ψ, and thus define Rozansky-Witten invariants $b_\Gamma(X)$ for quaternionic-Kähler manifolds X.

There ought to be some interesting consequences to this approach. Since we are using R_Q and not the full curvature tensor, we don't expect to get characteristic numbers, as in the hyperkähler case. However, if the scalar curvature s is of fixed sign, we might generate *inequalities* between these Rozansky-Witten invariants and characteristic numbers. For example, it is conjectured that all positive (i.e. $s > 0$) quaternionic-Kähler manifolds are symmetric (known as Wolf spaces), and this has been proved in dimensions 4, 8, and 12. In dimension 12, bounds on the $\hat{A}$-genus play an important role in the proof; it is tempting to believe that equations like (4), suitably generalised to the quaternionic-Kähler case, would be useful in tackling the conjecture in higher dimensions.

There is an alternative approach to defining invariants if we wish to stay closer to authentic characteristic numbers. We first recall some facts about characteristic classes of quaternionic-Kähler manifolds (see [25], Section 8). Denote by u minus the second Chern class $-c_2(H)$ of the bundle H. Then u and the Chern classes of E are well-defined elements of the rational cohomology ring of X (this is true even when ϵ is non-zero). Moreover, $4u$ is actually integral. From

$$\mathrm{ch}(TX \otimes_{\mathbb{R}} \mathbb{C}) = \mathrm{ch}(E)\mathrm{ch}(H)$$

we see that the first Pontrjagin class of X is

$$p_1 = 2(ku - c_2(E)).$$

If we look at the left hand side of equation (3) we see that we only need c_2 and the 4-form $\omega\bar{\omega}$ to define $b_{\Theta^k}(X)$. Recall that for a hyperkähler manifold the odd Chern classes vanish and the even Chern classes are equivalent to the Pontrjagin classes, so we can replace c_2 by p_1. On a quaternionic-Kähler manifold X there is a natural 4-form Ω which is parallel and hence closed. One way to define Ω is to take local almost-complex structures $\{I, J, K\}$ behaving like the quaternions ($IJ = K$), and corresponding 2-forms defined by $\omega_I(v, w) = g(Iv, w)$, etc. Then

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K$$

is well-defined up to overall scale, and parallel (even though the local almost-complex structures and 2-forms are not). Thus in the quaternionic-Kähler case we can replace $\omega\bar{\omega}$ in equation (3) by Ω, and hence define

$$b_{\Theta^k}(X) = \frac{2^k k^k}{k!} \frac{(\int_X p_1[\Omega]^{k-1})^k}{(\int_X [\Omega]^k)^{k-1}}.$$

The right hand side is invariant under rescaling of Ω. In fact there is a normalised version of Ω (involving the scalar curvature s) which represents the cohomology class u, so the definition above is equivalent to

$$b_{\Theta^k}(X) = \frac{2^k k^k}{k!} \frac{(\int_X p_1 u^{k-1})^k}{(\int_X u^k)^{k-1}}.$$

The denominator is known as the quaternionic volume, and is positive.

Clearly this expression is just one of a family of expressions involving integrals of Pontrjagin classes and the cohomology class u. In the hyperkähler case (using Chern numbers and $[\omega\bar{\omega}]$) all of these expressions are Rozansky-Witten invariants, though there is no reason to expect that the converse is true. So it may be that we don't get a full generalisation of Rozansky-Witten invariants to quaternionic-Kähler manifolds in this way. Perhaps more importantly, we no longer have the machinery of graph homology which enabled us to prove relations such as equation (4).

4 Hyperkähler manifolds and Lie algebras

The primary examples of Vassiliev weight systems are those coming from complex semisimple Lie algebras and superalgebras. It was once conjectured that in fact *all* weight systems were of this form, but this was shown to be false by Vogel [32].

From the point of view of Vassiliev theory, then, the most obvious question is whether the Rozansky-Witten weight systems are really *new*, that is, lying outside the span of the Lie algebraic weight systems. To resolve this issue it would probably be necessary to understand how the RW weight systems behave under the action of Vogel's algebra Λ.

A more general problem is to understand the common properties of hyperkähler manifolds and Lie algebras. There is in fact an *algebraic* way to unite them, described in [22], where it is shown that (roughly) the derived category of coherent sheaves on X^{4n} is the category of modules over the holomorphic tangent sheaf T, which is a Lie algebra in this category. Thus, a hyperkähler manifold *can* be thought of as giving rise to a Lie algebra – but not in the usual category of vector spaces.

Studying the analogy between the derived category and a representation category seems fruitful. There are similiarities: the derived category possesses a Duflo isomorphism, namely "theorem of the M. Kontsevich on complex manifold" [16]. But there are also basic differences: the derived category is not semisimple. It would be interesting to study whether there are analogues of Adams operations, character formulae, Harish-Chandra isomorphisms, etc. in the derived category; whether there are classes of sheaves (exceptional, stable, etc.) or bundles which behave like irreducible objects; and whether there is a meaningful interpretation of the "quantum dimensions" of sheaves arising from the Rozansky-Witten partition function of the unknot.

In the converse direction, is it possible to realise the Lie algebraic weight systems as the Rozansky-Witten invariants of suitable hyperkähler manifolds? These would presumably have to be *infinite-dimensional* manifolds, because a Lie algebra defines weight systems in all degrees, whereas a manifold X^{4k} defines weight systems only in degree less than or equal to k. A potentially useful observation is that the based loop space $\Omega G^{\mathbb{C}}$ of a *complex* semisimple Lie group is a hyperkähler manifold; the idea to look at the loop space (or perhaps the classifying space BG) is also suggested by Kontsevich's description [15] of a "shifted" Lie algebra as a formal symplectic manifold. The formal shift in grading is closely related to the shift used in forming the bar complex to obtain a model of the cohomology of the loop space.

Returning again to gravitational instantons, we have seen a tiny piece of numerical evidence suggesting that the invariants $b_\Gamma(X)$ corresponding to these spaces should be related to the weight systems of the corresponding Lie algebras (i.e. of types A, D, and E). Of course the ALE spaces are four-dimensional and so there is only one invariant $b_\Theta(X)$; to obtain invariants in all degrees we would need to consider also the Hilbert schemes of points on X, or more generally moduli spaces of instantons on X. These spaces can also be obtained by the hyperkähler quotient construction: see Kronheimer and Nakajima [18].

A slightly different way to look for connections between hyperkähler manifolds and Lie algebras is to work in terms of the associated TQFTs, rather than just weight systems. Each TQFT is determined by an underlying *ribbon category* $\mathcal{C}$. In Witten's Chern-Simons theory [33], $\mathcal{C}$ is the category of representations of a quantum group at a root of unity (see Reshetikhin and Turaev [21]), whereas in Rozansky-Witten theory we find [23] that $\mathcal{C}$ is essentially the derived category of coherent sheaves on X, with a non-standard ribbon category structure. Might there then be a manifold whose derived category is one of these representation categories? This seems somewhat outrageous, because one would not expect the derived categories to be semisimple, or even abelian. One could at least hope for some extension of the work of Kapranov and Vasserot [14], who described the derived category of the ALE spaces in purely algebraic terms. Note also that Freed, Hopkins and Teleman [7] have shown that the representation ring (Verlinde algebra) of the quantum group appears geometrically as a twisted K-group; it is tempting to wonder whether the underlying representation category might not arise as some kind of underlying twisted derived category of sheaves (such things have been defined by Caldararu [6]).

A totally different approach to connecting the two worlds would be to define a theory of *equivariant* Rozansky-Witten invariants, in which the G-equivariant invariants of a point give rise to Lie algebra weight systems. Quite what properties such a theory should have is unclear: it should perhaps be related more to the hyperkähler quotient construction than to the usual homotopy quotient underlying equivariant cohomology.

5 Further problems

Problem 1 *Derive new constraints of the form (4) for hyperkähler manifolds.*

Remarks We hope that the existence of the TQFT structure, which allows the computation of partition functions in many different ways, will yield new identities involving the characteristic classes of hyperkähler manifolds. It would be particularly useful to give a formula for the norm of the curvature of a general holomorphic bundle, bounding the Chern numbers of such bundles on X.

Problem 2 *Are the invariants $b_\Gamma(X)$ rational, or even integral?*

Remarks Current evidence suggests that, correctly normalised, the weight systems for compact hyperkähler manifolds take *integral* values on trivalent diagrams, and are at least rational for non-compact manifolds. Might the integrality be a consequence of an alternative, enumerative-geometric "counting" definition of the invariants?

Problem 3 *What is the meaning of the $\hat{A}^{\frac{1}{2}}$-genus of a manifold?*

Remarks This genus appears in the identity (4). The $\hat{A}^{\frac{1}{2}}$-class appears also in Kontsevich's Duflo isomorphism for complex manifolds, and in physics, in the study of D-brane charge [1]. One would assume that these occurrences are somehow related. Is there some natural class of manifolds for which the genus is integral? In other words, might it be the index of an elliptic operator which can only be defined under certain geometrical conditions? (It is *not* actually integral for hyperkähler manifolds — see [26].)

Problem 4 *What is the relation between RW invariants and Chern numbers?*

Remarks We can consider various classes of invariants, each a generalisation of the previous one: linear combinations of honest Chern numbers; generalised Chern numbers involving powers of the symplectic form and its conjugate; RW invariants; "big Chern classes" [13]. The linear combinations of Chern numbers are contained in rational functions of Chern numbers, which are determined by the Chern classes. Which of these inclusions is strict? Are there other interesting ones?

Problem 5 *Explain the "characteristic numbers" of non-compact examples.*

Remarks The partition functions $Z_X(S^1 \times S^2)$ and $Z_X(S^1 \times S^1 \times S^1)$ are, for any TQFT, the dimensions of certain vector spaces, and hence integral. In RW theory on a compact manifold X, they are the *Todd genus* and *Euler characteristic* of X, respectively. For non-compact X they seem to be rational: for example $Z_{X_{\mathrm{AH}}}(S^1 \times S^2) = -\frac{1}{12}$ according to Rozansky and Witten (at least up to sign, about which there are some difficulties). Might such numbers in fact be the regularised dimensions of infinite-dimensional vector spaces?

Problem 6 *Sen's conjecture on the cohomology of monopole spaces.*

Remarks The monopole space, mentioned earlier, can also be described as a certain space of rational functions on the Riemann sphere. The *Sen conjecture* (see [28]) makes certain predictions about the L^2-cohomology of this space, in particular that there is a certain $SL(2, \mathbb{Z})$ action on it. Might this action be derivable from TQFT?

Problem 7 *Define RW invariants for singular hyperkähler manifolds.*

Remarks It might be helpful to be able to handle orbifolds, in order to better understand the behaviour of RW invariants of hyperkähler quotients and quotients by finite groups. If the equivariant theory goes through, might there be localisation formulae for the curvature integrals?

Problem 8 *Compute the invariants for some non-Kähler holomorphic symplectic manifolds.*

Remarks It would be interesting to calculate RW invariants for non-Kähler holomorphic symplectic manifolds, such as Douady spaces of the Kodaira surface (in fact the invariants are zero for these) or Guan's examples [9], and to try to work out whether such invariants are independent of choice (if any) of complex structure.

Problem 9 *Try to extend the invariants to other holonomy groups.*

Remarks Why stop with quaternionic-Kähler manifolds? If we take the viewpoint that Rozansky-Witten invariants are special characteristic classes arising out of a reduction of structure group on a Riemannian manifold, then it is natural to try to find analogues for all special holonomy groups.

Here is the idea. First examine the symmetries of the curvature tensor and decide what kind of graphical vertex it corresponds to. Decide on what kind of graphs index the possible self-combinations of the curvature and the other special structural forms (e.g. on a G_2-manifold, the canonical 3- and 4-forms.) Then attempt to compute the universal graphical relations amongst such combinations; that is, find out what kind of graph homology is appropriate.

For example, on any Kähler manifold (holonomy $U(k)$) one is free to use trivalent *trees* to combine the curvature with itself, and the IHX relation is satisfied in cohomology. This is part of Kapranov's L_∞ structure. Graphs with one loop can be created by taking trace, but to glue up more legs would require a bilinear form, as in the holomorphic symplectic case. On a *Calabi-Yau* manifold (holonomy $SU(k)$) there is a canonical volume form, which would be pictured as a k-valent vertex. Whether or not the appropriate graph complex is ever rich enough to be useful, we do not know.

Problem 10 *Prove that S^6 is not complex.*

Remarks It is a famous problem to show that the 6-sphere, though almost-complex, has no (integrable) complex structure. Perhaps the L_∞ structure on the Dolbeault complex of a complex manifold (due to Kapranov, and a basic ingredient in RW theory) might provide a new approach to a contradiction.

Problem 11 *Explore Kontsevich's generalisation of the theory.*

Remarks Kontsevich gave a tremendous generalisation of the RW formalism, taking in foliated *real* manifolds with a transverse symplectic form, and in particular flat symplectic fibre bundles. This construction is completely unexplored; perhaps a sheaf-theoretic approach could help in calculating some examples, starting with the case of surface bundles over surfaces.

Problem 12 *Investigate the use of higher graph cohomology.*

Remarks Kapranov's L_∞ structure allows the construction of weight systems in higher graph cohomology, that is, numerical evaluations of graphs with vertices of valence bigger than three. Little is known about the graph homology spaces in positive degree, primarily because the usual Lie algebraic methods are useless for studying them.

Problem 13 *Investigate the functoriality of RW invariants.*

Remarks In contrast to the usual theory of characteristic classes, the RW theory does not have a theory of *classifying spaces*, unless one uses Kontsevich's Gelfand-Fuchs cohomology framework. A basic problem is that hyperkähler manifolds don't form a reasonable category (there are almost no morphisms), and so comparing the invariants of different manifolds is very hard. Is there any way to get around this?

Problem 14 *Compare Chern-Simons theory and RW theory as sigma-models.*

Remarks Chern-Simons theory is a TQFT defined by Witten [33] whose partition function is given by an integral over the space of all connections on the trivial principal G-bundle over a 3-manifold. Such connections can be thought of as pullbacks of the universal connection on the bundle $EG \to BG$ via smooth maps $M \to BG$, and so in these terms CS theory resembles the RW theory with $X = BG$ (compare this with the remarks in section 4 about the loop space). The CS integrand can be defined by extending the map $M \to BG$ over a 4-manifold W whose boundary is M, and computing the integral over W of the pullback of the fundamental (Pontrjagin) 4-form on BG. In the RW case one could try to do something similar, at least for a null-homologous $M \to X$, by using the fundamental 4-form $\omega\bar{\omega}$. Can these analogies be made more precise, and if so is there a fruitful exchange of techniques?

Problem 15 *What are the trajectories of the RW action functional?*

Remarks In Chern-Simons theory, the action functional can be viewed as a Morse function on the space of G-connections on the 3-manifold M. Its critical points are the flat connections, and the gradient flow lines (corresponding to instantons on $M \times \mathbb{R}$) can be used to define the Floer homology of M. In RW theory, the action functional is defined on smooth maps $\phi : M^3 \to X$, and its stationary points are the constant maps. Can one make sense of the trajectories of the functional, thought of as maps : $M \times \mathbb{R} \to X$, and does this lead to a Floer homology of some kind?

Problem 16 *Can RW weight systems detect orientation?*

Remarks There are many problems on the subject of the RW TQFT, and its interaction with the theories of Vassiliev and quantum invariants. These will be addressed elsewhere [23], but perhaps the most straightforward is as follows. The Vassiliev algebra $\mathcal{A}$ has an involution induced by reversing orientations of the preferred circles of all spanning diagrams. An open question is whether this involution is the identity or not; equivalently, whether Vassiliev invariants cannot (or can) detect the orientation of knots. Weight systems arising from Lie algebras are incapable of distinguishing between diagrams and their reverses, but it is not at all clear how to prove that RW weight systems share this property. Thus it is conceivable that they could be used to find non-reversible diagrams, if indeed such things exist.

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