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# QHI, 3-manifolds scissors congruence classes and the volume conjecture

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**Abstract** This is a survey of our work on Quantum Hyperbolic Invariants (QHI) of 3-manifolds. We explain how the theory of scissors congruence classes is a powerful geometric framework for QHI and for a 'Volume Conjecture' to make sense.

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## 1 Introduction

This text is based on the talk that the second author gave at the workshop *Invariants of Knots and 3-Manifolds* (RIMS, Kyoto 2001, September 17 – September 21), completed by some "private" talks he gave at the same occasion. It reports on our joint works in progress. We refer to [2, 3] for more details; in fact, in [3] we develop the ideas of sections 7–9 of [2], with some important differences in the way they are concretized. Here we content ourselves with providing precise definitions and statements. This is summarized as follows.

Let $(W, L, \rho)$ be a triple formed by a smooth compact closed oriented 3-manifold $W$, a non-empty link $L$ in $W$ and a flat principal $B$-bundle $\rho$ on $W$. We denote by $B$ the Borel subgroup of upper triangular matrices of $SL(2, \mathbb{C})$.

One associates to $(W, L, \rho)$ a *$\mathcal{D}$-scissors congruence class* $\mathfrak{c}_{\mathcal{D}}(W, L, \rho)$ which belongs to a *(pre)-Bloch-like group* $\mathcal{P}(\mathcal{D})$ built on suitably decorated tetrahedra. The class $\mathfrak{c}_{\mathcal{D}}(W, L, \rho)$ may be represented geometrically by any *$\mathcal{D}$-triangulation* of $(W, L, \rho)$ and it depends on the topology and the geometry of the triple $(W, L, \rho)$. For any odd integer $N > 1$ and for any $\mathcal{D}$-triangulation $\mathcal{T}$, one defines a *reduction* $\mathrm{mod}(N)$ $\mathcal{T}_N$ of $\mathcal{T}$. It is obtained via a "quantization" procedure using the cyclic representation theory of a quantum Borel subalgebra

 

$\mathcal{W}_N$ of $U_q(sl(2,\mathbb{C}))$ specialized at the root of unit $\omega_N = \exp(2\pi i/N)$. One of the main contributions of this work is to have pointed out the relationship between this representation theory and $B$-flat bundles, which are encoded in $\mathcal{T}$ by so called simplicial "full" 1-cocycles - see Section 2. Basically, this relationship relies on the theory of quantum coadjoint action of [6], which should ultimately lead to generalizations of the QHI, replacing $B$ by other algebraic Lie groups.

One also defines a family of complex valued *Quantum Hyperbolic Invariants* (QHI) $K_N(W,L,\rho)$, which have *state sum* expressions $K(\mathcal{T}_N)$ based on any $\mathcal{T}_N$. The elementary building blocks of $K(\mathcal{T}_N)$ are the *cyclic* $6j$*-symbols* of $\mathcal{W}_N$. Roughly speaking $K_N(W,L,\rho)$ may be considered as a function of the class $\mathfrak{c}_{\mathcal{D}}(W,L,\rho)$, which we shall write $K_N(\mathfrak{c}_{\mathcal{D}}(W,L,\rho))$ (strictly speaking this is not completely correct - see the end of §5). It turns out that when $\rho$ is the trivial flat $B$-bundle one recovers the topological invariant conjectured in [7].

The asymptotic behaviour of $K_N(W,L,\rho)$ when $N \to \infty$ should depend on the $\mathcal{D}$-scissors congruence class $\mathfrak{c}_{\mathcal{D}}(W,L,\rho)$. In fact, to any triple $(W,L,\rho)$ one can associate also an $\mathcal{I}$*-scissors congruence class* $\mathfrak{c}_{\mathcal{I}}(W,L,\rho)$ that belongs to $\mathcal{P}(\mathcal{I})$, which is an enriched version of the classical (pre)-Bloch group built on hyperbolic ideal tetrahedra. The class $\mathfrak{c}_{\mathcal{I}}(W,L,\rho)$ may be represented by an $\mathcal{I}$-triangulation $\mathcal{T}_{\mathcal{I}}$ of $(W,L,\rho)$, which is obtained by means of an explicit *idealization* of any $\mathcal{D}$-triangulation of the triple. Moreover, the explicit state sum expression $K(\mathcal{T}_N)$ of $K_N(W,L,\rho)$ tells us that

$$\lim_{N\to\infty} (2\pi/N^2)\log\left(|K_N(W,L,\rho)|\right) = G(\mathcal{T}_{\mathcal{I}}) \ , \tag{1}$$

where $|\ |$ denotes the modulus of a complex number, and $G$ basically depends on the geometry of the ideal tetrahedra of $\mathcal{T}_{\mathcal{I}}$, and on the portion of the 1-skeleton of the triangulation of $W$ which triangulates the link $L$. As $\mathcal{T}_{\mathcal{I}}$ is arbitrary, roughly speaking again, $G$ may be considered as a function $G(\mathfrak{c}_{\mathcal{I}}(W,L,\rho))$ of the $\mathcal{I}$-scissors congruence class.

Following [11, 12], there exists a refined version $\widehat{\mathcal{P}(\mathcal{I})}$ of the classical (pre)-Bloch group such that, by using hyperbolic ideal triangulations of a non-compact complete and finite volume hyperbolic 3-manifold $M$ one can define a scissors congruence class $\widehat{\beta}(M) \in \widehat{\mathcal{P}(\mathcal{I})}$. Moreover, one has

$$R(\widehat{\beta}(M)) = i(Vol(M) + iCS(M)) \mod \left((\pi^2/2)\ \mathbb{Z}\right) \ , \tag{2}$$

where $CS$ is the Chern-Simons invariant and $R : \widehat{\mathcal{P}(\mathcal{I})} \to \mathbb{C}/\left((\pi^2/2)\ \mathbb{Z}\right)$ is a natural lift of the classical Rogers dilogarithm on $\widehat{\mathcal{P}(\mathcal{I})}$.

Starting from any $\mathcal{T}_{\mathcal{I}}$ as above, one can also define a refined $\mathcal{I}$-class $\hat{\mathfrak{c}}_{\mathcal{I}}(W,L,\rho)$ and a *dilogarithmic invariant*

$$R(W,L,\rho) := R(\hat{\mathfrak{c}}_{\mathcal{I}}(W,L,\rho)) \mod \left((\pi^2/2)\ \mathbb{Z}\right) \ .$$

There are strong structural relations between the classes $\mathfrak{c}_{\mathcal{I}}(W,L,\rho)$, $\hat{\mathfrak{c}}_{\mathcal{I}}(W,L,\rho)$ and $\widehat{\beta}(M)$. These relations and the actual asymptotic expansion of the cyclic $6j$-symbols support a formulation for triples $(W,L,\rho)$ of a so-called *Volume Conjecture*, which predicts, in particular, that

$$G(\mathfrak{c}_{\mathcal{I}}(W,L,\rho)) = \mathrm{Im}\ R(\hat{\mathfrak{c}}_{\mathcal{I}}(W,L,\rho)) \ .$$

In section 8 we discuss an extension of this conjecture, involving the whole $K_N$ (not only $|K_N|$) and $R$. We stress that the transition from $\mathcal{T}$ to $\mathcal{T}_{\mathcal{I}}$ is explicit and geometric and does not involve any "optimistic" computation. On the other hand the actual identification of $G(\mathfrak{c}_{\mathcal{I}}(W,L,\rho))$ with Im $R(\hat{\mathfrak{c}}_{\mathcal{I}}(W,L,\rho))$ still sets serious analytic problems.

**Acknowledgements** We thank the referee for having forced us to correct an early wrong formulation of the Complex Volume Conjecture 8.2.

## 2 $\mathcal{D}$-tetrahedra

We fix a *base* tetrahedron $\Delta$ embedded in $\mathbb{R}^3$, with the natural cell-decomposition. We orient $\mathbb{R}^3$ by stipulating that the standard basis is positive, and $\Delta$ is oriented in accordance with it. We consider $\Delta$ up to orientation-preserving cellular self-homeomorphisms which induce the identity on the set of vertices.

A $\mathcal{D}$-decoration on $\Delta$ is a triple $((b,*),z,c)$ where:

(1) $b$ is a *branching* on $\Delta$, that is a system of edge-orientations such that no 2-face inherits a coherent orientation on its boundary. It turns out that exactly one vertex is a source and one vertex is a sink. Every simplex of the natural triangulation of the boundary of $\Delta$ inherits a "name" from the branching $b$: if one denotes by $\mathcal{V}(\Delta)$ the set of vertices, these are named by the natural ordering map

$$\{0,1,2,3\} \rightarrow \mathcal{V}(\Delta) \qquad i \rightarrow v_i$$

such that $[v_i,v_j]$ is an oriented edge of $(\Delta,b)$ if and only if $j>i$. Any other $j$-simplex is named by means of the names of its vertices. The 2-faces can be equivalently named in terms of the opposite vertices. One can select the ordered triple of oriented edges

$$(e_0 = [v_0,v_1],\ e_1 = [v_1,v_2],\ e_2 = -[v_2,v_0] = [v_0,v_2])$$

which are the edges of the face opposite to the vertex $v_3$. For every edge $e$ of $\Delta$, one denotes by $e'$ the *opposite* edge. A branching induces an orientation on $\Delta$ defined by the basis $(e'_0,\ e'_1,\ e'_2)$, considered as an ordered triple of vectors at $v_3$. If this branching-orientation agrees with the fixed orientation of $\Delta$ the branching is said *positive*, and *negative* otherwise - see Figure 1. We will write $*(\Delta, b)$, $* \in \{+,-\}$, to encode the sign of the $b$-orientation of $\Delta$. We stipulate that $\bar{*} = -*$ with the usual "sign rule". In a similar way, every simplex in $\partial\Delta$ inherits an orientation from $b$.

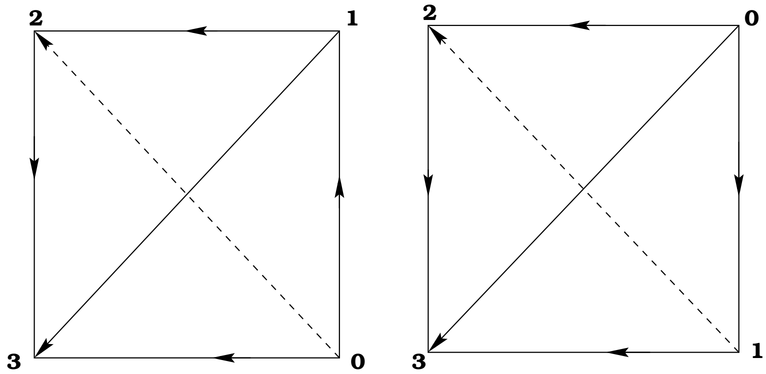


Figure 1: $+(\Delta, b)$ and $-(\Delta, b)$

(2) $z$ is a *full* $B$-valued 1-cocycle on $(\Delta, b)$, where $B$ is the Borel subgroup of upper triangular matrices of $SL(2,\mathbb{C})$. To specify $z$ one uses the edge-orientation given by the branching $b$. For each oriented edge $e$, $z(e)$ is an upper triangular matrix having $(t(e), 1/t(e))$ on the diagonal and upper diagonal entry equal to $x(e)$. So one can identify $z(e)$ with $(t(e), x(e))$. "Full" means that $x(e) \neq 0$ for every $e$.

(3) $c$ is an *integral charge* on $\Delta$. Let $\mathcal{E}(\Delta)$ denote the set of edges of $\Delta$; then $c : \mathcal{E}(\Delta) \to \mathbb{Z}$ is such that

- for every $e \in \mathcal{E}(\Delta)$, $c(e) = c(e')$;
- If $e_i$, $i = 0,\ 1,\ 2$, are the edges of any 2-face of $\Delta$ and $c_i = c(e_i)$, then

$$c_0 + c_1 + c_2 = 1\ .$$

It is useful (and pertinent - see §3) to look at $c(e)\pi$ as a dihedral angle; in this way the $c_i\pi$'s formally behave like the dihedral angles of hyperbolic ideal tetrahedra.

Let us call $\mathcal{D}$ the set of all $\mathcal{D}$-tetrahedra $*(\Delta, b, z, c)$. Denote by $\mathbf{S}_4$ the group of permutations on 4 elements. Changing the branching (i.e. permuting the order of the vertices) induces a natural action $p_{\mathcal{D}}$ of $\mathbf{S}_4$ on $\mathcal{D}$ with

$$p_{\mathcal{D}}(s, *(\Delta, b, z, c)) = \epsilon(s) * (\Delta, s(b), s(z), s(c)) \ , \tag{3}$$

where $\epsilon(s)$ is the signature of the permutation $s$, $s(b)$ is the new branching obtained by permutation of the vertex ordering, and for every $e \in \mathcal{E}(\Delta)$ we have $s(z)(e) = z(e)$ iff the edge $e$ keeps the same orientation and $s(z)(e) = z(e)^{-1}$ otherwise, and $s(c)(e) = c(e)$.

## 3 The (pre)-Bloch-like group $\mathcal{P}(\mathcal{D})$

Let $\mathbb{Z}[\mathcal{D}]$ be the free $\mathbb{Z}$-module generated by the $\mathcal{D}$-tetrahedra $*(\Delta, b, z, c)$; recall that we set $\bar{*}(\Delta, b, z, c) = (-1) * (\Delta, b, z, c)$.

In this section we shall describe a notion of $2 \leftrightarrow 3$ *$\mathcal{D}$-transit* between decorated triangulations such that every instance of $2 \leftrightarrow 3$ $\mathcal{D}$-transit produces a natural *five term* relation in $\mathbb{Z}[\mathcal{D}]$. By definition $\mathcal{P}(\mathcal{D}) = \mathbb{Z}[\mathcal{D}]/T(\mathcal{D})$ is the *(pre)-Bloch-like $\mathcal{D}$-group*, where $T(\mathcal{D})$ is the module generated by all these five term relations and by the relations (3).

The support of any $2 \leftrightarrow 3$ $\mathcal{D}$-transit is the usual $2 \leftrightarrow 3$ move on 3-dimensional triangulations. Let us specify how the decorations transit. In Figure 2, forgetting the charge for a while, one can see an instance of *branching transit.* The branching transits are also carefully analyzed in [5], in terms of the dual viewpoint of "branched spines".

A *charge transit* is branching independent. Consider a $2 \leftrightarrow 3$ move $T_0 \leftrightarrow T_1$ relating two triangulations of some 3-manifold, and suppose that the tetrahedra involved in the move are endowed with integral charges. There are 9 edges $E_1, \ldots, E_9$ which are present in both $T_0$ and $T_1$ and a further edge $E_0$ which is present only in $T_1$. Also, for $i = 1, \ldots, 9$ , $E_i$ is an edge of exactly one tedrahedron of $T_k$ iff it is an edge of exactly two tetrahedra of $T_{k+1}$ (where $k \in \mathbb{Z}/2\mathbb{Z}$), and $E_0$ is an edge of exactly 3 tetrahedra of $T_1$. Denote by $\gamma_k(E_i)$, $i = 1, \ldots, 9$ , the sum of the integral charges at $E_i$ of the tetrahedra of $T_k$ which have $E_i$ among their edges (by convention put $\gamma_0(E_0) = 0$). The integral charges on $T_0$ and $T_1$ define a $2 \leftrightarrow 3$ *charge transit* iff they coincide for the tetrahedra not involved in the move, and the following relations are satisfied:

$$\gamma_0(E_i) = \gamma_1(E_i) \ , \quad i = 1, \ldots, 9 \ .$$

These linear relations imply $\gamma_1(E_0) = 2$, which together with the second condition in §2 (3) for the integral charges on $T_1$ read:

$$\theta(1) + \theta(2) + \theta(3) = 2 \quad ; \quad \theta(j) + \alpha(j) + \beta(j) = 1 \ , \quad j = 1, 2, 3 \ .$$

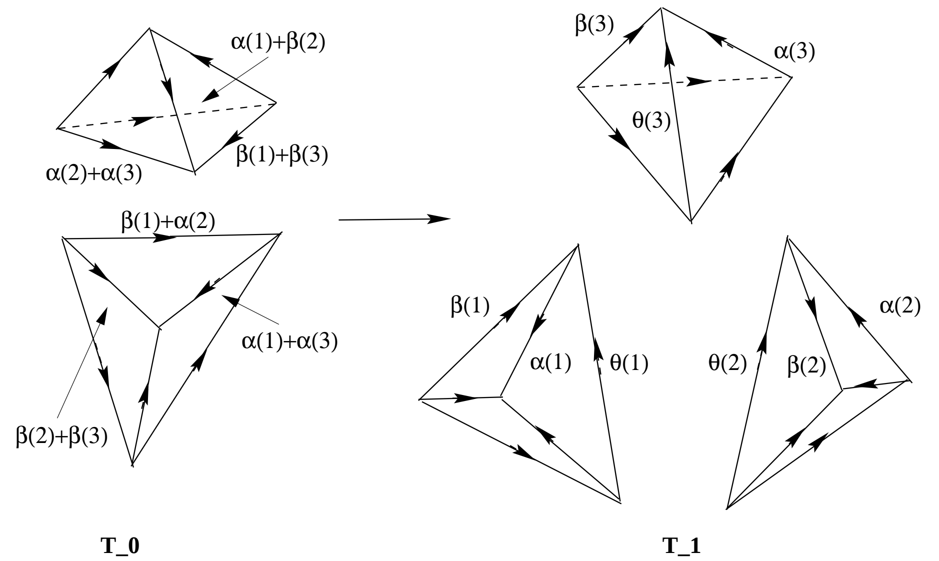


Figure 2: An instance of branching and charge transit

These two relations show that a charge transit is coherent with the above "dihedral angle" interpretation of the integral charges.

The last decoration component of a $\mathcal{D}$-decoration is the cocycle $z$. Let $T_0$, $T_1$ be as above and $z_0 \in Z^1(T_0;B)$. Then there is only one 1-cocycle $z_1$ in $Z^1(T_1;B)$ such that $z_0(E_i) = z_1(E_i)$ for $i = 1,\dots,9$ . It defines the *cocycle transit*, providing that both cocycles $z_0$ and $z_1$ are full. In Figure 3 one can see an instance of cocycle transit where, for the sake of simplicity, we have used cocycles in $Z^1(T_k;\mathbb{C}) \cong Z^1(T_k;Par(B))$, where $Par(B)$ is the parabolic subgroup of $B$ of matrices with unitary diagonal.

Any instance of $2 \to 3$ $\mathcal{D}$-transit induces a five term relation in $\mathbb{Z}[\mathcal{D}]$; note that we have to take into account the orientations (i.e. the signs) of the 5 involved tetrahedra in these relations.

# 4 The $\mathcal{D}$-scissors congruence class

Let $(W,L,\rho)$ be a triple formed by a smooth compact closed oriented 3-manifold $W$, a non-empty link $L$ in $W$ and a flat principal $B$-bundle $\rho$ on $W$. The triple

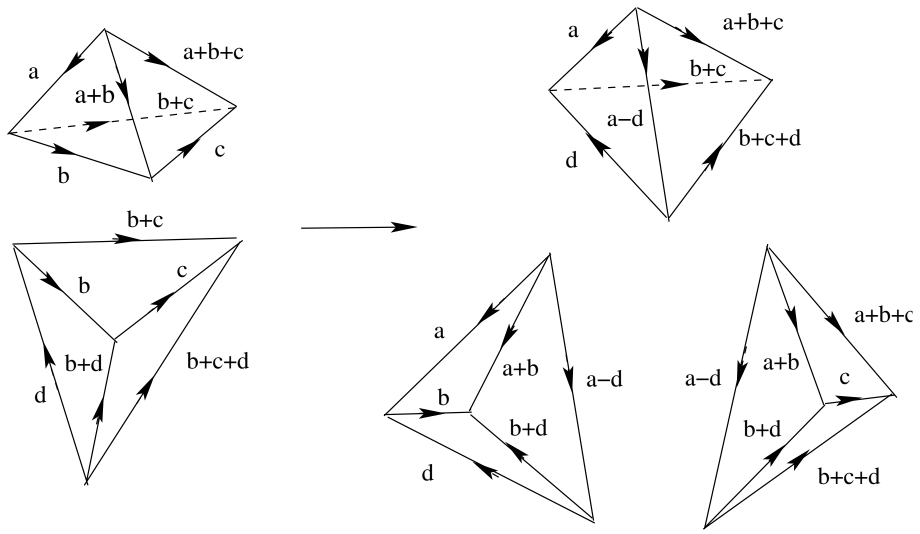


Figure 3: An instance of cocycle transit

is regarded up to orientation-preserving diffeomorphisms of $(W, L)$ which are flat bundle isomorphisms.

A *distinguished triangulation* $(T, H)$ of $(W, L)$ is a (singular) triangulation $T$ of $W$ such that $L$ is realized by a *Hamiltonian* sub-complex $H$ of $T$ that contains all the vertices. Such a triangulation can be interpreted as a certain finite family $\{\Delta_1, \dots, \Delta_k\}$ of copies of our base tetrahedron endowed with a set of 2-faces identification (pairing) rules. A $\mathcal{D}$*-decoration* $(b, z, c)$ on $(T, H)$ consists of a family of $\mathcal{D}$-tetrahedra $\{*_i(\Delta_i, b_i, z_i, c_i)\}$, $i = 1, \dots, k$, which is compatible with the face pairings; this means that

(1) the pairings respect the edge-orientations due to the branchings; thus they give us a global branching $b$ on $T$ (see [5]);

(2) identified edges have the same cocycle value, so that the $z_i$'s define a global $B$-valued full 1-cocycle $z$ on $T$;

(3) denote by $\mathcal{E} = \coprod_i \mathcal{E}(\Delta_i)$ the set of all edges of all $\Delta_i$'s; the system of integral charges $c_i$ can be considered as a global integral charge $c : \mathcal{E} \to \mathbb{Z}$.

Moreover, one imposes the following global constraints:

(4) a branching $b_i$ is positive if and only if the corresponding $b_i$-orientation on $\Delta_i$ agrees with the one of the manifold $W$;

(5) the global cocycle $z$ on $T$ represents the flat bundle $\rho$ on $W$: $\rho = [z]$;

(6) denote by $\mathcal{E}(T)$ the set of edges of $T$. There is a natural projection map $p : \mathcal{E} \to \mathcal{E}(T)$. Then one requires that for every $s \in \mathcal{E}(T) \setminus \mathcal{E}(H)$

$$\sum_{e \in p^{-1}(s)} c(e) = 2 \ ,$$

and for every $s \in \mathcal{E}(H)$

$$\sum_{e \in p^{-1}(s)} c(e) = 0 \ .$$

(7) to every $c$ which satisfies condition (6) one can associate an element $[c] \in H^1(W; \mathbb{Z}/2\mathbb{Z})$; one finally imposes that $[c] = 0$.

Note that, by property (6), any integral charge on $(T, H)$ actually encodes $H$. We say that that a triangulation $T$ of $W$ is *fullable* if it carries a full 1-cocycle representing the trivial flat $B$-bundle. This is equivalent to the fact that every edge of $T$ has two distinct vertices. If $T$ is fullable then for any flat $B$-bundle $\rho$ it carries a full 1-cocycle representing $\rho$. Finally $\mathcal{T} = (T, H, (b, z, c))$ is called a *$\mathcal{D}$-triangulation* of $(W, L, \rho)$ if $T$ is fullable and $z$ is full.

**Theorem 4.1** *For every triple $(W, L, \rho)$ there exist $\mathcal{D}$-triangulations.*

To any $\mathcal{D}$-triangulation $\mathcal{T}$ one can associate the formal sum $\mathfrak{c}_{\mathcal{D}}(\mathcal{T}) \in \mathbb{Z}[\mathcal{D}]$ of its decorated tetrahedra (all the coefficients being equal to 1).

**Theorem 4.2** *The equivalence class of $\mathfrak{c}_{\mathcal{D}}(\mathcal{T})$ in $\mathcal{P}(\mathcal{D})$ does not depend on the choice of $\mathcal{T}$. Thus it defines an element $\mathfrak{c}_{\mathcal{D}}(W, L, \rho) \in \mathcal{P}(\mathcal{D})$, which is called the* $\mathcal{D}$-scissors congruence class *(or* $\mathcal{D}$-class*) of the triple $(W, L, \rho)$.*

The proof of this theorem is similar to the proof of the invariance of the QHI state sums (see the next section). However, the proof requires more triangulation moves (such as the so called *bubble* move) than the only $2 \leftrightarrow 3$ move. A remarkable fact is that these further moves do not introduce new independent algebraic relations in $\mathbb{Z}[\mathcal{D}]$.

## 5 Quantum hyperbolic state sum invariants

Let $(W, L, \rho)$ be as in §4 and fix any $\mathcal{D}$-triangulation $\mathcal{T} = (T, H, (b, z, c))$ of $(W, L, \rho)$. Let $N > 1$ be an odd integer. Fix a determination of the $N$th-root which holds for all the matrix entries $t(e)$ and $x(e)$ of $z(e)$, for all the edges $e$ of $T$. The *reduction mod*$(N)$ $\mathcal{T}_N$ of $\mathcal{T}$ consists in changing the decoration of each edge $e$ of $T$ as follows:

- $(a(e) = t(e)^{1/N},\ y(e) = x(e)^{1/N})$ instead of $z(e) = (t(e), x(e))$;
- $c_N(e) = c(e)/2 \mod(N)$ instead of $c(e)$ (it makes sense because $N$ is odd).

Let us interpret this new decoration. All the details and justifications of the explicit formulas given below can be found in [1], see also the appendix of [2]. Consider a quantum Borel subalgebra $\mathcal{W}_N$ of $U_q(sl(2,\mathbb{C}))$, specialized at the root of unit $\omega_N = \exp(2\pi i/N)$. Each $(a(e), y(e))$ describes an irreducible $N$-dimensional *cyclic representation* $r_N(e)$ of this algebra.

Call $\mathcal{F}(T)$ the set of 2-faces of $T$. A $N$*-state* of $T$ is a function $\alpha : \mathcal{F}(T) \to \{0, 1, \dots, N-1\}$ (in fact one often identifies $\{0, 1, \dots, N-1\}$ and $\mathbb{Z}/N\mathbb{Z}$). The state $\alpha$ can be considered as a family of functions $\alpha_i : \mathcal{F}(\Delta_i) \to \{0, 1, \dots, N-1\}$ which are compatible with the face pairings.

Consider on each branched tetrahedron $(\Delta_i, b_i)$ of $\mathcal{T}$ the ordered triple of oriented edges $(e_0 = [v_0, v_1],\ e_1 = [v_1, v_2],\ e_2 = -[v_2, v_0])$ which are the opposite edges to the vertex $v_3$. The cocycle property of $z$ and the fullness assumption (this is crucial at this point, due to the algebraic structure of the cyclic representations of $\mathcal{W}_N$) imply that $r_N(e_0) \otimes r_N(e_1)$ coincides up to isomorphism with the direct sum of $N$ copies of $r_N(e_2)$. This set of data allows one to associate to every $*(\Delta_i, b_i, r_{N,i}, \alpha_i)$ a 6j-symbol $R(*(\Delta_i, b_i, r_{N,i}, \alpha_i)) \in \mathbb{C}$, that is a matrix element of a suitable "intertwiner" operator. The reduced charge $c_N$ is used to slightly modify this operator in order to get its (partial) invariance up to branching changes. In this way one gets the (partially) symmetrized *c-6j-symbols* $T(*(\Delta_i, b_i, r_{N,i}, c_i, \alpha_i)) \in \mathbb{C}$. We are now ready to define the state sums $H(\mathcal{T}_N)$, which are *weighted* traces. Denote by $V$ the number of vertices of $T$. Set

$$\Psi(\mathcal{T}_N) = \sum_{\alpha} \prod_{i} T(*(\Delta_i, b_i, r_{N,i}, c_i, \alpha_i))$$

$$H(\mathcal{T}_N) = \Psi(\mathcal{T}_N)\ N^{-V} \prod_{e \in \mathcal{E}(T) \setminus \mathcal{E}(H)} x(e)^{(N-1)/N}\ . \tag{4}$$

**Theorem 5.1** *Up to multiplication with $N$th-roots of unity, the scalar $H(\mathcal{T}_N)$ does not depend on the choice of $\mathcal{T}$. Hence $K(\mathcal{T}_N) := H(\mathcal{T}_N)^N$ defines an invariant $K_N(W, L, \rho)$ of the triple $(W, L, \rho)$.*

**Proposition 5.2** *Let $\rho = [z] = [(t, x)]$.*

- *For every $\lambda \neq 0$ set $\lambda\rho = [(t, \lambda x)]$. Then $K_N(W, L, \rho) = K_N(W, L, \lambda\rho)$ .*

- *Denote by $z^*$ the complex conjugate cocycle of $z$ and by $\rho^* = [z^*]$ the corresponding flat $B$-bundle. Let $-W$ be $W$ endowed with the opposite orientation. Then $K_N(-W, L, \rho^*) = K_N(W, L, \rho)^*$ .*

Kashaev proposed in [7] a conjectural purely topological invariant $K_N(W, L)$, which should have been expressed by a state sum as in (4). In fact, one eventually recognizes such a $K_N(W, L)$ as the special case of our $K_N(W, L, \rho)$ when $\rho$ is the *trivial* flat $B$-bundle on $W$ (although in [7] there are neither flat bundles, nor geometric interpretations nor existence results of all the datas).

The algebraic properties of the c-$6j$-symbols (the "pentagon relation" and so on) ensure the invariance of $K(\mathcal{T}_N)$ up to $\mathcal{D}$-transits supported by certain "bare" triangulation moves ($2 \leftrightarrow 3$ moves as defined in §3, "bubble moves",...). Rephrased in our setup, this was the main achievement of [7]. However, one cannot deduce the complete invariance of $K(\mathcal{T}_N)$ solely from this $\mathcal{D}$-transit invariance because it is difficult to connect by $\mathcal{D}$-transits two given $\mathcal{D}$-triangulations of a triple $(W, L, \rho)$. For example, "negative" $3 \to 2$ moves are in general not "brancheable", and full cocycles do not transit, in general, to full cocycles. Also the charge invariance relies deeply on the fact that the set of integral charges of any fixed distinguished triangulation is an integral lattice.

The nature of the ambiguity of $H(\mathcal{T}_N)$, up to $N$th-roots of unity, is not yet clear to the authors. The problem is due to the symmetrization procedure, which turns $6j$-symbols into c-$6j$-symbols. Indeed, the $6j$-symbols have a very subtle behaviour w.r.t. branchings: only the pentagon relations corresponding to a non-trivial proper subset of branched $2 \leftrightarrow 3$ moves are valid.

As the value of $K(\mathcal{T}_N)$ does not depend on the choice of $\mathcal{T}$ one would like to consider $K_N(W, L, \rho)$ as a function of the $\mathcal{D}$-class $\mathfrak{c}_\mathcal{D}(W, L, \rho)$. This is not completely correct because the face pairings between the $\mathcal{D}$-tetrahedra of $\mathcal{T}$ are not encoded in the representatives $\mathfrak{c}_\mathcal{D}(\mathcal{T})$ of $\mathfrak{c}_\mathcal{D}(W, L, \rho)$. Moreover, the states as well as the non-$\Psi(\mathcal{T}_N)$ factors in the right-hand side of (4) depend on the face pairings. This is a technical point which can be overcome as follows, by looking at $K(\mathcal{T}_N)$ as a function of a formal sum of "augmented" $\mathcal{D}$-tetrahedra.

A *$\widetilde{\mathcal{D}}$-tetrahedron* is of the form $*(\Delta, b, z, c, v^0, v^1, v^2)$ where $v^0, v^1, v^2$ are $\mathbb{N}$-valued functions defined on $\mathcal{V}(\Delta)$, $\mathcal{E}(\Delta)$, $\mathcal{F}(\Delta)$ respectively. Let $\Gamma \in \mathbb{Z}[\widetilde{\mathcal{D}}]$ be a formal sum of terms with coefficients all equal to 1. For every $\sigma : \mathbb{N} \to \mathbb{Z}/N\mathbb{Z}$ set $\alpha_i(\sigma) = \sigma \circ v_i^2$. Say that $\sigma$ and $\sigma'$ are "identified modulo $\Gamma$" if $\alpha_i = \alpha_i'$. Put

$$\Phi = \prod_{w \in \mathcal{V}(\Delta_i)} N^{-1/v_i^0(w)} \quad , \quad \Omega = \prod_{e \in \mathcal{E}(\Delta_i)} x(e)^{(N-1)/v_i^1(e)N}$$

$$\widetilde{T}(*(\Delta_i, b_i, r_{N,i}, c_i, \alpha_i(\sigma), v_i^0, v_i^1)) = T(*(\Delta_i, b_i, r_{N,i}, c_i, \alpha_i(\sigma)) \ \Phi \ \Omega \ .$$

Finally set

$$H(\Gamma) = \sum_{\sigma \in (\mathbb{Z}/N\mathbb{Z})^{\mathbb{N}}/\Gamma} \prod_i \widetilde{T}(*(\Delta_i, b_i, r_{N,i}, c_i, \alpha_i(\sigma), v_i^0, v_i^1)) \ .$$

Let $\mathcal{T}$ be a $\mathcal{D}$-triangulation. For every vertex $w$ of $\Delta_i$ set $v_i^0(w) = |p_0^{-1}p_0(w)| \in \mathbb{N}$, where $p_0$ is the identification map in $T$ of the vertices of the $\Delta_i$'s. For every edge $e$ of $\Delta_i$ set $v_i^1(e) = |p^{-1}p(e)| \in \mathbb{N}$, where $p : \mathcal{E} \to \mathcal{E}(T)$ is as in §4 (6). Finally order $\mathcal{F}(T)$ via an arbitrary $\mathbb{N}$-valued map which is compatible (for the face pairings) with $v^2$. In this way we get a $\widetilde{\mathcal{D}}$-triangulation $\widetilde{\mathcal{T}}$ with the associated $\mathfrak{c}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{T}}) \in \mathbb{Z}[\widetilde{\mathcal{D}}]$. It is clear that

$$K_N(W, L, \rho) = H(\mathfrak{c}_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{T}}))^N \ .$$

Following the lines of the above construction, it is not hard to define the notions of $\widetilde{\mathcal{D}}$-transit, (pre)-Bloch-like group $\mathcal{P}(\widetilde{\mathcal{D}})$ and $\widetilde{\mathcal{D}}$-class $\mathfrak{c}_{\widetilde{\mathcal{D}}}(W, L, \rho) \in \mathcal{P}(\widetilde{\mathcal{D}})$, so that one can eventually define a map $K_N : \mathcal{P}(\widetilde{\mathcal{D}}) \to \mathbb{C}$ such that

$$K_N(W, L, \rho) = K_N(\mathfrak{c}_{\widetilde{\mathcal{D}}}(W, L, \rho)) \ .$$

Moreover, there is a "forgetting map" $\mathcal{P}(\widetilde{\mathcal{D}}) \to \mathcal{P}(\mathcal{D})$ sending $\mathfrak{c}_{\widetilde{\mathcal{D}}}(W, L, \rho)$ to $\mathfrak{c}_{\mathcal{D}}(W, L, \rho)$. With this technical precision in mind, we shall say anyway that, roughly speaking, $K_N(W, L, \rho)$ depends on $\mathfrak{c}_{\mathcal{D}}(W, L, \rho)$.

## 6 The (pre)-Bloch like group $\mathcal{P}(\mathcal{I})$

Let us go back to our base tetrahedron $\Delta$. An $\mathcal{I}$-*decoration* on $\Delta$ is a triple $((b,*), w, c)$ where $(b,*)$ and $c$ are as in the $\mathcal{D}$-decorations and $w : \mathcal{E}(\Delta) \to \mathbb{C} \setminus \{0,1\}$ is such that :

- for every $e \in \mathcal{E}(\Delta)$, if $e'$ is opposite to $e$, then $w(e) = w(e')$;
- if $e_0,\ e_1,\ e_2$ are the edges of the face opposite to $v_3$ and $w_i = w(e_i)$, then $w_0 w_1 w_2 = -1$ and $w_0 w_1 - w_1 = -1$.

Clearly, $w' = w_1 = (1 - w_0)^{-1}$, $w'' = w_2 = (1 - w_1)^{-1}$ and $w = w_0 = (1 - w_2)^{-1}$, so that $(w, w', w'')$ is the *modular triple* of an oriented hyperbolic ideal tetrahedron $\bar{\Delta}$ in oriented $\mathbb{H}^3$. If $\mathrm{Im}(w) > 0$ then $\bar{\Delta}$ is oriented positively, and negatively otherwise. However, we do not require that the $b$-orientation of $*(\Delta, b, w, c)$ coincides with the sign of $\mathrm{Im}(w)$.

Figure 4 shows an instance of $2 \leftrightarrow 3$ $\mathcal{I}$-*transit*. Only the first members of the modular triples are indicated. Note that we are assuming that both on $T_0$ and $T_1$ we actually have modular triples; this means that the $\mathcal{I}$-transit is possible only if $x \neq y$. This fact is strictly related to the fullness requirement for $\mathcal{D}$-transits (see §7). Each instance of $\mathcal{I}$-transit operates on the branching and on the integral charges like a $\mathcal{D}$-transit. Recall that a modular triple determines and is determined by the dihedral angles at the edges of the corresponding ideal tetrahedra. Then, in terms of dihedral angles, modular triple transits are formally defined like integral charge transits. In another (equivalent) way, in a $2 \leftrightarrow 3$ $\mathcal{I}$-transit that splits $*(\Delta, b, w)$ into $*_1(\Delta_1, b_1, w_1)$ and $*_2(\Delta_2, b_2, w_2)$, we have $w(e)^* = w_1(e)^{*_1} w_2(e)^{*_2}$ (recall that each of $*, *_1$ and $*_2$ equals $\pm 1$).

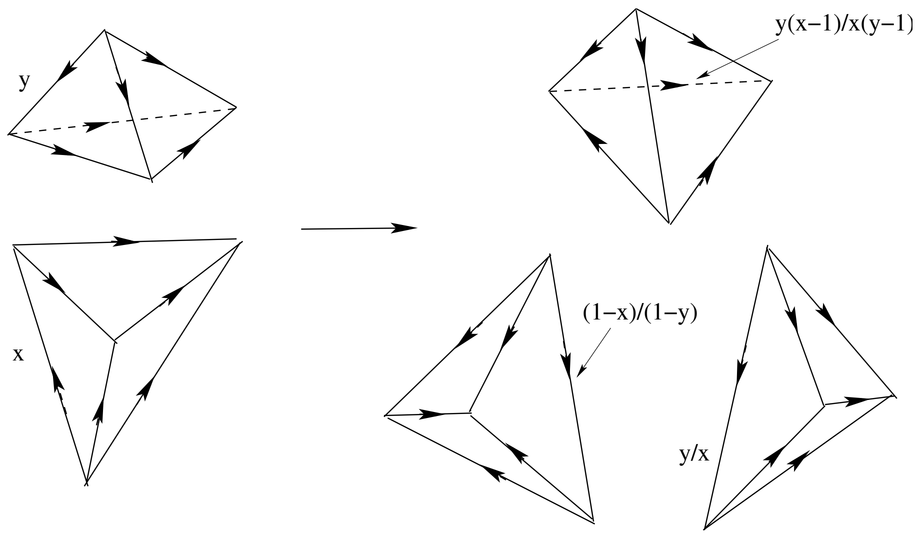


Figure 4: An instance of ideal transit

Denote by $\mathcal{I}$ the set of all $\mathcal{I}$-tetrahedra $*(\Delta, b, w, c)$. Any instance of $2 \leftrightarrow 3$ $\mathcal{I}$-transit produces a five term relation in the free $\mathbb{Z}$-module $\mathbb{Z}[\mathcal{I}]$. Also, there is a natural action $p_{\mathcal{I}}$ of $\mathbf{S}_4$ on $\mathcal{I}$ which acts as $p_{\mathcal{D}}$ in (3) on $b$, $*$ and $c$; moreover $s(w)(e) = w(e)^{\epsilon(s)}$. One defines the *(pre)-Bloch-like group* $\mathcal{P}(\mathcal{I})$ as the quotient of $\mathbb{Z}[\mathcal{I}]$ by the module generated by all the $2 \leftrightarrow 3$ $\mathcal{I}$-transit five terms relations and by the relations induced by $p_{\mathcal{I}}$.

## 7 Idealization of $\mathcal{D}$-tetrahedra

Let $*(\Delta,b,z,c) \in \mathcal{D}$, and $(e_0=[v_0,v_1],\ e_1=[v_1,v_2],\ e_2=-[v_2,v_0])$ be as in §2. The cocycle property of $z=(t,x)$ implies that

$$x(e_0)x(e_0')+x(e_1)x(e_1')+(-x(e_2)x(e_2')) := p_0+p_1+p_2=0\ . \qquad (5)$$

Define (indices $\mathrm{mod}(\mathbb{Z}_3)$):

$$F(*(\Delta,b,z,c)) = *(\Delta,b,w(z),c), \quad w_i=-p_{i+1}/p_{i+2}\ .$$

It is readily seen that $F(*(\Delta,b,z,c))$ belongs to $\mathcal{I}$; we call $F$ the *idealization map*. In fact, consider an oriented hyperbolic ideal tetrahedron $\bar{\Delta}$ with vertices $v_j \in S^2_\infty = \partial\bar{\mathbb{H}}^3$ and modular triple $(w_0,w_1,w_2)$. One can assume that every $v_j \in \mathbb{C} \subset \mathbb{C}\cup\{\infty\} = \partial\mathbb{H}^3$. If one looks at the $v_j$'s as defining a 0-cochain $u$ on $\bar{\Delta}$, then the usual *cross-ratio* expressions

$$w_0=[v_0:v_1:v_2:v_3]=\frac{(v_2-v_1)(v_3-v_0)}{(v_2-v_0)(v_3-v_1)}\ ,$$

etc., are compatible with the definition of $F$ by using the 1-cocycle given by $\delta(u)$. Note also that the idealization only depends on the "projective" class of the full cocycle, similarly to the behaviour of QHI in Proposition 5.2. Clearly, the map $F$ is onto; it extends linearly to $F:\mathbb{Z}[\mathcal{D}]\to\mathbb{Z}[\mathcal{I}]$. Remarkably one has

**Proposition 7.1** *The map $F$ induces a well-defined surjective homomorphism $\widehat{F}:\mathcal{P}(\mathcal{D})\to\mathcal{P}(\mathcal{I})$.*

## 8 Hyperbolic-like structures – A volume conjecture

A fundamental problem in QHI theory is to understand the asymptotic behaviour of the state sum invariants $K_N(W,L,\rho)$ when $N\to\infty$. Accordingly with the considerations of §5, this should depend on the $\mathcal{D}$-class of $(W,L,\rho)$. Note that one has

$$\lim_{N\to\infty}(2\pi/N^2)\log(|K_N(W,L,\rho)|) = \lim_{N\to\infty}(2\pi/N)\log(|\Psi(\mathcal{T}_N)|) \qquad (6)$$

This limit is finite and does not vanish iff $|K_N(W,L,\rho)|$ grows exponentially w.r.t. $N^2$. This is corroborated by the computation of the asymptotic behaviour of the c-$6j$-symbols; their exponential growth rate involves classical dilogarithm functions. Moreover, the explicit expression of the c-$6j$-symbols shows that

$$T(*(\Delta_i, b_i, r_{N,i}, c_i, \alpha_i)) = T(F(*(\Delta_i, b_i, r_{N,i}, c_i, \alpha_i))) \ . \tag{7}$$

If $\mathcal{T}$ is any $\mathcal{D}$-triangulation, then $F(\mathcal{T})$ is an $\mathcal{I}$-triangulation. Using it, one can define a conjugacy class of holonomy representations $\hat{\rho}_{\mathcal{T}} : \pi_1(W) \to PSL(2,\mathbb{C})$ equipped with piecewise straight equivariant maps from the universal covering of $W$ to $\bar{\mathbb{H}}^3$. Moreover, set $\mathfrak{c}_{\mathcal{I}}(F(\mathcal{T})) = \widehat{F}(\mathfrak{c}_{\mathcal{D}}(\mathcal{T})) \in \mathcal{P}(\mathcal{I})$. One has

**Theorem 8.1** *Both $\hat{\rho}_{\mathcal{T}}$ and $\mathfrak{c}_{\mathcal{I}}(F(\mathcal{T}))$ do not depend on $\mathcal{T}$. Hence they define respectively an* hyperbolic-like structure *$\hat{\rho}$ on $W$ which only depends on $(W,\rho)$, and an $\mathcal{I}$-*scissors congruence class *(shortly $\mathcal{I}$-*class*)* $\mathfrak{c}_{\mathcal{I}}(W,L,\rho)$.

The proof of Theorem 8.1 essentially follows from Theorem 4.2 and 7.1. It also uses the following remarkable geometric feature. Let $\{*(\Delta_i, b_i, w_i, c_i)\}$ be the family of $\mathcal{I}$-tetrahedra of $F(\mathcal{T})$. The union of $w_i$'s can be considered as a map $w$ defined on the set $\mathcal{E}$ of edges of all $\Delta_i$'s. Let $p : \mathcal{E} \to \mathcal{E}(T)$ be as in §4 (6). Then, for every $s \in \mathcal{E}(T)$, one has $\prod_{e \in p^{-1}(s)} w(e)^* = 1$.

If $\mathcal{T}$ is a $\mathcal{D}$-triangulation, using (7) one can rewrite (6) as

$$\lim_{N\to\infty} (2\pi/N^2) \log(|K_N(W,L,\rho)|) = G(F(\mathcal{T})) \ .$$

Here the right-hand side is a function of the hyperbolic ideal tetrahedra of $F(\mathcal{T})$ and of $H$, which is encoded by the charge in $\mathcal{T}$. As $G(F(\mathcal{T}))$ does not depend on the choice of $\mathcal{T}$, roughly speaking (see the discussion at the end of §5), $G$ is a function of the $\mathcal{I}$-class of $(W,L,\rho)$.

Starting from any idealized triangulation $F(\mathcal{T})$ of $(W,L,\rho)$ one can also define a refined $\mathcal{I}$-class $\hat{\mathfrak{c}}_{\mathcal{I}}(W,L,\rho)$. This class can be represented by a certain decorated triangulation $F'(\mathcal{T})$ which differs from $F(\mathcal{T})$ by adding a "combinatorial flattening" (see [12]) in the decoration. The trace of the latter on each tetrahedron of $T$ behaves as a signed charge, and it satisfies global constraints similar to conditions (6) and (7) in Section 4, but depending also on the moduli.

Following the comments in the Introduction, one then defines a *dilogarithmic invariant* as

$$R(W,L,\rho) := R(\hat{\mathfrak{c}}_{\mathcal{I}}(W,L,\rho)) \mod \left((\pi^2/2)\ \mathbb{Z}\right) \ .$$

A first formulation of the Volume Conjecture is:

**Conjecture 8.1** (Real Volume Conjecture) *For any $(W,L,\rho)$ we have*

$$\lim_{N\to\infty} (2\pi/N^2) \log\left(|K_N(W,L,\rho)|\right) = \mathrm{Im}\ R(\hat{\mathfrak{c}}_{\mathcal{I}}(W,L,\rho)) \ .$$

Conjecture 8.1 is in *formal* agreement with the current Volume Conjecture for hyperbolic knots in $S^3$ based on the coloured Jones polynomial $J_N$ ([8, 9],[10, 13]), because it is commonly accepted that $(J_N)^N$ is an instance of $K_N$. However, in our opinion, this has not yet been proved anywhere and the question of the relationship between $K_N$ and $J_N$ needs further investigation (see [4]).

A rough idea to extend the above conjecture for $K_N(W,L,\rho)$ (not only for its modulus) is to formally pass to an exponential version of Conjecture 8.1, and replace $\mathrm{Im}\, R(\hat{\mathfrak{c}}_{\mathcal{I}}(W,L,\rho))$ with $R(\hat{\mathfrak{c}}_{\mathcal{I}}(W,L,\rho))$. But $R(\hat{\mathfrak{c}}_{\mathcal{I}}(W,L,\rho))$ is only determined $\mathrm{mod}((\pi^2/2)\ \mathbb{Z})$. One avoids this ambiguity as follows:

**Conjecture 8.2** (Complex Volume Conjecture) *There exist invariants $C(W,L,\rho) \in \mathbb{C}\ \mathrm{mod}((\pi^2/2)\ \mathbb{Z})$ and $D = D(W,L,\rho) \in \mathbb{C}^*$ such that for any branches $R$ and $C$ of $R(\hat{\mathfrak{c}}_{\mathcal{I}}(W,L,\rho))$ and $C(W,L,\rho)$ respectively, we have*

$$K_N(W,L,\rho)^8 = \left[\exp\left(\frac{C+NR}{2i\pi}\right)\right]^{8N}\left(D+\mathcal{O}(\frac{1}{N})\right) .$$

Conjecture 8.2 says at first that $K_N(W,L,\rho)^8$ has an exponential growth rate. Assuming it, the fact that $\exp(4C/i\pi)$, $\exp(4R/i\pi)$ and $D$ are well-determined invariants of $(W,L,\rho)$ follows from the invariance of $K_N(W,L,\rho)$ and the uniqueness of the coefficients of asymptotic expansions (of Poincaré type). Moreover Conjecture 8.2 predicts that $R$ is a branch of $R(\hat{\mathfrak{c}}_{\mathcal{I}}(W,L,\rho))$.

At present, the nature of $C(W,L,\rho)$ and $D(W,L,\rho)$ is somewhat mysterious. There are no reasons to expect that, for instance, $C(W,L,\rho)=0$ or $D(W,L,\rho)=1$; in fact, their value could be related to the hard problem of finding an appropriate contour of integration in the stationary phase approach to the evaluation of the asymptotic behaviour of $K_N(W,L,\rho)$.

A natural complement to the conjecture is the problem of understanding the geometric meaning of the dilogarithmic invariant of a triple $(W,L,\rho)$.

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