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$(2+1)$-dimensional topological quantum field theory with a Verlinde basis and Turaev-Viro-Ocneanu invariants of 3-manifolds

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Abstract In this article, we discuss a $(2+1)$-dimensional topological quantum field theory, for short TQFT, with a Verlinde basis. As a conclusion of this general theory, we have a Dehn surgery formula. We show that Turaev-Viro-Ocneanu TQFT has a Verlinde basis. Several applications of this theorem are exposed. Based on Izumi's data of subfactors, we list several computations of Turaev-Viro-Ocneanu invariants for some 3-manifolds.

AMS Classification 57R56, 81T45; 46L37, 57M25, 57N10

Keywords Turaev-Viro-Ocneanu TQFT, Verlinde basis, Dehn surgery formula, tube algebra

1 Introduction

At the beginning of the 1990's, Turaev-Viro-Ocneanu TQFT [7] was introduced by A. Ocneanu by using a type II_1 subfactor with finite index and finite depth as a generalization of Turaev-Viro TQFT [11] which was derived from the quantum group $U_q(sl(2,\mathbb{C}))$ at certain roots of unity. K. Suzuki and the second author [9] have found a Verlinde basis for the Turaev-Viro-Ocneanu TQFT from an E_6-subfactor and have computed the invariant for several 3-manifolds including lens spaces $L(p,q)$, where p,q are less than or equal to 12. An interesting consequence of their research is that the Turaev-Viro-Ocneanu invariant distinguishes orientations for specific manifolds. It is known that the Turaev-Viro invariant cannot distinguish orientations since for an orientable closed 3-manifold it coincides with the square of absolute value of the Reshetikhin-Turaev invariant [10]. These facts let us expect that Turaev-Viro-Ocneanu invariant has a lot of more useful information than the Turaev-Viro invariant.

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After this introduction, two expository sections of subfactors and the Turaev-Viro-Ocneanu invariant are in order.

In Section 4, we introduce the notion of a $(2+1)$-dimensional TQFT with a "Verlinde basis". This is a special case of a $(2+1)$-dimensional TQFT Z with extra assumptions for the representations of the action of the mapping class group of $S^1 \times S^1$ on $Z(S^1 \times S^1)$. From this general theory, we can afford a useful Dehn surgery formula of 3-manifold invariants. Actually, we have many useful formulas for calculating the Turaev-Viro-Ocneanu invariant of some concrete 3-manifolds, for example, lens spaces and Brieskorn 3-manifolds.

In Section 5, we state a theorem that the Turaev-Viro-Ocneanu TQFT has a Verlinde basis in our sense. Hence, we have the Dehn surgery formula. We list several applications derived from this theorem.

In Section 6, we show that S and T-matrices in Izumi's sector theory associated with the Longo-Rehren subfactor and S and T-matrices in the Turaev-Viro-Ocneanu $(2+1)$-dimensional TQFT are complex conjugate. Namely, for any subfactor there is a complex conjugate isomorphism as matrix representations of $SL_2(\mathbb{Z})$. In the last part of this article, we concretely compute Turaev-Viro-Ocneanu invariants of lens spaces and homology 3-spheres based on Izumi's data [6] of S and T-matrices.

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2 Subfactors and fusion algebras

In this section, we will have a quick exposition of subfactor theory. For more precise treatment, see [4] for instance. Let us start with the definition of von Neumann algebras. Let H be a Hilbert space and $B(H)$ be the set of bounded linear operators on H. Recall that $B(H)$ is closed under the adjoint operation: for an operator $a \in B(H)$, there exists the operator a^* such that $\langle a\xi, \eta \rangle = \langle \xi, a^*\eta \rangle$ holds for every vectors ξ, $\eta \in H$. With this involution, $B(H)$ becomes a $*$-algebra. Let M be a $*$-subalgebra of $B(H)$. This M is a *von Neumann algebra* if M is closed under the weak topology. (Namely, in the weak topology, a sequence of operators $\{a_n\}$ converges to a if for arbitrary vectors ξ, η in H, $\langle a_n\xi, \eta \rangle \to \langle a\xi, \eta \rangle$ holds.) A von Neumann algebra M is called a *factor* if $M \cap M'$ is $\mathbb{C}1_M$, where 1_M is the identity element of M and M' means the set of operators which commute with each element of M in $B(H)$. Remark that every von Neumann algebra is decomposed into a direct integral of factors. Factors are classified to three types, type I, type II and type III. We are mainly interested in type II and type III. (von Neumann factors belonging to Type I are of the form $B(H)$ for some H.) We do not get into further on factors, but we mainly use a type II_1 factor, which is a type II factor with the unique normal normalized trace.

As we see in the following example, we can have *subfactors* of a factor.

Example 2.1 Let A_n be $\bigotimes_{k=1}^{n} M_2(\mathbb{C})$ and B_n be $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \bigotimes_{k=1}^{n-1} M_2(\mathbb{C})$. Then trivially B_n is a subalgebra of A_n. Besides, we can embed A_n into A_{n+1} sending x to $x \otimes 1$. In a similar way, embed B_n into B_{n+1}. Then, taking the inductive limit of A_n and taking the weak closure induced from the trace on $\cup_{n=1}^{\infty} A_n$, we have a factor of type II_1 $A = \overline{\cup_{n=1}^{\infty} A_n}$ and a subfactor of A, $B = \overline{\cup_{n=1}^{\infty} B_n}$. This is a simple example of subfactors. Those factors are approximated by the finite dimensional algebras. In such a case, we call our factor AFD(= Approximately Finite Dimensional) in brief.

Now, let us define Jones index for a type II_1 subfactor $N \subset M$. The bigger factor can be considered as a left N-module ${}_N M$ and this left module is projective. So it can be projectively decomposed. Assume the number of direct summands is finite. (Otherwise we think that Jones index takes the value infinity.) Then, left N-module ${}_N M$ is isomorphic to ${}_N N \oplus \cdots \oplus {}_N N \oplus {}_N N p$, where p is a projection in N. Then, Jones index $[M : N]$ is defined to be (the number of ${}_N N$ in the direct sum) + $\mathrm{tr}(p)$, where tr is the unique trace on M. For instance, $[A : B] = 4$ in the previous example. Jones index is sometimes denoted by $\dim_N M$.

For a subfactor $N \subset M$, note that M can be viewed as N-M, M-N, M-M and N-N bimodule. Using this fact, we are going to produce new bimodules. Let ${}_P X_Q$ be a P-Q bimodule, where $P, Q \in \{M, N\}$. Then, we define the Q-P bimodule ${}_Q \bar{X}_P$ in the following manner. As a Hilbert space, it is the conjugate Hilbert space of X. The Q-P action is defined by $q \cdot \bar{\xi} \cdot p = \overline{p^* \cdot \xi \cdot q^*}$, where $\xi \in X$, $p \in P$ and $q \in Q$. This Q-P bimodule ${}_Q \bar{X}_P$ is called the conjugate bimodule of ${}_P X_Q$. For instance, $\overline{{}_N M_M} = {}_M M_N$. Another operation is the relative tensor product of bimodules. This is similar to the tensor product of algebraic bimodules over a certain ring except the completion as a Hilbert space. We say that a P-Q bimodule ${}_P X_Q$ is irreducible if the set of bounded P-Q linear maps $\mathrm{End}({}_P X_Q)$ is isomorphic to the scalar $\mathbb{C}$. Set $g = {}_N M_M$. Take relative tensor products $g \underset{M}{\otimes} \bar{g} \underset{N}{\otimes} \cdots \underset{N}{\otimes} g$. Then this can be decomposed into the irreducible bimodules $\bigoplus_i m_i \, {}_N X_{iM}$, where m_i is the multiplicity of the irreducible bimodule ${}_N X_{iM}$ in the relative tensor products. (Recall that this is very much similar to representation theory of a compact group.) Finally, the dimension function $[{}_P X_Q]$ of ${}_P X_Q$ is defined to be $[{}_P X_Q] = (\dim_P X)(\dim X_Q) \in [1, \infty]$. (Note that this corresponds to the square of " quantum dimension" in different literatures.)

A set of four kinds of irreducible bimodules with the dimension function obtained from a subfactor $N \subset M$ with finite Jones index, closed under the operations, relative tensor product, conjugation and direct sum is called the *graded fusion rule algebra* associated with $N \subset M$. If the number of unitary equivalence classes of four kinds of irreducible bimodules is finite, then subfactor $N \subset M$ is said to have *finite depth.*

For type III factors, it is more natural to use endomorphisms than bimodules. Let M be a type III factor. Take endomorphisms ρ, $\sigma \in \mathrm{End}(M)$. We have a natural unitary equivalence relation between two endomorphisms in such a way that $u\rho(x)u^* = \sigma(x)$, where u is a unitary in M. Equivalence classes of $\mathrm{End}(M)$ are called *sectors.* For sectors there are notions of direct sum, composition($=$ "tensor product"), conjugation, irreducibility and statistical dimension($=$ "quantum dimension"), which play the roles of the corresponding operations for the bimodules obtained from a type II_1 subfactor. In both cases of type II_1 subfactors and type III factors, it is a certain C^*-tensor category that we have finally constructed.

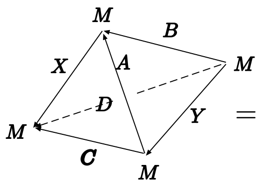
3 Turaev-Viro-Ocneanu $(2+1)$-dimensional TQFT

Let $N \subset M$ be a subfactor of type II_1 with finite Jones index and finite depth. As we have seen in the previous section, we have the graded fusion rule algebra associated with this subfactor. Then, a *quantum 6j-symbol* is defined as a composition $\xi_4 \cdot (\xi_3 \otimes id) \cdot (id \otimes \xi_1)^* \cdot \xi_2^*$ of the following four intertwiners, i.e., bounded M-M linear homomorphisms

$$\begin{array}{ccc} X \underset{M}{\otimes} A \underset{M}{\otimes} Y & \xrightarrow{id\otimes\xi_1} & X \underset{M}{\otimes} B \\ {\scriptstyle \xi_3\otimes id}\big\downarrow & & \big\downarrow{\scriptstyle \xi_2} \\ C \underset{M}{\otimes} Y & \xrightarrow{\xi_4} & D \end{array}$$

where, X, Y, A, B, C and D are irreducible M-M bimodules and ξ_i's are orthonormal basis of intertwiners. (For irreducible bimodules X, Y and Z, we introduce an inner product $\langle \xi, \eta \rangle$ for two intertwiners ξ and η in $\mathrm{Hom}(X \underset{M}{\otimes} Y, Z)$ by $\xi \cdot \eta^*$.)

Let us consider a quantum $6j$-symbol as a value of a tetrahedron in the following way. In the picture, each vertex, edge and face correspond to II_1 factor M, an irreducible bimodule and an intertwiner, respectively.



$$= [B]^{-\frac{1}{4}}[C]^{-\frac{1}{4}}\ \xi_4 \cdot (\xi_3 \otimes id) \cdot (id \otimes \xi_1)^* \cdot \xi_2^*$$

Let V be a closed 3-manifold with triangulation $\mathcal{T}$ and let $\mathcal{M}$ be the C^*-tensor category of M-M bimodules obtained from a subfactor $N \subset M$. Then, the Turaev-Viro-Ocneanu invariant $Z_{\mathcal{M}}(V, \mathcal{T})$ is defined in the following way. First, assign a II_1 factor M for each vertex. Then, assign an irreducible bimodule for each edge. Finally, assign an intertwiner for each face of a tetrahedron in $\mathcal{T}$. We denote the set of edges in $\mathcal{T}$ by E. We denote all the possible assignment of edge colorings and face colorings by e and φ, respectively. Note that we have a complex number for each tetrahedron which is induced from a quantum $6j$-symbol. We denote this value of a tetrahedron τ by $W(\tau; e, \varphi)$. Multiply all tetrahedra and take summations over all possible assignments of face colorings and edge colorings:

$$Z_{\mathcal{M}}(V, \mathcal{T}) = \lambda^{-a} \sum_{e} (\prod_{E} [X]^{\frac{1}{2}}) \sum_{\varphi} \prod_{\tau} W(\tau; e, \varphi)$$

where $[X] = (\dim_M X)(\dim X_M)$, $\lambda = \sum_{{}_M X_M} [X]$ (called the *global index*) and a is the number of vertices in $\mathcal{T}$.

This $Z_{\mathcal{M}}(V, \mathcal{T})$ turns out to be independent of the choice of triangulations. Hence, it is a topological invariant of V called the *Turaev-Viro-Ocneanu invariant*. We may simply write $Z_{\mathcal{M}}(V)$ for this invariant. One can extend this construction to a (2+1)-dimensional unitary TQFT. We again use $Z_{\mathcal{M}}$ to express the Turaev-Viro-Ocneanu TQFT.

4 $(2+1)$-dimensional TQFT with a Verlinde basis

In this section we briefly explain a general theory of $(2+1)$-dimensional TQFT with a Verlinde basis. For a $(2+1)$-dimensional TQFT with a Verlinde basis, we have a Dehn surgery formula of the 3-manifold invariant induced from the TQFT. By applying the Dehn surgery formula to some specific 3-manifolds including lens spaces and Brieskorn 3-manifolds, we get formulas such that the invariants of them can be calculated from the S and T-matrices associated with the TQFT.

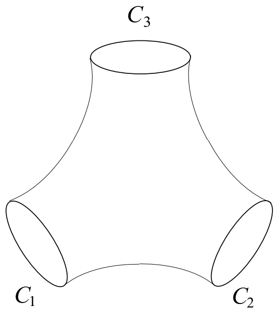
Let Z be a (2+1)-dimensional TQFT and $\{v_i\}_{i=0}^m$ a basis of $Z(S^1\times S^1)$. Then a family of framed link invariants $J(L; i_1, \cdots, i_r) \in \mathbb{C},\ i_1, \cdots, i_r \in \{0, 1, \cdots, m\}$ is defined as follows.

Let L be a framed link in S^3 with r-components $L_1, \cdots, L_r$ and X the link exterior of L in S^3, that is $X = \overline{S^3 - N(L_1) \cup \cdots \cup N(L_r)}$, where $N(L_i)$ is a tubular neighbourhood of L_i. Then we have a cobordism $W_L := (X; \partial N(L_1) \cup \cdots \cup \partial N(L_r), \emptyset)$. This cobordism induces a linear map $Z_L : Z(S^1 \times S^1)^{\otimes m} \longrightarrow \mathbb{C}$. By using this linear map Z_L, a complex number $J(L; i_1, \cdots, i_r)$ is defined by

$$J(L; i_1, \cdots, i_r) := Z_L(v_{i_1} \otimes \cdots \otimes v_{i_r})$$

for each $i_1, \cdots, i_r \in \{0, 1, \cdots, m\}$. It is easily seen that $J(L; i_1, \cdots, i_r)$ is an invariant of framed links in S^3.

Let Z be a $(2+1)$-dimensional TQFT. Taking the cobordism $W_{\text{fusion}} := (Y \times S^1; \Sigma_1 \sqcup \Sigma_2, \Sigma_3)$, where Y is the compact oriented surface depicted in the figure and $\Sigma_i = C_i \times S^1$ for $i = 1, 2, 3$, we get a linear map $Z_{W_{\text{fusion}}} : Z(S^1 \times S^1) \otimes Z(S^1 \times S^1) \longrightarrow Z(S^1 \times S^1)$ from the axiom for TQFT [2, 10]. This map $Z_{W_{\text{fusion}}}$ defines an associative and unital product on $Z(S^1 \times S^1)$. It can be easily shown that this algebra is commutative and the identity element of it is given by $Z_{W_0}(1)$, where $W_0 := (D^2 \times S^1; \phi, S^1 \times S^1)$. We call this algebra the *fusion algebra associated with* Z.



Any (2+1)-dimensional TQFT Z induces an action of the mapping class group of an oriented closed surface Σ on the vector space $Z(\Sigma)$ [2, 10]. In case of $\Sigma = S^1 \times S^1$, the mapping class group of it is isomorphic to the group $SL_2(\mathbb{Z})$ of integral 2×2-matrices with determinant 1, that is generated by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ with relations $S^4 = I,\ (ST)^3 = S^2$. The matrices S and T correspond to the orientation preserving homeomorphisms from $S^1 \times S^1$ to $S^1 \times S^1$ which are defined by $S(z, w) = (w, \bar{z})$ and $T(z, w) = (z, zw),\ (z, w) \in$

$S^1 \times S^1$, respectively, where we regard S^1 as the set of complex numbers of absolute value 1. The actions of S and T on $Z(S^1 \times S^1)$ are denoted by $Z(S)$ and $Z(T)$, respectively.

Definition 4.1 Let Z be a unitary $(2+1)$-dimensional TQFT. An orthonormal basis $\{v_i\}_{i=0}^m$ of $Z(S^1 \times S^1)$ is called a *Verlinde basis* if the following conditions are satisfied.

(i) v_0 is the identity element of the fusion algebra.

(ii) $Z(S)$ is presented by a unitary and symmetric matrix, and $Z(T)$ is presented by a unitary and diagonal matrix with respect to $\{v_i\}_{i=0}^m$.

(iii) $Z(S)^2 v_i \in \{v_i\}_{i=0}^m$ for all $i \in \{0,1,\cdots,m\}$, and $Z(S)^2 v_0 = v_0$.

(iv) We write $Z(S)v_i = \sum\limits_{j=0}^m S_{ji} v_j$, $S_{ji} \in \mathbb{C}$ for all $i \in \{0,1,\cdots,m\}$. Then

(a) $S_{0i} \neq 0$ for all $i \in \{0,1,\cdots,m\}$

(b) $N_{ij}^k := \sum\limits_{l=0}^m \dfrac{S_{il}S_{jl}\overline{S_{lk}}}{S_{0l}}$ is a non-negative integer for all $i,j,k \in \{0,1,\cdots,m\}$

(c) $\{N_{ij}^k\}_{i,j,k=0,1,\cdots,m}$ coincide with the structure constants of the fusion algebra with respect to $\{v_i\}_{i=0}^m$: $v_i v_j = \sum\limits_{k=0}^m N_{ij}^k v_k$.

By using the gluing axiom [2, 10] in TQFT, we have the following proposition.

Proposition 4.2 *Let Z be a $(2+1)$-dimensional TQFT with Verlinde basis $\{v_i\}_{i=0}^m$. Let M be a closed oriented 3-manifold. If M is obtained from S^3 by Dehn surgery along a framed link $L = L_1 \cup \cdots \cup L_r \subset S^3$, then $Z(M)$ is given by the formula*

$$Z(M) = \sum_{i_1,\cdots,i_r=0}^m S_{i_1,0} \cdots S_{i_r,0} J(L; i_1, \cdots, i_r).$$

We apply the above proposition to the case where M is the lens space $L(p,q)$, $q = 1,2$ or the Brieskorn 3-manifold $M(p,q,r) = \{(u,v,w) \in \mathbb{C}^3 \mid u^p + v^q + w^r = 0,\ |u|^2 + |v|^2 + |w|^2 = 1\}$, where $p,q,r \geq 2$. Since these manifolds are obtained from S^3 by Dehn surgery along framed links depicted in the figure on the next page, we obtain the following formulas.

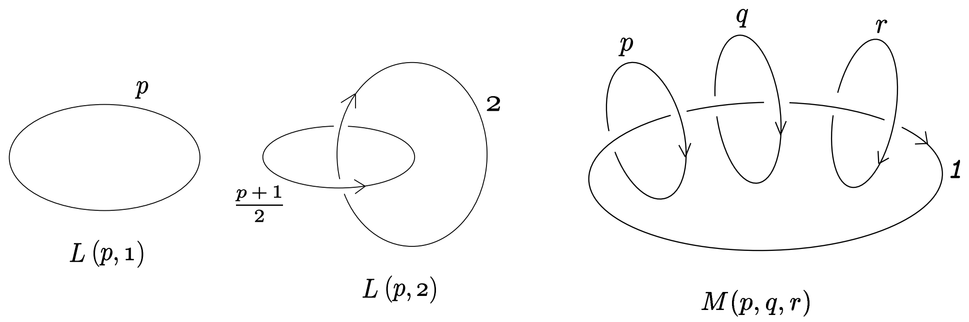
$$Z(L(p,1)) = \sum_{i=0}^m t_i^p S_{i0}^2 \quad (p \in \mathbb{N}),$$

$$Z(L(p,2)) = \sum_{i,j=0}^{m} t_i^{\frac{p+1}{2}} t_j^2 S_{i0} S_{j0} S_{ij} \quad (p \in \mathbb{N} \text{ is odd}),$$

$$Z(M(p,q,r)) = \sum_{i,j,k,l=0}^{m} t_i^p t_j^q t_k^r t_l \frac{S_{i0} S_{j0} S_{k0} S_{il} S_{jl} S_{kl}}{S_{l0}}$$

where $Z(T)v_i = t_i v_i$ $(i = 0, 1, \cdots, m)$.

These formulas are helpful to compute the values of Turaev-Viro-Ocneanu invariants from several concrete subfactors in the section 6.



5 Applications to Turaev-Viro-Ocneanu TQFT

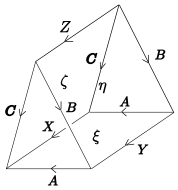
Let $N \subset M$ be an AFD II_1 subfactor with finite Jones index and with finite depth. Then, we can construct a new subfactor out of this, which is called the *asymptotic subfactor*

$$M \vee M^{op} \subset \vee_{n=0}^{\infty} M_n = M_\infty$$

where $N \subset M \subset M_1 \subset \cdots \subset M_n \subset \cdots$ is the Jones tower, $\vee$ means "generated by" and M^{op} is the opposite algebra of M. Let us denote the C^*-tensor category of M_∞-M_∞ bimodules obtained from the asymptotic subfactor by $\mathcal{M}_\infty$. It turns out that $\mathcal{M}_\infty$ is modular in the sense of Turaev. It is sometimes easier to treat another tensor category than $\mathcal{M}_\infty$ itself. Instead of $\mathcal{M}_\infty$, we introduce the *tube system* associated with $N \subset M$, which is equivalent to $\mathcal{M}_\infty$ as modular categories. Originally, the tube system was introduced by A. Ocneanu to analyze the asymptotic subfactors [7], [8].

First, we introduce the *tube algebra* $\mathrm{Tube}(\mathcal{M})$, due to Ocneanu. As a linear space $\mathrm{Tube}(\mathcal{M})$ is equal to $\oplus_{X,Y,A}\mathrm{Hom}(X \underset{M}{\otimes} A, A \underset{M}{\otimes} Y)$. The product structure is given by Turaev-Viro-Ocneanu invariant of the join $A * S^1$ of the annulus A and the circle S^1 as follows:

$$X\overset{A}{\underset{A}{\boxed{\;\xi\;}}}Y \, . \, X'\overset{B}{\underset{B}{\boxed{\;\eta\;}}}Z = \delta_{Y,X'}\frac{[Y]^{\frac{1}{2}}}{[X]^{\frac{1}{4}}[Z]^{\frac{1}{4}}}\sum_{\xi,\eta,\zeta} Z\Bigg($$

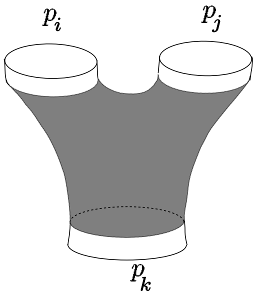


$$\Bigg) X\overset{C}{\underset{C}{\boxed{\;\zeta\;}}}Z$$

Indeed, it turns out that $\mathrm{Tube}(\mathcal{M})$ is a finite dimensional C^*-algebra. So, $\mathrm{Tube}(\mathcal{M}) \cong \oplus_{i=0}^{d} M_{n_i}(\mathbb{C})$. Let $\{p_0, \cdots, p_d\}$ be minimal projections in each direct summand of $\mathrm{Tube}(\mathcal{M})$.

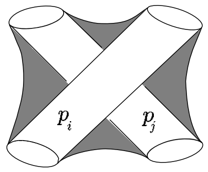
Definition 5.1 The *tube system* associated with $N \subset M$ is defined to be a modular category with objects, morphisms and braidings in the following: Objects are the $\mathbb{C}$-linear span of minimal projections of $\mathrm{Tube}(\mathcal{M})$. Morphisms are defined in the following way: $\mathrm{Hom}(p_i, p_j) = \begin{cases} \mathbb{C} & i = j \\ \{0\} & i \neq j \end{cases}$

An element of $\mathrm{Hom}(p_i \otimes p_j, p_k)$ is a vector in the Hilbert space of a surface in the following picture.

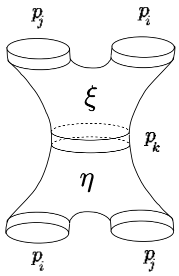


Finally, a unitary braiding $\epsilon_{i,j} \in \mathrm{Hom}(p_i \otimes p_j, p_j \otimes p_i)$ is defined by the following formula:

$$\epsilon_{ij} = \sum_{k,\xi,\eta} Z_{\mathcal{M}}\Bigg($$



$$\Bigg)$$



The following theorem is due to Ocneanu.

Theorem 5.2 *$Z_{\mathcal{M}}(S^1 \times S^1)$ is naturally identified with the center of $\mathrm{Tube}(\mathcal{M})$.*

Thanks to this theorem, one can consider the mapping class group of the torus $S^1 \times S^1$ as represented on the center of Tube$(\mathcal{M})$. Then, we have the following theorem.

Theorem 5.3 *$\{p_0, \cdots, p_d\}$ is a Verlinde basis of $Z(S^1 \times S^1)$.*

Hence, for a closed oriented 3-manifold V obtained from Dehn surgery along a link L, we have the Dehn surgery formula for the Turaev-Viro-Ocneanu TQFT:

$$Z_{\mathcal{M}}(V) = \sum_{i_1, \cdots, i_m = 0}^{d} S_{0i_1} \cdots S_{0i_m} J(L; i_1, \cdots, i_m)$$

where $Z(S)p_i = \sum_{j=0}^{d} S_{ij} p_j$, $(i = 0, \cdots, d)$.

There are some applications of this theorem and we list them as corollaries.

Note that, since the tube system is a modular category in the sense of Turaev, we can make a Reshetikhin-Turaev invariant of 3-manifolds out of it.

Corollary 5.4 *Let V be a closed oriented 3-dimensional manifold. Then, Turaev-Viro-Ocneanu invariant of V is equal to the Reshetikhin-Turaev invariant constructed from the tube system.*

We remark that A. Ocneanu also has the above statement in [8].

Corollary 5.5 *The Dijkgraaf-Witten invariant [3, 12] of finite group G with 3-cocycle ω is equal to Altshuler-Coste invariant [1] of the quantum double $D^{\omega}(G)$.*

Corollary 5.6 *If we further assume our C^*-tensor category $\mathcal{M}$ is modular, then $Z_{\mathcal{M}}(V) = \tau_{\mathcal{M}}(V)\overline{\tau_{\mathcal{M}}(V)}$, where $\tau_{\mathcal{M}}(V)$ is the Reshetikhin-Turaev invariant of a closed oriented 3-dimensional manifold V constructed from $\mathcal{M}$ and the bar means the complex conjugation.*

The last corollary follows because our tube system splits into $\mathcal{M} \otimes \overline{\mathcal{M}}$ under the assumption that $\mathcal{M}$ is modular.

6 Computations of Turaev-Viro-Ocneanu invariants

In the analysis of the Longo-Rehren subfactor, which corresponds to the center construction in the sense of Drinfel'd, M. Izumi formulated a tube algebra in terms of sectors for a finite closed system of endomorphisms of a type III subfactor in a slightly different way [5], although Izumi's tube algebra is isomorphic to Ocneanu's one as C^*-algebras. He also explicitly gave an action of $SL_2(\mathbb{Z})$ on the center of the tube algebra in the language of sectors, and derived some formulas about Turaev-Viro-Ocneanu invariants of lens spaces for concretely given subfactors. However, it is not clear how S and T-matrices in Izumi's sector theory associated with the Longo-Rehren subfactor and S and T-matrices in Turaev-Viro-Ocneanu (2+1)-dimensional TQFT are related. Actually, we can prove that for any subfactor an isomorphism between two tube algebras in Izumi's sector theory and Ocneanu's one, which is derived from Turaev-Viro-Ocneanu $(2+1)$-dimensional TQFT, induces a conjugate linear isomorphism as matrix representations of $SL_2(\mathbb{Z})$. Hence, we can compute a plenty of Turaev-Viro-Ocneanu invariants of 3-manifolds based on Izumi's data [6] of S and T-matrices.

As a summary of the above argument, we state the following proposition.

Proposition 6.1 *The representation of $SL_2(\mathbb{Z})$ constructed from the Turaev-Viro-Ocneanu TQFT are complex conjugate to ones introduced by Izumi.*

We have the following lists by partially using the Maple software for analytical computation in mathematics. In the list, we use the notation $Z_{\mathcal{M}}(V)$ in such a way that it means that the Turaev-Viro-Ocneanu invariant of a 3-manifold V constructed from the data $\mathcal{M}$, which is obtained from a subfactor $N \subset M$. (See also Section 3 for these notations.)

<u>the $D_5^{(1)}$-subfactor</u>

$$Z_{D_5^{(1)}}(L(3,1)) = \overline{Z_{D_5^{(1)}}(L(3,2))} = \frac{1+2w^2}{6}, \text{ where } w^3 = 1$$

$$Z_{D_5^{(1)}}(L(5,1)) = Z_{D_5^{(1)}}(L(5,2)) = Z_{D_5^{(1)}}(L(7,1)) = Z_{D_5^{(1)}}(L(7,2)) = \frac{1}{6}$$

<u>an E_6-subfactor</u> [9]

$$Z_{E_6}(L(p,1)) = \frac{1}{12}\{((-1)^p+1)e^{-\frac{p\pi i}{3}} + 2e^{-\frac{5p\pi i}{6}} + i^p + 2(-1)^p + 5\}$$

$$Z_{E_6}(L(p,2)) = \frac{1}{4} + \frac{(-1)^{\frac{p+1}{2}} i}{12} - \frac{\sqrt{3}+i}{12} e^{-\frac{(p+1)\pi i}{6}}$$

(p,q,r)	$(2,3,5)$	$(2,3,7)$	$(2,5,7)$	$(3,5,7)$
$Z_{E_6}(M(p,q,r))$	$\frac{(6+2\sqrt{3})+(3-3\sqrt{3})i}{12}$	$\frac{(6+2\sqrt{3})+(3-3\sqrt{3})i}{12}$	$\frac{-\sqrt{3}+9+6i}{12}$	$\frac{2-\sqrt{3}i}{2}$

the generalized E_6-subfactor with $G = \mathbb{Z}/3\mathbb{Z}$ — see footnote[1]

$$Z_{E_6,\mathbb{Z}/3\mathbb{Z}}(L(3,1)) = \overline{Z_{E_6,\mathbb{Z}/3\mathbb{Z}}(L(3,2))} = \frac{7-\sqrt{7}i}{14}$$

$$Z_{E_6,\mathbb{Z}/3\mathbb{Z}}(L(5,1)) = Z_{E_6,\mathbb{Z}/3\mathbb{Z}}(L(5,2)) = \frac{7-\sqrt{21}}{42}$$

$$Z_{E_6,\mathbb{Z}/3\mathbb{Z}}(L(7,1)) = Z_{E_6,\mathbb{Z}/3\mathbb{Z}}(L(7,2)) = \frac{1+\sqrt{3}i}{6}$$

the generalized E_6-subfactor with $G = \mathbb{Z}/4\mathbb{Z}$

$$Z_{E_6,\mathbb{Z}/4\mathbb{Z}}(L(3,1)) = Z_{E_6,\mathbb{Z}/4\mathbb{Z}}(L(3,2)) = \frac{2+\sqrt{2}}{16}$$

$$Z_{E_6,\mathbb{Z}/4\mathbb{Z}}(L(5,1)) = Z_{E_6,\mathbb{Z}/4\mathbb{Z}}(L(5,2)) = \frac{2+\sqrt{2}}{16}$$

$$Z_{E_6,\mathbb{Z}/4\mathbb{Z}}(L(7,1)) = Z_{E_6,\mathbb{Z}/4\mathbb{Z}}(L(7,2)) = \frac{2-\sqrt{2}}{16}$$

the generalized E_6-subfactor with $G = \mathbb{Z}/5\mathbb{Z}$ — see footnote[1]

$$Z_{E_6,\mathbb{Z}/5\mathbb{Z}}(L(3,1)) = Z_{E_6,\mathbb{Z}/5\mathbb{Z}}(L(3,2)) = \frac{1-\sqrt{5}}{10}$$

$$Z_{E_6,\mathbb{Z}/5\mathbb{Z}}(L(5,1)) = \frac{1}{3},\ Z_{E_6,\mathbb{Z}/5\mathbb{Z}}(L(5,2)) = \frac{2}{3}$$

$$Z_{E_6,\mathbb{Z}/5\mathbb{Z}}(L(7,1)) = Z_{E_6,\mathbb{Z}/5\mathbb{Z}}(L(7,2)) = \frac{3+\sqrt{5}}{30}$$

[1]Values corrected 21 March 2003

the generalized E_6-subfactor with $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

$$Z_{E_6, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}(L(3,1)) = Z_{E_6, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}(L(3,2)) = \frac{2+\sqrt{2}}{16}$$

$$Z_{E_6, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}(L(5,1)) = Z_{E_6, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}(L(5,2)) = \frac{2+\sqrt{2}}{16}$$

$$Z_{E_6, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}(L(7,1)) = Z_{E_6, \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}(L(7,2)) = \frac{2-\sqrt{2}}{16}$$

the Haargerup subfactor

$$Z_{Haagerup}(L(7,1)) = Z_{Haagerup}(L(7,2)) = \frac{13+3\sqrt{13}}{78}$$

$$Z_{Haagerup}(M(2,3,5)) = -\frac{\sqrt{13}}{26} + \frac{7}{6}$$

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