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Skein module deformations of elementary moves on links

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Abstract This paper is based on my talks ("Skein modules with a cubic skein relation: properties and speculations" and "Symplectic structure on colorings, Lagrangian tangles and its applications") given in Kyoto (RIMS), September 11 and September 18 respectively, 2001. The first three sections closely follow the talks: starting from elementary moves on links and ending on applications to unknotting number motivated by a skein module deformation of a 3-move. The theory of skein modules is outlined in the problem section of these proceedings (see [12, 26, 36]).

In the first section we make the point that despite its long history, knot theory has many elementary problems that are still open. We discuss several of them starting from the Nakanishi's 4-move conjecture. In the second section we introduce the idea of *Lagrangian tangles* and we show how to apply them to elementary moves and to rotors. In the third section we apply $(2,2)$-moves and a skein module deformation of a 3-move to approximate unknotting numbers of knots. In the fourth section we introduce the Burnside groups of links and use these invariants to resolve several problems stated in Section 1.

AMS Classification 57M27; 20D99

Keywords Knot, link, skein module, n-move, rational move, algebraic tangle, Lagrangian tangle, rotor, unknotting number, Fox coloring, Burnside group, branched cover

1 Elementary moves on links: history and conjectures

Knot theory is more than two hundred years old; the first scientists who considered knots as mathematical objects were A Vandermonde (1771) and CF Gauss (1794). However, despite the impressive growth of the theory, there are

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simply formulated yet fundamental questions, to which we do not know answers. These problems are not just interesting puzzles but they lead to very interesting theory (structure).

The oldest of such problems is the Nakanishi 4-move conjecture, formulated in 1979 [17, 22, 24]. Recall that an n-move on a link is a local change of the link illustrated in Figure 1.1. In our convention the part of the link outside of the disk, in which the move takes part, is unchanged.

Two unoriented links are said to be n-move equivalent if there is a sequence of $\pm n$-moves which converts one link to the other.

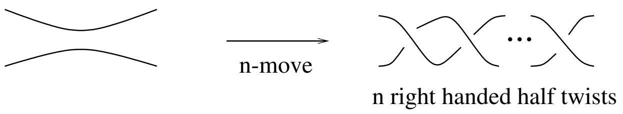


Figure 1.1: n-move

Conjecture 1.1 (Nakanishi, 1979) *Every knot is* 4*-move equivalent to the trivial knot.*

The conjecture holds for closed 3-braids and 2-algebraic links (i.e. algebraic links in the Conway sense) [17]. Because the concept of algebraic links [6, 3] and its generalizations will be used often later in the paper, let us recall the definition [30].

Definition 1.2

(i) The set of n-algebraic tangles is the smallest family of n-tangles which satisfies:
(0) Any n-tangle with 0 or 1 crossing is n-algebraic.
(1) If A and B are n-algebraic tangles then $r^i(A) * r^j(B)$ is n-algebraic; r denotes here the rotation of a tangle along the z-axis by the angle $\frac{2\pi}{2n}$, and * denotes (horizontal) composition of tangles.

(ii) If in the condition (1), B is restricted to tangles with no more than k crossings, we obtain the family of (n,k)-algebraic tangles.

(iii) If a link, L, is obtained from an (n,k)-algebraic tangle (resp. n-algebraic tangle) by connecting its endpoints without introducing any new crossings then L is called an (n,k)-algebraic (resp. n-algebraic) link.

Two examples of 2 and 4-algebraic tangles are shown in Figure 1.2.

The inductive character of our definition allows us to show:

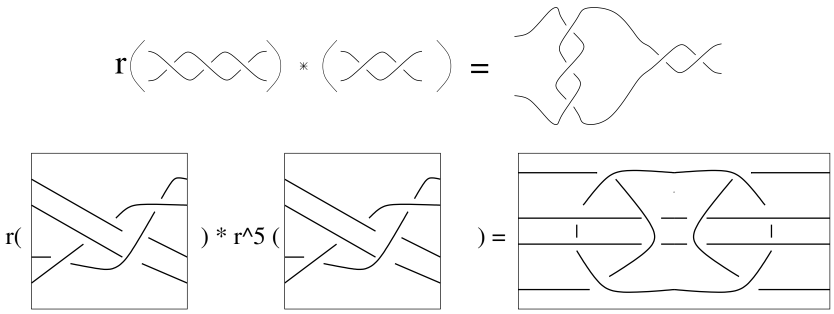


Figure 1.2

Proposition 1.3 [28]

(a) *Every 2-algebraic tangle without a closed component can be reduced by 4-moves to one of the following six 2-tangles, Figure 1.3.*

(b) *Every 2-algebraic knot can be reduced by 4-moves to the trivial knot.*

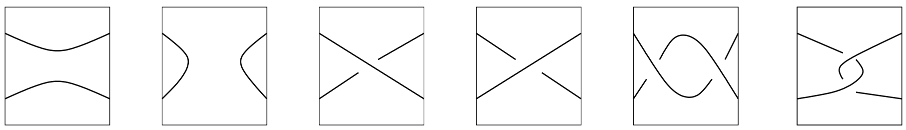


Figure 1.3

In 1994 Nakanishi began to suspect that the $(2,1)$-cable of the trefoil knot cannot be simplified by 4-moves [17]. However N.Askitas was able to simplify this knot [2]. Askitas, in turn, suspects the $(2,1)$-cable of the figure eight knot as the simplest counter-example to the Nakanishi 4-move conjecture.

Not every link can be reduced to a trivial link by 4-moves, in particular the linking matrix modulo 2 is preserved by 4-moves. Furthermore Nakanishi demonstrated that the Borromean rings cannot be reduced to the trivial link of 3-components [23].

Kawauchi expressed the question for links as follows:

Problem 1.4 [17]

(i) Is it true that if two links are link-homotopic then they are 4-move equivalent?

(ii) In particular, is it true that every 2-component link is 4-move equivalent to the trivial link of two components or to the Hopf link?

We can use an inductive argument to show that any 2-component 2-algebraic link is 4-move equivalent to a trivial link or a Hopf link. We also have proved that the answer to Kawauchi's question is affirmative for closed 3-braids [28].

Nakanishi identified the "half" 2-cabling of the Whitehead link, $\mathcal{W}$, Figure 1.5, as the simplest link which he could not reduce by 4-moves but which is link homotopy equivalent to the trivial link[1].

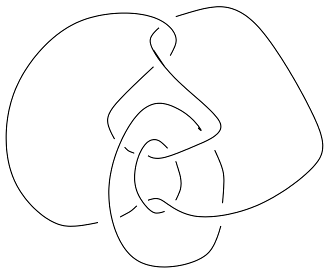


Figure 1.5

The second oldest and the most studied elementary moves question is the Montesinos-Nakanishi 3-move conjecture.

Conjecture 1.5 *Any link is 3-move equivalent to a trivial link.*

Nakanishi first considered the conjecture in 1981. J. Montesinos analyzed 3-moves before, in connection with 3-fold dihedral branch coverings [20].

The conjecture has been proved to be valid for several classes of links: by Y.Nakanishi for links up to 10 crossings and Montesinos links, by J.Przytycki for links up to 11 crossings, 2-algebraic links and closed 3-braids, by Q.Chen for links up to 12 crossings, closed 4-braids and closed 5-braids with the exception of the class of $\hat{\gamma}$ – the square of the center of the fifth braid group ($\gamma = (\sigma_1\sigma_2\sigma_3\sigma_4)^{10}$), by Przytycki and T.Tsukamoto for 3-algebraic links (including 3-bridge links), and by Tsukamoto for $(4,5)$-algebraic links (including 4-bridge links), [17, 5, 30, 35].

Nakanishi presented in 1994, an example which he could not reduce by 3-moves: the 2-parallel of the Borromean rings, L_{2BR}. The link L_{2BR} has 24 crossings and $\hat{\gamma}$ can be reduced to a link of 20 crossings (Figure 1.6) which is the smallest link not reduced yet by 3-moves[2].

[1]In June of 2002 we have shown that this example cannot, in fact, be reduced by 4-moves, [9]. We discuss this result in Section 4.

[2]We proved in [8] that neither $\hat{\gamma}$ nor L_{2BR} can be reduced by 3-moves to a trivial link; see Section 4.

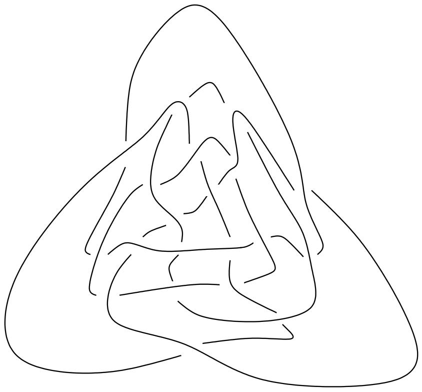
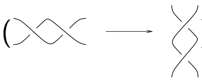
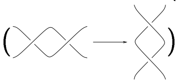


Figure 1.6: L_{2BR} and the Chen's link

Not every link can be simplified using 5-moves, but every 5-move is a combination of more delicate $(2,2)$-moves, or, more precisely of a $(2,2)$-move () and its mirror image $(-2,-2)$-move ().

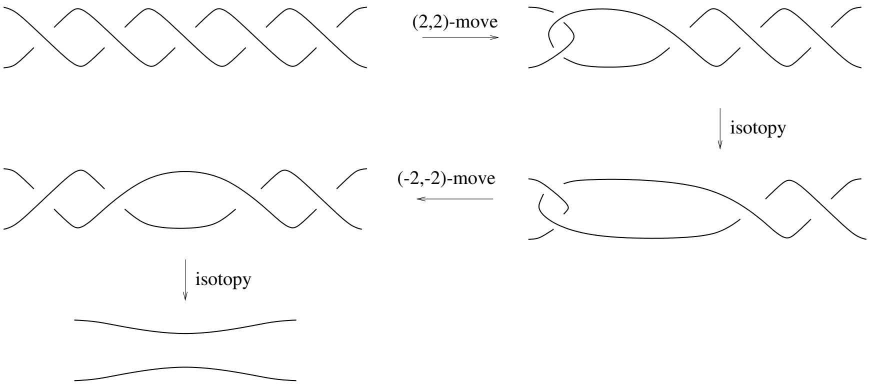


Figure 1.7: 5-move as a combination of $\pm(2,2)$-moves

Conjecture 1.6 (Harikae, Nakanishi, Uchida, 1992) *Every link is* $(2,2)$*-move equivalent to a trivial link.*

Conjecture 1.6 has been established for several classes of links.

Lemma 1.7 [28] *Every 2-algebraic link is* $(2,2)$*-move equivalent to a trivial link.*

Lemma 1.8 [28] *Every link, up to 9 crossings, is $(2,2)$-move equivalent either to a trivial link or to 9_{40} or 9_{49} knots*[3] *, or to their mirror images $\bar{9}_{40}$ and $\bar{9}_{49}$.*

Lemma 1.9 [28] *Every closed 3-braid is $(2,2)$-move equivalent to a trivial link or to the closure of the braids $(\sigma_1^2\sigma_2^{-1})^3$ or $\sigma_1^2\sigma_2^2\sigma_1^{-2}\sigma_2^2\sigma_1^2\sigma_2^{-2}$, or their mirror images.*

Notice that the knot 9_{49} and the closure of $(\sigma_1^2\sigma_2^{-1})^3$ (i.e. the link 9_{40}^2 in Rolfsen's notation [31]) are related by a $(2,2)$-move; Figure 1.8. Similarly, the knot 9_{40} and the closure of the 3-braid $\sigma_1^2\sigma_2^2\sigma_1^{-2}\sigma_2^2\sigma_1^2\sigma_2^{-2}$ are $(2,2)$-move equivalent [28].

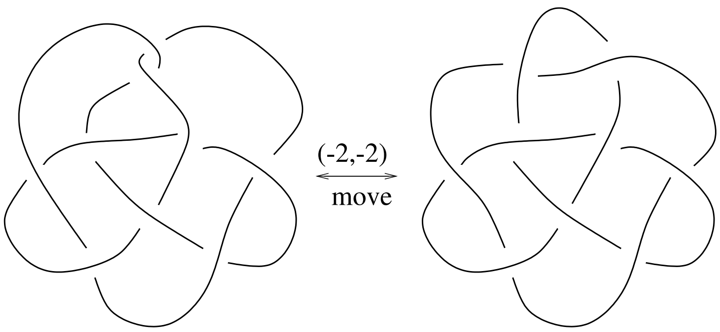


Figure 1.8

Similarly as in the case of 5-moves, not every link can be reduced via 7-moves to a trivial link. The 7-move is however a combination of $(2,3)$-moves[4] which might be sufficient for a reduction. We say that two links are $(2,3)$-move equivalent if there is a sequence of $\pm(2,3)$-moves and their inverses, $\pm(3,2)$-moves, which converts one to the other.

Problem 1.10 (Kir) Is every link $(2,3)$-move equivalent to a trivial link?

The answer is affirmative for 2-algebraic links. Furthermore every 2-algebraic tangle can be reduced to one of the 8 "basic" algebraic tangles (with possible additional trivial components) presented in Figure 1.10.

Six months ago, in March of 2000, presenting the talk "Open problems in knot theory that everyone can try to solve" at the conference in Cuautitlan I tried

[3] We showed in [9] that the knots 9_{40} and 9_{49} are not $(2,2)$-move equivalent to trivial links; see Section 4.

[4] To be precise, a 7-move is a combination of a $(-3,-2)$- and a $(2,3)$-move; compare Figures 1.9 and 1.7.

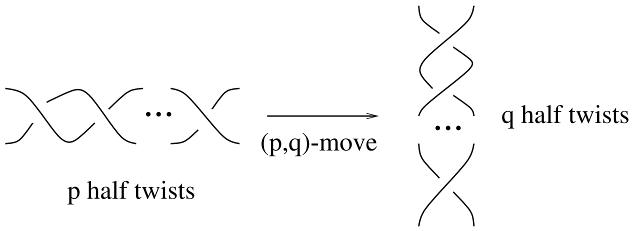


Figure 1.9

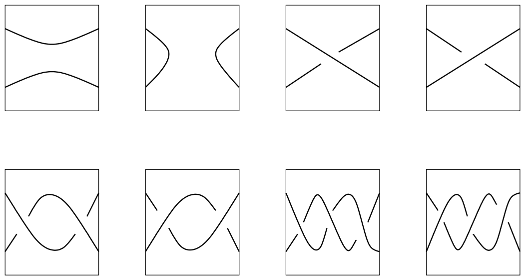


Figure 1.10

to extend the range of classical unknotting moves, so I proposed the following question.

Problem 1.11 Can every link be reduced to a trivial link by $(2,5)$ and $(4,-3)$ moves, their inverses and mirror images?

As before the question was motivated by the observation that the answer is positive for 2-algebraic links. In the proof we noticed that every 2-algebraic tangle can be reduced to one of the 12 "basic" algebraic tangles (with possible additional trivial components) presented in Figure 1.11.

We can ask: how to generalize further our problem? Notice that $(2,5)$ and $(4,-3)$ moves are equivalent to $2+\frac{1}{5}=\frac{11}{5}$ and $4+\frac{1}{-3}=\frac{11}{3}$ rational moves, respectively. Figure 1.12 illustrates the fact that an (s,q)-move is equivalent to the rational $s+\frac{1}{q}=\frac{sq+1}{q}$-move.

These suggest the following extension of the problem using rational moves[5].

Problem 1.12 Let p be a fixed prime number, then

[5]In each $\frac{p}{q}$-rational move, a 0-tangle is replaced by a rational $\frac{p}{q}$-tangle.

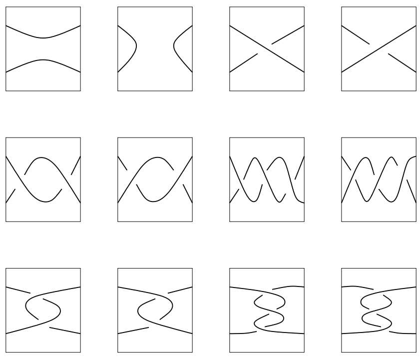


Figure 1.11

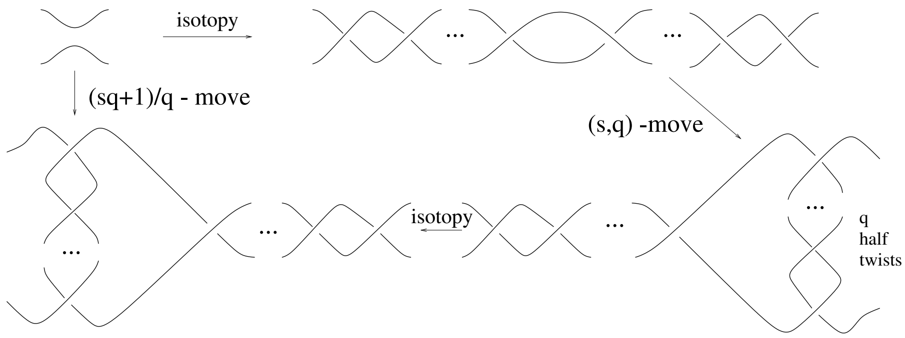


Figure 1.12: (s, q)-move as $\frac{sq+1}{q}$-move

(i) Is it true that every link can be reduced to a trivial link by rational $\frac{p}{q}$-moves (q any integer).[6]

(ii) Is there a function $f(n,p)$ such that any n-tangle can be reduced to one of "basic" $f(n,p)$ n-tangles (allowing additional trivial components) by rational $\frac{p}{q}$-moves.

The method of "Lagrangian tangles" [10] has allowed us to prove that $f(n,p) \geq \Pi_{i=1}^{n-1}(p^i + 1)$. We will discuss the method in the second section.

[6] We proved in [9] that the answer is negative: for $p \geq 5$ the closure of the 3-braid $(\sigma_1\sigma_2)^6$ cannot be reduced; compare Section 4.

2 Symplectic structures, Lagrangian tangles and Rotors

I will illustrate here how the symplectic structure can be used to answer the "classical" question about homology of double branched covers (or Fox colorings). I will describe how symplectic structures came into the picture and finally use them to analyze how homology of double branched covers changes after rotation [1, 10, 27].

Definition 2.1 *Fox colorings*

(i) We say that a link (or a tangle) diagram is k-colored if every arc is colored by one of the numbers $0, 1, ..., k-1$ (forming a group Z_k) in such a way that at each crossing the sum of the colors of the undercrossings is equal to twice the color of the overcrossing modulo k; see Figure 2.1.

(ii) The set of k-colorings forms an abelian group, denoted by $Col_k(D)$. The cardinality of the group will be denoted by $col_k(D)$. For an n-tangle T each Fox k-coloring of T yields a coloring of boundary points of T and we have the homomorphism $\psi : Col_k(T) \to Z_k^{2n}$.

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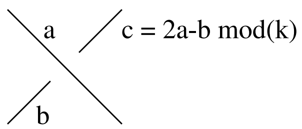


Figure 2.1

It is a pleasant exercise to show that $Col_k(D)$ is unchanged by Reidemeister moves and by k-moves (Figure 2.2).

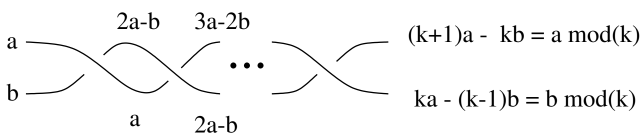


Figure 2.2

It is known that the module $Col_k(T) = H^1(M_T^{(2)}, Z_k) \oplus Z_k$ where $M_T^{(2)}$ denotes the double branched cover of D^3 with the tangle, T, as the branch set (compare [25]).

I was curious for a long time how to characterize images $\psi(Col_k(T))$. I was sure that such a description could be used to analyze elementary moves on links[7]. Below I describe the solution in which spaces $\psi(Col_k(T))$ are identified as Lagrangians in some symplectic spaces [10].

Consider $2n$ points on a circle (or a square) and a field $\mathbf{Z}_p$ of p-colorings of a point. The colorings of $2n$ points form $\mathbf{Z}_p^{2n}$ linear space. Let $e_1,\dots,e_{2n}$ be its basis,

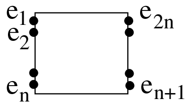


Figure 2.3

$e_i=(0,\dots,1,\dots,0)$, where 1 occurs in the i-th position. Let $\mathbf{Z}_p^{2n-1}\subset\mathbf{Z}_p^{2n}$ be the subspace of vectors $\sum a_ie_i$ satisfying $\sum(-1)^ia_i=0$ (alternating condition). Consider the basis $f_1,\dots,f_{2n-1}$ of $\mathbf{Z}_p^{2n-1}$ where $f_k=e_k+e_{k+1}$. Consider a skew-symmetric form ϕ on $\mathbf{Z}_p^{2n-1}$ of nullity 1 given by the matrix

$$\phi=\begin{pmatrix} 0 & 1 & 0 & \dots \\ -1 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & -1 & 0 \end{pmatrix}$$

that is

$$\phi(f_i,f_j)=\begin{cases} 0 & \text{if } |j-i|\neq 1 \\ 1 & \text{if } j=i+1 \\ -1 & \text{if } j=i-1. \end{cases}$$

[7] At the time I was giving my talks in Kyoto I had high hopes that the Fox 3-colorings are the only obstructions to 3-move equivalence of links. I extended this hope to tangles. From May of 2000 I was very much involved in an analysis of the structure of the space of 3-colorings of tangles, $Col_3(T)$. In particular, we had two invariants of 3-move equivalence classes of tangles: the space $Col_3(T)$ and its image under the map $\psi: Col_3(T)\to Z_3^{2n}$, where Z_3^{2n} represents the space of colorings of $2n$ boundary points of T. By interpreting tangles as Lagrangian subspaces in the symplectic space Z_3^{2n-2} we were able to prove that there are exactly $\Pi_{i=1}^{n-1}(3^i+1)$ different images $\psi(Col_3(T))$. Obstructions to the Montesinos-Nakanishi conjecture, constructed in February of 2002 (see Section 4), show however that there are more 3-move equivalence classes of n-tangles than predicted by 3-coloring invariants [8]. To understand these classes is the major task.

Notice that the vector $e_1+e_2+\ldots+e_{2n}$ $(= f_1+f_3+\ldots+f_{2n-1} = f_2+f_4+\ldots+f_{2n})$ is ϕ-orthogonal to any other vector. If we consider $\mathbf{Z}_p^{2n-2} = \mathbf{Z}_p^{2n-1}/\mathbf{Z}_p$, where the subspace $\mathbf{Z}_p$ is generated by $e_1+\ldots+e_{2n}$, that is, $\mathbf{Z}_p$ consists of monochromatic (i.e. trivial) colorings, then ϕ descends to the symplectic form $\hat{\phi}$ on $\mathbf{Z}_p^{2n-2}$. Now we can analyze isotropic subspaces of $(\mathbf{Z}_p^{2n-2}, \hat{\phi})$, that is subspaces on which $\hat{\phi}$ is 0 ($W \subset \mathbf{Z}_p^{2n-2}, \phi(w_1,w_2) = 0$ for $w_1, w_2 \in W$). The maximal isotropic ($(n-1)$-dimensional) subspaces of $\mathbf{Z}_p^{2n-2}$ are called Lagrangian subspaces (or maximal totally degenerated subspaces) and there are $\prod_{i=1}^{n-1}(p^i+1)$ of them.

We have $\psi : Col_pT \to \mathbf{Z}_p^{2n}$. Our local condition on Fox colorings (Figure 2.1) guarantees that for any tangle T, $\psi(Col_pT) \subset \mathbf{Z}_p^{2n-1}$. Furthermore, the space of trivial colorings, $\mathbf{Z}_p$, always lays in Col_pT. Thus ψ descents to $\hat{\psi} : Col_pT/\mathbf{Z}_p \to \mathbf{Z}_p^{2n-2} = \mathbf{Z}_p^{2n-1}/\mathbf{Z}_p$. Now we have the fundamental question: Which subspaces of $\mathbf{Z}_p^{2n-2}$ are yielded by n-tangles? We answer this question below.

Theorem 2.2 [10] *$\hat{\psi}(Col_pT/\mathbf{Z}_p)$ is a Lagrangian subspace of $\mathbf{Z}_p^{2n-2}$ with the symplectic form $\hat{\phi}$.*

The natural question would be whether every Lagrangian subspace can be realized by a tangle. The answer is negative for $p = 2$ and positive for $p > 2$ [10]. As a corollary we obtain a fact which was considered to be difficult before, even for 2-tangles.

Corollary 2.3 *For any p-coloring of a tangle boundary satisfying the alternating property (i.e. an element of $\mathbf{Z}_p^{2n-1}$) there is an n-tangle and its p-coloring yielding the given coloring on the boundary. In other words: $\mathbf{Z}_p^{2n-1} = \bigcup_T \psi_T(Col_p(T))$. Furthermore, the space $\psi_T(Col_p(T))$ is n-dimensional.*

A few months ago we have found a rather spectacular application of the "Lagrangian tangles" method to rotors of links [10]. We have given criteria when n-rotation preserves the space of Fox p-colorings, $Col_p(L)$.

Definition 2.4 [1] Consider an n-tangle, that is a part of the link diagram, placed in the regular n-gon with 2n boundary points (n inputs and n outputs). We say that this n-tangle is an n-rotor if it has a rotational symmetry, that is the tangle is invariant with respect to rotation along z-axis by the angle $\frac{2\pi}{n}$; see Figure 2.4.

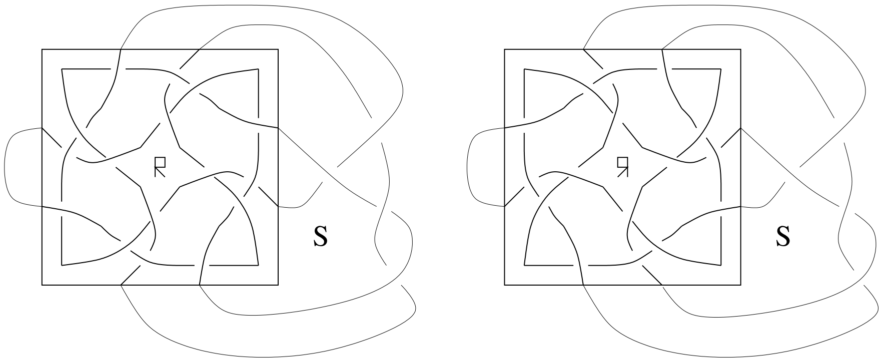


Figure 2.4

Theorem 2.5 [10] *Let L be a link diagram with an n-rotor part R. Let the rotant $\rho(L)$, be obtained from L by rotating R around the y-axis by the angle π and keeping the stator, $L-R$, unchanged. Assume that either $n=p$, where p is a prime number, or n is co-prime to p and such that there exists an s with $p^s \equiv -1 \bmod n$. Then the space $Col_p(L)$ is preserved by any n-rotation.*

The technique we use in the proof of the theorem is to analyze eigenspaces of the symplectic space Z_p^{2n-2} with respect to rotation. We obtain conditions under which the Lagrangian subspace which is invariant under rotation is also invariant under dihedral "flype" ρ. Figure 2.4 illustrates a pair of 4-rotants which have different space of Fox 5-colorings, Z_5^2 and Z_5^3, respectively. One can also compute from Figure 2.4 that $ker\psi = Z_5$, therefore there exists a nontrivial 5-coloring which is equal to 0 on the boundary of R[8].

We can construct analogous examples if $gcd(p-1,n) > 2$ [10].

3 Unknotting number from a skein module deformation of moves

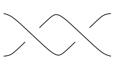
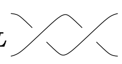
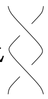
We discuss, in this section, how certain substitutions in skein module deformation of elementary moves define weighted Fox colorings and can be used to find unknotting numbers of links[9].

[8] It follows from the fact that any rotation preserves the determinant of L which is equal to the order of $H_1(M_L^{(2)}, Z)$ ([27]). Thus if a rotation changes the space of

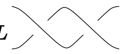
Figure 3.1

We describe here the case of the 2-move whose deformation leads to the Kauffman polynomial of links and speculate about the deformation of a 3-move, cubic polynomial and weighted 7-colorings.

In order to find a relation between the Kauffman polynomial and the Fox 5-colorings, we study first how the Kauffman polynomial is changed under $\pm(2,2)$-moves[10]. We look for substitutions for a and x such that

Condition 3.1 *F_L  $= cF_L$  and F_L  $= c'F_L$  for some constants c and c'.*

From the definition we get:

F_L  $= xF_{L_+} - F_{L_0} + a^{-1}xF_{L_\infty}$, and F_L  $= -xF_{L_+} + (a^{-1}x + x^2)F_{L_0} + (-1+x^2)F_{L_\infty}$.

If we assume Condition 3.1 and compare coefficients of elementary tangles (L_+, L_0, L_∞) we obtain: $c = c' = -a^2 = -1$ and $x^2 + ax - 1 = 0$. These lead to $x = \frac{-a\pm\sqrt{5}}{2}$ (if we let $x = b + b^{-1}$ then $x^2 + ax - 1 = b^2 + ab + 1 + ab^{-1} + b^{-2} = b^{-2}(\frac{b^5-a}{b-a})$ so b is a 5th primitive root of 1 for $a = 1$, and 10th primitive root of 1 for $a = -1$). Let us assume that $a = 1$ and $x = \frac{-1+\sqrt{5}}{2} = 2cos(2\pi/5)$. Then

5-coloring then always $ker(\psi) \neq \{0\}$.

[9]We follow the Traczyk's idea how to use invariants of links satisfying skein relations to approximate the unknotting numbers of links. Traczyk described this for the Jones polynomial $V_L(e^{2\pi i/6})$ [34, 25] and in this case $|Col_3(L)| = 3|V_L(e^{2\pi i/6})|^2$. A.Stoimenov implemented the method for the Brandt-Lickorish-Millett-Ho polynomial [33].

[10]The Kauffman polynomial is the Laurent polynomial invariant of framed links satisfying the skein relation $F_{L_+}(a,x) + F_{L_-}(a,x) = x(F_{L_0}(a,x) + F_{L_\infty}(a,x))$ (Figure 3.1), and the framing relation $F_{L^{(1)}}(a,x) = aF_L(a,x)$, where $L^{(1)}$ denotes a link obtained from L by one positive twist of the framing of L.

$F_{T_n} = (\sqrt{5})^{n-1}$ and for links $(2,2)$-equivalent to trivial links one has immediately that $Col_5(L) = 5(F_L(1, 2cos(2\pi/5))^2$. To see this one should compare initial data (for trivial links) and use the fact that both sides of the equation are preserved by $\pm(2,2)$-moves. In fact the equality holds in general.

Lemma 3.2 *For every link L we have $Col_5(L) = 5(F_L(1, 2cos(2\pi/5))^2$.*

Proof As mentioned before $Col_5(L) = H_1(M_L^{(2)}; Z_5) \oplus Z_5$. Furthermore Jones proved ([15, 32]), analyzing the Goeritz matrix of L, that $(F_L(1, 2cos(2\pi/5))^2$ is the order of $H_1(M_L^{(2)}; Z_5)$. The diagrammatic proof of Lemma 3.2 was given in [14] but due to the untimely death of Francois Jaeger the paper is not published yet (compare the remark on p. 283 in [25]). □

From the fact that every $\pm(2,2)$-move is changing the sign of $F_L(1, 2cos(2\pi/5))$ and from Theorem 3.4 we obtain the following very elementary but very powerful statement (we formulate it before Theorem 3.4 because of its very elementary character)[11]:

Corollary 3.3 [29]

(i) *If a knot K can be reduced to the trivial link of two components by an even number of $\pm(2,2)$-moves then the unknotting number $u(K)$ is at least two.*

(ii) *If a knot K can be reduced to the trivial link of n components by k $\pm(2,2)$-moves then $u(K) \geq n + \frac{(-1)^{n-k}-1}{2}$.*

From Corollary 3.3 we get that $u(7_4) = 2$, $u(8_8) = 2$ and $u(8_{16}) = 2$. Namely, first we check by inspection that two crossing changes trivialize our knots. In Figure 3.2 we illustrate how to reduce the knot 7_4 to T_2 by four $\pm(2,2)$-moves (two 5-moves). In Figure 3.3 the reduction of the knot 8_8 by two $\pm(2,2)$-moves is illustrated. Finally in Figure 3.4 the reduction of the knot 8_{16} by an even number (six) of $\pm(2,2)$-moves is illustrated.

[11] Our method is related to that used by J R Rickard, a student of W B R Lickorish in his unfinished PhD thesis where he shows that the knot 8_{16} has the unknotting number equal to 2. Rickard died from cancer before finishing his PhD thesis. The short outline of his work is given in [18, 17].

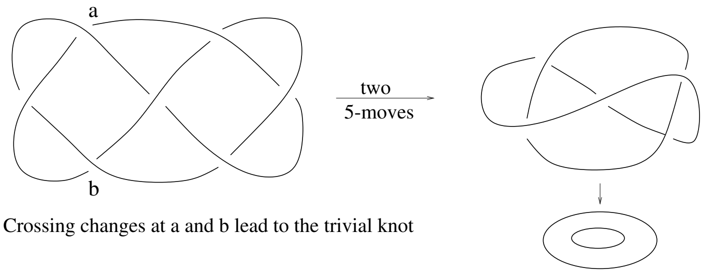


Figure 3.2

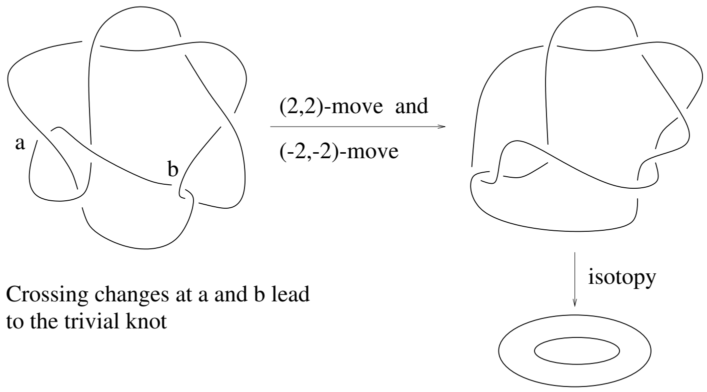


Figure 3.3

For the knot 9_{49} one should use Traczyk method in full its generality[12]. As we noted before, for $a = 1, x = 2cos(2\pi/5)$, we have $F_L = \pm\sqrt{\frac{1}{5}col_5(L)} = \epsilon(L)\sqrt{5}^{\lambda(L)}$ where $\epsilon(L) = \pm 1$ and $\lambda(L)$ is an integer.

Theorem 3.4 [29, 33] *For a knot K, its Kauffman skein quadruplet, K_+, K_-, K_0, K_∞ and invariants $u(K)$, $\epsilon(K)$ and $\lambda(K)$ we have the following properties of the unknotting number:*

(i) $u(K) \geq \lambda(K)$.

(ii) *If* $u(K_+) = \lambda(K_+)$ *and* $u(K_-) < u(K_+)$ *then* $u(K_-) = \lambda(K_-) = \lambda(K_+) - 1$ *and* $\epsilon(K_-) = -\epsilon(K_+) = \epsilon(K_0) = \epsilon(K_\infty)$.

(iii) *If* $\epsilon(K) = -(-1)^{\lambda(K)}$ *then* $u(K) > \lambda(K)$.

[12] The authors of [29] tried, but failed to simplify the knot 9_{49} to a trivial link by $(2,2)$-moves. Hence, Corollary 3.3 could not be used. The full reason for this is explained in Theorem 4.5(3).

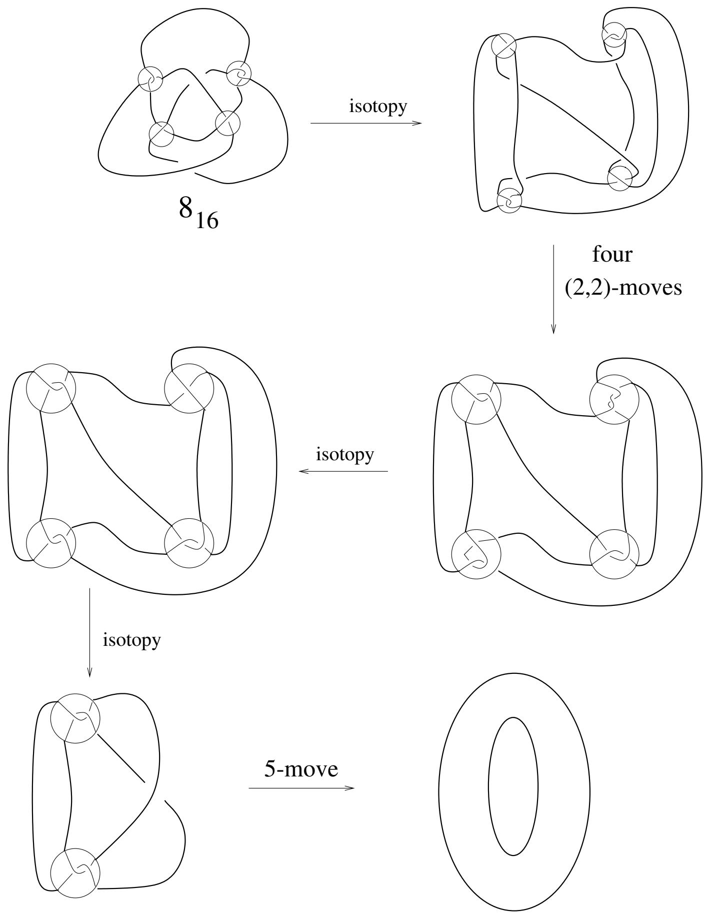


Figure 3.4

Proof Part (i) is a special case of the Wendt theorem, the first nontrivial result concerning unknotting number [40] (compare Lemma 2.2(h) of [25])[13]. Assumptions of part (ii) guarantee that $\lambda(K_-) = \lambda(K_+) - 1$ as $\lambda(K)$ can be changed at most by 1 when K is modified by a crossing change. Thus $u(K_-) = \lambda(K_-)$. Now we have two cases to consider:

(a) $\epsilon(K_+) = \epsilon(K_-)$. We put $\epsilon(K_+) = \epsilon(K_-) = 1$ (the case of -1 being

[13] $\lambda(K) = log_5 Col_5(K) - 1$ and a crossing change can change dimension of $Col_5(K)$ at most by one. This can be proven quickly by observing that the relations for $L_+ =$ are $a = c$ and $a - b + c - d = 0$, and the relations for $L_- =$ are $b = d$ and $a - b + c - d = 0$. Thus the linear spaces $Col_5(L_+)$ and $Col_5(L_-)$ differ by at most one equation, so their dimensions differ by at most one.

analogous). Then we have

$$\sqrt{5}^{\lambda(K_+)} + \sqrt{5}^{\lambda(K_-)} = (\frac{-1+\sqrt{5}}{2})(\epsilon(K_0)\sqrt{5}^{\lambda(K_0)} + \epsilon(K_\infty)\sqrt{5}^{\lambda(K_\infty)})$$

Therefore $(\sqrt{5}+1)\sqrt{5}^{\lambda(K_-)} = (\frac{-1+\sqrt{5}}{2})(\epsilon(K_0)\sqrt{5}^{\lambda(K_0)} + \epsilon(K_\infty)\sqrt{5}^{\lambda(K_\infty)})$
Thus $(3+\sqrt{5})\sqrt{5}^{\lambda(K_-)} = \epsilon(K_0)\sqrt{5}^{\lambda(K_0)} + \epsilon(K_\infty)\sqrt{5}^{\lambda(K_\infty)}$ which is impossible (just compare coefficients of 1 and $\sqrt{5}$ in the formula).

(b) $\epsilon(K_+) = -\epsilon(K_-)$. This case leads to part (iii). Furthermore, the formula we obtain is as follows:

$$\epsilon(K_+)(\sqrt{5}-1)\sqrt{5}^{\lambda(K_-)} = (\frac{-1+\sqrt{5}}{2})(\epsilon(K_0)\sqrt{5}^{\lambda(K_0)} + \epsilon(K_\infty)\sqrt{5}^{\lambda(K_\infty)})$$

Thus $2\epsilon(K_+)\sqrt{5}^{\lambda(K_-)} = (\epsilon(K_0)\sqrt{5}^{\lambda(K_0)} + \epsilon(K_\infty)\sqrt{5}^{\lambda(K_\infty)})$ Which holds iff $\lambda(K_-) = \lambda(K_0) = \lambda(K_\infty)$ and $-\epsilon(K_-) = \epsilon(K_+) = \epsilon(K_0) = \epsilon(K_\infty)$. This completes our proof of Theorem 3.4. □

Corollary 3.5 *The knot 9_{49} has the unknotting number equal to 3.*

Proof We have $F_{9_{49}}(1, \frac{-1+\sqrt{5}}{2}) = -5 = -\sqrt{5}^2$. Thus $\epsilon(9_{49}) = -1$ and by Theorem 3.2(iii), $u(9_{49}) > 2$. We can easily check by inspection that the knot 9_{49} can be unknotted by 3 crossing changes thus $u(9_{49}) = 3$. □

It is convenient to reformulate Theorem 3.4 so it applies to links. Consider, after [21], the metric space of links of n components, with the distance, $u(L_1, L_2)$, defined to be the minimal number of crossing changes needed to convert L_1 to L_2.

Theorem 3.6 [29]

$$u(L_1, L_2) \geq |\lambda(L_2) - \lambda(L_1)| + \frac{|\epsilon(L_1)\epsilon(L_2) - (-1)^{\lambda(L_2)-\lambda(L_1)}|}{2}.$$

Example 3.7 The distance between the trefoil knot and the figure eight knot is equal to two (i.e. $u(3_1, 4_1) = 2$).

We have, of course, that $u(3_1, 4_1) \leq 2$. Furthermore $F_{3_1}(1, 2cos(2\pi/5)) = -1$ and $F_{4_1}(1, 2cos(2\pi/5)) = -\sqrt{5}$. Thus $\epsilon(3_1) = \epsilon(4_1) = -1$, $\lambda(4_1) = \lambda(3_1) + 1 = 1$. Therefore by Theorem 3.6 on has $u(3_1, 4_1) \geq 1 + 1 = 2$ and finally $u(3_1, 4_1) = 2$.

We predict that Fox 7-colorings should be related with cubic skein modules of S^3. In turn the cubic skein module should be useful to analyze unknotting number for knots with nontrivial Fox 7-colorings. We analyzed with M.Veve cubic skein invariants, preserved, up to constant, by $\pm(2,3)$ moves. We didn't get results similar to that for the Kauffman or Jones polynomials. We speculate that the method does not work because instead of a constant one should probably use the invariant related to Z_7 Witt class of the Goeritz form of a link. This line of research awaits exploration.

4 Burnside groups of links

Five months after my talks in Kyoto, several elementary move conjectures have been solved, or more precisely, disproved, using Burnside groups of links. I describe below the joint work with my student Mietek Dąbkowski. Our tool is a noncommutative analogue of Fox n-colorings which we call the n'th Burnside group of a link, $B_L(n)$, [8, 9].

Definition 4.1 The nth Burnside group of a link is the quotient of the fundamental group of the double branched cover of S^3 with the link as the branch set divided by all relations of the form $w^n = 1$. Succinctly: $B_L(n) = \pi_1(M_L^{(2)})/(w^n)$.

Notice that for the trivial link of k components, T_k, one has $B_{T_k}(n) = B(k-1,n)$ where $B(k-1,n)$ is the classical Burnside group of $k-1$ generators and exponent n.

For practical applications it is very convenient to have a diagrammatic (and local) description of the groups $\pi_1(M_L^{(2)})$ and $B_L(n)$. Such a presentation, using the core group idea of Bruck [4] and Joice [16] was given in [11, 39][14].

Definition 4.2 Let D be a diagram of a link L. We define the associated core group $\Pi_D^{(2)}$ of D by the following presentation: generators of $\Pi_D^{(2)}$ correspond to arcs of the diagram. Any crossing v_s yields the relation $r_s = y_i y_j^{-1} y_i y_k^{-1}$ where y_i corresponds to the overcrossing and y_j, y_k correspond to the undercrossings at v_s (see Figure 4.1).

In the above presentation of $\Pi_L^{(2)}$ one relation can be dropped since it is a consequence of others.

[14]The presentation was also found before, in 1976, by Victor Kobelsky a student of O.Viro, but this was never published [38].

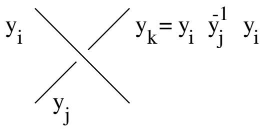


Figure 4.1

The relation to the fundamental group of a double branch cover, mentioned before, is formulated below (an elementary proof, working for links and tangles, using only Wirtinger presentation was given in [25]).

Theorem 4.3 (Wada)

$$\Pi_D^{(2)} = \pi_1(M_L^{(2)}) * Z.$$

Furthermore, if we put $y_i = 1$ *for any fixed generator, then* $\Pi_D^{(2)}$ *reduces to* $\pi_1(M_L^{(2)})$.

We introduced Burnside groups in order to analyze elementary moves on links.

Theorem 4.4 [9] $B_L(n)$ *is preserved by rational* $\frac{n}{q}$*-moves. In particular* n*-moves preserve* $B_L(n)$.

Theorem 4.4 has been used to find obstructions to several Conjectures described in Section 1.

Theorem 4.5 [8, 9]

(1) *The third Burnside groups of the 2-parallel of the Borromean rings and of the Chen's link are different than third Burnside groups of trivial links. In particular we have* $|B_{Chen\ link}(3)| = 3^{10}$, $|B_{T_5}(3)| = 3^{14}$, $|B_{2BR}(3)| = 3^{21}$ *and* $|B_{T_6}(3)| = 3^{25}$.[15]

(2) *The fourth Burnside group of the "half" 2-cabling of the Whitehead link,* $\mathcal{W}$, *is different then the group of the trivial link of 3 component. We have:* $|B_{\mathcal{W}}(4)| = 2^{10} \neq 2^{12} = B(2,4) = B_{T_3}(4)$. *For the Borromean rings we get* $|B_{BR}(4)| = 2^5$ *which gives a simple proof of the Nakanishi result that the Borromean rings are not 4-move equivalent to a trivial link.*

[15] It is still an open problem whether the 2-parallel of the Borromean rings is 3-move equivalent to the Chen's link with an additional trivial component; both links have third Burnside groups of order 3^{21}.

(3) *The links 9_{40} and 9_{49} are not $(2,2)$-move equivalent to trivial links*[16]*. We compare the third terms, $L_3(\ )$, of the graded Lie algebra associated to the lower central series of groups*[17] *and show that $L_3(B_{9_{40}}(5)) = L_3(B_{9_{49}}(5)) \neq L_3(B_{T_3}(5)) = L_3(B(2,5)) = Z_5 \oplus Z_5$.*

(4) *For p prime, $p > 3$, the closure of the 3-braid $(\sigma_1\sigma_2)^6$ cannot be reduced to a trivial link by $\frac{p}{q}$-moves. We show that the obstruction lies in the third term of the graded Lie algebra associated to the lower central series of the p'th Burnside group.*

Notice that Theorem 4.5(1) combined with the fact that 3-algebraic links are 3-move equivalent to trivial links [30] implies that the 2-parallel of the Borromean rings and the Chen's link are not 3-algebraic. The fact that not every link is 3-algebraic is new. The question whether for any n there is a link which is not n algebraic still remains open. Similarly, Theorem 4.5(3) combined with Lemma 1.7 implies that the knots 9_{40} and 9_{49} are not 2-algebraic[18].

We were unable to use our method in the case when the abelianization of the n'th Burnside group (i.e. $H_1(M_L^2, Z_n)$) is a cyclic group. In particular Nakanishi 4-move conjecture remains open as well as 2-component version of the Kawauchi 4-move question (Problem 1.4(ii)). In the same vain the question whether the knot 8_{18} is $(2,3)$-move equivalent to a trivial link remains open.

Acknowledgments I would like to thank organizers of the Kyoto workshop for their hospitality.

[16]It is still an open problems whether 9_{40} and 9_{49} are $(2,2)$-move equivalent or whether the Burnside group $B_{9_{40}}(5)$ is isomorphic to $B_{9_{49}}(5)$.

[17]The lower central series of a group G ($G_1 = G, G_2 = [G,G], ..., G_n = [G_{n-1},G]$) yields the associated graded Lie ring of the group: $L = L_1 \oplus L_2 \oplus ... \oplus L_i \oplus ...$ where $L_i = G_i/G_{i+1}$. The Lie bracket in L corresponds to the group bracket $[g,h] = g^{-1}h^{-1}gh$,[37].

[18]The first proof that the knot 9_{49} is not 2-algebraic was by showing that the 2-fold branched cover of $(S^3, 9_{49})$ is a hyperbolic 3-manifold. In fact, it is the manifold I suspected from 1983 to have the smallest volume among oriented hyperbolic 3-manifolds [13, 17, 19].

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