Geometry & Topology Monographs
Volume 4: Invariants of knots and 3-manifolds (Kyoto 2001)
Pages 295–302

Torsions of 3-manifolds

Vladimir Turaev

Abstract We give a brief survey of abelian torsions of 3-manifolds.

AMS Classification 57M27; 57Q10

Keywords Torsions, 3-manifolds

Introduction

This paper is a brief survey of my work on abelian torsions of 3-manifolds. In 1976 I introduced an invariant, $\tau(M)$, of a compact smooth (or PL-) manifold M of any dimension (see [21] and references therein). This invariant is a sum of several Reidemeister torsions of M numerated by characters of the (finite abelian) group $\mathrm{Tors}H$ where $H = H_1(M)$. This invariant lies in a certain extension of the group ring $\mathbf{Z}[H]$ and is defined up to multiplication by ± 1 and elements of H. In the case $\dim M = 3$ one can be more specific: if $b_1(M) \geq 2$ then $\tau(M) \in \mathbf{Z}[H]/\pm H$; if $b_1(M) = 0$ then $\tau(M) \in \mathbf{Q}[H]/\pm H$; if $b_1(M) = 1$, then $\tau(M)$ can be expanded as a sum of an element of $\mathbf{Z}[H]$ and a certain standard fraction. Classically, the Reidemeister torsions are used to distinguish homotopy equivalent but not simply homotopy equivalent spaces like lens spaces or their connected sums. The study of $\tau(M)$ was motivated by its connections with the Alexander-Fox invariants of M. The present interest to this invariant is motivated by its connections to the Seiberg-Witten invariants.

To get rid of the ambiguity in the definition of $\tau(M)$ one needs to involve additional structures on M. In [18] I introduced a refined version $\tau(M, e, \omega)$ of $\tau(M)$ depending on the choice of a so-called Euler structure e on M and a homology orientation ω of M (this is an orientation in the real vector space $H_*(M; \mathbf{R})$). An Euler structure on M is a non-singular tangent vector field on M considered up to homotopy and an arbitrary modification in a small neighborhood of a point. The set $\mathrm{Eul}(M)$ of Euler structures on M has a natural involution $e \mapsto e^{-1}$ transforming the class of a non-singular vector field to the class of the opposite vector field. If $\chi(M) = 0$, then the group $H = H_1(M)$ acts freely and transitively on $\mathrm{Eul}(M)$ so that $|\mathrm{Eul}(M)| = |H|$.

The invariant $\tau(M,e,\omega)$ has no indeterminacy and $\tau(M) = \pm H\,\tau(M,e,\omega)$ for all e,ω. The torsions of various Euler structures on M are computed from each other via $\tau(M,he,\omega) = h\,\tau(M,e,\omega)$ for any $h \in H$.

In this paper we shall assume that M is a closed connected oriented 3-manifold. It has a homology orientation ω_M defined by a basis $([pt], b, b^*, [M])$ in $H_*(M;\mathbf{R})$ where $[pt] \in H_0(M;\mathbf{R})$ is the homology class of a point, b is an arbitrary basis in $H_1(M;\mathbf{R})$, b^* is the basis in $H_2(M;\mathbf{R})$ dual to b with respect to the (non-degenerate) intersection form $H_1(M;\mathbf{R}) \times H_2(M;\mathbf{R}) \to \mathbf{R}$, and finally $[M] \in H_3(M;\mathbf{R})$ is the fundamental class of M. The homology orientation ω_M does not depend on the choice of b. We shall write $\tau(M,e)$ for $\tau(M,e,\omega_M)$ where $e \in \mathrm{Eul}(M)$. If $-M$ is M with opposite orientation, then $\tau(-M,e) = (-1)^{b_1(M)+1}\,\tau(M,e)$.

The torsion $\tau(M,e)$ satisfies a fundamental duality formula

$$\overline{\tau(M,e)} = \tau(M,e^{-1})$$

where the overbar denotes the conjugation in the group ring sending group elements to their inverses.

The torsion $\tau(M)$ determines the first elementary ideal $E(\pi) \subset \mathbf{Z}[H]$ of the fundamental group $\pi = \pi_1(M)$. In particular if $b_1(M) \geq 1$, then $E(\pi) = \tau(M,e)I^2$ where e is any Euler structure on M and $I \subset \mathbf{Z}[H]$ is the augmentation ideal. This implies that $\tau(M,e)$ determines the Alexander-Fox polynomial $\Delta(M) = \Delta(\pi) \in \mathbf{Z}[H/\mathrm{Tors}H]$. The torsion $\tau(M)$ can be viewed as a natural lift of $\Delta(M)$ to $\mathbf{Z}[H]$. In contrast to $\Delta(M)$, the torsion $\tau(M)$ in general is not determined by $\pi_1(M)$; indeed, it distinguishes lens spaces with the same π_1.

The torsion $\tau(M)$ can be rewritten in terms of a numerical *torsion function* T_M on the set $\mathrm{Eul}(M)$, see [19]. This function takes values in $\mathbf{Z}$ if $b_1(M) \neq 0$ and in $\mathbf{Q}$ if $b_1(M) = 0$. If $b_1(M) \neq 1$, then τ and T_M are related by the formula

$$\tau(M,e) = \sum_{h\in H} T_M(he)\,h^{-1}$$

for any $e \in \mathrm{Eul}(M)$. For $b_1(M) = 1$, there is a similar but a little more complicated formula. The torsion function has a finite support and satisfies the identity $T_M(e) = T_M(e^{-1})$ for all $e \in \mathrm{Eul}(M)$.

For $b_1(M) = 0$, we have $\sum_{e\in\mathbf{Eul}(M)} T_M(e) = 0$. For $b_1(M) \geq 1$, the number $\sum_{e\in\mathbf{Eul}(M)} T_M(e)$ is essentially the Casson-Walker-Lescop invariant $\lambda(M) \in \mathbf{Q}$:

$$\sum_{e\in\mathbf{Eul}(M)} T_M(e) = \begin{cases} (-1)^{b_1(M)+1}\lambda(M), & \text{if } b_1(M) \geq 2, \\ \lambda(M) + |\mathrm{Tors}H|/12, & \text{if } b_1(M) = 1. \end{cases}$$

Connections to the Seiberg-Witten theory

The Seiberg-Witten invariant of M is a numerical function, SW_M, on the set of $Spin^c$-structures on M, see for instance [14], [11], [12] and references therein. For a $Spin^c$-structure e on M, the integer $SW_M(e)$ is the algebraic number of solutions, called monopoles, to a certain system of differential equations associated with e. This number coincides with the 4-dimensional SW-invariant of the $Spin^c$-structure $e \times 1$ on $M \times S^1$.

The invariants SW_M and $\tau(M)$ turn out to be equivalent (at least up to sign). The first step in this direction was made by Meng and Taubes [14] who observed that SW_M determines the Alexander-Fox polynomial $\Delta(M)$. The equivalence between SW_M and $\tau(M)$ was established in [20] in the case $b_1(M) \geq 1$. The Euler structures on M are identified there with $Spin^c$-structures on M and it is proven that $SW_M(e) = \pm T_M(e)$ for all $e \in \mathrm{Eul}(M)$. A similar result for rational homology spheres was recently obtained by Nicolaescu [15].

For any $e \in Spin^c(M)$, the number $T_M(e) = \pm SW_M(e)$ can be viewed as the Euler characteristic of the Seiberg-Witten-Floer monopole homology of M associated with e (see [12]). The same number appears also as the Euler characteristic of the Floer-type homology of M associated with e by Ozsváth and Szabó, see [16].

Surgery formulas

The definition of τ is based on the methods of the theory of torsions, specifically, cellular chain complexes, coverings, etc. The definition of SW is analytical. These definitions are not always suitable for explicit computations. We outline a surgery formula for T_M suitable for computations.

We first give a surgery description of Euler structures ($=$ $Spin^c$-structures) on 3-manifolds. To this end we introduce a notion of a charge. A *charge* on an oriented link $L = L_1 \cup ... \cup L_m \subset S^3$ is an m-tuple $(k_1, ..., k_m) \in \mathbf{Z}^m$ such that for all $i = 1, ..., m$,

$$k_i \equiv 1 + \sum_{j \neq i} lk(L_i, L_j) \pmod 2$$

where lk is the linking number in S^3. A charge k on L determines an Euler structure, e_k^M, on any 3-manifold M obtained by surgery on L, see [22].

The surgery formula computes $T_M(e_k^M)$ in terms of the framing and linking numbers of the components of L, and the Alexander-Conway polynomials of L

and its sublinks. Thus, the algebraic number of monopoles can be computed (at least up to sign) in terms of classical link invariants. For simplicity, we state here the surgery formula only in the case of algebraically split links and the first Betti number ≥ 2.

Let $L = L_1 \cup ... \cup L_m \subset S^3$ be an oriented algebraically split link (i.e., $lk(L_i, L_j) = 0$ for all $i \neq j$). Recall the Alexander-Conway polynomial $\nabla_L \in \mathbf{Z}[t_1^{\pm 1}, ..., t_m^{\pm 1}]$, see [3]. Since L is algebraically split, ∇_L is divisible by $\prod_{i=1}^m (t_i^2 - 1)$ in $\mathbf{Z}[t_1^{\pm 1}, ..., t_m^{\pm 1}]$. We have a finite expansion

$$\nabla_L / \prod_{i=1}^m (t_i^2 - 1) = \sum_{l=(l_1,...,l_m)\in \mathbf{Z}^m} z_l(L)\ t_1^{l_1} ... t_m^{l_m}$$

where $z_l(L) \in \mathbf{Z}$.

Let M be obtained by surgery on L with framing $f = (f_1, ..., f_m) \in \mathbf{Z}^m$. Denote by J_0 the set of all $j \in \{1, ..., m\}$ such that $f_j = 0$. For a set $J \subset \{1, ..., m\}$, denote the link $\cup_{j\in J} L_j$ by L^J. Put $|J| = \mathrm{card}(J)$ and suppose that $|J_0| \geq 2$. Then for any charge $k = (k_1, ..., k_m)$ on L,

$$\begin{aligned} &T_M(e_k^M) \\ &= (-1)^{m+1} \sum_{J_0 \subset J \subset \{1,...,m\}} (-1)^{|J\setminus J_0|} \prod_{j\in J\setminus J_0} \mathrm{sign}(f_j) \sum_{l\in \mathbf{Z}^J, l=-k\, (\mathrm{mod}\, 2f)} z_l(L^J). \end{aligned} \tag{1}$$

Here the sum goes over all sets $J \subset \{1, ..., m\}$ containing J_0. The sign $\mathrm{sign}(f_j) = \pm 1$ of f_j is well defined since $f_j \neq 0$ for $j \in J \setminus J_0$. The formula $l \in \mathbf{Z}^J, l = -k\, (\mathrm{mod}\, 2f)$ means that l runs over all tuples of integers numerated by elements of J such that $l_j = -k_j\, (\mathrm{mod}\, 2f_j)$ for all $j \in J$. By $|J| \geq |J_0| \geq 2$, the algebraically split link L^J has ≥ 2 components so that $z_l(L^J)$ is a well defined integer. Only a finite number of these integers are non-zero and therefore the sum in (1) is finite. This sum obviously depends only on $k (\mathrm{mod}\, 2f)$; the Euler structure e_k^M also depends only on $k (\mathrm{mod}\, 2f)$.

For a link $L = L_1 \cup ... \cup L_m$ which is not algebraically split, the polynomial ∇_L can be divided by $\prod_{i=1}^m (t_i^2 - 1)$ in a certain quotient of $\mathbf{Z}[t_1^{\pm 1}, ..., t_m^{\pm 1}]$. This leads to a surgery formula for an arbitrary L, see [22]. Formula (1) and its generalizations to non-algebraically split links yield similar formulas for the Alexander polynomial $\Delta(M)$ and the Casson-Walker-Lescop invariant of M (in the case $b_1(M) \neq 0$).

Moments of the torsion function

Every $e \in \mathrm{Eul}(M)$ has a characteristic class $c(e) \in H = H_1(M)$ defined as the unique element of H such that $e = c(e) e^{-1}$. This class is the first obstruction to the existence of a homotopy between a vector field representing e and the opposite vector field. For any $x_1, ..., x_m \in H^1(M; \mathbf{R})$, we define the corresponding *m-th moment* of T_M by

$$\langle T_M \,|\, x_1, ..., x_m \rangle = \sum_{e \in \mathbf{Eul}(M)} T_M(e) \prod_{i=1}^{m} \langle c(e), x_i \rangle$$

where on the right-hand side $\langle\, , \rangle$ is the evaluation pairing $H \times H^1(M; \mathbf{R}) \to \mathbf{R}$. It turns out that if $m \leq b_1(M) - 4$, then $\langle T_M \,|\, x_1, ..., x_m \rangle = 0$. In particular, if $b_1(M) \geq 4$ then $\sum_e T_M(e) = 0$.

Interesting phenomena occur for $m = b_1(M) - 3$. Set $b = b_1(M)$. If b is even then $\langle T_M \,|\, x_1, ..., x_{b-3} \rangle = 0$ for any $x_1, ..., x_{b-3} \in H^1(M; \mathbf{R})$. For odd b, the number $\langle T_M \,|\, x_1, ..., x_{b-3} \rangle$ is determined by the cohomology ring of M with coefficients in $\mathbf{Z}$. We state here a special case of this computation. Recall that an element of $H^1(M)$ is *primitive* if it is divisible only by ± 1. If $b \geq 3$ is odd then for any primitive $x \in H^1(M)$,

$$\langle T_M \,|\, \underbrace{x, ..., x}_{b-3} \rangle = 2^{b-3}(b-3)!\, |\mathrm{Tors} H| \det g_x \tag{2}$$

where g_x is the skew-symmetric bilinear form on the lattice $H^1(M)/\mathbf{Z}x$ defined by $g_x(y, z) = (x \cup y \cup z)([M])$ for $y, z \in H^1(M)/\mathbf{Z}x$. Note that $\det g_x = (\mathrm{Pf}(g_x))^2 \geq 0$. This computation implies for instance that if x is dual to the homology class of a (singular) closed oriented surface in M of genus $\leq (b-3)/2$, then $\sum_e T_M(e) \langle c(e), x \rangle^{b-3} = 0$. For $b = 3$, formula (2) gives

$$\sum_e T_M(e) = |\mathrm{Tors} H|\, ((x \cup y \cup z)([M]))^2 \tag{3}$$

where x, y, z is any basis of $H^1(M)$. Formula (3) and the equality $\sum_e T_M(e) = \lambda(M)$ yield Lescop's computation of $\lambda(M)$ for $b_1(M) = 3$.

Basic Euler structures and the Thurston norm

An Euler structure $e \in \mathrm{Eul}(M)$ is said to be *basic* if $T_M(e) \neq 0$. The set of basic Euler structures is closely related to the Thurston norm, see [17]. Recall that the Thurston norm of $s \in H^1(M)$ is defined by

$$||s||_T = \min_S \{\chi_-(S)\}$$

where S runs over closed oriented embedded (not necessarily connected) surfaces in M dual to s and $\chi_-(S) = \sum_i \max(-\chi(S_i), 0)$ where the sum runs over all components S_i of S. Then for any $s \in H^1(M)$ and any basic Euler structure e on M,

$$||s||_T \geq |\langle c(e), s\rangle|. \tag{4}$$

This inequality is a cousin of the classical Seifert inequality which says that the genus of a knot in S^3 is greater than or equal to the half of the span of its Alexander polynomial. The inequality (4) is a 3-dimensional version of the (much deeper) adjunction inequality in dimension 4. A weaker version of (4) involving $\Delta(M)$ rather than $\tau(M)$ appeared in [13]. For more general homological estimates of the Thurston norm, see [4], [23]. For analogous estimates in the Seiberg-Witten theory in dimension 3, see [1], [6], [7], [8]. Similar estimates appear also in the Ozsváth-Szabó theory [16].

Examples

Let M be the total space of an oriented circle bundle over a closed connected orientable surface Σ of genus $g \geq 0$. Let $e_\pm \in \mathrm{Eul}(M)$ be represented by the non-singular vector field on M tangent to the fibers of the bundle $M \to \Sigma$ in the positive (resp. negative) direction. Observe that $e_- = (e_+)^{-1}$ and $c(e_-) = e_-/e_+ = t^{2g-2}$ where $t \in H = H_1(M)$ is the homology class of the fiber S^1. We claim that

$$\pm\tau(M, e_-) = \pm(t-1)^{2g-2}. \tag{5}$$

Here we do not (homologically) orient M and consider the torsion up to sign. Applying (5) to the opposite orientation of the fibers, we obtain that $\pm\tau(M, e_+) = \pm(t^{-1}-1)^{2g-2}$. The same formula follows from (5) and the duality for τ.

The Thurston norm for M can be easily computed since most (if not all) generators of $H_2(M)$ are represented by tori. For any $s \in H_1(M;\mathbf{R})$, we have $||s||_T = |\langle t, s\rangle|$. In particular if the bundle $M \to \Sigma$ is non-trivial then the Thurston norm is identically 0. As an exercise, the reader may compute the torsion function for M (at least up to sign) and check (4). Similar computations are available in the Seiberg-Witten theory, see [2].

In particular, if M is the 3-torus $S^1 \times S^1 \times S^1$ and $e \in \mathrm{Eul}(M)$ is represented by the non-singular vector field on M tangent to the fibers of the obvious projection $M \to S^1 \times S^1$, then $\pm\tau(M, e) = \pm 1$. This and formula (3) imply that $\tau(M, e) = 1$ for any orientation of M.

Realization

The realization problem for τ consists in finding necessary and sufficient conditions for a pair (a finitely generated abelian group, an element of its group ring) to be realizable as the first homology group and the torsion τ of a closed connected oriented 3-manifold. In spirit of Levine's [10] realization theorem for the Alexander polynomial of links in S^3, we have the following partial result.

Let H be a finitely generated abelian group and $\lambda \in \mathbf{Z}[H]$ be symmetric with $\operatorname{aug}(\lambda) = 1$*. If a pair (H, τ) is realizible then so is $(H, \lambda\tau)$.*

Here λ is said to be *symmetric* if $\overline{\lambda} = \lambda$. For example, since the torsion of a 3-torus is 1, we obtain that any symmetric $\lambda \in \mathbf{Z}[\mathbf{Z}^3]$ with $\operatorname{aug}(\lambda) = 1$ is realizable as the torsion τ of a closed oriented 3-manifold with $H_1 = \mathbf{Z}^3$. It follows from the Bailey theorem (see [5]) and the surgery formula outlined above that any symmetric $\lambda \in \mathbf{Z}[\mathbf{Z}^2]$ is realizable as the torsion τ of a closed oriented 3-manifold with $H_1 = \mathbf{Z}^2$.

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Institut de Recherche Mathématique Avancée, Université Louis Pasteur - C.N.R.S.
7 rue René Descartes, F-67084 Strasbourg, France

Email: turaev@math.u-strasbg.fr

Received: 27 November 2001