Problems on invariants of knots and 3-manifolds

Edited by T. OHTSUKI

Invariants of Knots and 3-Manifolds
Kyoto 2001

Preface

The workshop and seminars on "Invariants of knots and 3-manifolds" was held at Research Institute for Mathematical Sciences, Kyoto University in September 2001. There were 25 talks in the workshop in September 17–21, and there were 27 talks in the seminars in the other weeks of September. Each speaker was requested to give his/her open problems in a short problem session after his/her talk, and many interesting open problems were given and discussed by the speakers and participants in the workshop and the seminars. Contributors of the open problems were also requested to give kind expositions of history, background, significance, and/or importance of the problems. This problem list was made by editing these open problems and such expositions.[1]

Since the interaction between geometry and mathematical physics in the 1980s, many invariants of knots and 3-manifolds have been discovered and studied. The discovery and analysis of the enormous number of these invariants yielded a new area: the study of invariants of knots and 3-manifolds (from another viewpoint, the study of the sets of knots and 3-manifolds). Recent works have almost completed the topological reconstruction of the invariants derived from the Chern-Simons field theory, which was one of main problems of this area. Further, relations among these invariants have been studied enough well, and these invariants are now well-organized. For the future developments of this area, it might be important to consider various streams of new directions;[2] this is a reason why the editor tried to make the problem list expository. The editor hopes this problem list will clarify the present frontier of this area and assist readers when considering future directions.

The editor will try to keep up-to-date information on this problem list at his web site.[3] If the reader knows a (partial) solution of any problem in this list, please let him[4] know it.

February, 2003

T. Ohtsuki

The logo for the workshop and the seminars was designed by N. Okuda.

[1]Open problems on the Rozansky-Witten invariant were written in a separate manuscript [349]. Some fundamental problems are quoted from other problem lists such as [188], [220], [262], [285], [286], [388, Pages 571–572].

[2]For example, directions related to other areas such as hyperbolic geometry via the volume conjecture and the theory of operator algebras via invariants arising from 6j-symbols.

[3]`http://www.kurims.kyoto-u.ac.jp/~tomotada/proj01/`

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Problems on invariants of knots and 3-manifolds

Edited by T. Ohtsuki

Abstract This is a list of open problems on invariants of knots and 3-manifolds with expositions of their history, background, significance, or importance. This list was made by editing open problems given in problem sessions in the workshop and seminars on "Invariants of Knots and 3-Manifolds" held at Kyoto in 2001.

AMS Classification 20F36, 57M25, 57M27, 57R56; 13B25, 17B10, 17B37, 18D10, 20C08, 20G42, 22E99, 41A60, 46L37, 57M05, 57M50, 57N10, 57Q10, 81T18, 81T45

Keywords Invariant, knot, 3-manifold, Jones polynomial, Vassiliev invariant, Kontsevich invariant, skein module, quandle, braid group, quantum invariant, perturbative invariant, topological quantum field theory, state-sum invariant, Casson invariant, finite type invariant, LMO invariant

0 Introduction

The study of quantum invariants of links and three-manifolds has a strange status within topology. When it was born, with Jones' 1984 discovery of his famous polynomial [186], it seemed that the novelty and power of the new invariant would be a wonderful tool with which to resolve some outstanding questions of three-dimensional topology. Over the last 16 years, such hopes have been largely unfulfilled, the only obvious exception being the solution of the Tait conjectures about alternating knots (see for example [281]).

This is a disappointment, and particularly so if one expects the role of the quantum invariants in mathematics to be the same as that of the classical invariants of three-dimensional topology. Such a comparison misses the point that most of the classical invariants were *created* specifically in order to distinguish between things; their definitions are mainly intrinsic, and it is therefore clear what kind of topological properties they reflect, and how to attempt to use them to solve topological problems.

Chapter 0 was written by J. Roberts.

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Quantum invariants, on the other hand, should be thought of as having been *discovered*. Their construction is usually indirect (think of the Jones polynomial, defined with reference to *diagrams* of a knot) and their existence seems to depend on very special kinds of algebraic structures (for example, R-matrices), whose behaviour is closely related to three-dimensional combinatorial topology (for example, Reidemeister moves). Unfortunately such constructions give little insight into what kind of topological information the invariants carry, and therefore into what kind of applications they might have.

Consequently, most of the development of the subject has taken place in directions away from classical algebraic and geometric topology. From the earliest days of the subject, a wealth of connections to different parts of mathematics has been evident: originally in links to operator algebras, statistical mechanics, graph theory and combinatorics, and latterly through physics (quantum field theory and perturbation theory) and algebra (deformation theory, quantum group representation theory). It is the investigation of these outward connections which seems to have been most profitable, for the two main frameworks of the modern theory, that of Topological Quantum Field Theory and Vassiliev theory (perturbation theory) have arisen from these.

The TQFT viewpoint [16] gives a good interpretation of the cutting and pasting properties of quantum invariants, and viewed as a kind of "higher dimensional representation theory" ties in very well with algebraic approaches to deformations of representation categories. It ties in well with geometric quantization theory and representations of loop groups [17]. In its physical formulation via the Chern-Simons path-integral (see Witten [403]), it even offers a *conceptual* explanation of the invariants' existence and properties, but because this is not rigorous, it can only be taken as a heuristic guide to the properties of the invariants and the connections between the various approaches to them.

The Vassiliev theory (see [25, 226, 383]) gives geometric definitions of the invariants in terms of integrals over configuration spaces, and also can be viewed as a classification theory, in the sense that there is a universal invariant, the Kontsevich integral (or more generally the Le-Murakami-Ohtsuki invariant [249]), through which all the other invariants factor. Its drawback is that the integrals are very hard to work with – eight years passed between the definition and calculation [383] of the Kontsevich integral of the *unknot*!

These two frameworks have revealed many amazing properties and algebraic structures of quantum invariants, which show that they are important and interesting pieces of mathematics in their own right, whether or not they have applications in three-dimensional topology. The structures revealed are precisely those which can, and therefore must, be studied with the aid of three-dimensional pictures and a topological viewpoint; the whole theory should therefore be considered as a new kind of algebraic topology specific to three dimensions.

Perhaps the most important overall goal is simply to *really understand* the topology underlying quantum invariants in three dimensions: to relate the "new algebraic topology" to more classical notions and obtain good intrinsic topological definitions of the invariants, with a view to applications in three-dimensional topology and beyond.

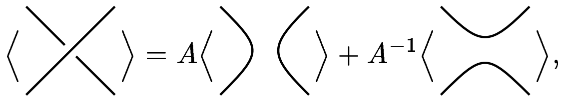
The problem list which follows contains detailed problems in all areas of the theory, and their division into sections is really only for convenience, as there are very many interrelationships between them. Some address unresolved matters or extensions arising from existing work; some introduce specific new conjectures; some describe evidence which hints at the existence of new patterns or structures; some are surveys on major and long-standing questions in the field; some are purely speculative.

Compiling a problem list is a very good way to stimulate research inside a subject, but it also provides a great opportunity to "take stock" of the overall state and direction of a subject, and to try to demonstrate its vitality and worth to those outside the area. We hope that this list will do both.

1 Polynomial invariants of knots

1.1 The Jones polynomial

The *Kauffman bracket* of unoriented link diagrams is defined by the following recursive relations,



$$\langle \bigcirc\ D \rangle = (-A^2 - A^{-2})\langle D \rangle \qquad \text{for any diagram } D,$$

$$\langle \text{the empty diagram } \emptyset \rangle = 1,$$

where three pictures in the first formula imply three links diagrams, which are identical except for a ball, where they differ as shown in the pictures. The *Jones polynomial* $V_L(t)$ of an oriented link L is defined by

$$V_L(t) = (-A^2 - A^{-2})^{-1}(-A^3)^{-w(D)}\langle D \rangle\Big|_{A^2=t^{-1/2}} \in \mathbb{Z}[t^{1/2}, t^{-1/2}],$$

where D is a diagram of L, $w(D)$ is the writhe of D, and $\langle D \rangle$ is the Kauffman bracket of D with its orientation forgotten. The Jones polynomial is an isotopy invariant of oriented links uniquely characterized by

$$\begin{aligned} & t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{1/2} - t^{-1/2})V_{L_0}(t), && (1) \\ & V_O(t) = 1, \end{aligned}$$

where O denotes the trivial knot, and L_+, L_-, and L_0 are three oriented links, which are identical except for a ball, where they differ as shown in Figure 1. It is shown, by (1), that for any knot K, its Jones polynomial $V_K(t)$ belongs to $\mathbb{Z}[t, t^{-1}]$.

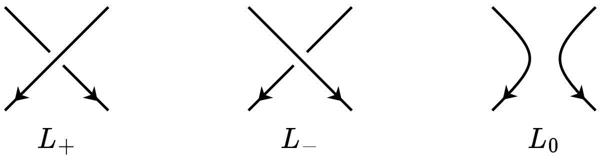


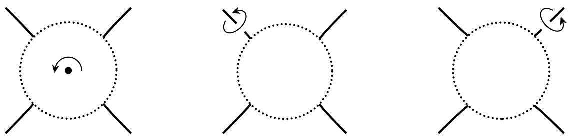
Figure 1: Three links L_+, L_-, L_0

1.1.1 Does the Jones polynomial distinguish the trivial knot?

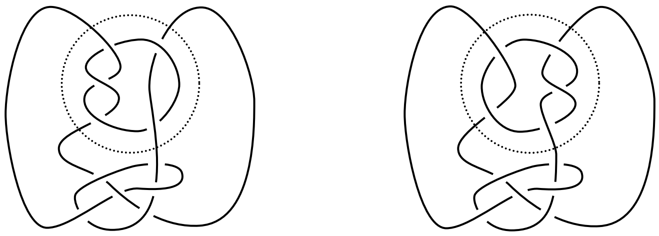
Problem 1.1 ([188, Problem 1]) *Find a non-trivial knot K with $V_K(t) = 1$.*

Remark It is shown by computer experiments that there are no non-trivial knots with $V_K(t) = 1$ up to 17 crossings of their diagrams [102], and up to 18 crossings [407]. See [52] (and [53]) for an approach to find such knots by using representations of braid groups.

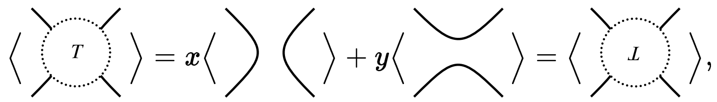
Remark Two knots with the same Jones polynomial can be obtained by mutation. A *mutation* is a relation of two knots, which are identical except for a ball, where they differ by π rotation of a 2-strand tangle in one of the following ways (see [12] for mutations).



For example, the Conway knot and the Kinoshita-Terasaka knot are related by a mutation.



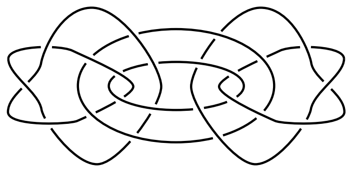
They have the same Jones polynomial, because their diagrams have the same writhe and the Kauffman bracket of the tangle shown in the dotted circle can be presented by



with some scalars x and y.

Remark The Jones polynomial can be obtained from the Kontsevich invariant through the weight system $W_{sl_2,V}$ for the vector representation V of sl_2 (see, e.g. [321]). Problem 1.1 might be related to the kernel of $W_{sl_2,V}$.

Remark Some links with the Jones polynomial equal to that of the corresponding trivial links are given in [115]. For example, the Jones polynomial of the following link is equal to the Jones polynomial of the trivial 4-component link.



Remark (X.-S. Lin [262]) Use Kontsevich integral to show the existence of a non-trivial knot with trivial Alexander-Conway polynomial. This might give us some hints to Problem 1.1.

1.1.2 Characterization and interpretation of the Jones polynomial

Problem 1.2 ([188, Problem 2]) *Characterize those elements of* $\mathbb{Z}[t,t^{-1}]$ *of the form* $V_K(t)$.

Remark [188] The corresponding problem for the Alexander polynomial has been solved; it is known that a polynomial $f(t) \in \mathbb{Z}[t,t^{-1}]$ is equal to the Alexander polynomial of some knot K if and only if $f(1) = 1$ and $f(t) = f(t^{-1})$. The formulas $V_K(1) = 1$ and $V_K(\exp \frac{2\pi\sqrt{-1}}{3}) = 1$ are obtained by the skein relation (1). These formulas give weak characterizations of the required elements.

Remark (X.-S. Lin [262]) The Mahler measure (see [119] for its exposition) of a polynomial $F(x) = a\prod_i (x - \alpha_i) \in \mathbb{C}[x]$ is defined by

$$m(F) = \log|a| + \sum_i \log\max\{1, |\alpha_i|\} = \int_0^1 \log|F(e^{2\pi\sqrt{-1}\theta})|d\theta.$$

The Mahler measure can be defined also for a Laurent polynomial similarly. Is it true that $m(V_K) > 0$ for the Jones polynomial V_K of a knot K, if K is a non-trivial knot?

Problem 1.3 *Find a 3-dimensional topological interpretation of the Jones polynomial of links.*

Remark The Alexander polynomial has a topological interpretation such as the characteristic polynomial of $H_1(\widetilde{S^3 - K};\mathbb{Q})$ of the infinite cyclic cover of the knot complement $S^3 - K$, where $H_1(\widetilde{S^3 - K};\mathbb{Q})$ is regarded as a $\mathbb{Q}[t,t^{-1}]$-module by regarding t as the action of the deck transformation on $\widetilde{S^3 - K}$.

Remark In the viewpoint of mathematical physics, Witten [403] gave a 3-dimensional interpretation of the Jones polynomial of a link by a path integral including a holonomy along the link in the Chern-Simons field theory.

Remark Certain special values of the Jones polynomial have some interpretations. The formulas $V_L(1) = (-2)^{\#L-1}$ and $V_L(\exp\frac{2\pi\sqrt{-1}}{3}) = 1$ are shown by the skein relation (1), where $\#L$ denotes the number of components of L. It is known that $|V_L(-1)|$ is equal to the order of $H_1(M_{2,L})$ if its order is finite, and 0 otherwise. Here, $M_{2,L}$ denotes the double branched cover of S^3 branched along L. It is shown, in [290], that $V_L(\sqrt{-1}) = (-\sqrt{2})^{\#L-1}(-1)^{\mathrm{Arf}(L)}$ if $\mathrm{Arf}(L)$ exists, and 0 otherwise. It is shown, in [257], that $V_L(\exp\frac{\sqrt{-1}\pi}{3}) = \pm\sqrt{-1}^{\#L-1}\sqrt{-3}^{\dim H_1(M_{2,L};\mathbb{Z}/3\mathbb{Z})}$. If ω is equal to a 2nd, 3rd, 4th, 6th root of unity, the computation of $V_L(\omega)$ can be done in polynomial time of the number of crossings of diagrams of L by the above interpretation of $V_L(\omega)$. Otherwise, $V_L(\omega)$ does not have such a topological interpretation, in the sense that computing $V_L(\omega)$ of an alternating link L at a given value ω is #P-hard except for the above mentioned roots of unity (see [183, 399]).

Problem 1.4 (J. Roberts) *Why is the Jones polynomial a polynomial?*

Remark (J. Roberts) A topological invariant of knots should ideally be defined in an intrinsically 3-dimensional fashion, so that its invariance under orientation-preserving diffeomorphisms of S^3 is built-in. Unfortunately, almost all of the known constructions of the Jones polynomial (via R-matrices, skein relations, braid groups or the Kontsevich integral, for example) break the symmetry, requiring the introduction of an axis (Morsification of the knot) or plane of projection (diagram of the knot). I believe that the "perturbative" construction via configuration space integrals [381], whose output is believed to be essentially equivalent to the Kontsevich integral, is the only known intrinsic construction.

In the definitions with broken symmetry, it is generally easy to see that the result is an integral Laurent polynomial in q or $q^{\frac{1}{2}}$. In the perturbative approach, however, we obtain a formal power series in $\hbar$, and although we know that it ought to be the expansion of an integral Laurent polynomial under the substitution $q = e^{\hbar}$, it seems hard to prove this directly. A related observation is that the analogues of the Jones polynomial for knots in 3-manifolds other than S^3 are *not* polynomials, but merely functions from the roots of unity to algebraic integers. What is the special property of S^3 (or perhaps $\mathbb{R}^3$) which causes this behaviour, and why does the variable q seem natural only when one breaks the symmetry?

The typical raison d'etre of a Laurent polynomial is that it is a character of the circle. (In highbrow terms this is an example of "categorification", but it is also belongs to a concrete tradition in combinatorics that to prove that something is a non-negative integer one should show that it is the dimension of a vector space.) The idea that the Jones polynomial is related to K-theory [402] and that it ought to be the S^1-equivariant index of some elliptic operator defined using the special geometry of $\mathbb{R}^3$ or S^3 is something Simon Willerton and I have been pondering for some time. As for the meaning of q, Atiyah suggested the example in equivariant K-theory

$$K_{SO(3)}(S^2) \cong K_{S^1}(pt) = \mathbb{Z}[q^{\pm 1}],$$

in to make the first identification *requires* a choice of axis in $\mathbb{R}^3$. (This would suggest looking for an $SO(3)$-equivariant S^2-family of operators.)

Problem 1.5 (J. Roberts) *Is there a relationship between values of Jones polynomials at roots of unity and branched cyclic coverings of a knot?*

Problem 1.6 (J. Roberts) *Is there a relationship between the Jones polynomial of a knot and the counting of points in varieties defined over finite fields?*

Remark (J. Roberts) These two problems prolong the "riff in the key of q": the amusing fact that traditional, apparently independent uses of that letter, denoting the number of elements in a finite field, the deformation parameter $q = e^{\hbar}$, the variable in the Poincaré series of a space, the variable in the theory of modular forms, etc. turn out to be related.

The first problem addresses a relationship which holds for the Alexander polynomial. For example, the order of the torsion in H_1 of the n-fold branched cyclic cover equals the product of the values of the Alexander polynomial at all the nth roots of unity. It's hard not to feel that the variable q has some kind of meaning as a deck translation, and that the values of the Jones polynomial at roots of unity should have special meanings.

The second has its roots in Jones' original formulation of his polynomial using Hecke algebras. The Hecke algebra $H_n(q)$ is just the Hall algebra of double cosets of the Borel subgroup inside $SL(n, \mathbb{F}_q)$; the famous quadratic relation $\sigma^2 = (q-1)\sigma + q$ falls naturally out of this. Although the alternative definition of $H_n(q)$ using generators and relations extends to allow q to be any complex number (and it is then the roots of unity, at which $H_n(q)$ is not semisimple, which are the obvious special values), it might be worth considering whether Jones polynomials at prime powers $q = p^s$ have any special properties.

Ideally one could try to find a topological definition of the Jones polynomial (perhaps only at such values) which involves finite fields. The coloured Jones polynomials of the unknot are quantum integers, which count the numbers of points in projective spaces defined over finite fields; might those for arbitrary knots in S^3 count points in other varieties? Instead of counting counting points, one could consider Poincaré polynomials, as the two things are closely related by the Weil conjectures.

One obvious construction involving finite fields is to count representations of a fundamental group into a finite group of Lie type, such as $SL(n, \mathbb{F}_q)$. Very much in this vein, Jeffrey Sink [369] associated to a knot a zeta-function formed from the counts of representations into $SL(2, \mathbb{F}_{p^s})$, for fixed p and varying s. His hope, motivated by the Weil conjectures, was the idea that the $SU(2)$ Casson invariant might be related to such counting. For such an idea to work, it is probably necessary to find some way of counting representations with signs, or at least to enhance the counting in some way. Perhaps the kind of twisting used in the Dijkgraaf-Witten theory [108] could be used.

Problem 1.7 (J. Roberts) *Define the Jones polynomial intrinsically using homology of local systems.*

Remark (J. Roberts) The Alexander polynomial of a knot can be defined using the twisted homology of the complement. In the case of the Jones polynomial, no similar direct construction is known, but the approach of Bigelow [55] is tantalising. He shows how to construct a representation of the braid group B_{2n} on the twisted homology of the configuration space of n points in the $2n$-punctured disc, and how to use a certain "matrix element" of this representation to obtain the Jones polynomial of a knot presented as a plait. Is there any way to write the same calculation directly in terms of configuration spaces of n points in the knot complement, for example?

Problem 1.8 (J. Roberts) *Study the relation between the Jones polynomial and Gromov-Witten theory.*

Remark (J. Roberts) The theory of pseudo-holomorphic curves or "Gromov-Witten invariants" has been growing steadily since around 1985, in parallel with the theory of quantum invariants in three dimensional topology. During that time it has come to absorb large parts of modern geometry and topology, including symplectic topology, Donaldson/Seiberg-Witten theory, Floer homology, enumerative algebraic geometry, etc. It is remarkable that three-dimensional TQFT has remained isolated from it for so long, but finally there is a connection, as explained in the paper by Vafa and Gopakumar [149] (though prefigured by Witten [404]), and now under investigation by many geometers. The basic idea is that the HOMFLY polynomial can be reformulated as a generating function counting pseudo-holomorphic curves in a certain Calabi-Yau manifold, with boundary condition a Lagrangian submanifold associated to the knot. (This is the one place where the HOMFLY and not the Jones polynomial is essential!) The importance of this connection can hardly be overestimated, as it should allow the exchange of powerful techniques between the two subjects.

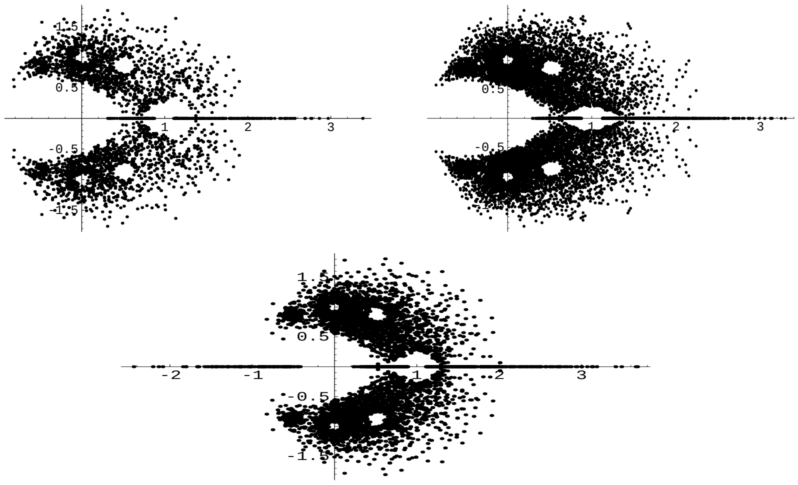


Figure 2: The upper pictures show the distribution of zeros of the Jones polynomial for alternating knots of 11 and 12 crossings [262]. The lower picture shows the distribution of zeros of the Jones polynomial for 12 crossing non-alternating knots [262]. See [262] for further pictures for alternating knots with 10 and 13 crossings.

1.1.3 Numerical experiments

The following problem might characterize the form of the Jones polynomial of knots in some sense.

Problem 1.9 (X.-S. Lin) *Describe the set of zeros of the Jones polynomial of all (alternating) knots.*

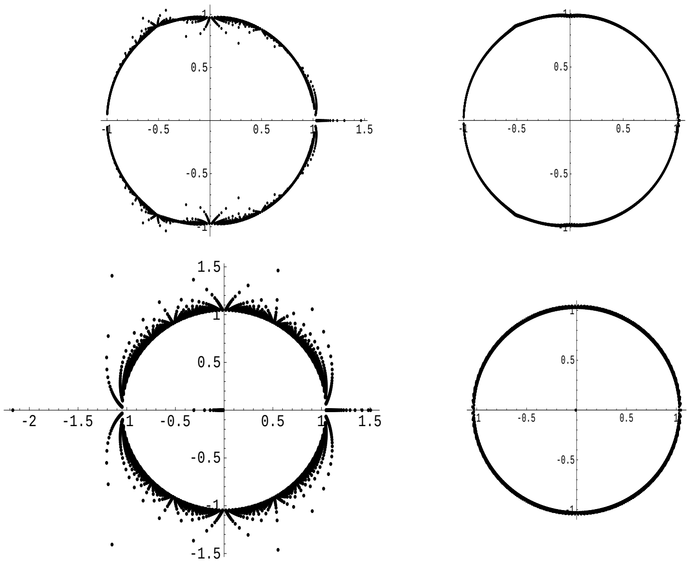


Figure 3: The upper pictures show the distribution of zeros of the Jones polynomial for n-twist knots, with n from 1 to 50 and from 51 to 100, respectively [262]. The lower pictures show the distribution of zeros of the Jones polynomial for $(2, 2n-1)$ torus knots, with n from 1 to 50 and from 51 to 100, respectively [262]. See [262] for further pictures for $(3, 3n+1)$ and $(3, 3n+2)$ torus knots.

Remark (X.-S. Lin) The plottings in Figure 2 numerically describe the set of zeros of the Jones polynomial of many knots. Similar plottings are already published in [405] for some other infinite families of knots for which the Jones polynomial is known explicitly. See also [84] for some other plottings.

Remark (X.-S. Lin) It would be a basic problem to look into the zero distribution of the family of polynomials with bounded degree such that coefficients are all integers and coefficients sum up to 1, and compare it with the zero distribution of the Jones polynomial on the collection of (alternating) knots with bounded crossing number. The paper [317] discusses the zero distribution of the family of polynomials with 0,1 coefficients and bounded degree. It is particularly interesting to compare the plotting shown in this paper with the plottings in Figures 2 and 3 for the zeros of the Jones polynomials.

Problem 1.10 (N. Dunfield) *Find the relationship between the hyperbolic volume of knot complements and* $\log V_K(-1)$ *(resp.* $\log V_K(-1)/\log \deg V_K(t)$*).*

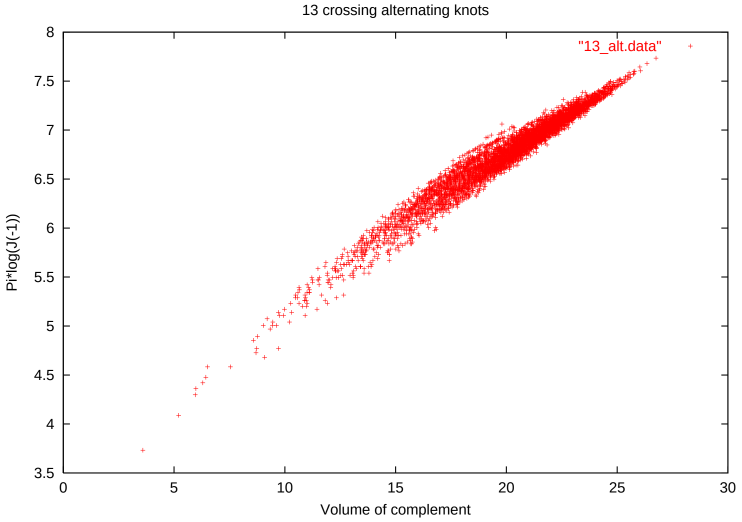


Figure 4: The distribution of pairs of the hyperbolic volume of knot complements and $\pi \log V_K(-1)$ for alternating knots with 13 crossings [112].

Remark (N. Dunfield [112]) $V_K(-1)$ is just $\Delta_K(-1)$, which is the order of the torsion in the homology of the double cover of S^3 branched over K. $\log V_K(-1)$ is one of the first terms of the volume conjecture (Conjecture 1.19). Figure 4 suggests that for alternating knots with a fixed number of crossings, $\log V_K(-1)$ is almost a linear function of the volume.

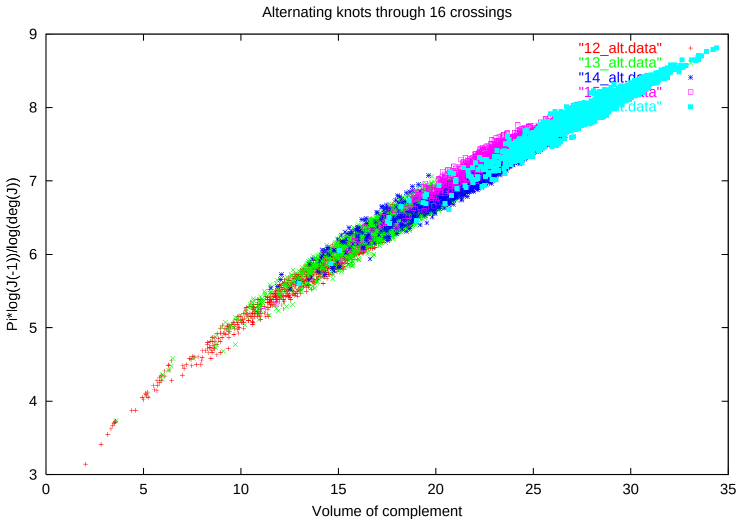
Figure 5 suggests that there should be an inequality

$$\frac{\log V_K(-1)}{\log \deg V_K(t)} < a \cdot \mathrm{vol}(S^3 - K) + b$$

for some constants a and b. For 2-bridge knots, Agol's work on the volumes of 2-bridge knots [1] can be used to prove such an inequality with $a = b = 2/v_3$ (here, v_3 is the volume of a regular ideal tetrahedron).

1.1.4 Categorification of the Jones polynomial

Khovanov [217, 218] defined certain homology groups of a knot whose Euler characteristic is equal to the Jones polynomial, which is called the *categorification* of the Jones polynomial. See also [32] for an exposition of it.



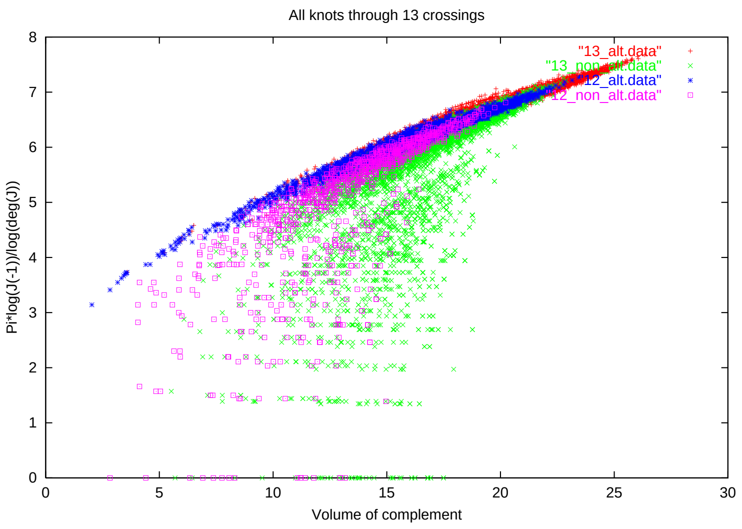


Figure 5: The distributions of pairs of the hyperbolic volume of knot complements and $\pi \log V_K(-1)/\log \deg V_K(t)$. The upper picture is for all alternating knots with 12 and 13 crossings and samples of alternating knots with 14, 15, and 16 crossings, and the lower picture is for all knots with 13 or fewer crossings [112].

Problem 1.11 *Understand Khovanov's categorification of the Jones polynomial.*

Problem 1.12 *Categorify other knot polynomials.*

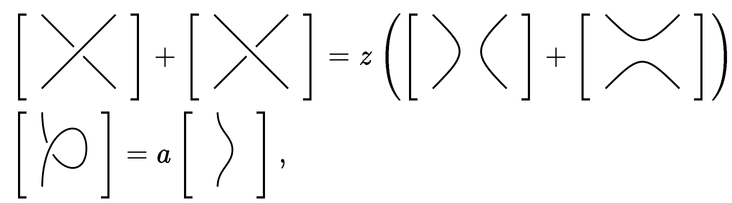
Remark (M. Hutchings) There does exist a categorification of the Alexander polynomial, or more precisely of $\Delta_K(t)/(1-t)^2$, where $\Delta_K(t)$ denotes the (symmetrized) Alexander polynomial of the knot K. It is a kind of Seiberg-Witten Floer homology of the three-manifold obtained by zero surgery on K. One can regard it as $\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$ graded, although in fact the column whose Euler characteristic gives the coefficient of t^k is relatively $\mathbb{Z}/2k\mathbb{Z}$ graded.

1.2 The HOMFLY, Q, and Kauffman polynomials

The *skein polynomial* (or the *HOMFLY polynomial*) $P_L(l,m) \in \mathbb{Z}[l^{\pm1}, m^{\pm1}]$ of an oriented link L is uniquely characterized by

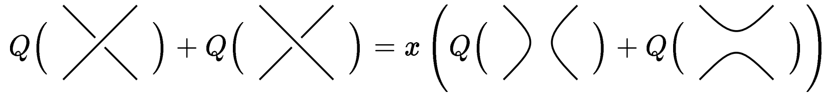
$$l^{-1}P_{L_+}(l,m) - lP_{L_-}(l,m) = mP_{L_0}(l,m),$$
$$P_O(l,m) = 1,$$

where O denotes the trivial knot, and L_+, L_-, and L_0 are three oriented links, which are identical except for a ball, where they differ as shown in Figure 1. For a knot K, $P_K(l,m) \in \mathbb{Z}[l^{\pm2}, m]$. The *Kauffman polynomial* $F_L(a,z) \in \mathbb{Z}[a^{\pm1}, z^{\pm1}]$ of an oriented link L is defined by $F_L(a,z) = a^{-w(D)}[D]$ for an unoriented diagram D presenting L (forgetting its orientation), where $[D]$ is uniquely characterized by



$$[O] = 1.$$

For a knot K, $F_K(a,z) \in \mathbb{Z}[a^{\pm1}, z]$. The *Q polynomial* $Q_L(x) \in \mathbb{Z}[x^{\pm1}]$ of an unoriented link L is uniquely characterized by



$$Q(O) = 1.$$

It is known that

$$\begin{aligned} V_L(t) &= P_L(t, t^{1/2} - t^{-1/2}) = F_L(-t^{-3/4}, t^{1/4} + t^{-1/4}), \\ \Delta_L(t) &= P_L(1, t^{1/2} - t^{-1/2}), \\ Q_L(z) &= F_L(1, z), \end{aligned}$$

where $\Delta_L(t)$ denotes the Alexander polynomial of L. The variable m of $P_L(l, m)$ is called the *Alexander variable.* See, e.g. [206, 255], for details of this paragraph.

Let the *span* of a polynomial denote the maximal degree minus the minimal degree of the polynomial.

Problem 1.13 (A. Stoimenow) *Does the Jones polynomial V admit only finitely many values of given span? What about the Q polynomial or the skein, Kauffman polynomials (when fixing the span in both variables)?*

Remark (A. Stoimenow) It is true for the skein polynomial when bounding the canonical genus (for which the Alexander degree of the skein polynomial is a lower bound by Morton), in particular it is true for the skein polynomial of homogeneous links [97]. It is true for the Jones, Q and Kauffman F polynomial of alternating links (for F more generally for adequate links). One cannot bound the number of different links, at least for the skein and Jones polynomial, because Kanenobu [192] gave infinitely many knots with the same skein polynomial.

Problem 1.14 (A. Stoimenow) *Why are the unit norm complex numbers α for which the value $Q_K(\alpha)$ has maximal norm statistically concentrated around $e^{11\pi\sqrt{-1}/25}$?*

Remark (A. Stoimenow) The maximal point of $|Q_K(e^{2\pi\sqrt{-1}t})|$ for $t \in [0, 1)$ is statistically concentrated around $t = 11/50$. This was revealed by an experiment in an attempt to estimate the asymptotical growth of the coefficients of the Q polynomial. There seems no difference in the behaviour of alternating and non-alternating knots.

Problem 1.15 (M. Kidwell, A. Stoimenow) *Let K be a non-trivial knot, and let W_K be a Whitehead double of K. Is then*

$$\deg_m P_{W_K}(l, m) = 2 \deg_z F_K(a, z) + 2\,?$$

Remark (A. Stoimenow) It is true for K up to 11 crossings. $\deg_m P_{W_K}(l,m)$ is independent on the twist of W_K if it is > 2 by a simple skein argument.

Update Gruber [157] showed that, if K is a prime alternating knot and W_K is its untwisted Whitehead double, then $\deg_m P_{W_K}(l,m) \leq 2\deg_z F_K(a,z) + 2$.

Problem 1.16 (E. Ferrand, A. Stoimenow) *Is for any alternating link L,*

$$\sigma(L) \geq \min\deg_l\big(P_L(l,m)\big) \geq \min\deg_a\big(F_L(a^{-1},z)\big)\ ?$$

Remark (A. Stoimenow) The second inequality is conjectured by Ferrand [125] (see also comment on Problem 1.18), and related to estimates of the Bennequin numbers of Legendrian knots. As for the first inequality, by Cromwell [97] we have $\min\deg_l\big(P_L(l,m)\big) \leq 1-\chi(L)$ and classically $\sigma(L) \leq 1-\chi(L)$.

Problem 1.17 (A. Stoimenow) *If ∇_k is the coefficient of z^k in the Conway polynomial and $c(L)$ is the crossing number of a link L, is then*

$$\big|\nabla_k(L)\big| \leq \frac{c(L)^k}{2^k\, k!}\ ?$$

Remark (A. Stoimenow) The inequality is non-trivial only for L of $k+1, k-1, \ldots$ components. It is also trivial for $k=0$, easy for $k=1$ (∇_1 is just the linking number of 2 component links) and proved by Polyak-Viro [331] for *knots* and $k=2$. There are constants C_k with

$$\big|\nabla_k(L)\big| \leq C_k\, c(L)^k\,,$$

following from the proof (due to [27, 370] for knots, due to [375] for links) of the Lin-Wang conjecture [263] for links, but determining C_k from the proof is difficult. Can the inequality be proved by Kontsevich-Drinfel'd, say at least for knots, using the description of the weight systems of ∇ of Bar-Natan and Garoufalidis [34]? More specifically, one can ask whether the $(2,n)$-torus links (with parallel orientation) attain the maximal values of ∇_k. One can also ask about the shape of C_k for other families of Vassiliev invariants, like $\frac{d^k}{dt^k}V_L(t)\big|_{t=1}$.

Problem 1.18 (A. Stoimenow) *Does $\min\deg_a\big(F_L(a^{-1},z)\big) \leq 1-\chi(L)$ hold for any link L? If $u(K)$ is the unknotting number of a knot K, does $\min\deg_a\big(F_K(a^{-1},z)\big) \leq 2u(K)$ hold for any knot K?*

Remark (A. Stoimenow) For the common lower bound of $2u$ and $1-\chi$ for knots, $2g_s$, there is a 15 crossing knot K with $2g_s(K) < \min\deg_a\big(F_K(a^{-1},z)\big)$. Morton [285] conjectured long ago that $1-\chi(L) \geq \min\deg_l\big(P_L(l,m)\big)$. There are recent counterexamples, but only of 19 to 21 crossings. Ferrand [125] observed that very often $\min\deg_l\big(P_K(l,m)\big) \geq \min\deg_a\big(F_K(a^{-1},z)\big)$ (he conjectures it in particular always to hold for alternating K), so replacing '$\min\deg_a\big(F(a^{-1},z)\big)$' for '$\min\deg_l\big(P_K(l,m)\big)$' enhances the difficulty of Morton's problem (the counterexamples are no longer such).

1.3 The volume conjecture

In [196] R. Kashaev defined a series of invariants $\langle L\rangle_N \in \mathbb{C}$ of a link L for $N = 2,3,\cdots$ by using the quantum dilogarithm. In [198] he observed, by formal calculations, that

$$2\pi \cdot \lim_{N\to\infty} \frac{\log\langle L\rangle_N}{N} = \mathrm{vol}(S^3 - L)$$

for $L = K_{4_1}, K_{5_2}, K_{6_1}$, where $\mathrm{vol}(S^3-L)$ denotes the hyperbolic volume of S^3-L. Further, he conjectured that this formula holds for any hyperbolic link L. In 1999, H. Murakami and J. Murakami [296] proved that $\langle L\rangle_N = J_N(L)$ for any link L, where $J_N(L)$ denotes the N-colored Jones polynomial[5] of L evaluated at $e^{2\pi\sqrt{-1}/N}$.

Conjecture 1.19 (The volume conjecture, [198, 296]) *For any knot K,*

$$2\pi \cdot \lim_{N\to\infty} \frac{\log|J_N(K)|}{N} = v_3||S^3 - K||, \tag{2}$$

where $||\cdot||$ denotes the simplicial volume and v_3 denotes the hyperbolic volume of the regular ideal tetrahedron.

Remark For a hyperbolic knot K, (2) implies that

$$2\pi \cdot \lim_{N\to\infty} \frac{\log|J_N(K)|}{N} = \mathrm{vol}(S^3 - K).$$

Remark [296] Both sides of (2) behave well under the connected sum and the mutation of knots. Namely,

$$\begin{aligned}&||S^3 - (K_1\#K_2)|| = ||S^3 - K_1|| + ||S^3 - K_2||,\\ &J_N(K_1\#K_2) = J_N(K_1)J_N(K_2),\end{aligned}$$

[5] This is the invariant obtained as the quantum invariant of links associated with the N-dimensional irreducible representation of the quantum group $U_q(sl_2)$.

and $J_N(K)$ and $||S^3 - K||$ do not change under a mutation of K. For details see [296] and references therein.

Remark The statement of the volume conjecture for a link L should probably be the same statement as (2) replacing K with L. It is necessary to assume that L is not a split link, since $J_N(L) = 0$ for a split link L (then, the left hand side of (2) does not make sense).

Example It is shown [200] that for a torus link L

$$\lim_{N\to\infty} \frac{\log\langle L\rangle_N}{N} = 0,$$

which implies that (2) is true for torus links.

Remark Conjecture 1.19 has been proved for the figure eight knot K_{4_1} (see [295] for an exposition). However, we do not have a rigorous proof of this conjecture for other hyperbolic knots so far. We explain its difficulty below, after a review of a proof for K_{4_1}.

We sketch a proof of Conjecture 1.19 for the figure eight knot K_{4_1}; for a detailed proof see [295]. It is known that

$$J_N(K_{4_1}) = \sum_{n=0}^{N-1} (q)_n (q^{-1})_n, \tag{3}$$

where we put $q = e^{2\pi\sqrt{-1}/N}$ and

$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n), \qquad (q)_0 = 1.$$

As N tends to infinity fixing n/N in finite, the asymptotic behaviour of the absolute value of $(q)_n$ is described by

$$\begin{aligned}\log|(q)_n| &= \sum_{k=1}^{n} \log\Big(2\sin\frac{\pi k}{N}\Big) = \frac{N}{\pi}\int_0^{n\pi/N} \log(2\sin t)dt + O(\log N)\\ &= -\frac{N}{2\pi}\mathrm{Im}\big(\mathrm{Li}_2(e^{2\pi n\sqrt{-1}/N})\big) + O(\log N),\end{aligned}$$

where Li_2 denotes the *dilogarithm function* defined on $\mathbb{C} - \{x \in \mathbb{R} \mid x > 1\}$ by

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} = -\int_0^z \frac{\log(1-s)}{s} ds.$$

Noting that each summand of (3) is real-valued, we have that

$$J_N(K_{4_1}) = \sum_{0\le n<N} \exp\Big(\frac{N}{2\pi}\mathrm{Im}\big(\mathrm{Li}_2(e^{-2\pi n\sqrt{-1}/N}) - \mathrm{Li}_2(e^{2\pi n\sqrt{-1}/N})\big) + O(\log N)\Big).$$

The asymptotic behaviour of this sum can be described by the maximal point z_0 of $\mathrm{Im}\big(\mathrm{Li}_2(1/z) - \mathrm{Li}_2(z)\big)$ on the unit circle $\{z\in\mathbb{C}\mid |z|=1\}$. In fact this z_0 is a critical point of $\mathrm{Li}_2(1/z)-\mathrm{Li}_2(z)$ in $\mathbb{C}$, and hence $\mathrm{Im}\big(\mathrm{Li}_2(1/z_0)-\mathrm{Li}_2(z_0)\big)$ gives the hyperbolic volume of $S^3 - K_{4_1}$. Therefore, the conjecture holds in this case.

Next, we sketch a formal argument toward Conjecture 1.19 for the knot K_{5_2}. Following [198], we have that

$$J_N(K_{5_2}) = \sum_{0\le m\le n<N} \frac{(q)_n^2}{(q)_m^*} q^{-m(n+1)},$$

where the asterisk implies the complex conjugate. By applying the formal approximation[6]

$$(q)_n \underset{?}{\sim} \exp\Big(\frac{N}{2\pi\sqrt{-1}}\big(\mathrm{Li}_2(1) - \mathrm{Li}_2(e^{2\pi n\sqrt{-1}/N})\big)\Big), \tag{4}$$

$$(q)_n^* \underset{?}{\sim} \exp\Big(\frac{N}{2\pi\sqrt{-1}}\big(\mathrm{Li}_2(e^{-2\pi n\sqrt{-1}/N}) - \mathrm{Li}_2(1)\big)\Big),$$

we have that

$$J_N(K_{5_2}) \underset{?}{\sim} \sum_{0\le m\le n<N} \exp\Big(\frac{N}{2\pi\sqrt{-1}}\big(\frac{\pi^2}{2} - 2\mathrm{Li}_2(e^{2\pi n\sqrt{-1}/N}) - \mathrm{Li}_2(e^{-2\pi m\sqrt{-1}/N}) + \frac{2\pi n}{N}\frac{2\pi m}{N}\big)\Big).$$

Further, by formally replacing[7] the sum with an integral putting $t=n/N$ and $s=m/N$, we have that

$$\begin{aligned} J_N(K_{5_2}) &\underset{??}{\sim} N^2\int_{0\le s\le t\le 1} \exp\frac{N}{2\pi\sqrt{-1}}\big(\frac{\pi^2}{2} - 2\mathrm{Li}_2(e^{2\pi\sqrt{-1}t}) \\ &\qquad - \mathrm{Li}_2(e^{-2\pi\sqrt{-1}s}) + 2\pi t\cdot 2\pi s\big)\,ds\,dt \qquad (5)\\ &= -\frac{N^2}{4\pi^2}\int \exp\frac{N}{2\pi\sqrt{-1}}\big(\frac{\pi^2}{2} - 2\mathrm{Li}_2(z) - \mathrm{Li}_2(\frac{1}{w}) - \log z\log w\big)\frac{dw}{w}\frac{dz}{z}, \end{aligned}$$

[6] It might be difficult to justify this approximation in a usual sense, since the argument of $(q)_n$, given by $(q)_n = |(q)_n|\cdot q^{-n(n+1)/2}(-\sqrt{-1})^n$, changes discretely and quickly near the limit.

[7] It might be seriously difficult to justify this replacement, since there is a large parameter N in the power of the summand, which exponentially contributes the summand.

where the second integral is over the domain $\{(z,w) \in \mathbb{C}^2 \mid |z| = |w| = 1,\ 0 \le \arg(w) \le \arg(z) \le 2\pi\}$, and the equality is obtained by putting $z = e^{2\pi\sqrt{-1}t}$ and $w = e^{2\pi\sqrt{-1}s}$. By applying the saddle point method[8] the asymptotic behaviour might be described by a critical value of

$$\frac{\pi^2}{2} - 2\mathrm{Li}_2(z) - \mathrm{Li}_2(\frac{1}{w}) - \log z \log w. \tag{6}$$

Since a critical value of this function gives a hyperbolic volume of $S^3 - K_{5_2}$, this formal argument suggests Conjecture 1.19 for K_{5_2}.

It was shown by Yokota [409], following ideas due to Kashaev [196] and Thurston [382], that the hyperbolic volume of the complement of any hyperbolic knot K is given by a critical value of such a function as (6), which is obtained from a similar computation of $J_N(K)$ as above.

Problem 1.20 *Justify the above arguments rigorously.*

Remark The asymptotic behaviour of $J_N(K)$ might be described by using quantum invariants of $S^3 - K$. We have some ways to compute the asymptotic behaviour of such a quantum invariant, say, when K is a fibered knot (in this case, $S^3 - K$ is homeomorphic to a mapping torus of a homeomorphism of a punctured surface), and when we choose a simplicial decomposition of (a closure of) $S^3 - K$. For details, see remarks of Conjecture 7.12.

The following conjecture is a complexification of the volume conjecture (Conjecture 1.19).

Conjecture 1.21 (H. Murakami, J. Murakami, M. Okamoto, T. Takata, Y. Yokota [297]) *For a hyperbolic link L,*

$$2\pi\sqrt{-1} \cdot \lim_{N\to\infty} \frac{\log J_N(L)}{N} = \mathrm{CS}(S^3 - L) + \sqrt{-1}\mathrm{vol}(S^3 - L)$$

for an appropriate choice of a branch of the logarithm, where CS and vol denote the Chern-Simons invariant and the hyperbolic volume respectively. Moreover,

$$\lim_{N\to\infty} \frac{J_{N+1}(L)}{J_N(L)} = \exp\Big(\frac{1}{2\pi\sqrt{-1}}\big(\mathrm{CS}(S^3 - L) + \sqrt{-1}\mathrm{vol}(S^3 - L)\big)\Big). \tag{7}$$

Remark It is shown [297], by formal calculations (such as (4) and (5)), that Conjecture 1.21 is "true" for $K_{5_2}, K_{6_1}, K_{6_3}, K_{7_2}, K_{8_9}$ and the Whitehead link.

[8] The saddle point method in multi-variables is not established yet. This might be a technical difficulty.

Remark The statement for non-hyperbolic links should probably be the same statement, replacing $\mathrm{vol}(S^3 - L)$ with $v_3||S^3 - L||$. Note that, if L is not hyperbolic, then it is also a problem (see Problem 7.16) to find an appropriate definition of $\mathrm{CS}(S^3 - L)$, which might be given by (7). It is necessary to assume that L is not a split link, since $J_N(L) = 0$ for a split link L.

Remark (H. Murakami) Zagier [410] gave a conjectural presentation of the asymptotic behaviour of the following sum,

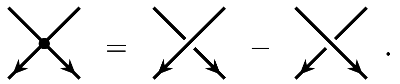
$$J_N(K_{3_1}) = \sum_{k=0}^{N-1} (q)_k \underset{N\to\infty}{\sim} \exp\Big(-\frac{\pi\sqrt{-1}}{12}(N-3+\frac{1}{N})\Big)N^{3/2} + \sum_{k\geq 0} \frac{b_k}{k!}\big(-\frac{2\pi\sqrt{-1}}{N}\big)^k$$

for some series b_k. This suggests that $\lim \frac{\log J_N(K_{3_1})}{N}$ should be $-\pi\sqrt{-1}/12$.

Problem 1.22 (H. Murakami) *For a torus knot K, calculate $\mathrm{CS}(S^3 - K)$ (giving an appropriate definition of it) and calculate $\lim \frac{\log J_N(K)}{N}$ (fixing an appropriate choice of a branch of the logarithm).*

2 Finite type invariants of knots

Let R be a commutative ring with 1 such as $\mathbb{Z}$ or $\mathbb{Q}$. We denote by $\mathbb{K}$ the set of isotopy classes of oriented knots. A *singular knot* is an immersion of S^1 into S^3 whose singularities are transversal double points. We regard singular knots as in $R\mathbb{K}$ by removing each singularity linearly by



Let $\mathcal{F}_d(R\mathbb{K})$ denote the submodule of $R\mathbb{K}$ spanned by singular knots with d double points, regarding them as in $R\mathbb{K}$. Then, we have a descending series of submodules,

$$R\mathbb{K} = \mathcal{F}_0(R\mathbb{K}) \;\supset\; \mathcal{F}_1(R\mathbb{K}) \;\supset\; \mathcal{F}_2(R\mathbb{K}) \;\supset\; \cdots.$$

An R-homomorphism $v : R\mathbb{K} \to R$ (or, a homomorphism $\mathbb{Z}\mathbb{K} \to A$ for an abelian group A) is called a *Vassiliev invariant* (or a *finite type invariant*) of degree d if $v|_{\mathcal{F}_{d+1}(R\mathbb{K})} = 0$. See [33] for many references of Vassiliev invariants.

A trivalent vertex of a graph is called *vertex-oriented* if a cyclic order of the three edges around the trivalent vertex is fixed. A *Jacobi diagram*[9] on an oriented 1-manifold X is the manifold X together with a uni-trivalent graph such that univalent vertices of the graph are distinct points on X and each trivalent vertex is vertex-oriented. The *degree* of a Jacobi diagram is half the number of univalent and trivalent vertices of the uni-trivalent graph of the Jacobi diagram. We denote by $\mathcal{A}(X;R)$ the module over R spanned by Jacobi diagrams on X subject to the AS, IHX, and STU relations shown in Figure 6, and denote by $\mathcal{A}(X;R)^{(d)}$ the submodule of $\mathcal{A}(X;R)$ spanned by Jacobi diagrams of degree d. There is a canonical surjective homomorphism

$$\mathcal{A}(S^1;R)^{(d)}/\mathrm{FI} \to \mathcal{F}_d(R\mathbb{K})/\mathcal{F}_{d+1}(R\mathbb{K}), \tag{8}$$

where FI is the relation shown in Figure 6. This map is known to be an isomorphism when $R = \mathbb{Q}$ (due to Kontsevich). For a Vassiliev invariant $v : R\mathbb{K} \to R$ of degree d, its weight system $\mathcal{A}(S^1;R)^{(d)}/\mathrm{FI} \to R$ is defined by the map (8).

[9] A Jacobi diagram is also called a *web diagram* or a *trivalent diagram* in some literatures. In physics this is often called a *Feynman diagram*.

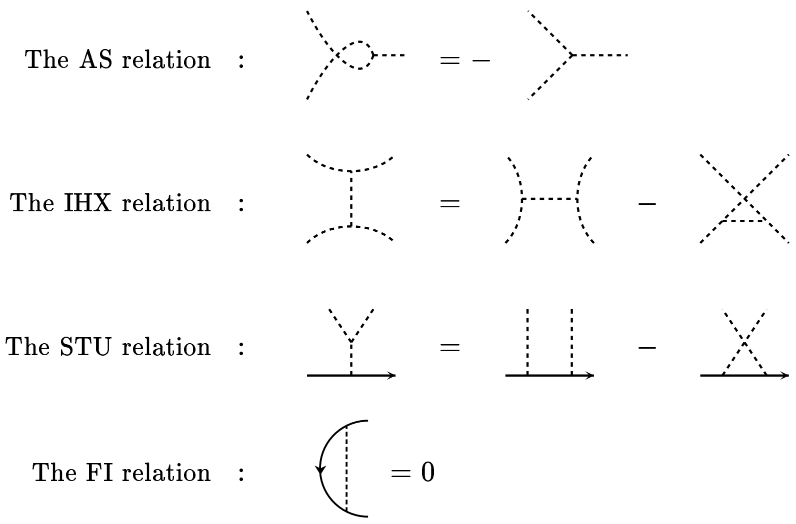


Figure 6: The AS, IHX, STU, and FI relations

2.1 Torsion and Vassiliev invariants

Let R be a commutative ring with 1, say $\mathbb{Z}/n\mathbb{Z}$. Then, $\mathbb{Q}$-, $\mathbb{Z}$-, R-valued Vassiliev invariants and their weight systems and the Kontsevich invariant form the following commutative diagram.

$$\begin{tikzcd}
\mathcal{A}(S^1;\mathbb{Q})/\mathrm{FI} \arrow[d, "\mathrm{proj}"] & & \mathbb{K} \arrow[ll, "\text{Kontsevich invariant}"'] \arrow[d] & \\
\mathcal{A}(S^1;\mathbb{Q})^{(d)}/\mathrm{FI} \arrow[r, "\mathrm{isom}"] & \mathcal{F}_d(\mathbb{QK})/\mathcal{F}_{d+1}(\mathbb{QK}) \arrow[r, "\subset"] & \mathbb{QK}/\mathcal{F}_{d+1}(\mathbb{QK}) \arrow[r] & \mathbb{Q} \\
\mathcal{A}(S^1;\mathbb{Z})^{(d)}/\mathrm{FI} \arrow[u, "\cdot\otimes\mathbb{Q}"] \arrow[r, "\mathrm{surj}"] \arrow[d, "\mathrm{proj}"] & \mathcal{F}_d(\mathbb{ZK})/\mathcal{F}_{d+1}(\mathbb{ZK}) \arrow[u, "\cdot\otimes\mathbb{Q}"] \arrow[r, "\subset"] \arrow[d, "\mathrm{proj}"] & \mathbb{ZK}/\mathcal{F}_{d+1}(\mathbb{ZK}) \arrow[u, "\cdot\otimes\mathbb{Q}"'] \arrow[r] \arrow[d, "\mathrm{proj}"'] & \mathbb{Z} \arrow[u, "\cdot\otimes\mathbb{Q}"'] \arrow[d, "\mathrm{proj}"'] \\
\mathcal{A}(S^1;R)^{(d)}/\mathrm{FI} \arrow[r, "\mathrm{surj}"] & \mathcal{F}_d(R\mathbb{K})/\mathcal{F}_{d+1}(R\mathbb{K}) \arrow[r, "\subset"] & R\mathbb{K}/\mathcal{F}_{d+1}(R\mathbb{K}) \arrow[r] & R
\end{tikzcd}$$

Here, the right horizontal maps are derived from Vassiliev invariants, and the compositions of horizontal maps are their weight systems.

Conjecture 2.1 ([220, Problem 1.92 (N)]) *$\mathcal{F}_d(\mathbb{Z}\mathbb{K})/\mathcal{F}_{d+1}(\mathbb{Z}\mathbb{K})$ is torsion free for each d.*

Remark (see [220, Remark on Problem 1.92 (N)]) Goussarov has checked the conjecture for $d \leq 6$. It has been checked that $\mathcal{F}_d(\mathbb{Z}\mathbb{K})/\mathcal{F}_{d+1}(\mathbb{Z}\mathbb{K})$ has no 2-torsion for $d \leq 9$ by Bar-Natan, and for $d \leq 12$ in [224].

Remark If this conjecture was true, then $\mathbb{Z}$-valued and $\mathbb{Q}$-valued Vassiliev invariants carry exactly the same information about knots. Moreover, any $(\mathbb{Z}/n\mathbb{Z})$-valued Vassiliev invariants would be derived from $\mathbb{Z}$-valued Vassiliev invariants.

Conjecture 2.2 *$\mathcal{A}(S^1;\mathbb{Z})$ is torsion free.*

Remark (T. Stanford) This conjecture would imply Conjecture 2.1 because of the Kontsevich integral. However, it is possible that there is torsion in $\mathcal{A}(S^1,\mathbb{Z})^{(d)}$ which is in the kernel of the map (8).

Conjecture 2.3 (X.-S. Lin [262]) *Let R be a commutative ring with 1, say $\mathbb{Z}/2\mathbb{Z}$. Every weight system $\mathcal{A}(S^1;R)^{(d)}/FI \to R$ is induced by some Vassiliev invariant $R\mathbb{K} \to R$.*

Remark If the map (8) is an isomorphism and $\mathcal{F}_d(R\mathbb{K})/\mathcal{F}_{d+1}(R\mathbb{K})$ is a direct summand of $R\mathbb{K}/\mathcal{F}_{d+1}(R\mathbb{K})$, then this conjecture is true (see the diagram at the beginning of this section).

Remark When $R = \mathbb{Q}$, this conjecture is true, since the composition of the Kontsevich invariant and a weight system gives a Vassiliev invariant, which induces the weight system. If the Kontsevich invariant with coefficients in R would be constructed (see Problem 3.7), this conjecture would be true.

Remark (T. Stanford) The chord diagram module $\mathcal{A}(\downarrow\downarrow,\mathbb{Z})$ corresponds to finite-type invariants of two-strand string links. Jan Kneissler and Ilya Dogolazky (see [109]) showed that there is a 2-torsion element in $\mathcal{A}(\downarrow\downarrow,\mathbb{Z})^{(5)}/\mathrm{FI}$ (see Figure 7). I have done recent calculations (to be written up soon) which show that there is no $\mathbb{Z}/2\mathbb{Z}$-valued invariant of string links corresponding to this torsion element. Thus there is a $\mathbb{Z}/2\mathbb{Z}$ weight system $\mathcal{A}(\downarrow\downarrow,\mathbb{Z}/2\mathbb{Z})/\mathrm{FI} \to \mathbb{Z}/2\mathbb{Z}$ which is not induced by a $\mathbb{Z}/2\mathbb{Z}$-valued finite-type invariant. So for string links, Conjecture 2.1 is false.

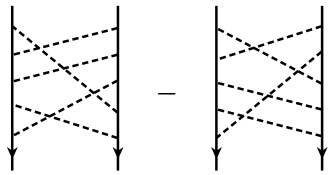


Figure 7: A 2-torsion element in $\mathcal{A}(\downarrow\downarrow;\mathbb{Z})$ due to Dogolazky–Kneissler

(T. Stanford) Note that the Kontsevich integral works (for rational invariants) for string links just as well as for knots. Since this calculation shows that there is no $\mathbb{Z}/2\mathbb{Z}$ Kontsevich integral for string links, it suggests that there is no $\mathbb{Z}/2\mathbb{Z}$ Kontsevich integral for knots.

Question 2.4 (T. Stanford) *The Dogolazky-Kneissler 2-torsion element in $\mathcal{A}(\downarrow\downarrow,\mathbb{Z})$ (see Figure 7) can be embedded into a chord diagram in $\mathcal{A}(S^1,\mathbb{Z})$ in many ways. Such an embedding will always produce an element $x \in \mathcal{A}(S^1,\mathbb{Z})$ with $2x = 0$. Is it possible to produce such an x which is nontrivial? If so, this would give a counterexample to Conjecture 2.3.*

2.2 Do Vassiliev invariants distinguish knots?

Conjecture 2.5 *Vassiliev invariants distinguish oriented knots. (See Conjecture 3.2 for an equivalent statement of this conjecture.)*

Remark Two knots with the same Vassiliev invariant up to an arbitrarily given degree can be obtained; see [324] and Goussarov-Habiro theory [152, 153, 165]. Hence, we need infinitely many Vassiliev invariants to show this conjecture.

Problem 2.6 *Does there exists a non-trivial oriented knot which can not be distinguished from the trivial knot by Vassiliev invariants? (See Problem 3.3 for an equivalent problem.)*

Remark The volume conjecture (Conjecture 1.19) suggests that the answer is no; see [296].

Conjecture 2.7 (see [220, Problem 1.89 (B)]) *For any oriented knot K, no Vassiliev invariants distinguish K from $-K$. (See Conjecture 3.4 for an equivalent statement of this conjecture.)*

Remark ([220, Remark on Problem 1.89]) The first oriented knot which is different from its reverse is 8_{17}. It is known that no Vassiliev invariants of degree ≤ 9 can distinguish a knot from its reverse.

Remark This conjecture is reduced to the problem to find $D \in \mathcal{A}(S^1)$ with $D \neq -D$, where $-D$ is D with the opposite orientation of S^1. If such a D existed, the conjecture would fail. Such a D has not been known so far.

Remark Kuperberg [235] showed that all Vassiliev invariants either distinguish all oriented knots, or there exist prime, unoriented knots which they do not distinguish.

2.3 Can Vassiliev invariants detect other invariants?

(T. Stanford) Let $h_G(K)$ be the number of homomorphisms from the fundamental group of the complement of a knot K to a finite group G. This is not a Vassiliev invariant [114]. $h_{\mathfrak{S}_3}(K)$ of the 3rd symmetric group $\mathfrak{S}_3$ is presented by the number of 3-colorings of K, and $h_{D_5}(K)$ of the dihedral group D_5 of order 10 is presented by the number of 5-colorings of K. These are determined by the Jones and Kauffman polynomials, respectively (see the remark of Problem 4.16), and therefore are determined by invariants of finite type. In fact, by the usual power-series expansions of the Jones and Kauffman polynomials, we see that $h_{\mathfrak{S}_3}$ and h_{D_5} are the (pointwise) limits of respective sequences of finite-type invariants.

Question 2.8 (T. Stanford) *Can we approximate h_G by Vassiliev invariants for other G than dihedral groups?*

Remark (T. Stanford) It is known (due to W. Thurston) that knot groups are residually finite. So if h_G can be approximated by finite-type invariants for all finite groups G, then Vassiliev invariants would distinguish the unknot.

Remark (T. Stanford) If p is a prime, then there exists a nontrivial p-coloring of a knot K, and hence a nontrivial representation of the fundamental group of K into the dihedral group D_p of order $2p$, if and only if $\Delta_K(-1)$ is divisible by p. Thus the Alexander polynomial contains information about h_{D_p}, though it may not determine h_{D_p} completely. Suppose that G is a finite, non-abelian group, not isomorphic to D_p. Even if we cannot approximate h_G by finite-type invariants, it would at least be interesting to know whether finite-type invariants provide any information at all about h_G.

Remark Let $h_X(K)$ denote the number of homomorphisms from the knot quandle of a knot K to a finite quandle X. The number $h_G(K)$ can be presented by the sum of $h_X(K)$ for subquandles X of the conjugation quandle of G. In this sense, it is a refinement of Question 2.8 to approximate h_X of finite quandles X by Vassiliev invariants. It is known [181] that $h_X(K)$ for certain Alexander quandles X can be presented by the ith Alexander polynomial of K.

Problem 2.9 (X.-S. Lin [262]) *Is the knot signature the limit of a sequence of Vassiliev invariants?*

Remark It is known [104] that the signature of knots is not a Vassiliev invariant.

2.4 Vassiliev invariants and crossing numbers

Let v_2 and v_3 be $\mathbb{R}$-valued Vassiliev invariants of degree 2 and 3 respectively normalized by the conditions that $v_2(\overline{K}) = v_2(K)$ and $v_3(\overline{K}) = -v_3(K)$ for any knot K and its mirror image $\overline{K}$ and that $v_2(K_{\overline{3}_1}) = v_3(K_{\overline{3}_1}) = 1$ for the right trefoil knot $K_{\overline{3}_1}$. They are primitive Vassiliev invariants, and the image of $v_2 \times v_3$ is equal to $\mathbb{Z} \times \mathbb{Z} \subset \mathbb{R} \times \mathbb{R}$.

Problem 2.10 (N. Okuda [325]) *Describe the set*

$$\Big\{ \big(\frac{v_2(K)}{n^2}, \frac{v_3(K)}{n^3} \big) \in \mathbb{R} \times \mathbb{R} \;\Big|\; K \textit{ has a knot diagram with } n \textit{ crossings} \Big\}. \tag{9}$$

Remark Willerton [401] observed that the set of $(v_2(K), v_3(K))$ for knots K with a (certain) fixed crossing number gives a fish-like graph. This fish-like graph is discussed in [103] from the point of view of the Jones polynomial.

Remark (N. Okuda) It is shown by Okuda [325] (the right inequality of (10) is due to [331]) that, if a knot K has a diagram with n crossings, then

$$-\Big\lfloor \frac{n^2}{16} \Big\rfloor \le v_2(K) \le \Big\lfloor \frac{n^2}{8} \Big\rfloor, \tag{10}$$

$$|v_3(K)| \le \Big\lfloor \frac{n(n-1)(n-2)}{15} \Big\rfloor, \tag{11}$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. It follows that the set (9) is included in the rectangle $[-1/16, 1/8] \times [-1/15, 1/15]$. It is a

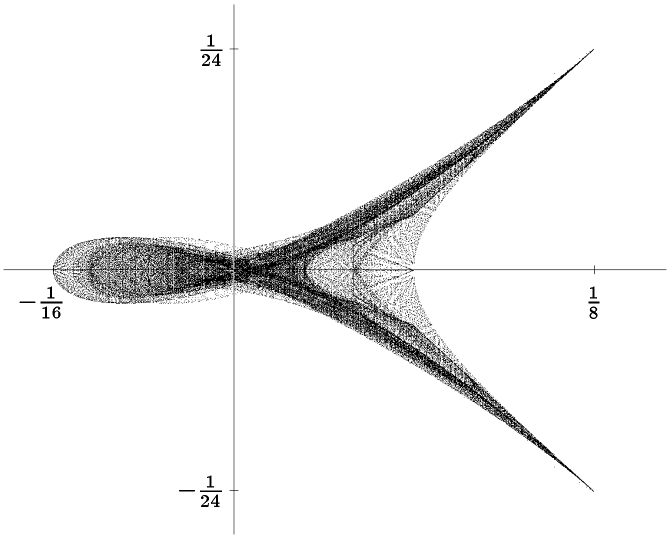
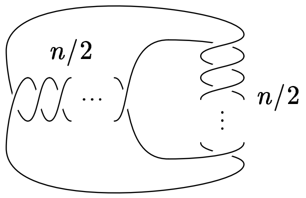


Figure 8: The plottings of the set (9) for some family of knots [325]

problem to describe the smallest domain including this set. The plottings in Figure 8 numerically describe the set (9) for a large finite subset of a certain infinite family of knots. Okuda [325] identified the boundary of the domain including this set for this infinite family of knots. This boundary is given by curves parameterized by some polynomials of degree 2 (for the v_2-coordinate) and of degree 3 (for the v_3-coordinate). Further, the points $(1/8, \pm 1/24)$ are the limits of the points given by the $(2, n)$ torus knot and its mirror image. The point $(-1/16, 0)$ is the limit of the points given by the knots



for n divisible by 4, where each twisting part has $n/2$ crossings. These knots gives the bounds of (10), while the inequality in (11) might not be best possible (see Conjecture 2.11 below).

Remark (O. Viro) The experimental data (in Figure 8) suggest that there might exists an additional invariant(s) which together with v_2, v_3, and n satisfy an algebraic equation(s) such that the set (9) is the projection of the algebraic set defined by the equation(s).

Conjecture 2.11 (S. Willerton [401]) *Let v_3 be as above. If a knot K has a diagram with n crossings, then*

$$|v_3(K)| \leq \left\lfloor \frac{n(n^2-1)}{24} \right\rfloor.$$

Remark It is shown in [401] that the $(2,n)$ torus knot gives the equality of this formula.

2.5 Dimensions of spaces of Vassiliev invariants

We denote by $\mathcal{A}(S^1;R)_{\mathrm{conn}}$ the submodule of $\mathcal{A}(S^1;R)$ spanned by Jacobi diagrams with connected uni-trivalent graphs. As a graded vector space $\mathcal{A}(S^1;\mathbb{Q})$ is isomorphic to the symmetric tensor algebra of $\mathcal{A}(S^1;\mathbb{Q})_{\mathrm{conn}}$. A Vassiliev invariant v is called *primitive* if $v(K_1\#K_2) = v(K_1)+v(K_2)$ for any oriented knots K_1 and K_2. The degree d subspace of $\mathcal{A}(S^1;\mathbb{Q})_{\mathrm{conn}}$ is dual to the dth graded vector space for $\mathbb{Q}$-valued primitive Vassiliev invariants.

Problem 2.12 *Determine the dimension of the space of primitive Vassiliev invariants of each degree d. Equivalently, determine the dimension of the space $\mathcal{A}(S^1;\mathbb{Q})^{(d)}_{\mathrm{conn}}$ for each d.*

d	0	1	2	3	4	5	6	7	8	9	10
dim $\mathcal{A}(S^1)^{(d)}_{\mathrm{conn}}$	0	1	1	1	2	3	5	8	12	18	27
dim $\mathcal{A}(S^1)^{(d)}$	1	1	2	3	6	10	19	33	60	104	184
dim $\mathcal{A}(S^1)^{(d)}/\mathrm{FI}$	1	0	1	1	3	4	9	14	27	44	80

d	11	12	13	14
dim $\mathcal{A}(S^1)^{(d)}_{\mathrm{conn}}$	39	55	≥ 78	≥ 108
dim $\mathcal{A}(S^1)^{(d)}$	316	548	≥ 932	≥ 1591
dim $\mathcal{A}(S^1)^{(d)}/\mathrm{FI}$	132	232	≥ 384	≥ 659

Table 1: Some dimensions given in [67, 224]

Remark The dimension of $\mathcal{A}(S^1;\mathbb{Q})^{(d)}_{\mathrm{conn}}$ can partially be computed as follows.

Let $\mathcal{B}$ denote the vector space over $\mathbb{Q}$ spanned by vertex-oriented uni-trivalent graphs subject to the AS and IHX relations, and let $\mathcal{B}^{(d)}_{\mathrm{conn}}$ denote the subspace of $\mathcal{B}$ spanned by connected uni-trivalent graphs with $2d$ vertices.

It is known that $\mathcal{A}(S^1;\mathbb{Q})^{(d)}_{\mathrm{conn}}$ is isomorphic to $\mathcal{B}^{(d)}_{\mathrm{conn}}$ by (21). Let $\mathcal{B}^{(d,u)}_{\mathrm{conn}}$ be the subspace of $\mathcal{B}^{(d)}_{\mathrm{conn}}$ spanned by uni-trivalent graphs with u univalent vertices (hence, with $2d-u$ trivalent vertices), and $\beta_{d,u}$ its dimension. Then, the dimension of $\mathcal{A}(S^1;\mathbb{Q})^{(d)}_{\mathrm{conn}}$ is presented by $\sum_{u\geq 2}\beta_{d,u}$.

Bar-Natan [28] gave a table of $\beta_{d,u}$ for $d\leq 9$ and for some other (d,u) by computer.

The series of $\beta_{k,k}$ is given as follows. The direct sum $\oplus_k\mathcal{B}^{(k,k)}_{\mathrm{conn}}$ is isomorphic to the polynomial ring $\mathbb{Q}[x^2]$ as a graded vector space by (23); in other words, it is spanned by "wheels". Hence, the series of $\beta_{k,k}$ is presented by the following generating function,

$$\sum_{k\geq 0}\beta_{k,k}t^k=\frac{1}{1-t^2}.$$

That is, $\beta_{k,k}=1$ if k is even, and 0 otherwise.

The series of $\beta_{k+1,k}$ is given as follows. The direct sum $\oplus_k\mathcal{B}^{(k+1,k)}_{\mathrm{conn}}$ is isomorphic to $\mathbb{Q}[\sigma_2,\sigma_3^2]$ as a graded vector space by (25), where σ_i denotes the i-th elementary symmetric polynomial in some variables. Hence, the series of $\beta_{k+1,k}$ is presented by the following generating function,

$$\sum_{k\geq 0}\beta_{k+1,k}t^k=\frac{1}{(1-t^2)(1-t^6)}.$$

The series of $\beta_{k+2,k}$ is presented by

$$\sum_{k\geq 0}\beta_{k+2,k}t^k=\frac{1}{(1-t^2)(1-t^4)(1-t^6)},$$

since $\oplus_k\mathcal{B}^{(k+2,k)}_{\mathrm{conn}}$ is isomorphic, as a graded vector space, to $\mathbb{Q}[\sigma_2,\sigma_3^2,\sigma_4]$ with elementary symmetric polynomials in some variables by (27).

It is conjectured [101] that the series of $\beta_{k+3,k}$ would be presented by

$$\sum_{k\geq 0}\beta_{k+3,k}t^k\overset{?}{=}\frac{1+t^2+t^8-t^{10}}{(1-t^2)(1-t^4)(1-t^6)(1-t^{10})}.$$

It has been shown that $\beta_{d,u}=0$ for $d\leq 9$ and for $d\leq u+2$. However, it is conjectured yet for other (d,u).

A conjecture of a two-variable generating function for the series of $\beta_{d,u}$ with two parameters d and u is given in [67].

$\beta_{d,u}$	$u=2$	$u=4$	$u=6$	$u=8$	$u=10$	$u=12$	$u=14$	total
$d=1$	1							1
$d=2$	1							1
$d=3$	1							1
$d=4$	1	1						2
$d=5$	2	1						3
$d=6$	2	2	1					5
$d=7$	3	3	2					8
$d=8$	4	4	3	1				12
$d=9$	5	6	5	2				18
$d=10$	6	8	8	4	1			27
$d=11$	8	10	11	8	2			39
$d=12$	9	13	15	12	5	1		55
$d=13$	≥ 11	≥ 16	≥ 20	≥ 18	≥ 10	3		≥ 78
$d=14$	≥ 13	≥ 19	≥ 25	≥ 26	≥ 17	7	1	≥ 108

Table 2: A table of $\beta_{d,u}$ [67, 224]

Remark An asymptotic evaluation of a lower bound of $\dim\mathcal{A}(S^1)^{(d)}_{\mathrm{conn}}$ was given in [87]; $\dim\mathcal{A}(S^1)^{(d)}_{\mathrm{conn}}$ grows at least as $d^{\log d}$ when $d\to\infty$. Further, it was improved in [101]; $\dim\mathcal{A}(S^1)^{(d)}_{\mathrm{conn}}$ grows at least as $e^{c\sqrt{d}}$ for any $c<\pi\sqrt{2/3}$ when $d\to\infty$.

Remark Upper bounds of $\dim\mathcal{A}(S^1)^{(d)}_{\mathrm{conn}}$ were given $\dim\mathcal{A}(S^1)^{(d)}_{\mathrm{conn}}\leq(d-1)!$ in [86] and $\dim\mathcal{A}(S^1)^{(d)}_{\mathrm{conn}}\leq(d-2)!/2$ (for $d>5$) in [313]. Stoimenow [373] introduced the number ξ_d of "regular linearized chord diagrams", and showed that $\dim\mathcal{A}(S^1)^{(d)}/\mathrm{FI}\leq\xi_d$. Further, he showed that ξ_d is asymptotically at most $d!/1.1^d$, which was improved by $d!/(2\ln 2+o(1))$ in [63]. Furthermore, Zagier [410] showed that

$$\sum_{n=0}^{\infty}(1-q)(1-q^2)\cdots(1-q^n)=\sum_{d=0}^{\infty}\xi_d(1-q)^d\in\mathbb{Z}[[1-q]], \tag{12}$$

and that

$$\xi_d\sim\frac{d!\sqrt{d}}{(\pi^2/6)^d}\Big(C_0+\frac{C_1}{d}+\frac{C_2}{d^2}+\cdots\Big)$$

with $C_0=12\sqrt{3}\pi^{-5/2}e^{\pi^2/12}\approx 2.704$, $C_1\approx -1.527$, $C_2\approx -0.269$. It follows that the asymptotic growth of $\dim\mathcal{A}(S^1)^{(d)}/\mathrm{FI}$ is at most $O(d!\sqrt{d}(\pi^2/6)^{-d})$.

d	0	1	2	3	4	5	6	7	8	9	10
dim $\mathcal{A}(S^1)^{(d)}/\mathrm{FI}$	1	0	1	1	3	4	9	14	27	44	80
ξ_d	1	1	2	5	15	53	217	1014	5335	31240	201608

Table 3: Upper bounds ξ_d of dim$\mathcal{A}(S^1)^{(d)}/\mathrm{FI}$ (see [373])

2.6 Milnor invariants

(T. Stanford) Fix k, and consider k-strand string links. Let V_n be the subspace of rational-valued finite-type invariants of order $\leq n$ (of k-strand string links). Let $M_n \subset V_n$ be the subspace of Milnor invariants and products of Milnor invariants. It is known that in general M_n is a proper subspace of V_n.

Question 2.13 (T. Stanford) *Does M_n have an interesting complementary space in V_n? Consider, for example, the space $N_n \subset V_n$ of invariants v with the property that $v(L) = 0$ for any string link L such that $\pi_1(B^3 - L)$ is free. Is N_n nontrivial? Do N_n and M_n together span V_n?*

Here is some background and motivation.

When considering finite-type invariants of string links, the first ones that come to mind are the Milnor invariants. These were defined by Milnor [283] in 1954 as numbers associated to links. They are not quite invariants of links, in the usual sense, because of some indeterminacy. They are, however, well-defined as invariants of string links, and this point of view was taken by Habegger and Lin [163]. After Vassiliev's work appeared, Bar-Natan [26] and Lin [261] showed (independently) that the Milnor invariants are finite-type invariants. Habegger and Masbaum [164] showed that on the chord diagram level, the Milnor invariants (including products of Milnor invariants) are exactly the ones that vanish on Jacobi diagrams that contain internal loops, and also that the Milnor invariants are the only rational-valued finite-type invariants of string links which are also concordance invariants.

String links may have local knots in the strands, and such knots are not detected by Milnor invariants. If a string link L has local knots, then $\pi_1(B^3 - L)$ is not free. Hence the question as to whether finite-type invariants can show that the complement of a string link is not free.

(M. Polyak) Let us review the constructions of Milnor $\overline{\mu}$-invariant in [89]. For a n-component link $L = L_1 \cup \cdots \cup L_n$, regard the homotopy class of L_n as in

$\pi_1\big(S^3-(L_1\cup\cdots\cup L_{n-1})\big)$, and write it in terms of meridians $m_1,\cdots,m_{n-1}$ of $L_1,\cdots,L_{n-1}$. Consider its Magnus expansion putting $m_i=1+X_i$ for non-commutative variables X_i. Then, Milnor's $\overline{\mu}$-invariant $\overline{\mu}_{i_1\cdots i_k,n}(L)$ is defined to be the coefficient of $X^{i_1}\cdots X^{i_k}$ in the expansion, which is an invariant under the assumption that the lower $\overline{\mu}$-invariants vanish. For example, $\mu_{1,2}$ is equal to the linking number $\mathrm{lk}(L_1,L_2)$ of L_1 and L_2. Further, if $\mu_{i,j}(L)=0$ for any i,j, then $\mu_{12,3}(L)=\mathrm{lk}(L_{12},L_3)$, where L_{12} denotes the link which is the intersection of Seifert surfaces of L_1 and L_2. In general, under the vanishing assumption of the lower $\overline{\mu}$-invariants, $\overline{\mu}_{12\cdots n-1,n}(L)=\mathrm{lk}(L_{12\cdots n-1},L_n)$ where $L_{12\cdots k}$ (for $k=2,3,\cdots,n-1$) denotes the link which is the intersection of Seifert surfaces of $L_{12\cdots k-1}$ and L_k.

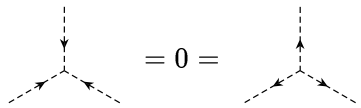
Problem 2.14 (M. Polyak) *Milnor's $\overline{\mu}$-invariants of string links can be defined similarly as above (see [329]). Find a topological presentation of a $\overline{\mu}$-invariant of string links (not assuming the vanishing of the lower $\overline{\mu}$-invariants).*

(1) *Show that $\mathrm{lk}(L_{12\cdots n-1},L_n)$ is well-defined in an appropriate sense.*

(2) *Identify it with $\overline{\mu}_{12\cdots n-1,n}(L)$.*

2.7 Finite type invariants of virtual knots

A *virtual knot* [203] is defined by a knot diagram with virtual crossings modulo Reidemeister moves. Finite type invariants of virtual knots were studied in [154], where their weight systems are defined on the space $\overrightarrow{\mathcal{A}}(X;R)/\overrightarrow{\mathrm{FI}}$ of arrow diagrams. Here an *arrow diagram* [330] is a chord diagram with oriented chords, and $\overrightarrow{\mathcal{A}}(X;R)$ denotes the module over a commutative ring R spanned by arrow diagrams on X subject to the 6T relation, and $\overrightarrow{\mathrm{FI}}$ denotes the oriented FI relation (see Figure 9 for these relations). It is known [330] that $\overrightarrow{\mathcal{A}}(X;R)$ is isomorphic to the module spanned by acyclic oriented Jacobi diagrams on X subject to the relations



and the $\overrightarrow{\mathrm{AS}}$, $\overrightarrow{\mathrm{IHX}}$, and $\overrightarrow{\mathrm{STU}}$ relations (see Figure 9).

Problem 2.15 *Let I denote an oriented interval.*

(1) *Determine the dimensions of $\overrightarrow{\mathcal{A}}(S^1;\mathbb{Q})^{(d)}$ and $\overrightarrow{\mathcal{A}}(I;\mathbb{Q})^{(d)}$ for each d.*

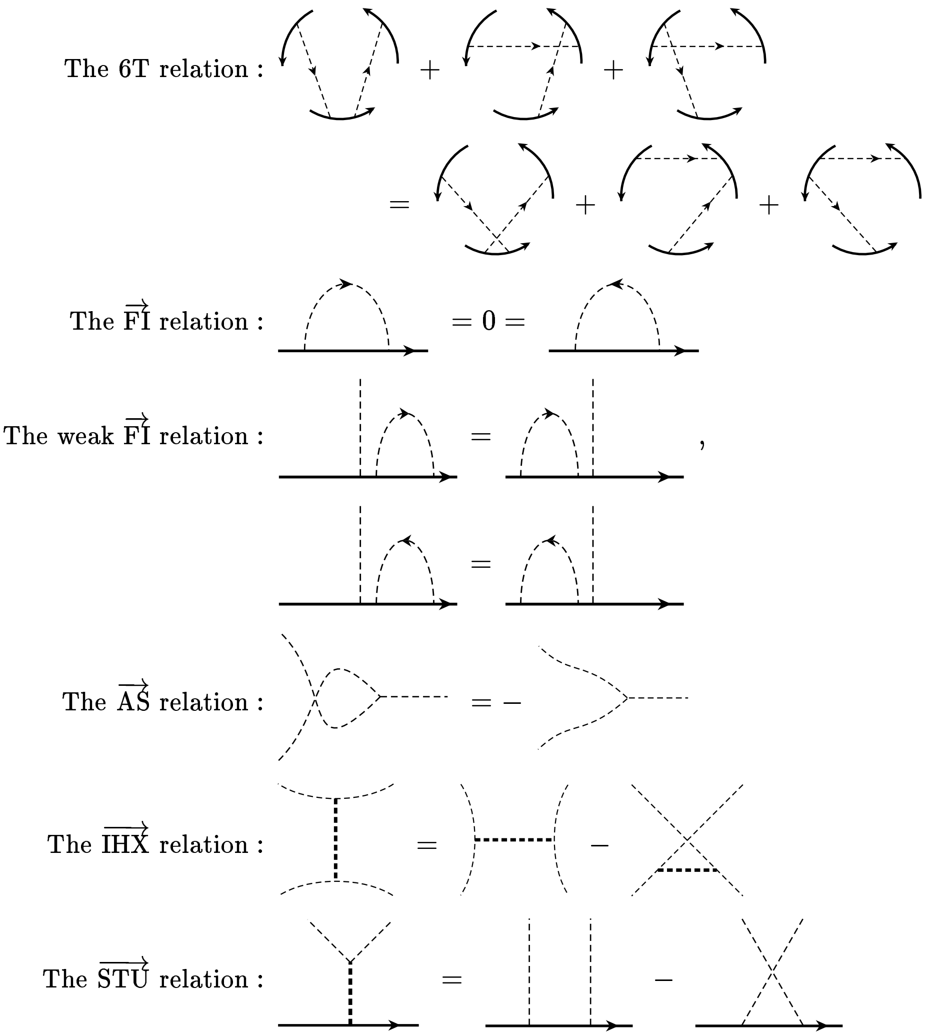


Figure 9: The 6T and the oriented FI, AS, IHX, and STU relations. Here, a thick dashed line implies the sum of the two orientations, and corresponding thin dashed lines of pictures in the same formula have the same (arbitrarily given) orientation.

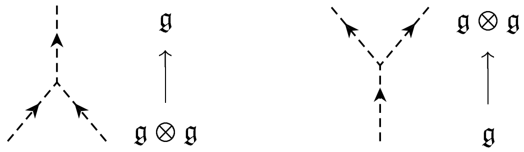
(2) *Determine the dimensions of* $\overrightarrow{\mathcal{A}}(S^1;\mathbb{Q})^{(d)}/\overrightarrow{FI}$ *and* $\overrightarrow{\mathcal{A}}(I;\mathbb{Q})^{(d)}/\overrightarrow{FI}$ *for each* d.

(3) *Determine the dimensions of* $\overrightarrow{\mathcal{A}}(S^1;\mathbb{Q})^{(d)}/$*(weak* $\overrightarrow{FI}$*) and* $\overrightarrow{\mathcal{A}}(I;\mathbb{Q})^{(d)}/$*(weak* $\overrightarrow{FI}$*) for each* d.

Remark It is shown by elementary computation that $\overrightarrow{\mathcal{A}}(S^1,\mathbb{Q})^{(2)}/\mathrm{FI}=0$ and that $\overrightarrow{\mathcal{A}}(I,\mathbb{Q})^{(2)}/\mathrm{FI}$ is a 2-dimensional vector space spanned by

Note that the dimensions of $\overrightarrow{\mathcal{A}}(S^1;\mathbb{Q})^{(d)}$ and $\overrightarrow{\mathcal{A}}(I;\mathbb{Q})^{(d)}$ differ unlike the unoriented case.

Remark Constructive weight systems on $\overrightarrow{\mathcal{A}}(X;R)$ can be defined by using Lie bialgebras (see, e.g. [111, 117], for Lie bialgebras), where the weight systems of the following diagrams

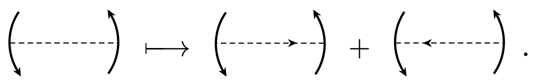


are defined to be the bracket and the co-bracket of a Lie bialgebra $\mathfrak{g}$. Such weight systems are helpful when we estimate lower bounds of the dimensions of the spaces $\mathcal{A}(X;R)$.

Conjecture 2.16 (M. Polyak) *The following two maps are injective,*

$$\mathcal{A}(I)^{(d)} \longrightarrow \overrightarrow{\mathcal{A}}(I)^{(d)}$$
$$\mathcal{A}(I)^{(d)}/FI \longrightarrow \overrightarrow{\mathcal{A}}(I)^{(d)}/\overrightarrow{FI},$$

where they are defined by



Remark If these maps are injective, then weight systems on $\mathcal{A}(I)^{(d)}$ and $\mathcal{A}(I)^{(d)}/\mathrm{FI}$ would be detected by weight systems on $\overrightarrow{\mathcal{A}}(I)^{(d)}$ and $\overrightarrow{\mathcal{A}}(I)^{(d)}/\overrightarrow{\mathrm{FI}}$;

in other words, the upper rightward map in the following diagram would be surjective.

$$\begin{array}{ccc}
\left\{\begin{array}{l}\text{degree } d \text{ weight systems}\\ \text{for long virtual knots}\end{array}\right\} & \longrightarrow & \left\{\begin{array}{l}\text{degree } d \text{ weight systems}\\ \text{for classical knots}\end{array}\right\} \\
\uparrow & & \uparrow \\
\left\{\begin{array}{l}\text{degree } d \text{ finite type invariants}\\ \text{for long virtual knots}\end{array}\right\} & \longrightarrow & \left\{\begin{array}{l}\text{degree } d \text{ Vassiliev invariants}\\ \text{for classical knots}\end{array}\right\}
\end{array}$$

Hence, this conjecture follows from Conjecture 2.17 below, which implies that the lower rightward map in the above diagram is surjective.

Conjecture 2.17 [154] *Every Vassiliev invariant of classical knots can be extended to a finite type invariant of long virtual knots. (See also Problem 3.9.)*

2.8 Finite type invariants derived from local moves

One aspect of the study of knot invariants is the study of the set of knots. A local move and finite type invariants derived from it might give an approach of this study.

A *local move* is a move between two knots, which are identical except for a ball, where they differ as shown in both sides of a move in Figure 10. Let R be a commutative ring with 1, and $\mathbb{K}$ the set of isotopy classes of oriented knots, as before. For a local move $\mathfrak{m}$, we define $\mathcal{F}_d(R\mathbb{K},\mathfrak{m})$ as follows. Let K be an oriented knot with d disjoint balls $B_1, B_2, \cdots, B_d$ such that K is as shown in one side of $\mathfrak{m}$ in each B_i. For any subset $S \subset \{1, 2, \cdots, d\}$, we denote by K_S the knot obtained from K by applying $\mathfrak{m}$ in each B_i for $i \in S$. We define $\mathcal{F}_d(R\mathbb{K},\mathfrak{m})$ to be the submodule of $R\mathbb{K}$ spanned by

$$\sum_S (-1)^{\#S} K_S \tag{13}$$

for any K with d balls, where $\#S$ denotes the number of elements of S, and the sum runs over all subsets S of $\{1, 2, \cdots, d\}$. Then, we have a descending series of submodules,

$$R\mathbb{K} = \mathcal{F}_0(R\mathbb{K},\mathfrak{m}) \supset \mathcal{F}_1(R\mathbb{K},\mathfrak{m}) \supset \mathcal{F}_2(R\mathbb{K},\mathfrak{m}) \supset \cdots.$$

Note that $\mathcal{F}_d(R\mathbb{K}) = \mathcal{F}_d(R\mathbb{K},\times)$ for a crossing change "$\times$". An R-homomorphism $v : R\mathbb{K} \to R$ is called a *finite type invariant of* $\mathfrak{m}$*-degree* d, or an $\mathfrak{m}$ *finite type invariant of degree* d, if $v|_{\mathcal{F}_{d+1}(R\mathbb{K},\mathfrak{m})} = 0$.

A crossing change "×" :

A double crossing change "××" :

A # move :

A pass move :

A Δ move :

A doubled delta move $\Delta\Delta$:

An n-gon move :

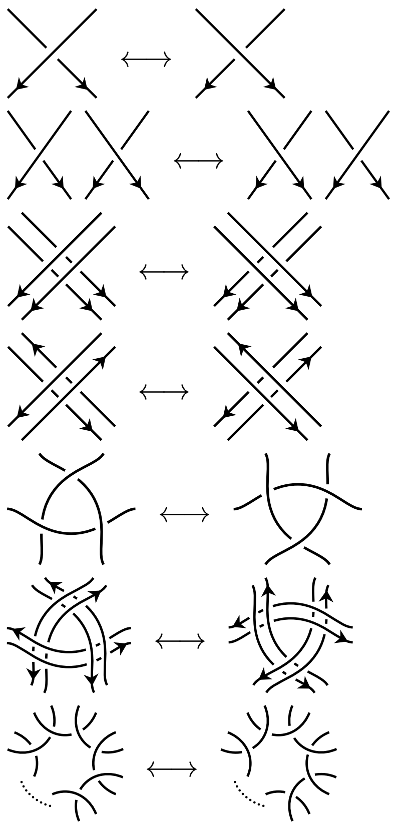


Figure 10: Some local moves among oriented knots. The strands of both sides of a Δ move and an n-gon move have any orientations such that corresponding strands from opposite sides of the moves are oriented in the same way. Each side of an n-gon move has n strands.

It is a fundamental problem of finite type invariants to calculate the corresponding graded spaces, which would enable us to identify finite type invariants in some sense.

Problem 2.18 *Calculate $\mathcal{F}_d(\mathbb{ZK}, \mathfrak{m})/\mathcal{F}_{d+1}(\mathbb{ZK}, \mathfrak{m})$, letting $\mathfrak{m}$ be a local move such as*

(1) *a # move,*

(2) *a pass move,*

(3) *a Δ move,*

(4) *an n-gon move.*

Remark It is known that crossing change, double crossing change, # move (see [289]), Δ move (see [298]), n-gon move (see [2]) are unknotting operations, i.e. any oriented knot can be related to the trivial knot by a sequence of isotopies and each of these moves. Hence, $\mathcal{F}_0(\mathbb{Z}\mathbb{K}, \mathfrak{m})/\mathcal{F}_1(\mathbb{Z}\mathbb{K}, \mathfrak{m}) \cong \mathbb{Z}$ for these moves $\mathfrak{m}$.

It is known [202] that Arf invariant gives the bijection

$$\{\text{knots}\}/(\text{pass move}) \longrightarrow \mathbb{Z}/2\mathbb{Z}.$$

Hence, $\mathcal{F}_0(\mathbb{Z}\mathbb{K}, \text{pass move})/\mathcal{F}_1(\mathbb{Z}\mathbb{K}, \text{pass move}) \cong \mathbb{Z} \oplus \mathbb{Z}$.

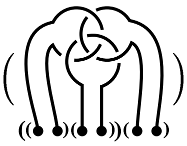
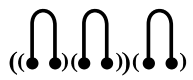
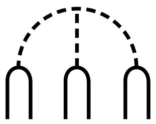
Remark Δ finite type invariants were introduced in [280]; see also [372].

Remark (K. Habiro) The following relations hold,

$$\begin{aligned} \mathcal{F}_{2d}(\mathbb{Z}\mathbb{K}, \times) \ &\supset\ \mathcal{F}_d(\mathbb{Z}\mathbb{K}, \Delta) \ \supset\ \mathcal{F}_{3d}(\mathbb{Z}\mathbb{K}, \times), \\ \mathcal{F}_d(\mathbb{Z}\mathbb{K}, \times) \ &\supset\ \mathcal{F}_d(\mathbb{Z}\mathbb{K}, \#) \ \supset\ \mathcal{F}_d(\mathbb{Z}\mathbb{K}, \Delta). \end{aligned}$$

These relations imply that $\mathfrak{m}$ finite type invariants are Vassiliev invariants, and Vassiliev invariants are $\mathfrak{m}$ finite type invariants, for $\mathfrak{m} = \#, \Delta$. Further, the rank of $\mathcal{F}_d(\mathbb{Z}\mathbb{K}, \mathfrak{m})/\mathcal{F}_{d+1}(\mathbb{Z}\mathbb{K}, \mathfrak{m})$ is finite for these $\mathfrak{m}$.

Remark For the Kontsevich invariant Z (introduced in Chapter 3), we have that

$Z\Big($  $\Big) - Z\Big($  $\Big) =$  $+ \left(\begin{array}{c} \text{terms of} \\ \text{higher degrees} \end{array} \right)$,

where two tangles in the left hand side are related by a Δ move. Hence, the image of

$$\mathcal{F}_d(\mathbb{Q}\mathbb{K}, \Delta) \longrightarrow \mathcal{F}_{2d}(\mathbb{Q}\mathbb{K}) \longrightarrow \mathcal{F}_{2d}(\mathbb{Q}\mathbb{K})/\mathcal{F}_{2d+1}(\mathbb{Q}\mathbb{K}) \cong \mathcal{A}(S^1; \mathbb{Q})^{(2d)}$$

is equal to the subspace of $\mathcal{A}(S^1; \mathbb{Q})^{(2d)}$ spanned by Jacobi diagrams on S^1 whose uni-trivalent graphs are disjoint unions of d dashed Y graphs.

Remark Finite type invariants derived from a double crossing change were introduced in [13], to study finite type invariants of links with a fixed linking matrix. For knots, they are equal to Vassiliev invariants, that is, $\mathcal{F}_d(\mathbb{Z}\mathbb{K}; \times\times) = \mathcal{F}_d(\mathbb{Z}\mathbb{K}, \times)$.

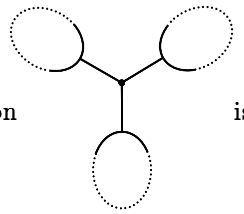
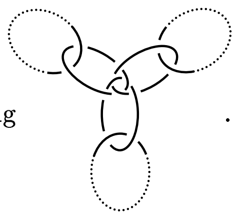
The surgery on  is defined to be the surgery along  .

Figure 11: Definition of the surgery on a Y graph. Dotted lines imply strands possibly knotting and linking. Three circles (partially dotted) in the left picture are called *leaves*.

(Y. Ohyama) In the case all arcs in a Δ move are contained in the same component, it is called a *self* Δ *move*. If two links can be transformed into each other by a finite sequence of self Δ moves, they are said to be Δ link homotopic.

Problem 2.19 (Y. Ohyama) *Find necessary and sufficient conditions for two μ-component links ($\mu > 2$) to be Δ link homotopic.*

Remark (Y. Ohyama) For a μ-component link $K = K_1 \cup K_2 \cup \ldots \cup K_\mu$, let $\delta_1 = a_{\mu-1}(K)$ and $\delta_2 = a_{\mu+1}(K) - a_{\mu-1}(K) \times (\sum_{i=1}^{\mu} a_2(K_i))$ for the coefficient $a_i(K)$ of the term z^i in the Conway polynomial of K.

It is known [278, 298] that two knots (or links) can be transformed into each other by a finite sequence of Δ moves if and only if they have the same number of components, and, for properly chosen orders and orientations, they have the same linking numbers between the corresponding components. In particular, if two links are Δ link homotopic, then their δ_1 coincide. Further, it is known [306] that if two μ component links are Δ link homotopic, then their δ_2 coincide. These are necessary conditions of this problem.

Moreover, for 2-component links, a pair of δ_1 and δ_2 is a faithful invariant of Δ link homotopy. Namely, for two 2-component links, they are Δ link homotopic if and only if their δ_1 and δ_2 coincide [307]. This gives a required condition of this problem for 2-components links.

2.9 Loop finite type invariants

The *loop-degree* of a Jacobi diagram on S^1 is defined to be half of the number given by the number of trivalent vertices minus the number of univalent vertices of the uni-trivalent graph of the Jacobi diagram. The filtration of $\mathcal{A}(S^1)$ given by loop-degrees is related to a filtration of $\mathbb{QK}$ through the Kontsevich invariant. The theory of the corresponding filtration in $\mathbb{Z}(\mathbb{MK})$ (given below) is developed in [142] (noting that this definition also appears in the September 1999 version of [232]).

We denote by $\mathbb{MK}$ the set of pairs (M,K) such that M is an integral homology 3-sphere and K is an oriented knot in M. Consider a move between two pairs (M,K) and (M',K') in $\mathbb{MK}$ such that (M',K') is obtained from (M,K) by surgery on a Y graph (see Figure 11) embedded in $M-K$ whose leaves have linking number zero with K. We call this move a *loop move*. Finite type invariants of degree d derived from a loop move by (13) are called *loop finite type invariants of degree d*, or *finite type invariants of loop-degree d*. We denote the corresponding submodule of $R(\mathbb{MK})$ by $\mathcal{F}_l(\mathbb{Z}(\mathbb{MK});\mathrm{loop})$.

A doubled delta move $\Delta\Delta$ (see Figure 10) was introduced by Naik-Stanford [304] as a move characterizing S-equivalence classes; two knots are *S-equivalent* if they are indistinguishable by Seifert matrices. A doubled delta move $\Delta\Delta$ can be presented by a surgery on such a Y graph as above. Thus, we have the map $\mathcal{F}_l(R\mathbb{K};\Delta\Delta)\to\mathcal{F}_l(R(\mathbb{MK});\mathrm{loop})$, taking a knot K to $(S^3,K)\in\mathbb{MK}$. Hence, a loop finite type invariant gives a $\Delta\Delta$ finite type invariant.

Problem 2.20 *Let R be a commutative ring with 1, say, $\mathbb{Z}$ or $\mathbb{Q}$.*

(1) *Describe the spaces $\mathcal{F}_l(R(\mathbb{MK});loop)/\mathcal{F}_{l+1}(R(\mathbb{MK});loop)$.*

(2) *Describe the spaces $\mathcal{F}_l(R\mathbb{K};\Delta\Delta)/\mathcal{F}_{l+1}(R\mathbb{K};\Delta\Delta)$.*

(3) *Describe the image of the above map $\mathcal{F}_l(R\mathbb{K};\Delta\Delta)\to\mathcal{F}_l(R(\mathbb{MK});loop)$.*

Remark (A. Kricker) It follows by a short argument from [304] and [278] that the following map taking a pair (M,K) to a Seifert matrix of K in M is bijective,

$$\mathbb{MK}/(\text{loop move}) \xrightarrow{=} \{\text{S-equivalence classes of Seifert matrices}\}. \tag{14}$$

(This implies that K and K' are related by a sequence of doubled delta moves if and only if (S^3,K) and (S^3,K') are related by a sequence of loop moves.) Hence, $\mathcal{F}_0(\mathbb{Z}(\mathbb{MK});\mathrm{loop})/\mathcal{F}_1(\mathbb{Z}(\mathbb{MK});\mathrm{loop})$ is isomorphic to the module over $\mathbb{Z}$ freely spanned by S-equivalence classes. Moreover, by (14), we have that $\mathbb{Z}(\mathbb{MK})=\oplus_s\mathbb{Z}(\mathbb{MK}_s)$, where the sum runs over all S-equivalence classes s. Further,

$$\mathcal{F}_l(\mathbb{Z}(\mathbb{MK});\mathrm{loop})/\mathcal{F}_{l+1}(\mathbb{Z}(\mathbb{MK});\mathrm{loop})=\bigoplus_s\mathcal{F}_l(\mathbb{Z}(\mathbb{MK}_s);\mathrm{loop})/\mathcal{F}_{l+1}(\mathbb{Z}(\mathbb{MK}_s);\mathrm{loop}).$$

Hence, the problem (1) splits into problems of describing the direct summands on the right hand sides: describe the spaces

$$\mathcal{F}_l(\mathbb{Z}(\mathbb{MK}_s);\mathrm{loop})/\mathcal{F}_{l+1}(\mathbb{Z}(\mathbb{MK}_s);\mathrm{loop})$$

for each S-equivalence class s. For the S-equivalence class u including the unknot, $\mathcal{F}_l(\mathbb{Q}(\mathbb{MK}_u);\mathrm{loop})/\mathcal{F}_{l+1}(\mathbb{Q}(\mathbb{MK}_u);\mathrm{loop})$ is isomorphic to $\mathcal{A}^{\mathbb{Z}[t^{\pm 1}]}(\emptyset;\mathbb{Q})^{(\mathrm{loop}\ l)}$ by the map (30) of the loop expansion of the Kontsevich invariant (see also [142]); for the definition of the space $\mathcal{A}^{\mathbb{Z}[t^{\pm 1}]}(\emptyset;\mathbb{Q})^{(\mathrm{loop}\ l)}$ see Section 3.9.

Remark A surgery on a Y graph in the definition of loop finite type invariants lifts to a surgery of the infinite cyclic cover of the knot complement, which does not change its homology. Hence, it is shown, topologically, that all coefficients of the Alexander polynomial are finite type invariants of loop-degree 0.

It follows that all coefficients of the Alexander polynomial are finite type invariants of $\Delta\Delta$-degree 0. It can also be shown from the fact that the Alexander polynomial can be defined by the Seifert matrix of a knot, which is unchanged by finite type invariants of $\Delta\Delta$-degree 0 as shown in [304].

The Alexander polynomial is universal among Vassiliev invariants which are of finite type of $\Delta\Delta$-degree 0; more precisely, $\log \Delta_K(e^{\hbar})$ as a power series of $\hbar$ is universal among $\mathbb{Q}$-valued primitive Vassiliev invariants which are of finite type of $\Delta\Delta$-degree 0. An equivalent statement has been shown in [299], using Vassiliev invariants of S-equivalence classes of Seifert matrices.

Remark As shown in [304] we have a bijection,

$$\{\text{knots}\}/\Delta\Delta \xrightarrow{=} \{\text{S-equivalence classes}\},$$

by taking a knot to its S-equivalence class. Hence, $\mathcal{F}_0(\mathbb{ZK};\Delta\Delta)/\mathcal{F}_1(\mathbb{ZK};\Delta\Delta)$ is isomorphic to the module over $\mathbb{Z}$ freely spanned by S-equivalence classes.

Remark (A. Kricker) The dual space of

$$\frac{\mathcal{F}_l(\mathbb{K}\otimes\mathbb{Q};\Delta\Delta)\cap\mathcal{F}_d(\mathbb{K}\otimes\mathbb{Q};\times)}{\big((\mathcal{F}_{l+1}(\mathbb{K}\otimes\mathbb{Q};\Delta\Delta)\cap\mathcal{F}_d(\mathbb{K}\otimes\mathbb{Q};\times)\big)+\big(\mathcal{F}_l(\mathbb{K}\otimes\mathbb{Q};\Delta\Delta)\cap\mathcal{F}_{d+1}(\mathbb{K}\otimes\mathbb{Q};\times)\big)}$$

is isomorphic to the subspace of $\mathcal{B}$ spanned by connected uni-trivalent graphs of degree d and of loop-degree l, i.e. the space $\mathcal{B}^{(d,d-l)}_{\mathrm{conn}}$ in the notation given in a remark in Problem 2.12.

(A. Kricker) Let $\mathbb{MK}$ denote the set of pairs (M,K) such that M is an integral homology 3-sphere and K is an oriented knot in M, as before. A *mod p loop* *move* in $\mathbb{MK}$ is defined to be a surgery on a Y graph (see Figure 11) such that each leaf has linking number 0 modulo p with the knot. We consider the question: what are the mod p loop move equivalence classes of knots?

To state the conjecture below, we give some notation. Consider a pair (M, K) of an integral homology 3-sphere M and a knot K in M. Let $\Sigma^p_{(M,K)}$ be the p-fold branched cyclic cover of (M, K), and assume that $\Sigma^p_{(M,K)}$ is a rational homology 3-sphere. Observe that there is an action of $\mathbb{Z}/p\mathbb{Z}$ on the homology group $H_1(\Sigma^p_{(M,K)}; \mathbb{Z})$ (induced from the covering transformations). Observe also that the linking pairing on the torsion of $H_1(\Sigma^p_{(M,K)}; \mathbb{Z})$ (which is the whole group) is invariant under the action of $\mathbb{Z}/p\mathbb{Z}$. Here, the linking pairing on the torsion of $H_1(N; \mathbb{Z})$ of a 3-manifold N is the map $\mathrm{Tor}\big(H_1(N; \mathbb{Z})\big) \otimes \mathrm{Tor}\big(H_1(N; \mathbb{Z})\big) \to \mathbb{Q}/\mathbb{Z}$ taking $\alpha \otimes \beta$ to $1/n$ times the algebraic intersection of F and β, where F is a compact surface bounding $n\alpha$ for some non-zero integer n.

Conjecture 2.21 (A. Kricker) *Take (M_1, K_1) and (M_2, K_2) of the above sort. Then, there exists a $(\mathbb{Z}/p\mathbb{Z})$-equivariant isomorphism $\phi : H_1(\Sigma^p_{(M_1,K_1)}; \mathbb{Z}) \to H_1(\Sigma^p_{(M_2,K_2)}; \mathbb{Z})$ preserving the linking pairing if and only if (M_1, K_1) is equivalent to (M_2, K_2) by a finite sequence of mod p loop moves.*

Remark (A. Kricker) The case of $p = 1$ would recover Matveev's theorem [278]: two closed 3-manifolds M and N are equivalent by a finite sequence of surgeries on Y graphs if and only if there is an isomorphism $H_1(M; \mathbb{Z}) \to H_1(N; \mathbb{Z})$ preserving the linking pairing on the torsion.

Also, the limit as $p \to \infty$ should recover a theorem due to Naik-Stanford [304]: two knots are equivalent by a finite sequence of loop moves if and only if they have isometric Blanchfield pairings. (Recall that the Blanchfield pairing is the equivariant linking pairing on the universal cyclic cover.)

2.10 Goussarov-Habiro theory for knots

Related to Vassiliev invariants of knots, equivalence relations among knots have been studied by Goussarov [152, 153] and Habiro [165], which is called the Goussarov-Habiro theory for knots. These equivalence relations are helpful for us to study structures of the set of knots.

The *C_d-equivalence*[10] $(d = 1, 2, 3, \cdots)$ among oriented knots is the equivalence relation generated by either of the following relations,

[10] The C_d-equivalence is also called the $(d-1)$*-equivalence* (due to Goussarov) in some literatures.

(1) C_d-move, i.e. surgery along a tree clasper with d trivalent vertices whose leaves are disc-leaves [165],

(2) relation on a certain collection of d crossing changes (Goussarov's $(d-1)$-equivalence) [150, 151, 153],

(3) surgery by an element in the dth group in the lower central series of pure braid group [371],

(4) capped grope cobordism of class d [94].

It is known that these relations generate the same equivalence relation among knots. The C_d-equivalence is defined among links, string links, $\cdots$, in the same way.

It is known [165] that there exists a natural surjective homomorphism

$$\mathcal{A}(S^1;\mathbb{Z})^{(d)}_{\mathrm{conn}} \longrightarrow \{K \underset{C_d}{\sim} O\}/_{\underset{C_{d+1}}{\sim}} \tag{15}$$

such that the tensor product of this map and $\mathbb{Q}$ is an isomorphism, where O denotes the trivial knot. In particular, $\{K \underset{C_d}{\sim} O\}/_{\underset{C_{d+1}}{\sim}}$ forms an abelian group with respect to the connected sum of knots, and hence, so does $\{\text{knots}\}/_{\underset{C_{d+1}}{\sim}}$.

Conjecture 2.22 *The map (15) is an isomorphism.*

This conjecture might be reduced to Conjecture 2.2 and the following conjecture.

Conjecture 2.23 $\{K \underset{C_d}{\sim} O\}/_{\underset{C_{d+1}}{\sim}}$ *is torsion free for each d.*

Remark Conjecture 2.2 implies this conjecture, since the surjective homomorphism (15) gives a $\mathbb{Q}$-isomorphism.

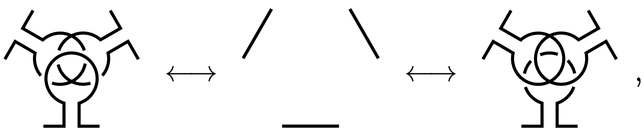
It is known [152, 371, 165] that two knots K and K' are C_d-equivalent if and only if $v(K) = v(K')$ for any A-valued Vassiliev invariant v of degree $< d$ for any abelian group A. In fact, a natural quotient map $\{\text{knots}\} \to \{\text{knots}\}/_{\underset{C_d}{\sim}}$ is a Vassiliev invariant of degree $< d$, which classifies C_d-equivalence classes of knots.

Conjecture 2.24 (K. Habiro [165], see also [153, "Theorem 5"]) *Two m-strand string links L and L' are C_d-equivalent if and only if $v(L) = v(L')$ for any A-valued finite type invariant v of degree $< d$ for any abelian group A.*

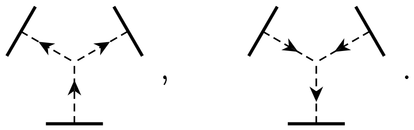
Remark (M. Polyak) The corresponding assertion for links does not hold; note that $\{\text{links}\}/\underset{C_d}{\sim}$ does not (naturally) form a group. Recall that $\{\text{knots}\}/\underset{C_d}{\sim}$ forms an abelian group, which guarantees the corresponding assertion for knots, as mentioned above. The set $\{m\text{-strand string links}\}/\underset{C_d}{\sim}$ forms a group with respect to the composition of string links, though it is not abelian.

Problem 2.25 (M. Polyak) *Establish the Goussarov-Habiro theory for virtual knots.*

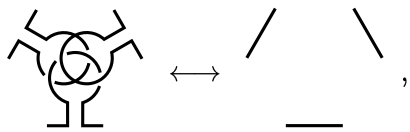
Remark Polyak suggested that the following moves,



(which appear in [154]) might play a similar role as the C_2-move plays among knots. They are related to the following diagrams respectively,



Further, Habiro suggested that the move,



should be added to the above moves. It is a problem to define a sequence of equivalence relations among virtual knots (an extension of the C_d-move) which induces finite type invariants of virtual knots. Are there surjective homomorphisms from certain modules of arrow graphs (oriented Jacobi diagrams) to the graded sets derived from such equivalence relations?

(K. Habiro) We denote by $\mathbb{MK}$ the set of pairs (M, K) such that M is an integral homology 3-sphere and K is an oriented knot in M. The *HL_d-equivalence* (*homology d-loop equivalence*) in $\mathbb{MK}$ is the equivalence relation generated by either of the following relations,

(1) surgery on a tree clasper with d trivalent vertices with null-homologous leaves,

(2) surgery on a graph clasper with d trivalent vertices with null-homologous leaves,

(3) surgery by an element of the dth lower central series subgroup of the Torelli group of compact connected surfaces embedded in a null-homologous way.

Here, "null-homologous" means null-homologous in knots complements. These relations generate the same equivalence relation in $\mathbb{MK}$.

Problem 2.26 (K. Habiro) *Describe the abelian group* $\{(M,K) \underset{HL_d}{\sim} (S^3, \mathit{unknot})\}/_{\underset{HL_{d+1}}{\sim}}$ *for each* d.

Remark (K. Habiro) Two pairs (M,K) and (M',K') in $\mathbb{MK}$ are HL_d-equivalent if and only if $v(M,K) = v(M',K')$ for any A-valued loop finite type invariant v of loop degree $< d$ for any abelian group A. Thus, the HL-equivalence gives the Goussarov-Habiro theory for loop finite type invariants.

The *homotopy d-loop equivalence* is defined by using "null-homotopic leaves" instead of "null-homologous leaves" in the definition of the HL_d-equivalence. These equivalences might be related to the rational Z invariant Z^{rat}. The homotopy loop equivalence relates ($\mathbb{Z}HS$, boundary link) to ($\mathbb{Z}HS$, boundary link). A high loop-degree part of Z^{rat} might be invariant under the homotopy loop equivalence.

The quotient set $\mathbb{MK}/\underset{HL_1}{\sim}$ can be identified with the commutative monoid of S-equivalence classes of Seifert matrices. (See a remark of Problem 2.20.)

Define the equivalence relation HL'_d among knots in S^3 to be the equivalence relation generated by surgery on a tree clasper with d trivalent vertices with null-homologous leaves in the complement of a knot such that at least one leaf bounds a disc with zero intersection number with the knot. Then, there exists a split exact sequence,

$$\{\text{knots in } S^3\}/\underset{HL'_d}{\sim} \longrightarrow \mathbb{MK}/\underset{HL_d}{\sim} \longrightarrow \{\mathbb{Z}HS\text{'s}\}/\underset{Y_d}{\sim},$$

where the first map takes a knot K to (S^3, K) and the second map is the map forgetting knots.

A refinement of Problem 2.26 is to consider the graded sets of the double sequence given by the C_d-equivalence and the HL_n-equivalence.

2.11 Other problems

(D. Bar-Natan)[11] Is there a Hilbert's Nullstellensatz for finite type invariants of links?

Let k be an algebraically closed field and let I be an ideal in the polynomial ring $k[x_1, \cdots, x_n]$. The Hilbert Nullstellensatz (see e.g. [113]) says that the ideal of polynomials in $k[x_1, \cdots, x_n]$ that vanish on the variety defined by the common zeros of all polynomials in I is the radical of I.

Problem 2.27 (D. Bar-Natan) *Is there a similar statement for finite type invariants of links? Let I be an ideal in the algebra V of finite type invariants of links. Let Z be the set of links that are annihilated by all members of I, and let J be the ideal in V of all invariants that vanish on Z. Clearly, J always contains the radical of I. Are they always equal?*

Example (D. Bar-Natan) Let I be the ideal generated by linking numbers. In this case, Z is the set of algebraically split links. Is it true that every finite type invariant that vanishes on algebraically split links is a sum of multiples of linking numbers? I believe it is true, and I believe it follows from the results of Appleboim [13], but I'm afraid Appleboim's paper is incomplete and while I believe it I cannot vouch for its validity.

Remark (D. Bar-Natan) One may also ask, "what is the Zariski closure of a given set of links?". I believe that in the light of the paragraphs above the meaning of this question should be clear. I know of at least one interesting example: In [312] Ng shows that the Zariski closure of the set of ribbon knots is the set of knots whose Arf invariant vanishes.

Is the similarity index of two different knots finite?
(M.-J. Jeong, C.-Y. Park)

K. Habiro and T. Stanford independently showed that for each positive integer n, two knots K and L have the same values for any Vassiliev invariants of type $< n$ if and only if they are LCS_n-equivalent. Y. Ohyama introduced triviality index of knots and K. Taniyama extended this to the similarity index of links; see [324]. Ohyama showed that if two knots are n-similar then they have the same value for any Vassiliev invariants of type $< n$. It is not difficult to see that two knots are n-similar if they are LCS_n-equivalent. D. Bar-Natan

[11] This part is a quotation from
`http://www.math.toronto.edu/~drorbn/Misc/Nullstellensatz/`

gave a problem whether Vassiliev invariants can distinguish all of knots or not. This problem is equivalent to the problem, whether the similarity index of any two different knots have finite similarity index. We will give a new criterion to calculate the similarity index of knots and, based on this, raise problems to calculate similarity index. For example, for two given knots, which knot invariants will give the best upper bound to calculate the similarity index of knots, along our above new result? As a partial problem, can we show that the triviality index of a non-trivial knot is finite by using our results?

Polynomial invariants and Vassiliev invariants
(M.-J. Jeong, C.-Y. Park)

In 1993, J.S. Birman and X.-S. Lin [57] showed that, after a suitable change of variables, each coefficient of the Jones, HOMFLY and Kauffman polynomial is a Vassiliev invariant. So we can obtain various Vassiliev invariants from the derivatives of knot polynomials.

In 2001, by using some specific kinds of tangles, we gave two operations $^{-}$ and * operations to get new polynomial invariants from a given Vassiliev invariant. These new polynomial invariants are also Vassiliev invariants. So we can obtain various Vassiliev invariants from the coefficients of these polynomial invariants.

Let V_n be the space of Vassiliev invariants of degrees $\leq n$. For $A_n \subset V_n$, let (A_n) be the set of Vassiliev invariants obtained from A_n by using finite numbers of $^{-}$ and * operations repeatedly.

Problem 2.28 (M.-J. Jeong, C.-Y. Park) *Find a minimal finite subset A_n of V_n such that* $\mathrm{span}(A_n) = V_n$.

3 The Kontsevich invariant

The framed Kontsevich invariant $Z(L) \in \mathcal{A}(\sqcup^l S^1; \mathbb{Q})$ of an oriented framed link L with l components is defined by using monodromy along solutions of the formal version of the KZ equation. Forgetting its framing, the Kontsevich invariant $Z(L)$ of an oriented link L is defined in $\mathcal{A}(\sqcup^l S^1; \mathbb{Q})/\mathrm{FI}$. The Kontsevich invariant is universal among quantum invariants in the sense that the quantum $(\mathfrak{g}, R)$ invariant recovers from the Kontsevich invariant through the weight system substituting a Lie algebra $\mathfrak{g}$ and its representation R into Jacobi diagrams. Moreover, the Kontsevich invariant is universal among Vassiliev invariants in the sense that each coefficient of the Kontsevich invariant is a Vassiliev invariant and any Vassiliev invariant can be presented by a linear sum of coefficients of the Kontsevich invariant.

3.1 Calculation of the Kontsevich invariant

Problem 3.1 *For each oriented knot K, calculate the Kontsevich invariant $Z(K)$ for all degrees.*

Remark For each d the degree d part of $Z(K)$ is a Vassiliev invariant. Hence, it is algorithmically possible to calculate it in a finite procedure. It is a problem to calculate $Z(K)$ for all degrees.

Remark D. Bar-Natan, T. Le, and D. Thurston [38] gave the following presentation of the Kontsevich invariant of the trivial knot O,

$$\log_{\sqcup} Z(O) = \frac{1}{2} \log \frac{\sinh(x/2)}{x/2}, \tag{16}$$

where x is an element in $\mathcal{B}$ (see (22)), and $\mathcal{B}$ is a space isomorphic to $\mathcal{A}(S^1)$ (see (21)). The Kontsevich invariant of a cable knot of a knot K can be calculated by applying a cabling formula [38] to the Kontsevich invariant of K. The Kontsevich invariant of the connected sum of knots is given by the connected sum of the Kontsevich invariant of the knots. Hence, we can calculate the Kontsevich invariant of knots obtained from the trivial knot by finite sequences of cabling and connected sum. To calculate the Kontsevich invariant of other knots in a combinatorial way, we probably need an associator, whose combinatorial direct presentation for all degrees is not known yet (see Problem 3.13).

3.2 Does the Kontsevich invariant distinguish knots?

Conjecture 3.2 *The Kontsevich invariant distinguishes oriented knots. (See Conjecture 2.5 for an equivalent statement of this conjecture.)*

Remark Kuperberg [235] showed that all finite type invariants either distinguish all oriented knots, or there exist prime, unoriented knots which they do not distinguish.

Problem 3.3 *Does there exists a non-trivial oriented knot K such that $Z(K) = Z(O)$ for the trivial knot O? (See Problem 2.6 for an equivalent problem.)*

Conjecture 3.4 $Z(K) = Z(-K)$ *for any oriented knot K, where $-K$ denotes K with the opposite orientation. (See Conjecture 2.7 for an equivalent statement of this conjecture.)*

3.3 Characterization and interpretation of the Kontsevich invariant

The space $\mathcal{A}(S^1)$ is an algebra with the product given by connected sum of Jacobi diagrams on S^1. Since the Kontsevich invariant $Z(K)$ of a knot K is group-like in $\mathcal{A}(S^1)$, its logarithm $\log Z(K)$ belongs to $\mathcal{A}(S^1)_{\mathrm{conn}}$, where $\mathcal{A}(S^1)_{\mathrm{conn}}$ denotes the vector subspace of $\mathcal{A}(S^1)$ spanned by Jacobi diagrams on S^1 with connected uni-trivalent graphs.

Problem 3.5 *Characterize those elements of $\hat{A}(S^1)_{conn}$ of the form $\log Z(K)$, or those elements of $\mathcal{B}_{conn}$ of the form $\log_{\sqcup} Z(K)$.*

Remark If the Kontsevich invariant was injective, this problem would be a step of the classification problem of knots. It is known (see, for example, [321]) that those elements of $\mathcal{A}(S^1)^{(\leq d)}_{\mathrm{conn}}$ of the form of the degree $\leq d$ part of $\log Z(K)$ forms a lattice, which is isomorphic to the lattice in $\mathcal{A}(S^1)_{\mathrm{conn}}$ spanned by Jacobi diagrams over $\mathbb{Z}$, and that the coefficients of $\log Z(K)$ are invariants which are independent to each other. Hence, it would be meaningful to characterize the form of infinite sums of coefficients of $\log Z(K)$, resp. $\log_{\sqcup} Z(K)$.

$W_{\mathfrak{g},R}\big(Z(K)\big)$ is a polynomial in $q^{\pm 1/2N}$ for any simple Lie algebra $\mathfrak{g}$ and its representation R, where N is the determinant of the Cartan matrix of $\mathfrak{g}$ (see [246]), since it is equal to the quantum $(\mathfrak{g}, R)$ invariant of K. This somehow characterizes the form of $Z(K)$.

The loop expansion characterizes the infinite sum of subsequences of $\log_{\sqcup} Z(K)$ in each loop-degrees; see (24), (26), and (28) in the cases of low loop-degrees. Since the image of the Kontsevich invariant is a countable set, there should be more restrictive properties.

Problem 3.6 (J. Roberts) *Give a good topological construction of the Kontsevich integral.*

Remark (J. Roberts) The Kontsevich integral is, in my opinion, the deepest part of the existing theory of quantum invariants, and it has two (conjecturally) equivalent formulations, each with its mysteries.

(a). In Kontsevich's original formulation of his integral, the part relating to *braids* is reasonably well-understood: it can be described using configuration spaces of points in the plane, the Knizhnik-Zamolodchikov equation, 1-minimal models in rational homotopy theory, Chen's iterated integrals and Magnus expansions. The fact that this actually extends to a *knot* invariant does not seem to appear naturally in these pictures, however. Passing from braids to (Morsified) knots suggests thinking about configuration spaces of varying numbers of points in the plane, and allowing some kind of annihilation and creation of pairs. Is there some way to utilise such spaces? (A related question is Problem 3.14.)

(b). In the perturbative integral formulation, the diagrammatic power series is introduced as a formal device for keeping track of which linear combinations of the individual (non-invariant) coefficient integrals give give knot invariants. It isn't really clear from this point of view why this series should turn out to have good properties such as multiplicativity, Kricker/Rozansky rationality, etc. Is there an "all-in-one" definition?

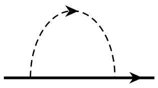
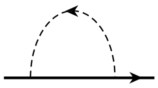
3.4 The Kontsevich invariant in a finite field

Problem 3.7 *Construct the Kontsevich invariant (i.e. a universal Vassiliev invariant) with coefficients in a finite field.*

Remark If we could find a solution (R, Φ) of the pentagon and hexagon relations with coefficients in a finite field, such a solution would give a combinatorial construction of the Kontsevich invariant with coefficients in that field. In this case we can not put $R = \exp\left(\downarrow\text{- - -}\downarrow /2 \right)$ unlike the case of $\mathbb{Q}$ coefficients, because p^{-1} of the order p of the field appears in the expansion of the exponential.

3.5 The Kontsevich invariant in arrow diagrams

Conjecture 3.8 (D. Bar-Natan, A. Haviv)

$\iota(Z(O)) = \text{closure}\ (\exp (\frac{1}{2} ($  $-$  $)))$,

where $Z(O)$ denotes the Kontsevich invariant of the trivial knot (see [35]) and ι is the map of Conjecture 2.16.

Remark (D. Bar-Natan, A. Haviv) This conjecture is true in any semi-simple Lie algebra.

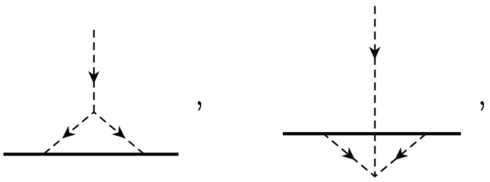
Problem 3.9 (M. Polyak) *Construct the "Kontsevich invariant" (i.e. a universal finite type invariant) of virtual knots in $\overrightarrow{\mathcal{A}}(I)$. (See also Conjecture 2.17.)*

Remark (M. Polyak) It is shown by Goussarov (see [154]) that there exists a Gauss diagram formula for any Vassiliev invariant of classical knots. His proof is an algorithmical proof, assuming the existence of such a Vassiliev invariant, and does *not* give a new proof of Kontsevich theorem "any weight system can be integrated to an invariant of knots". It would be nice to have a new direct combinatorial proof, which would imply Kontsevich theorem. Then, it would work for virtual knots.

Remark (M. Polyak) It is known (see, for example, [321]) that quantum invariants of knots can be defined by using quasi-triangular quasi-Hopf algebras with associators Φ. When $\Phi = 1$, such definition can naturally extend for virtual knots. However, when $\Phi \neq 1$ (as in the combinatorial definition of the Kontsevich invariant of classical knots), this extension does not work.

Problem 3.10 (D. Thurston) *Construct a series of configuration space integrals whose value is in $\overrightarrow{\mathcal{A}}(I)$ so that it gives all finite type invariants of virtual knots.*

Remark (D. Thurston) A technical difficulty is to kill the hidden strata of the configuration spaces (see also Problem 3.11). A way to kill a hidden strata is to use an involution on the strata, but, in this case, such an involution takes the following left diagram to the right diagram,



where the right diagram is equal to 0 by definition, while the left one is not necessarily equal to 0.

(M. Polyak) Each of the following three approaches gives all Vassiliev invariants.

- Construction of the Kontsevich invariant using monodromy along solutions of the KZ equation.
- Configuration space integrals motivated by perturbative Chern-Simons theory.
- Gauss diagram formulas, which count configurations of crossings of knot diagrams.

The invariants derived from these three approaches are expected to be naturally equivalent in the following sense.[12 13] The integral of the second approach gives an integral presentation of the mapping degree of a certain map on a configuration space, and it is shown in the degree 2 case [331] that the invariants of the first and third approaches can be obtained by localizing the integral presentation with respect to appropriate volume forms on the target space. A technical difficulty to show this in a general degree is to compute the localization on the "hidden strata"; it is a part of the boundary of a configuration space, whose contribution to the derivative of the integral is killed by an involution on the strata.

Problem 3.11 (M. Polyak) *Find another way to kill the hidden strata, so that the above three approaches can naturally present the mapping degree of the same map.*

[12] S. Poirier [328] showed the equivalence between the invariants derived from the first and second approaches, under the assumption of the vanishing of anomaly, by comparing these invariants for quasi-tangles (see Question 3.12).

[13] D. Thurston suggests that Etingof–Kazhdan R matrices [117] might be helpful to relate the invariants derived from the first and third approaches.

3.6 The Chern-Simons series of configuration space integrals

Question 3.12 (C. Lescop) *Is the Kontsevich integral of a (zero-framed) knot equal to the Chern-Simons series of configuration space integrals of the same knot (with Gauss integral 0)?*

The (normalized) Chern-Simons series of configuration space integrals is a universal Vassiliev knot invariant that admits a natural and beautiful symmetric definition that will be given below before describing the present situation of this question that was first raised by Kontsevich in [225].

In 1833, Carl Friedrich Gauss defined the first example of a *configuration space integral* for an oriented two-component link. Let us formulate his definition in a modern language. Consider an embedding

$$L : S^1_1 \sqcup S^1_2 \hookrightarrow \mathbb{R}^3$$

of the disjoint union of two circles $S^1 = \{z \in \mathbb{C} \text{ s.t. } |z| = 1\}$ into $\mathbb{R}^3$. With an element (z_1, z_2) of $S^1_1 \times S^1_2$ that will be called a *configuration*, we may associate the oriented direction $\Psi((z_1, z_2))$ of the vector $\overrightarrow{L(z_1)L(z_2)}$. $\Psi((z_1, z_2)) \in S^2$. Thus, we have associated a map

$$\Psi : S^1_1 \times S^1_2 \longrightarrow S^2$$

from a compact oriented 2-manifold to another one with our embedding. This map has an integral degree $\deg(\Psi)$ that can be defined in several equivalent ways. For example, it is the number of preimages of a regular value of Ψ counted with signs that can easily be computed from a regular diagram of our two-component link as

$$\deg(\Psi) = \sharp \times_{1\;2} - \sharp \times_{2\;1} = \sharp \times_{2\;1} - \sharp \times_{1\;2}\, .$$

It can also be defined as the following *configuration space integral*

$$\deg(\Psi) = \int_{S^1 \times S^1} \Psi^*(\omega)$$

where ω is the homogeneous volume form on S^2 such that $\int_{S^2} \omega = 1$. It is obvious that this integral degree, that depends continuously on our embedding, is an isotopy invariant; and the reader has recognized that $\deg(\Psi)$ is nothing but the linking number of the two components of L.

Section 3.6 was written by C. Lescop.

We can again follow Gauss and associate the following similar *Gauss integral* $I(K)$ to a C^∞ embedding $K : S^1 \hookrightarrow \mathbb{R}^3$. Here, we consider the configuration space $C = S^1 \times]0, 2\pi[$, and the map

$$\Psi : C \longrightarrow S^2$$

that maps (z_1, η) to the oriented direction of $\overrightarrow{K(z_1)K(z_1 e^{i\eta})}$, and we set

$$I(K) = \int_C \Psi^*(\omega).$$

This Gauss integral is NOT an isotopy invariant, and it can be seen as an exercise that it takes any real value on any given isotopy class of knots.

However, we can follow Guadagnini, Martellini and Mintchev and associate configuration space integrals to our embedding K and to any Jacobi diagram on the circle Γ without small loop like . A *configuration* of such a diagram is an embedding c of the set $U \cup T$ of its vertices into $\mathbb{R}^3$ whose restriction to the set U of univalent vertices factors through the knot embedding K so that the factorization induces the cyclic order of U. Denote the set of these configurations by $C(K;\Gamma)$. $C(K;\Gamma)$ is an open submanifold of $(S^1)^U \times (\mathbb{R}^3)^T$. Denote the set of dashed edges of Γ by E, and fix an orientation for these edges. Then we can define the map $\Psi : C(K;\Gamma) \longrightarrow \left(S^2\right)^E$ whose projection to the S^2 factor indexed by an edge from a vertex v_1 to a vertex v_2 is the direction of $\overrightarrow{c(v_1)c(v_2)}$. This map Ψ is again a map between two orientable manifolds that have the same dimension, namely the number of dashed half-edges of Γ, and we can write the *configuration space integral:*

$$I(K;\Gamma) = \int_{C(K;\Gamma)} \Psi^*(\Lambda^E \omega).$$

For example, if θ denotes the Jacobi diagram , then $I(K;\theta) = I(K)$. Bott and Taubes have proved that this integral is convergent [65]. Thus, this integral is well-defined up to sign. In fact, an orientation of the trivalent vertices of Γ provides $I(K;\Gamma)$ with a well-defined sign[14] such that the product $I(K;\Gamma)[\Gamma] \in \mathcal{A}(S^1;\mathbb{R})$ does not depend on the vertex orientation of Γ.

[14]Since S^2 is equipped with its standard orientation, it is enough to orient $C(K;\Gamma) \subset (S^1)^U \times (\mathbb{R}^3)^T$ in order to define this sign. This will be done by providing the set of the natural coordinates of $(S^1)^U \times (\mathbb{R}^3)^T$ with some order up to an even permutation. This set is in one-to-one correspondence with the set of dashed half-edges of Γ, and the vertex-orientation of the trivalent vertices provides a natural preferred such one-to-one correspondence up to some (even!) cyclic permutations of three half-edges meeting at a trivalent vertex. Fix an order on E, then the set of half-edges becomes ordered by (origin of the first edge, endpoint of the first edge, origin of the second edge, ..., endpoint of the last edge), and this order orients $C(L;\Gamma)$. As an exercise, check that the sign of $I(K;\Gamma)[\Gamma]$ does depend neither on our choices nor on the vertex orientation of Γ.

Now, the *perturbative expansion of the Chern-Simons theory for knots in* $\mathbb{R}^3$ is the following sum running over all the Jacobi diagrams without small loops and without vertex orientation:

$$Z_{\mathrm{CS}}(K) = \sum \frac{I(K;\Gamma)}{\sharp\mathrm{Aut}\Gamma}[\Gamma] \quad \in \mathcal{A}(S^1;\mathbb{R})$$

where $\sharp\mathrm{Aut}\Gamma$ is the number of automorphisms of Γ as a uni-trivalent graph whose univalent vertices are cyclically ordered, but without vertex-orientation for the trivalent vertices. The degree one part of Z_{CS} is $\frac{I(K;\theta)}{2}$ and therefore Z_{CS} is not invariant under isotopy. However, the evaluation[15] of Z_{CS} at representatives of knots with null Gauss integral is an isotopy invariant that is a universal Vassiliev invariant of knots [65, 5, 381, 328]. Now, the still open question raised by Kontsevich in [225] is: *Is the Kontsevich integral of a zero framed representative of a knot K equal to the above series of configuration space integrals of a representative of K with Gauss integral 0?*

This question has been reduced by Sylvain Poirier [328] to the computation of the following constant in $\mathcal{A}(S^1;\mathbb{R}) = \mathcal{A}([0,1];\mathbb{R})$ that is called the Bott and Taubes *anomaly*. In order to define the anomaly, replace the above knot K by a straight line D, and consider a Jacobi diagram Γ on the oriented line. Define $C(D;\Gamma)$ and Ψ as before. Let $\hat{C}(D;\Gamma)$ be the quotient of $C(D;\Gamma)$ by the translations parallel to D and by the positive homotheties, then Ψ factors through $\hat{C}(D;\Gamma)$ that has two dimensions less. Now, allow D to run among all the oriented lines through the origin of $\mathbb{R}^3$ and define $\hat{C}(\Gamma)$ as the total space of the fibration over S^2 where the fiber over the direction of D is $\hat{C}(D;\Gamma)$. Ψ becomes a map between two smooth oriented[16] manifolds of the same dimension. Then we can again define

$$I(\Gamma) = \int_{\hat{C}(\Gamma)} \Psi^*(\Lambda^E \omega).$$

Now, the anomaly is the following sum running over all Jacobi diagrams on the oriented lines (again without vertex-orientation and without small loop):

$$\alpha = \sum \frac{I(\Gamma)}{\sharp\mathrm{Aut}\Gamma}[\Gamma] \quad \in \mathcal{A}([0,1];\mathbb{R}).$$

[15] Actually, this evaluation is equal to $Z_{\mathrm{CS}}(K)\exp(-\frac{I(K;\theta)}{2}\alpha)$ for any representative K, where $\alpha \in \mathcal{A}([0,1];\mathbb{R})$ is the Bott and Taubes anomaly.

[16] $\hat{C}(\Gamma)$ carries a natural smooth structure and can be oriented as follows: orient $C(D;\Gamma)$ as before, orient $\hat{C}(D;\Gamma)$ so that $C(D;\Gamma)$ is locally homeomorphic to the oriented product (translation vector of the oriented line, ratio of homothety) $\times \hat{C}(D;\Gamma)$ and orient $\hat{C}(\Gamma)$ as the local product base $\times$ fiber.

Its degree one part is

$$\alpha_1 =$$  .

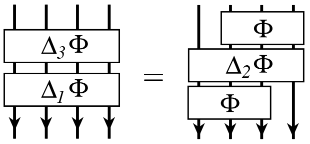
It is not hard to see that for any integer n, $\alpha_{2n} = 0$. In [328], Sylvain Poirier proved that if all α_i vanish for $i \geq 2$, then the answer to the above Kontsevich question is YES, and he computed $\alpha_3 = 0$. He also computed $\alpha_5 = 0$ with the help of Maple. In [252], it is proved that α is a combination of diagrams with two univalent vertices. Poirier also gave an equivalent definition of the anomaly that allows one to see that, for any $i > 1$, α_i is a combination of diagrams with at least 6 univalent vertices.

As a corollary, all coefficients of the HOMFLY polynomial properly normalized that are Vassiliev invariants of degree less than seven can be explicitly written as combinations of the above configuration space integrals. A positive answer to the Kontsevich question would allow one to express any canonical Vassiliev invariant as an explicit combination of the above configuration space integrals.

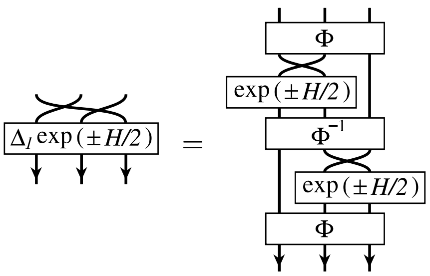
G. Kuperberg and D. Thurston have constructed a universal finite type invariant for homology spheres as a series of configuration space integrals similar to the above Chern-Simons series in [237]. Their construction yields two natural questions that are stated in Question 11.9.

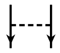
3.7 Associators

An *associator* Φ is defined to be an invertible group-like element in $\mathcal{A}(\downarrow\downarrow\downarrow;\mathbb{C})$ satisfying that $\varepsilon_2\Phi = 1 \in \mathcal{A}(\downarrow\downarrow;\mathbb{C})$ and the following relations,

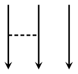
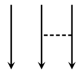


,



where we put $H =$  .

Here, Δ_i and ε_i are the comultiplication and the counit acting on the i-th solid line; see [29] for these notations. An associator is derived from a Drinfel'd series $\varphi(A,B)$ by

$\Phi = \varphi($  $,$  $) \in \mathcal{A}(\downarrow\downarrow\downarrow;\mathbb{C}),$ (17)

where a *Drinfel'd series* is an invertible group-like power series $\varphi(A,B)$ of non-commutative indeterminates A and B satisfying certain relations.

The Drinfel'd associator is given as follows. We consider the differential equation

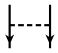
$$G'(z) = \frac{1}{2\pi\sqrt{-1}}\Big(\frac{A}{z} + \frac{B}{z-1}\Big)G(z), \tag{18}$$

for an analytic function G of the variable z, where $G(z)$ belongs to the formal power series ring $\mathbb{C}\langle\langle A,B\rangle\rangle$ of non-commutative indeterminates A and B. There exists unique solutions $G_{(\bullet\bullet)\bullet}$ and $G_{\bullet(\bullet\bullet)}$ of the above differential equation of the forms

$$G_{(\bullet\bullet)\bullet}(z) = f(z)z^{A/2\pi\sqrt{-1}}$$
$$G_{\bullet(\bullet\bullet)}(z) = g(1-z)(1-z)^{B/2\pi\sqrt{-1}}$$

where $f(z)$ and $g(z)$ are analytic functions with $f(0) = g(0) = 1 \in \mathbb{C}\langle\langle A,B\rangle\rangle$ defined in a neighborhood of $0 \in \mathbb{C}$. The power series $\varphi_{\mathrm{KZ}}(A,B) \in \mathbb{C}\langle\langle A,B\rangle\rangle$ is defined by $G_{(\bullet\bullet)\bullet} = G_{\bullet(\bullet\bullet)}\varphi_{\mathrm{KZ}}(A,B)$. The associator derived from $\varphi_{\mathrm{KZ}}(A,B)$ by (17) is called the *Drinfel'd associator*.

Problem 3.13 *Find a combinatorial direct presentation of an associator for all degrees, in particular, an associator with rational coefficients.*

Remark We still do not have a combinatorial direct presentation of any associator for all degrees. This implies that we still do not know a combinatorial direct presentation of the Kontsevich invariant of each knot for all degrees (except for the trivial knot); see Problem 3.1 and its remarks. Bar-Natan [29] showed a combinatorial degree-by-degree proof of the existence of solutions of the defining relations of a pair (R,Φ). Our definition of Φ follows from the defining relations when R is given by $\exp\big(\frac{1}{2}$  $\big)$.

Remark The only associator whose coefficients can be directly presented for all degrees so far is the Drinfel'd associator. We can present all degrees of the Drinfel'd associator by a limit of iterated integrals (see (19)) of by multiple zeta functions (see (20)). It is known [248] that all associators are related to each other by "twists", which are some actions of symmetric elements in $\mathcal{A}(\downarrow\downarrow;\mathbb{C})$ on associators.

Remark $\varphi_{\mathrm{KZ}}(A,B)$ is presented by the following limit ,

$$\varphi_{\mathrm{KZ}}(A,B)=\lim_{\varepsilon\to 0}\varepsilon^{-B/2\pi\sqrt{-1}}G_\varepsilon(1-\varepsilon)\varepsilon^{A/2\pi\sqrt{-1}}, \tag{19}$$

where we regard ε^x as

$$\varepsilon^x=\exp(x\log\varepsilon)=1+x\log\varepsilon+x^2\frac{(\log\varepsilon)^2}{2}+\cdots.$$

Further, G_ε is a solution of (18) given by

$$G_\varepsilon(1-\varepsilon)=1+\sum_{m=1}^{\infty}\int_{\varepsilon\le t_1\le\cdots\le t_m\le 1-\varepsilon}w(t_m)\cdots w(t_1)dt_1\cdots dt_m,$$

putting

$$w(t)=\frac{1}{2\pi\sqrt{-1}}\Big(\frac{A}{t}+\frac{B}{t-1}\Big).$$

Remark In [248], $\varphi_{\mathrm{KZ}}(A,B)$ is presented by

$$\varphi_{\mathrm{KZ}}(A,B)=1+\sum_{l=1}^{\infty}\sum_{\mathbf{a},\mathbf{b},\mathbf{p},\mathbf{q}}(-1)^{|\mathbf{b}|+|\mathbf{p}|}\eta(\mathbf{a}+\mathbf{p},\mathbf{b}+\mathbf{q})\binom{\mathbf{a}+\mathbf{p}}{\mathbf{p}}\binom{\mathbf{b}+\mathbf{q}}{\mathbf{q}} \times B^{|\mathbf{q}|}(A,B)^{(\mathbf{a},\mathbf{b})}A^{|\mathbf{p}|}, \tag{20}$$

where the second sum runs over $\mathbf{a},\mathbf{b},\mathbf{p},\mathbf{q}$ such that the sum of their length is equal to l and entries of them are non-negative integers. Here, the notations are given by

$$\eta(\mathbf{a},\mathbf{b})=\zeta(\underbrace{1,1,\cdots,1}_{a_1-1},b_1+1,\underbrace{1,1,\cdots,1}_{a_1-1},b_2+1,\cdots,\underbrace{1,1,\cdots,1}_{a_l-1},b_l+1),$$

$$|\mathbf{a}|=a_1+a_2+\cdots+a_l,$$

$$\binom{\mathbf{a}}{\mathbf{b}}=\binom{a_1}{b_1}\binom{a_2}{b_2}\cdots\binom{a_l}{b_l},$$

$$(A,B)^{(\mathbf{a},\mathbf{b})}=A^{a_1}B^{b_1}\cdots A^{a_l}B^{b_l}.$$

for $\mathbf{a}=(a_1,\cdots,a_l)$ and $\mathbf{b}=(b_1,\cdots,b_l)$, where the multiple zeta function is defined by

$$\zeta(a_1,a_2,\cdots,a_k)=\sum_{n_1<n_2<\cdots<n_k\in\mathbb{N}}n_1^{-a_1}n_2^{-a_2}\cdots n_k^{-a_k}.$$

In particular,

$$\varphi_{\mathrm{KZ}}(A,B) = 1+\frac{1}{24}[A,B]-\frac{\zeta(3)}{(2\pi\sqrt{-1})^3}([A,[A,B]]+[B,[A,B]])+\begin{pmatrix}\text{terms of}\\ \text{degree} \geq 4\end{pmatrix}.$$

Remark In [29], an associator with rational coefficients is given in low degrees by

$$\begin{aligned}\log\varphi(A,B) &= \frac{[A,B]}{48} - \frac{8[A,[A,[A,B]]]+[A,[B,[A,B]]]}{11520}\\ &+\frac{[A,[A,[A,[A,[A,B]]]]]}{60480}+\frac{[A,[A,[A,[B,[A,B]]]]]}{1451520}+\frac{13[A,[A,[B,[B,[A,B]]]]]}{1161216}\\ &+\frac{17[A,[B,[A,[A,[A,B]]]]]}{1451520}+\frac{[A,[B,[A,[B,[A,B]]]]]}{1451520}\\ &-(\text{interchange of } A \text{ and } B)\\ &+(\text{terms of degree} \geq 8).\end{aligned}$$

Problem 3.14 (J. Roberts) *Construct a rational Drinfel'd associator in the context of rational homotopy theory.*

Remark (J. Roberts) The theory of 1-minimal models provides a representation of the pure braid group, which is the fundamental group of the configuration space of distinct ordered points in $\mathbb{C}$, the "pure braid space" for short. This is the representation coming from the Kontsevich integral. A better way to describe it is as a representation of the fundamental *groupoid* of the pure braid space, using "basepoints at infinity" described by associations (bracketings) of the points. In this picture, the Drinfel'd associator is the image of a certain path which changes the basepoint. Is there a theory of 1-minimal models for fundamental groupoids which gives a straightforward construction of a (rational-valued) associator, as an alternative to the tricky iterative procedures of [29]?

3.8 Graph cohomology

Problem 3.15 (J. Roberts) *What is graph cohomology the cohomology of?*

Remark (J. Roberts) In the theory of quantum knot invariants such as the Jones polynomial, the topology and algebra (in this case, the group $SU(2)$) are entangled somewhat confusingly. Passing to the theory of finite type invariants,

Section 3.8 was written by J. Roberts.

they become separated: there is a purely topological part (the Kontsevich integral of a knot) and a purely algebraic part (the weight system associated to $SU(2)$) whose intermediary is the space of Jacobi diagrams.

Viewing this space as (part of) Kontsevich's *graph (co)homology* [226], we see that quantum invariants arise from a pairing between elements of graph cohomology and homology. But what actually is this cohomology? A good geometric interpretation of it might lead to better understanding of the topological and algebraic constructions involving it, and their composite.

Most of the intuition about graph cohomology has been built up from the algebraic side: it has been portrayed primarily as a kind of universal invariant theory for Lie algebras. Vogel has pursued this idea the furthest, but he also showed [395] that not all weight systems come from classical Lie algebras. In fact, the work of Rozansky and Witten [360] and Kapranov [194] demonstrates that compact holomorphic symplectic manifolds can be used instead of Lie algebras to define Vassiliev weight systems, and this gives quite a different perspective on graph cohomology, which Simon Willerton and I have been studying [348].

In a similar vein, Bar-Natan, Le and Thurston [383] have proved the so-called "wheeling conjectures", diagrammatic generalisations of the Duflo isomorphism of Lie theory. Their theorem is far too striking for a purely combinatorial interpretation to be satisfactory. Does it have a geometric interpretation?

Kontsevich [226] has given three topological interpretations of graph cohomology. The first is that it is the twisted cohomology of "outer space", the classifying space of the group of outer automorphisms of a free group. This is analogous to the fact that a certain complex of *fatgraphs* gives the cohomology of the moduli space of Riemann surfaces. The answer is unsatisfying because the natural geometric model for the classifying space is, unlike the Riemann moduli space, not a smooth orbifold, and if we are seeking geometric constructions underlying the various kinds of diagrammatic operations we encounter, smoothness would seem to be an essential property. Is there is a better model?

A second approach comes from configuration spaces of points in $\mathbb{R}^3$. The complex of graphs (with distinguished legs) maps to the de Rham complex of configuration spaces, and gives a model for its cohomology. This kind of viewpoint was exploited by Kontsevich (and Taubes, and Axelrod and Singer) in defining the perturbative invariants of 3-manifolds, and by Bott and Taubes [65] for knots.

In this context, Lie algebra weight systems are functionals on the cohomology of the configuration spaces, and might be thought of as homology classes, or even cycles. Hence the following problem, posed by Raoul Bott:

Problem 3.16 (R. Bott) *Give a geometric construction of these homology classes coming from Lie algebras.*

The third and currently best interpretation of graph cohomology is that it is the cohomology of an infinite-dimensional Lie algebra of formal Hamiltonian vector fields. Kontsevich uses this to explain (and vastly generalise) Rozansky-Witten weight systems in terms of Gelfand-Fuchs cohomology. Can this interpretation be employed on the topological rather than algebraic side? In other words, is there a construction involving knots and algebras of formal vector fields which yields the Kontsevich integral?

3.9 The loop expansion of the Kontsevich invariant

The loop expansion is the series of the rational presentations of the Kontsevich invariant in loop-degrees. It was conjectured by Rozansky [357]. The existence of such rational presentations has been proved by Kricker [231] (though such a rational presentation itself is not necessarily a knot invariant in a general loop degree). Further, Garoufalidis and Kricker [139] defined a knot invariant in any loop degree, from which such a rational presentation can be deduced.

We have three isomorphic algebras

$$\mathcal{A}(S^1) \cong \mathcal{B} \cong \mathcal{B}_{\sqcup}, \tag{21}$$

where the first isomorphism is the formal Poincare-Birkhoff-Witt isomorphism, and $\mathcal{B}$ has the product structure related, by the isomorphism, to the product structure of $\mathcal{A}(S^1)$ given by connected sum. Further, the second isomorphism is the wheeling isomorphism [35] between $\mathcal{B}$ and $\mathcal{B}_{\sqcup}$, where $\mathcal{B}_{\sqcup}$ is $\mathcal{B}$ as a space and has the product given by the disjoint union of uni-trivalent graphs.

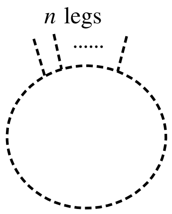
We denote by $\mathcal{B}_{\text{conn}}$ the vector subspace of $\mathcal{B}_{\sqcup}$ spanned by connected uni-trivalent graphs, and denote by $\mathcal{B}_{\text{conn}}^{(\text{loop } l)}$ the vector subspace of $\mathcal{B}_{\text{conn}}$ spanned by connected uni-trivalent graphs of loop-degree l, where the *loop-degree* of a uni-trivalent graph is defined to be half of the number given by the number of trivalent vertices minus the number of univalent vertices. Then,

$$\mathcal{B}_{\text{conn}} = \bigoplus_{l=0}^{\infty} \mathcal{B}_{\text{conn}}^{(\text{loop } l)}.$$

Each $\mathcal{B}_{\text{conn}}^{(\text{loop } l)}$ can be presented by using the polynomial rings in $H^1(G)$ for trivalent graphs G of loop-degree l subject to $\mathrm{Aut}(G)$ and the AS and IHX relations. We will present $\mathcal{B}^{(\mathrm{loop}\ l)}_{\mathrm{conn}}$ for $l = 0, 1, 2$ in this way, to state the loop expansion in these loop-degrees.

When $l = 0$, we have the map

$$\mathbb{Q}[x] \longrightarrow \mathcal{B}^{(\mathrm{loop}\ 0)}_{\mathrm{conn}}, \qquad x^n \longmapsto$$



, (22)

regarding x as a basis of $H^1(\text{circle})$. Since the orientation-reversing automorphism of S^1 takes x^n to $-x^n$ by the AS relation, the above map deduces the following isomorphism,

$$\mathcal{B}^{(\mathrm{loop}\ 0)}_{\mathrm{conn}} \cong \mathbb{Q}[x^2]. \tag{23}$$

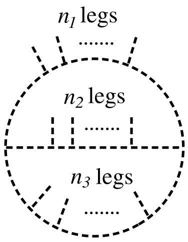
For a knot K,

$$\bigl(\log_{\sqcup} Z(K)\bigr)^{(\mathrm{loop}\ 0)} = \frac{1}{2}\log\frac{\sinh(x/2)}{x/2} - \frac{1}{2}\log\Delta_K(e^x), \tag{24}$$

where $\log_{\sqcup}$ is the logarithm in $\mathcal{B}_{\sqcup}$ regarding $Z(K)$ as in $\mathcal{B}_{\sqcup}$, and the left hand side is the summand of $\log_{\sqcup} Z(K) \in \mathcal{B}_{\mathrm{conn}}$ in $\mathcal{B}^{(\mathrm{loop}\ 0)}_{\mathrm{conn}}$. This development follows from the theory of [34]. See also [231, 139] (and references therein) for a recent direct calculation.

When $l = 1$, we have the map

$$\mathbb{Q}[x_1, x_2, x_3] \longrightarrow \mathcal{B}^{(\mathrm{loop}\ 1)}_{\mathrm{conn}}, \qquad x_1^{n_1} x_2^{n_2} x_3^{n_3} \longmapsto$$



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regarding $H^1(\theta\text{-graph})$ as the vector space spanned by x_1, x_2, x_3 subject to the relation $x_1 + x_2 + x_3 = 0$. Since $\mathrm{Aut}(\theta\text{-graph}) \cong \mathfrak{S}_2 \times \mathfrak{S}_3$, the above map deduces

$$\begin{aligned} \mathcal{B}^{(\mathrm{loop}\ 1)}_{\mathrm{conn}} &\cong \mathbb{Q}[x_1, x_2, x_3] \big/ (\mathfrak{S}_2 \times \mathfrak{S}_3,\ x_1 + x_2 + x_3 = 0) \\ &\cong \Bigl(\mathbb{Q}[x_1, x_2, x_3] \big/ (x_1 + x_2 + x_3 = 0)\Bigr)^{\mathfrak{S}_2 \times \mathfrak{S}_3} \\ &\cong \Bigl(\mathbb{Q}[\sigma_1, \sigma_2, \sigma_3] \big/ (\sigma_1 = 0)\Bigr)^{(\mathrm{even})} \cong \mathbb{Q}[\sigma_2, \sigma_3^2], \end{aligned} \tag{25}$$

where σ_i denotes the i-th elementary symmetric polynomial in x_1, x_2, and x_3. (To compute $\mathcal{B}^{(\mathrm{loop}\ 1)}_{\mathrm{conn}}$ in a precise argument, we must also consider the space of "dumbbell diagram" with legs. Since this space is injectively mapped to the right hand side of the above formula, we omit its computation here.) For a knot K there exists a polynomial $P^{\theta}_K(t_1,t_2,t_3)$, called *the 2-loop polynomial*, satisfying that

$$\big(\log_{\sqcup} Z(K)\big)^{(\text{loop } 1)} = \frac{P^{\theta}_K(e^{x_1},e^{x_2},e^{x_3})}{\Delta_K(e^{x_1})\Delta_K(e^{x_2})\Delta_K(e^{x_3})}. \tag{26}$$

The 2-loop polynomial $P^{\theta}_K(t_1,t_2,t_3)$ in t_1,t_2,t_3 satisfying $t_1t_2t_3=1$ is uniquely determined by each knot K. It is an invariant of K satisfying that $P^{\theta}_K(t_i^{\pm1},t_j^{\pm1},t_k^{\pm1}) = P^{\theta}_K(t_1,t_2,t_3)$ for any signs and any $\{i,j,k\}=\{1,2,3\}$.

Problem 3.17 *Find a topological construction of the 2-loop polynomial* P^{θ}_K.

Remark As in (24) the loop-degree 0 part of the Kontsevich invariant is presented by the Alexander polynomial, which can be constructed from the homology of the infinite cyclic cover of the knot complement. It is shown, in [142], that the "first derivative" of the 2-loop polynomial is given in terms of linking functions associated to the infinite cyclic cover of the knot complement. It is expected [142] that the 2-loop polynomial would be described in terms of invariants of the infinite cyclic cover of the knot complement.

Remark A table of the 2-loop polynomial for knots with up to 7 crossings is given by Rozansky [358]. See also a computer program [359], which calculates the 2-loop polynomial of each knot. For example,

$$\begin{aligned}
12P^{\theta}_{3_1}(t_1,t_2,\frac{1}{t_1t_2}) &= -t_1^2t_2+t_1^2,\\
12P^{\theta}_{4_1}(t_1,t_2,\frac{1}{t_1t_2}) &= 0,\\
12P^{\theta}_{5_1}(t_1,t_2,\frac{1}{t_1t_2}) &= 2t_1^4t_2^2-2t_1^4t_2+2t_1^4-t_1^2t_2+t_1^2.
\end{aligned}$$

The 2-loop polynomial for the torus knots is calculated independently by Marché [270] and Ohtsuki [322].

The following problem is a step to Problem 3.17.

Problem 3.18 (A. Kricker) *Let K_T be the knot obtained from a tangle T as shown in Figure 12. Find a presentation of the 2-loop polynomial $P^{\theta}_{K_T}$ of K_T by using the Kontsevich invariant $Z(T)$ of T.*

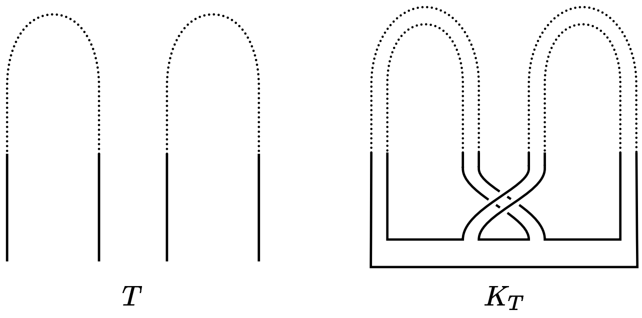
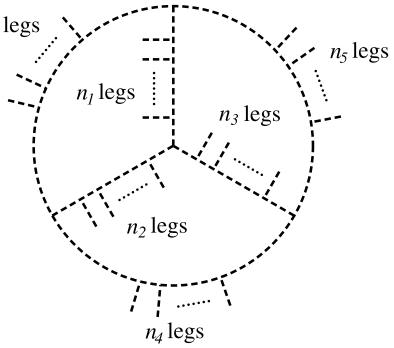


Figure 12: The knot K_T is obtained from the 2-parallel of a 2-strand tangle T by adding the tangle depicted in solid lines in the right picture. The dotted lines imply strands possibly knotted and linked in some fashion.

Remark (A. Kricker) $P^{\theta}_{K_T}$ might be presented by the degree ≤ 3 part of $Z(T)$. Generalize the presentation $\Delta_K(t) = \det(t^{1/2}S - t^{-1/2}S^T)$ of the Alexander polynomial $\Delta_K(t)$ by a Seifert matrix S of K.

When $l = 2$, we have the map

$$x_1^{n_1} x_2^{n_2} \cdots x_6^{n_6} \longmapsto$$



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which deduces the following isomorphism,

$$\mathcal{B}^{(\text{loop } 2)}_{\text{conn}} \cong \mathbb{Q}[x_1, x_2, \cdots, x_6] \big/ (\mathfrak{S}_4, x_1 + x_2 + x_3 = x_1 + x_6 - x_5 = 0 \\ x_2 + x_4 - x_6 = x_3 + x_5 - x_4 = 0).$$

Corresponding to faces of a tetrahedra, we put $y_1 = x_1 - x_2 - x_6$, $y_2 = x_2 - x_3 - x_4$, $y_3 = x_3 - x_1 - x_4$, and $y_4 = x_4 + x_5 + x_6$. Then,

$$\begin{aligned} \mathcal{B}^{(\text{loop } 2)}_{\text{conn}} &\cong \mathbb{Q}[y_1, y_2, y_3, y_4] \big/ (\mathfrak{S}_4, y_1 + y_2 + y_3 + y_4 = 0) \\ &\cong \Big(\mathbb{Q}[y_1, y_2, y_3, y_4] \big/ (y_1 + y_2 + y_3 + y_4 = 0) \Big)^{\mathfrak{S}_4}, \end{aligned}$$

where the action of $\tau \in \mathfrak{S}_4$ takes a polynomial $p(y_1, y_2, y_3, y_4)$ to $(\mathrm{sgn}\tau)p(y_{\tau(1)}, y_{\tau(2)}, y_{\tau(3)}, y_{\tau(4)})$. Hence,

$$\mathcal{B}_{\text{conn}}^{(\text{loop } 2)} \cong \Big(\mathbb{Q}[\sigma_2, \sigma_3, \sigma_4]\Big)^{(\text{even})} \cong \mathbb{Q}[\sigma_2, \sigma_3^2, \sigma_4], \tag{27}$$

where σ_i is the i-th elementary symmetric polynomial in y_1, y_2, y_3, and y_4. (To compute $\mathcal{B}_{\text{conn}}^{(\text{loop } 2)}$ in a precise argument, we need some more computations, which are omitted here.) For a knot K there exists a polynomial $P'_K(t_1, t_2, \cdots, t_6)$ satisfying that

$$\big(\log_{\sqcup} Z(K)\big)^{(\text{loop } 2)} = \frac{P'_K(e^{x_1}, e^{x_2}, \cdots, e^{x_6})}{\Delta_K(e^{x_1})\Delta_K(e^{x_2})\cdots\Delta_K(e^{x_6})}. \tag{28}$$

$P'_K(e^{x_1}, e^{x_2}, \cdots, e^{x_6})$ is uniquely determined by a knot K (hence, is an invariant of K) in the completion of $\mathbb{Q}[\sigma_2, \sigma_3^2, \sigma_4]$.

Problem 3.19 *Find a topological construction of the polynomial P'_K given above.*

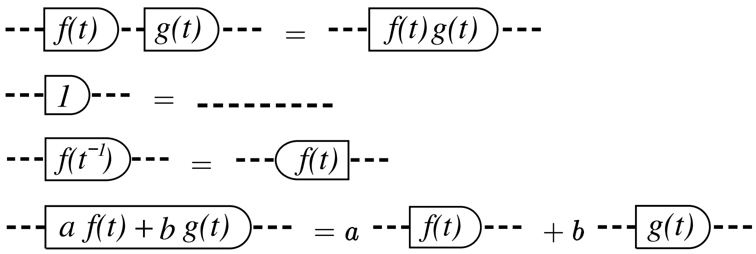


Figure 13: The multi-linear relations. Here, $f(t), g(t) \in S$, and a, b are scalars.

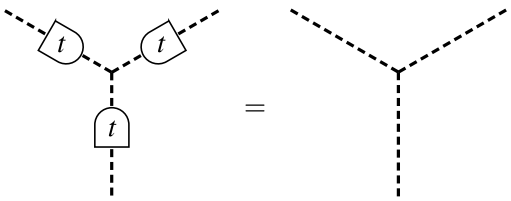
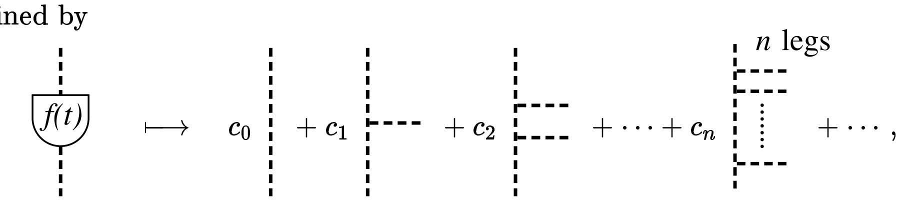


Figure 14: The push relation

The loop expansion in a general loop-degree is described as follows. Let R be a field, say $\mathbb{Q}$, and let S be a subring of $R(t)$ which is invariant under the involution $t \mapsto t^{-1}$, where t is an indeterminate. A *labeled Jacobi diagram on* $\emptyset$ is a vertex-oriented trivalent graph, whose edges are labeled by pairs of local orientations and elements of S. We define $\mathcal{A}^S(\emptyset; R)$ to be the vector space over R spanned by labeled Jacobi diagrams on $\emptyset$ subject to the AS, IHX, multilinear, and push relations (see Figures 13 and 14). The *loop-degree* of a labeled Jacobi diagram is half the number of trivalent vertices of the Jacobi diagram. For a polynomial $A(t)$ with $A(1) = 1$ and $A(t) = A(t^{-1})$, we have a map

$$\mathcal{A}^{\mathbb{Q}[t^{\pm 1}, 1/A(t)]}(\emptyset; \mathbb{Q}) \longrightarrow \mathcal{B}, \tag{29}$$

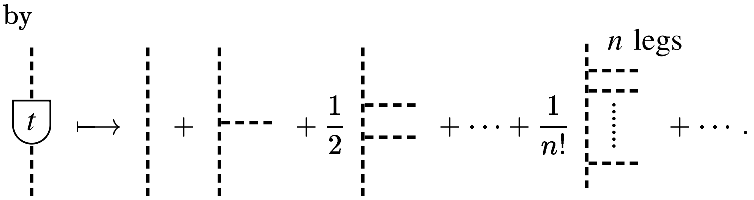
defined by



where $f(t) \in \mathbb{Q}[t^{\pm 1}, 1/A(t)]$ is written $f(e^h) = \sum_{k=0}^{\infty} c_k h^k$. In particular, the map

$$\mathcal{A}^{\mathbb{Q}[t^{\pm 1}]}(\emptyset; \mathbb{Q}) \longrightarrow \mathcal{B} \tag{30}$$

is defined by



The loop expansion of the Kontsevich invariant is described by the rational Z invariant $Z^{\mathrm{rat}}(K) \in \mathcal{A}^{\mathbb{Q}[t^{\pm 1}, 1/\Delta_K(t)]}(\emptyset; \mathbb{Q})$ which is taken to $\log_{\sqcup} Z(K)$ by the map (29). In particular, when $\Delta_K(t) = 1$, $Z^{\mathrm{rat}}(K) \in \mathcal{A}^{\mathbb{Q}[t^{\pm 1}]}(\emptyset; \mathbb{Q})$. (The existence of $Z^{\mathrm{rat}}(K)$ has been shown in [231], and the canonicality of $Z^{\mathrm{rat}}(K)$ has been shown in [139].)

Problem 3.20 *Find a topological construction of the loop-degree l part of the rational Z invariant $Z^{\mathrm{rat}}(K) \in \mathcal{A}^{\mathbb{Q}[t^{\pm 1}, 1/\Delta_K(t)]}(\emptyset; \mathbb{Q})$ of a knot K, for each l.*

Problem 3.21 *Find a basis of the space $\mathcal{A}^{\mathbb{Q}[t^{\pm 1}, 1/A(t)]}(\emptyset; \mathbb{Q})^{(\mathrm{loop}\ l)}$, for each l, where $A(t)$ is a polynomial with $A(1) = 1$ and $A(t) = A(t^{-1})$. In particular, find a basis of the space $\mathcal{A}^{\mathbb{Q}[t^{\pm 1}]}(\emptyset; \mathbb{Q})^{(\mathrm{loop}\ l)}$.*

Conjecture 3.22 [357, 139] *The map (29) is injective. In particular, the map (30) is injective.*

Remark If this conjecture is true, $Z^{\mathrm{rat}}(K)$ is determined by the Kontsevich invariant.

3.10 The Kontsevich invariant of links in $\Sigma \times [0,1]$

Let Σ be a closed oriented surface. We denote by $\mathcal{A}_\Sigma$ the algebra of chord diagrams on Σ. It is defined to be the vector space over $\mathbb{C}$ spanned by the homotopy classes of continuous maps from chord diagrams to Σ modulo 4T relations.

Problem 3.23 (T. Kohno) *Construct explicitly a universal invariant of finite type for links in $\Sigma \times [0,1]$ with values in $\mathcal{A}_\Sigma$.*

In the case of genus 0 the above problem is solved by Kontsevich integral. In higher genus case a suggestion for a construction of a universal invariant was given by Deligne at Oberwolfach meeting 1995. In the case of a punctured surface the problem was solved by Andersen, Mattes and Reshetikhin.

Let G be a simple Lie group and $\mathcal{M}^G(\Sigma)$ the moduli space of G flat connections on Σ. The space of smooth functions on $\mathcal{M}^G(\Sigma)$ denoted by $C(\mathcal{M}^G(\Sigma))$ has a structure of a Poisson algebra coming from a symplectic structure on $\mathcal{M}^G(\Sigma)$. The algebra $\mathcal{A}_\Sigma$ has also a Poisson algebra structure (see [8]). If each component of $\mathcal{A}_\Sigma$ is colored by a representation of G, then there is a natural Poisson algebra homomorphism

$$\tau : \mathcal{A}_\Sigma \to C(\mathcal{M}^G(\Sigma)).$$

Problem 3.23 is related to the following problem.

Problem 3.24 (T. Kohno) *Give a deformation quantization of the Poisson algebra $\mathcal{A}_\Sigma$ which descends to a deformation quantization of $C(\mathcal{M}^G(\Sigma))$.*

The above problem will give a new insight on quantization of $\mathcal{M}^G(\Sigma)$. It would also be interesting to investigate a relation to the geometric quantization of $\mathcal{M}^G(\Sigma)$.

Problem 3.25 (T. Kohno) *Clarify the relation between a deformation quantization of $C(\mathcal{M}^G(\Sigma))$ at a special parameter and the space of conformal blocks in WZW models.*

Section 3.10 was written by T. Kohno.

Problem 3.26 (T. Kohno) *Determine the image and the kernel of the above map τ.*

The space of conformal blocks in WZW model is defined as the space of coinvariant tensors in the following way. Let $p_1, \cdots, p_n$ be marked points on Σ and $H_1, \cdots, H_n$ be representations of the affine Lie algebra $\widehat{\mathfrak{g}}$. The space of conformal blocks is defined to be the set of linear forms

$$\phi : H_1 \otimes \cdots \otimes H_n \longrightarrow \mathbb{C}$$

invariant under the action of meromorphic functions with values in $\mathfrak{g}$ with poles at most at $p_1, \cdots, p_n$, where the action is defined by the Laurent expansion at these points. There is a twisted version of the above construction, where the above meromorphic functions are replaced by meromorphic sections of a $\mathfrak{g}$ local system.

Problem 3.27 (T. Kohno) *Compute the holonomy of the space of conformal blocks of the twisted WZW model. In particular, determine the action of the braid group of Σ on the space of conformal blocks for each G flat connection on Σ.*

There is also a notion of the algebra of chord diagrams on n strings with horizontal chord on Σ, which we shall denote by $\mathcal{A}_n(\Sigma)$.

Problem 3.28 (T. Kohno) *Let $P_n(\Sigma)$ denote the pure braid group of Σ with n strings. Does there exist an injective multiplicative homomorphism*

$$\theta : P_n(\Sigma) \to \mathcal{A}_n(\Sigma)$$

defined over $\mathbb{Q}$?

4 Skein modules

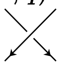
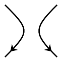
Skein module is an algebraic object associated to a manifold, usually constructed as a formal linear combination of embedded (or immersed) submanifolds, modulo locally defined relations. In a more restricted setting a *skein module*[17] is a module associated to a 3-dimensional manifold, by considering linear combinations of links in the manifold, modulo properly chosen (skein) relations. It is a main object of the *algebraic topology based on knots*. In the choice of relations one takes into account several factors:

(i) Is the module we obtain accessible (computable)?

(ii) How precise are our modules in distinguishing 3-manifolds and links in them?

(iii) Does the module reflect topology/geometry of a 3-manifold (e.g. surfaces in a manifold, geometric decomposition of a manifold)?

(iv) Does the module admit some additional structure (e.g. filtration, gradation, multiplication, Hopf algebra structure)? Is it leading to a Topological Quantum Field Theory (TQFT) by taking a finite dimensional quotient?

One of the simplest skein modules is a q-deformation of the first homology group of an oriented 3-manifold M, denoted by $\mathcal{S}_2(M;q)$. It is based on the skein relation (between oriented framed links in M):  $= q$  ; it also satisfies the framing relation  $= q$, where the diagrams in each formula imply framed links, which are identical except in a ball, where they differ as shown in the diagrams. Already this simply defined skein module "sees" non-separating surfaces in M. These surfaces are responsible for torsion part of the skein module [338].

There is more general pattern: most of analyzed skein modules reflect various surfaces in a manifold.

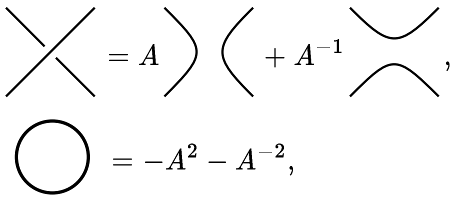
The best studied skein modules use skein relations which worked successfully in the classical knot theory (when defining polynomial invariants of links in $\mathbb{R}^3$).

The original version of Chapter 4 was written by J. H. Przytycki. It was revised by T. Ohtsuki following suggestions given by the referee. Based on it, Przytycki wrote this chapter.

[17] Alexander first wrote down the skein relation for his polynomial. Conway rediscovered the relation and placed in the abstract setting of "linear skein". He predicted the corresponding skein module for a tangle. General skein modules of 3-manifolds were first considered in 1987 by Przytycki and Turaev independently [333], [386].

4.1 The Kauffman bracket skein module

Let M be an oriented 3-manifold, and put $R = \mathbb{Z}[A^{\pm 1}]$. The *Kauffman bracket skein module* $S_{2,\infty}(M)$ of M is defined to be the R module spanned by unoriented framed links in M (including the empty link) subject to the relations



where three diagrams in the first formula imply three framed links, which are identical except in a ball, where they differ as shown in the diagrams. The Kauffman bracket gives an isomorphism between $S_{2,\infty}(S^3)$ and R. Thus, $S_{2,\infty}(M)$ is a generalization of the Jones polynomial (in its Kauffman bracket interpretation). The Kauffman bracket skein module is best understood among the Jones type skein modules. It can be interpreted as a quantization of the co-ordinate ring of the character variety of $SL(2,\mathbb{C})$ representations of the fundamental group of the manifold M, [71, 343, 74, 344].

Problem 4.1 *Calculate $S_{2,\infty}(M)$ for each oriented 3-manifold M. Find a convenient methodology to calculate it.*

Remark It is known that $S_{2,\infty}(L(p,q))$ of the lens space $L(p,q)$ is a free R module with $[p/2]+1$ generators [177], and that $S_{2,\infty}(S^1 \times S^2) \cong R \oplus \bigoplus_{i=1}^{\infty} R/(1-A^{2i+4})$ [178]. The Kauffman bracket skein modules are also calculated for I-bundles over surfaces [174, 333], the exteriors of $(2,n)$ torus knots [69], and Whitehead manifolds [179]. A connected sum formula is given in [340]. Skein modules at the 4th roots of unity are calculated in [366]. It is shown in [267] that $S_{2,\infty}(M_1 \cup_F M_2)$ for orientable 3-manifolds M_1 and M_2 with a common boundary F is expressed as a quotient module of a direct sum of tensor products of relative skein modules of M_1 and M_2.

Problem 4.2 (J. Przytycki) *Incompressible tori and 2-spheres in M yield torsion in $S_{2,\infty}(M)$ [339]. It is a question of fundamental importance whether other surfaces can yield torsion as well.*

Conjecture 4.3 *If every closed incompressible surface in M is parallel to ∂M, then $S_{2,\infty}(M)$ is torsion free.*

Remark The Kauffman bracket skein module of the 3-manifold obtained by an integral surgery along the trefoil knot is finitely generated if and only if the 3-manifold contains no essential surface [70].

The test case for the conjecture is the manifold $M = F_{0,3} \times S^1$, where $F_{0,3}$ is a 2-sphere with 3 holes, because it contains immersed π_1-injective torus.

Problem 4.4 (J. Przytycki) *Compute* $S_{2,\infty}(F_{0,3} \times S^1)$.

Problem 4.5 *Let* F *be a surface and* I *an interval. Describe the algebra* $S_{2,\infty}(F \times I)$.

Remark $S_{2,\infty}(F \times I)$ is an algebra (usually noncommutative). It is finitely generated algebra for a compact F [72], and has no zero divisors [344]. The center of the algebra is generated by boundary components of F [75, 344].

Problem 4.6 *Calculate the skein homology based on the Kauffman bracket skein relation.*

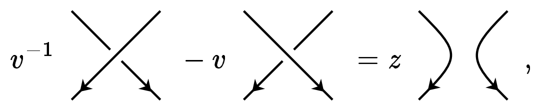
Remark The skein homology were introduced in [73] (see also [193]).

Problem 4.7 *We define the* sl_3 *skein module* $S^{sl_3}(M)$ *of an oriented 3-manifold* M *by the defining relations of the* sl_3 *linear skein [233, 323]. Calculate* $S^{sl_3}(M)$ *of each 3-manifold* M.

Remark The quantum sl_3 invariant of links gives an isomorphism between $S^{sl_3}(S^3)$ and the coefficient ring; see, e.g. [321]. Thus, $S^{sl_3}(M)$ gives a generalization of the quantum sl_3 invariant of links.

4.2 The Homflypt skein module

Let M be an oriented 3-manifold, and put $R = \mathbb{Z}[v^{\pm 1}, z^{\pm 1}]$. The *Homflypt skein module* $S_3(M)$ of M is defined to be the R module spanned by oriented links in M subject to the relation



where three diagrams in the formula imply three oriented links, which are identical except in a ball, where they differ as shown in the diagrams. The Homflypt polynomial gives an isomorphism between $S_3(S^3)$ and R. The Homflypt skein modules generalize skein modules based on Conway relation which were hinted by Conway. $S_3(M)$ is related to the algebraic set of $SL(n,\mathbb{C})$ representations of the fundamental group of the manifold M [367].

Problem 4.8 *Calculate $S_3(M)$ for each oriented 3-manifold M. Find a convenient methodology to calculate it.*

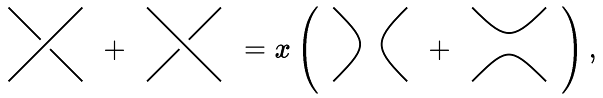
Remark It is known that $S_3(F \times I)$ is an infinitely generated free module [335], and that $S_3(S^1 \times S^2)$ is isomorphic to the direct sum of R and an R-torsion module [146]. A connected sum formula is given in [147].

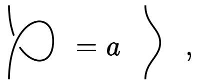
Problem 4.9 *Let F be a surface and I an interval. Describe the algebra $S_3(F \times I)$.*

Remark $S_3(F \times I)$ is a Hopf algebra (usually neither commutative nor cocommutative) [387, 335]. $S_3(F \times I)$ is a free module (as mentioned above) and can be interpreted as a quantization [386, 173, 387, 334].

4.3 The Kauffman skein module

Let M be an oriented 3-manifold, and put $R = \mathbb{Z}[a^{\pm 1}, x^{\pm 1}]$. The *Kauffman skein module* $S_{3,\infty}(M)$ of M is defined to be the R module spanned by unoriented framed links in M subject to the relations


(31)


(32)

where the diagrams in each formula imply framed links, which are identical except in a ball, where they differ as shown in the diagrams.

Problem 4.10 *Calculate $S_{3,\infty}(M)$ for each oriented 3-manifold M. Find a convenient methodology to calculate it.*

Remark $S_{3,\infty}(F \times I)$ is known to be a free module. The case of F being a torus was solved by Hoste, Kidwell and Turaev. It is calculated in [260] for a surface F with boundary. $S_{3,\infty}(S^1 \times S^2)$ is calculated in [412]. A connected sum formula is given in [411].

Problem 4.11 *Calculate the higher skein modules based on the Kauffman skein relation $W_i^{3,\infty}(M)$ and $\hat{W}^{3,\infty}(M)$ (see below for their definitions).*

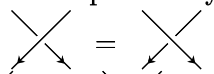
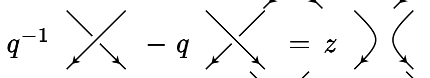
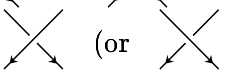
Remark The higher skein modules were introduced in [336]. They are discussed (in the case of the Conway skein triple) in [351, 258] and [9, 10]. In the case of the Kauffman skein relation, definitions are as follows: Let $R\mathbb{L}$ denote the free R module spanned by the ambient isotopy classes of unoriented framed links in an oriented 3-manifold M modulo the framing relation (32), where $R = Z[a^{\pm 1}, x^{\pm 1}]$. We regard singular links with a finite number of double points as elements in $R\mathbb{L}$ by replacing a double point with the difference of the two sides of (31). We introduce a (singular links) filtration $R\mathbb{L} = C_0 \supset C_1 \supset C_2 \supset C_3 \supset \cdots$, where the module C_i is generated by singular links with i double points. We define the ith higher Kauffman skein module as: $W_i^{3,\infty}(M) = R\mathbb{L}/C_{i+1}$ and the completed higher Kauffman skein module, $\hat{W}^{3,\infty}(M)$, as the completion of $R\mathbb{L}$ with respect to the filtration $\{C_i\}$.

Problem 4.12 *Construct invariants of 3-manifolds via a linear skein theory based on the Kauffman skein module.*

Remark It is known that quantum invariants of 3-manifolds can be constructed via linear skein theories based on the Kauffman bracket skein modules (see [255]) and the Homflypt skein modules [408].

Update Beliakova and Blanchet have done this [47].

4.4 The q-homotopy skein module

Let M be an oriented 3-manifold, and put $R = \mathbb{Z}[q^{\pm 1}, z]$. The *$q$-homotopy skein module* $HS^q(M)$ of M is defined to be the R module spanned by oriented links in M subject to the link homotopy relation  for self-crossings and the skein relation  for "mixed crossings", i.e. we assume that the two strings of ) of the skein relation belong to different components of the link.

We have an isomorphism between $HS^q(S^3)$ and $\mathbb{Z}[q^{\pm 1}, t, z]$, regarding t^k as the trivial link with k components, and this isomorphism is given by the linking numbers [341].

Problem 4.13 *Calculate $HS^q(M)$ for each 3-manifold M.*

Remark $HS^q(F\times I)$ is a quantization [175, 387, 341], and as noted by Kaiser it can be almost completely understood using singular tori technique of X.-S. Lin. $HS^q(M)$ is free if and only if $\pi_1(M)$ is abelian and $2b_1(M) = b_1(\partial M)$ [191].

4.5 The $(4,\infty)$ skein module

We generalize the Kauffman bracket and Kauffman skein modules by considering the general, unoriented skein relation $b_0L_0 + b_1L_1 + \cdots + b_{n-1}L_{n-1} + b_\infty L_\infty$ (see Figure 15). The first new case to analyze, $n = 4$, is described in this section. We call it the $(4,\infty)$ skein module and denote by $\mathcal{S}_{4,\infty}(M;R)$. This problem is very interesting even for $M = S^3$.

The definitions are as follows. Let M be an oriented 3-manifold, $\mathcal{L}_{fr}$ the set of unoriented framed links in M (including the empty knot, $\emptyset$) and R any commutative ring with unity. We fix a, b_0, b_3 to be invertible elements in R and fix b_1, b_2, b_∞ to be elements of R. Then we define the $(4,\infty)$ *skein module* as: $\mathcal{S}_{4,\infty}(M;R) = R\mathcal{L}_{fr}/I_{(4,\infty)}$, where $I_{(4,\infty)}$ is the submodule of $R\mathcal{L}_{fr}$ generated by the following two relations:

the $(4,\infty)$ skein relation: $\quad b_0L_0 + b_1L_1 + b_2L_2 + b_3L_3 + b_\infty L_\infty = 0,$

the framing relation: $\quad L^{(1)} = aL,$

where $L_0, \cdots, L_\infty$ are framed links which are identical except in a ball, where they differ as shown in Figure 15, and $L^{(1)}$ denotes a link obtained from L by adding $+1$ framing to some component of L.

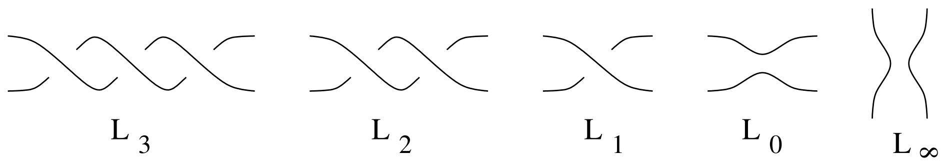


Figure 15: $L_3, \cdots, L_0, L_\infty$ are framed links which are identical except in a ball, where they differ as shown in the pictures. Links L_k for $k = 4, 5, \cdots$ are similarly defined.

Problem 4.14 (J. Przytycki)

(i) *Find generators of* $\mathcal{S}_{4,\infty}(S^3, R)$.

(ii) *For which parameters of the* $(4,\infty)$ *skein and framing relations, trivial links are linearly independent in* $\mathcal{S}_{4,\infty}(S^3; R)$?

(iii) *For which parameters of the* $(4,\infty)$ *skein and framing relations, the trivial knot is not representing a torsion element of* $\mathcal{S}_{4,\infty}(S^3, R)$?

A generalization of the Montesinos-Nakanishi conjecture [345] said that $\mathcal{S}_{4,\infty}(S^3, R)$ is generated by trivial links and that the $(4,\infty)$ skein module (suitably defined) for n-tangles is generated by $\prod_{i=1}^{n-1}(3^i+1)$ certain basic n-tangles. This would give a generating set for the $(4,\infty)$ skein module of S^3 or D^3 with $2n$ boundary points (for n-tangles). However, the Montesinos-Nakanishi 3-move conjecture has been disproved by M.Dabkowski and J.H.Przytycki in February 2002 [99] and [342]. Therefore $\prod_{i=1}^{n-1}(3^i+1)$ is only the lower bound for the number of generators.

In [345] we extensively analyze the possibilities that trivial links are linearly independent; if $b_\infty = 0$, then this may happen only if $b_0 b_1 = b_2 b_3$. These leads to the following conjecture (cases (1)–(2)):

Conjecture 4.15 (J. Przytycki, see [286])

(1) *There is a polynomial invariant of unoriented links,* $P_1(L) \in \mathbb{Z}[x,t]$ *which satisfies:*

(i) *Initial conditions:* $P_1(T_n) = t^n$, *where* T_n *is a trivial link of* n *components.*

(ii) *Skein relation* $P_1(L_0) + xP_1(L_1) - xP_1(L_2) - P_1(L_3) = 0$ *where* L_0, L_1, L_2, L_3 *is a standard, unoriented skein quadruple (*L_{i+1} *is obtained from* L_i *by a right-handed half twist on two arcs involved in* L_i*; compare Figure 15.)*

(2) *There is a polynomial invariant of unoriented framed links,* $P_2(L) \in \mathbb{Z}[A^{\pm 1}, t]$ *which satisfies:*

(i) *Initial conditions:* $P_2(T_n) = t^n$,

(ii) *Framing relation:* $P_2(L^{(1)}) = -A^3 P_2(L)$ *where* $L^{(1)}$ *is obtained from a framed link* L *by a positive half twist on its framing.*

(iii) *Skein relation:* $P_2(L_0) + A(A^2 + A^{-2})P_2(L_1) + (A^2 + A^{-2})P_2(L_2) + AP_2(L_3) = 0$.

(3) *There is a rational function invariant of unoriented framed links,* $P_3(L) \in \mathbb{Z}[a^{\pm 1}, x, y, (x + y + xy + y^2)^{-1}]$ *which satisfies:*

(i) *Initial conditions:* $P_3(T_n) = (\frac{-a^3(x+y+xy+x^2)+a^7(x+y+1)^2-a^{-1}}{x+y+xy+y^2})^{n-1}$,

(ii) *Framing relation:* $P_3(L^{(1)}) = aP_3(L)$,

(iii) *Skein relation:* $P_3(L_0)+axP_3(L_1)+a^2yP_3(L_2)-a^3(x+y+1)P_3(L_3) = 0$.

(4) *The invariant predicted in (1) (respectively (2) and (3)) is not uniquely defined (if it exists).*

Note that a solution to (3) becomes a solution to (1) under the substitution $a = 1$, $x = -y$ and that a solution to (3) becomes a solution to (2) under the substitution $a = -A^3$, $x = -1-A^{-4}$, $y = A^{-4}+A^{-8}$. As for the uniqueness of (4), note that all such invariants agree on trivial links and therefore they agree on the space spanned by trivial links in the related cubic skein module.

The above conjectures assume that $b_\infty = 0$ in our skein relation. Let consider the possibility that b_∞ is invertible in R. Using the "denominator" of our skein relation (the first line of Figure 16) we get the relation which allows to compute the effect of adding a trivial component to a link L (we write t^n for the trivial link T_n):

$$(a^{-3}b_3 + a^{-2}b_2 + a^{-1}b_1 + b_0 + b_\infty t)L = 0. \tag{33}$$

When considering the "numerator" of the relation and its mirror image (Figure 16) we obtain formulas for Hopf link summands, and because unoriented Hopf link is amphicheiral we can eliminate it from our equations to get the formula (34):

$$\begin{gathered} b_3(L\#H) + (ab_2 + b_1t + a^{-1}b_0 + ab_\infty)L = 0.\\ b_0(L\#H) + (a^{-1}b_1 + b_2t + ab_3 + a^2b_\infty)L = 0.\\ ((b_0b_1 - b_2b_3)t + (a^{-1}b_0^2 - ab_3^2) + (ab_0b_2 - a^{-1}b_1b_3) + b_\infty(ab_0 - a^2b_3))L = 0. \end{gathered} \tag{34}$$

It is possible that (33) and (34) are the only relations in the module. Precisely, we ask whether $\mathcal{S}_{4,\infty}(S^3;R)$ is the quotient ring $R[t]/(\mathcal{I})$ where t^i represents the trivial link of i components and $\mathcal{I}$ is the ideal generated by (33) and (34) for $L = t$. The substitution which realizes the relations is: $b_0 = b_3 = a = 1$, $b_1 = b_2 = x$, $b_\infty = y$. This may lead to the polynomial invariant of unoriented links in S^3 with values in $\mathbb{Z}[x,y]$ and the skein relation $L_3+xL_2+xL_1+L_0+yL_\infty = 0$.

Problem 4.16 (J. Przytycki) *For which coefficients of the $(4,\infty)$ skein relation is the number of Fox 7-colorings measured by the $(4,\infty)$ skein module?*

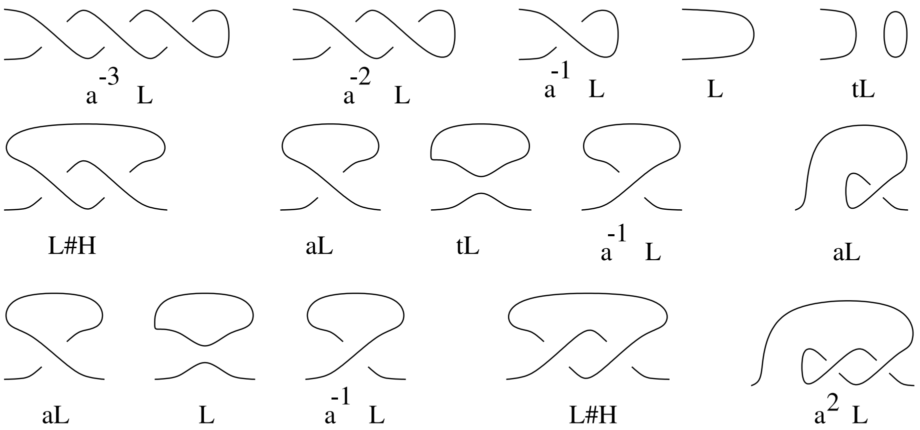
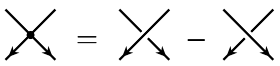


Figure 16

Remark We denote by $Col_p(L)$ the $(\mathbb{Z}/p\mathbb{Z})$-linear space (for p prime) of Fox p-colorings of a link L (for its definition, see [337]) and $col_p(L)$ denotes the cardinality of the space. It is known that $Col_p(L)$ can be identified with $H_1\big(M_2(L);\mathbb{Z}/p\mathbb{Z}\big)$, where $M_2(L)$ denotes the double cover of S^3 branched along L. Since the double covers of tangles defining $L_0, L_1, \cdots, L_{p-1}, L_\infty$ give all subspaces of $H_1(T^2;\mathbb{Z}/p\mathbb{Z})$ respectively (where T^2 is the double cover of $(S^2,$ 4 points$)$), col_p of those links are equal except for col_p of one link which is equal to p times the others [337]. This leads to the relation of type (p,∞). A relation between the Jones polynomial (or the Kauffman bracket) and $col_3(L)$ has the form: $col_3(L) = 3|V_L(e^{\pi\sqrt{-1}/3})|^2$ and a formula relating the Kauffman polynomial and $col_5(L)$ has the form: $col_5(L) = 5|F_L(1, e^{2\pi\sqrt{-1}/5} + e^{-2\pi\sqrt{-1}/5})|^2$. This seems to suggest the existence of a similar formula[18] for $col_7(L)$.

4.6 Other problems

We extend the family $\mathbb{K}$ of oriented knots in a 3-manifold M by singular knots, and resolve a singular crossing by . These allows us to define the Vassiliev-Goussarov filtration: $R\mathbb{K} = C_0 \supset C_1 \supset C_2 \supset C_3 \cdots$, where R is a commutative ring with unity and C_k is generated by knots with k

[18] François Jaeger told Przytycki that he knew how to get the space of Fox p-colorings from a short skein relation (of type $(\frac{p+1}{2},\infty)$). François died prematurely in 1997 and his proof has never been recorded.

singular points. Regarding the quotient $W_k(M) = R\mathbb{K}/C_{k+1}$ as an invariant of M, we call it the *kth Vassiliev-Goussarov skein module* of M. The completion of the space of knots with respect to the Vassiliev-Goussarov filtration, $\hat{R\mathbb{K}}$, is a *Hopf algebra* (for $M = S^3$). Functions dual to Vassiliev-Goussarov skein modules are called *finite type* or *Vassiliev invariants* of knots; see [336].

Problem 4.17 *Calculate $W_k(M)$ for each 3-manifold M.*

Remark When $M = S^3$, and coefficients are from $\mathbb{Q}$ then the graded space C_k/C_{k+1} can be described by chord diagrams of degree k; see Chapter 2.

Problem 4.18 *Define a skein module of 3-manifolds, and calculate it.*

Remark The quantum Hilbert space (or the space of conformal blocks) of $(S^2$, 4 points) is known to be finite dimensional. This is a reason why a quantum invariant of links satisfies a skein relation; it is a linear relation of tangles bounded by $(S^2$, 4 points) whose invariants are linearly dependent in the quantum Hilbert space. The quantum Hilbert space of a closed surface, say, a torus, is also known to be finite dimensional. Hence, a quantum invariant of 3-manifolds satisfies a "skein relation"; it should be a linear relation of 3-manifolds bounded by a surface. A skein module of 3-manifolds might be defined to be a module spanned by closed oriented 3-manifolds subject to a suitably chosen "skein relation" among 3-manifolds. It is a problem to define such a skein module which can be calculated.

5 Quandles

A *quandle* is a set X equipped with a binary operation $*$ satisfying the following 3 axioms.

(1) $x * x = x$ for any $x \in X$.

(2) For any $y, z \in X$ there exists a unique $x \in X$ such that $z = x * y$.

(3) $(x * y) * z = (x * z) * (y * z)$ for any $x, y, z \in X$.

The notions of subquandle, homomorphism, isomorphism, automorphism are appropriately defined. Each x in a quandle X defines a map $S_x : X \to X$ by $S_x(y) = y * x$. This map is an automorphism of X by the axioms (2) and (3). The *inner automorphism group* is a group of automorphisms generated by S_x ($x \in X$). An orbit under the action of the inner automorphism group on a quandle X is simply called an *orbit* of X. This forms a subquandle of X. A quandle is called *connected*[19] if the action of its inner automorphism group is transitive on it (i.e. if X has only one orbit). A quandle is called *simple* if every surjective homomorphism from the quandle is either an isomorphism or the constant map to the one-element quandle. The *dual quandle* of X is the set X with the dual binary operation given by $x \bar{*} y = S_y^{-1}(x)$.

The *conjugation quandle* of a group is the group with the binary operation $x * y = y^{-1}xy$. This kind of quandle is a prototype of quandles; the defining relations of a quandle are relations satisfied by the conjugation of a group. Any conjugacy class of a group is a subquandle of the conjugation quandle of the group. The *dihedral quandle* R_n of order n is the subquandle of the conjugation quandle of the dihedral group of order $2n$, consisting of reflections. An *Alexander quandle* is a quotient module $\mathbb{Z}[t^{\pm 1}]/J$, where t is an indeterminate and J is an ideal of the Laurent polynomial ring $\mathbb{Z}[t^{\pm 1}]$, equipped with the binary operation $x * y = tx + (1-t)y$. The dihedral quandle R_n is isomorphic to $\mathbb{Z}[t^{\pm 1}]/(n, t+1)$.

5.1 Classification of quandles

It was a classical problem in group theory to classify the isomorphism classes of groups of order n for each n. The following problem is a corresponding problem for connected quandles.

Chapter 5 was written by T. Ohtsuki, following suggestions and comments given by S. Kamada and M. Saito. Section 5.6 was added by C. Rourke and B.Sanderson.

[19] We call this property *connected* here following [190]. This is also called *weakly homogeneous* in some of the literature.

Problem 5.1 *Classify the isomorphism classes of connected quandles of order n for each positive integer n.*

See Table 4 for a list of connected quandles of order n for some n.

n	#	Connected quandles of order n	
		Self-dual	Not self-dual
1	1	A trivial quandle	
2	0		
3	1	R_3	
4	1	$\Lambda_2/(t^2+t+1)$	
5	3	R_5	$\Lambda_5/(t-2)$, its dual
6	2	2 subquandles of $\mathrm{Conj}(\mathfrak{S}_4)$	
7	5	R_7	$\Lambda_7/(t-2)$, $\Lambda_7/(t-3)$, their duals
8	≥ 3	An abelian extension of $\Lambda_2/(t^2+t+1)$	$\Lambda_2/(t^3+t+1)$, its dual
9	8	R_9, $\Lambda_3/(t^2-t+1)$, $R_3 \times R_3$, $\Lambda_3/(t^2+1)$	$\Lambda_9/(t-2)$, $\Lambda_3/(t^2+t-1)$, their duals
10	≥ 1	A subquandle of $\mathrm{Conj}(\mathfrak{S}_5)$	
11	9	R_{11}	$\Lambda_{11}/(t-a)$ $(a=2,3,\cdots,9)$
12	≥ 2	$R_3 \times \big(\Lambda_2/(t^2+t+1)\big)$, An icosahedral quandle	
13	11	R_{13}	$\Lambda_{13}/(t-a)$ $(a=2,3,\cdots,11)$
14	≥ 0		
15	≥ 4	$R_3 \times R_5$, A subquandle of $\mathrm{Conj}(\mathfrak{S}_5)$	$R_3 \times \big(\Lambda_5/(t-2)\big)$, its dual
$\vdots$			
Prime p	$p-2$	R_p	$\Lambda_p/(t-a)$ $(a=2,3,\cdots,p-2)$

Table 4: A table of some connected quandles. The second column shows the numbers of isomorphism classes of connected quandles of order n. We denote $\mathbb{Z}[t^{\pm 1}]/(n)$ by Λ_n. $\mathrm{Conj}(\mathfrak{S}_m)$ denotes the conjugation quandle of the mth symmetric group $\mathfrak{S}_m$. An *icosahedral quandle* is a quandle whose elements are the vertices of an icosahedron such that S_x of each element x is given by a rotation of the icosahedron centered at x.

Remark (M. Graña) It is shown, in [116] in terms of set theoretical solutions of the quantum Yang-Baxter equation, that a connected quandle of prime order p is isomorphic to the Alexander quandle $\mathbb{Z}[t^{\pm}]/(p, t-a)$ for some a. It is shown in [11, 308] that two connected Alexander quandles are isomorphic if and only if they are isomorphic as $\mathbb{Z}[t^{\pm 1}]$-modules. These give the classification of connected quandles of prime order shown in Table 4.

Remark (M. Graña) It is shown in [11] that a simple quandle of prime power order is an Alexander quandle; it is a finite field F such that t acts by multiplication by some primitive element w (i.e. w generates F as an algebra). Further, it is shown in [156] that a connected quandle of prime square order is an Alexander quandle. This gives the classification of connected quandles of order 9 shown in Table 4.

Remark S. Yamada made the list of isomorphism classes of quandles (and racks) of order ≤ 7 by computer search. The list of connected quandles of order ≤ 7 in Table 4 follows from it. S. Nelson [308] classifies the Alexander quandles of order ≤ 15; connected ones among them are listed in Table 4.

Remark The following modification of Problem 5.1 gives an algorithm to list connected quandles: classify the isomorphism classes of connected quandles fixing the conjugacy class of the union of S_x and an identity map. For a quandle X we denote by S_X the set of S_x $(x \in X)$, which is regarded as a subset of $\mathfrak{S}_n$ when X is of order n. The map $X \to \mathfrak{S}_n$, taking $x \mapsto S_x$, is often injective, though in general the map $X \to S_X$ is a quotient map, and the order of S_X divides n when X is connected. Let us investigate this problem in some simple cases.

Let X be a connected quandle of order n whose S_X includes $(12) \in \mathfrak{S}_n$. Then, for any i there is a sequence $1 = a_0, a_1, \cdots, a_k = i$ such that $(a_0a_1), (a_1a_2), \cdots \in S_X$ since X is connected. Further, since S_X is closed with respect to conjugation, S_X includes $(1i) \in \mathfrak{S}_n$, and hence any $(ij) \in \mathfrak{S}_n$. Therefore, $n = 3$, and X is isomorphic to the dihedral quandle R_3.

Let X be a connected quandle of order n whose S_X includes $(123) \in \mathfrak{S}_n$. Suppose that S_X further included $(145) \in \mathfrak{S}_n$. Then, since S_X is closed with respect to conjugation, S_X would include $(ijk) \in \mathfrak{S}_n$ for any $\{i,j,k\} \subset \{1,2,3,4,5\}$. This would contradict, since the order of S_X is at most n. Hence, $n = 4$, and X is isomorphic to the conjugation subquandle of $\mathfrak{A}_4$ consisting of (123), (134), (142), and (243), which is isomorphic to $\mathbb{Z}[t^{\pm 1}]/(2, t^2+t+1)$.

Let X be a connected quandle of order n whose S_X includes $(1234) \in \mathfrak{S}_n$. If S_X further included $(1567) \in \mathfrak{S}_n$, a contradiction would follow from a similar argument as above. Hence, it is sufficient to consider the cases that S_X include (1234) and either of (1256), (2156), (1526), (1536), or $(ijk5)$ for any $\{i,j,k\} = \{1,2,3\}$. It follows from some concrete computations that such a

X is isomorphic to either of the Alexander quandle $\mathbb{Z}[t^{\pm1}]/(5, t-2)$, its dual quandle, or the conjugation subquandle of $\mathfrak{S}_4$ including (1234).

5.2 Representations of knot quandles

Consider the conjugation quandle of the fundamental group $\pi_1(S^3 - K)$ of the complement of a knot K. The *reduced knot quandle* $\hat{Q}(K)$ is its subquandle generated by meridians of K. A *knot quandle*[20] $Q(K)$ is a quandle generated by meridians of K (for its precise definition, see [190]) which is almost equal to $\hat{Q}(K)$; to be precise, there is a surjective (almost, bijective) homomorphism $Q(K) \to \hat{Q}(K)$.

Homomorphisms to a fixed group/quandle are often called *representations.* It was said, before quantum invariants were discovered, that to count the numbers of representations of knot groups to a fixed finite group was a most powerful method to distinguish two given knots. The following problem is a refinement of it. A motivation is to construct a methodology to count the number of representations of a knot quandle to a fixed quandle of finite order.

Problem 5.2 *Describe (the number of) representations of a knot quandle to a fixed connected quandle of finite order, say, by using knot invariants known so far, or by reducing the problem to the case of smaller target quandles.*

Remark Since a knot quandle is connected, the image of a representation to a quandle X is included in an orbit of X, which forms a subquandle of X. Hence, the number of representations to X is equal to the sum of the numbers of representations to the quandles which are obtained as orbits of X. Repeating this procedure, the number of representations to X can be presented by the sum of the numbers of representations to certain connected quandles. Hence, it is sufficient to consider this problem when a target quandle is connected.[21]

Remark The problem to count the number of representations of a knot group to a fixed finite group can be reduced to Problem 5.2. Because it is equal to the number of representations of a knot quandle to the conjugation quandle of the group, and the problem to count it can be reduced to Problem 5.2 by the above remark.

[20] Knot quandle was introduced by Joyce [190] and independently by Matveev [277]; see [121] for an exposition.

[21] This argument is not available for the link case, since a link quandle is not connected.

Remark The number of representations of a knot quandle to an Alexander quandle can be presented by using the ith Alexander polynomials of the knot [181]. In particular, the number of representations to a dihedral quandle can be obtained as its corollary.

Remark Let X be a connected finite quandle, and let $h_X(K)$ denote the number of representations of the knot quandle of a knot K to X. Then, h_X is multiplicative with respect to connected sum of knots. It is known (see, for example, [321]) that any $\mathbb{Q}$-valued Vassiliev invariant is equal to a polynomial in some primitive Vassiliev invariants, where primitive Vassiliev invariants are additive with respect to connected sum of knots. Hence, h_X is not a Vassiliev invariant, unless it is constant. (See also [4] for another proof.)

Conjecture 5.3 *Let h_X be as above. Then, $\log h_X$ is not a Vassiliev invariant, unless it is constant.*

5.3 (Co)homology of quandles

Second cohomology classes of a quandle are used in order to define quandle cocycle invariants of knots. They are introduced as follows. Let A be an abelian group, written additively, and let $C^n(X;A)$ be the abelian group consisting of maps $X^n \to A$, where X^n denotes the direct product of n copies of X. We put

$$C^1_Q(X;A) = C^1(X;A),$$
$$C^2_Q(X;A) = \{f \in C^2(X;A) \mid f(x,x) = 0 \text{ for any } x \in X\},$$
$$C^3_Q(X;A) = \{g \in C^3(X;A) \mid g(x,x,y) = 0 \text{ and } g(x,y,y) = 0 \text{ for any } x,y \in X\}.$$

The coboundary operators $d_i : C^i_Q(X;A) \to C^{i+1}_Q(X;A)$ are given by

$$d_1 f(x,y) = f(x) - f(x*y),$$
$$d_2 g(x,y,z) = g(x,z) - g(x,y) - g(x*y,z) + g(x*z,y*z),$$

for $f \in C^1_Q(X;A)$ and $g \in C^2_Q(X;A)$. We define the second quandle cohomology group by $H^2_Q(X;A) = (\text{kernel}\, d_2)/(\text{image}\, d_1)$. It is known that $H^2_Q(X;A)$ is isomorphic to $\text{Hom}\big(H^Q_2(X);A\big)$ by the universal coefficient theorem, noting that $H^Q_1(X)$ is free abelian (see [82]). Here, $H^Q_2(X)$ denotes the second homology group of the dual complex of $\{C^\star_Q(X;\mathbb{Z}), d_\star\}$. See [82] for the definition of the nth quandle (co)homology group. Therefore, to obtain $H^2_Q(X;A)$ for any A, it is sufficient to compute $H^Q_2(X)$.

Connected quandle X	Order	$H_2^Q(X)$	$H_3^Q(X)$
R_3	3	0	$\mathbb{Z}/3\mathbb{Z}$
$\mathbb{Z}[t^{\pm 1}]/(2, t^2+t+1)$	4	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$
R_5	5	0	$\mathbb{Z}/5\mathbb{Z}$
$\mathbb{Z}[t^{\pm 1}]/(5, t-2)$		0	0
R_7	7	0	$\mathbb{Z}/7\mathbb{Z}$
$\mathbb{Z}[t^{\pm 1}]/(7, t-2)$		0	0
$\mathbb{Z}[t^{\pm 1}]/(7, t-3)$		0	0
$\mathbb{Z}[t^{\pm 1}]/(2, t^3+t+1)$	8	0	$\mathbb{Z}/2\mathbb{Z}$
R_9	9	0	$\mathbb{Z}/9\mathbb{Z}$
$\mathbb{Z}[t^{\pm 1}]/(9, t-2)$		0	$\mathbb{Z}/3\mathbb{Z}$
$\mathbb{Z}[t^{\pm 1}]/(3, t^2+1)$		$\mathbb{Z}/3\mathbb{Z}$	$(\mathbb{Z}/3\mathbb{Z})^3$
$\mathbb{Z}[t^{\pm 1}]/(3, t^2-t+1)$		$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/9\mathbb{Z}$
$\mathbb{Z}[t^{\pm 1}]/(3, t^2+t-1)$		0	0
$\mathbb{Z}[t^{\pm 1}]/(p, t-a)$ for any prime p and any $a \neq 0, 1 \in \mathbb{Z}/p\mathbb{Z}$	p	0	

Table 5: The cohomologies of the quandles, except for the last one, in the table are due to [264]. From a table in [264] we omit one of two dual quandles and quandles that are not connected (see remarks on Problem 5.6). The 2nd homology of $\mathbb{Z}[t^{\pm 1}]/(p, t-a)$ is due to [284]. See [264, 284] for computations of cohomology groups of some more quandles.

Problem 5.4 *Compute $H_2^Q(X)$ for each connected quandle X. More generally, find a convenient methodology to compute quandle (co)homology groups.*

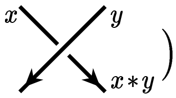
See Table 5 for some quandle homology groups given in [264]; see also [284] for computations of quandle cohomology groups of many Alexander quandles. There are maple programs [185] for computing quandle cohomology groups.

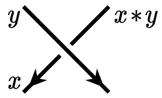
Remark We consider only connected quandles in this problem, since computations of quandle cocycle invariants of knots can be reduced to the cases of connected quandles (see a remark on Problem 5.6).

Problem 5.5 (J.S. Carter) *Compute $H_i^Q(\mathfrak{S}_n^m)$ of $\mathfrak{S}_n^m$ which denotes the quandle of the nth symmetric group with the binary operation given by $x * y = y^{-m} x y^m$.*

5.4 Quandle cocycle invariant

The quandle cocycle invariant, introduced in [79, 80], is defined as follows. For $\alpha \in H^2(X;A)$ we choose a 2-cocycle ϕ representing α. Any representation of a knot quandle $Q(K)$ to X is presented by a coloring of a knot diagram of K, where a *coloring* of an oriented knot diagram is a map of the set of over-arcs of it to X satisfying the condition depicted in the pictures of (35) at each crossing of the knot diagram. We define the weight of a crossing of a colored diagram by

$W\Big($ 

$\Big) = \phi(x,y) \in A,\qquad W\Big($ 

$\Big) = \phi(x,y)^{-1} \in A,\qquad (35)$

where we write A multiplicatively here. The *quandle cocycle invariant* of a knot K is defined by

$$\Phi_\alpha(K) = \sum_{\mathcal{C}} \prod_{\tau} W(\tau, \mathcal{C}) \in \mathbb{Z}[A],$$

where the sum runs over all coloring $\mathcal{C}$ of a diagram of K, and the product runs over all crossing τ of the diagram, and $\mathbb{Z}[A]$ denotes the group ring of A. $\Phi_\alpha(K)$ only depends on K and α.

Problem 5.6 *Compute the quandle cocycle invariant $\Phi_\alpha(K)$ of each knot K for a second cohomology class α of a connected quandle.*

Remark When $X = R_4$ (which is not connected), it is shown as follows (see also [81] for numerical computation) that $\Phi_\alpha(K) = 4$ for any K and α, though R_4 has non-trivial cohomology groups since $H_2^Q(R_4) = \mathbb{Z}^2 \oplus (\mathbb{Z}/2\mathbb{Z})^2$. The quandle R_4 has two orbits, which form subquandles isomorphic to T_2, where T_n denotes the trivial quandle (i.e. $x * y = x$ for any x, y) of order n. Further, T_2 has two orbits, which form subquandles isomorphic to T_1. Since $Q(K)$ is connected, any representation of $Q(K)$ to R_4 is trivial (i.e. a constant map). Hence, any coloring is trivial (i.e. colored by a single element of X). Since $\phi(x,x) = 0$ for any 2-cocycle ϕ, $\Phi_\alpha(K) = 4$ by definition.

When $X = \mathbb{Z}[t^{\pm 1}]/(9, t-4)$ (which is not connected), it follows from a similar argument (see also [81] for numerical computation) that $\Phi_\alpha(K) = 9$ for any K and α, noting that this X has three orbits, which form subquandles isomorphic to T_3.

In general, let $X_1, X_2, \cdots$ be the orbits of X. These form subquandles of X. We denote by $i_k : X_k \to X$ the inclusions. Then, it follows from a similar argument as above that $\Phi_\alpha(K) = \sum_k \Phi_{i_k^*\alpha}(K)$. Repeating this procedure, the computations of $\Phi_\alpha(K)$ of a knot K can be reduced to those for connected quandles.[22]

Remark The cohomology group $H^2_Q(\overline{X}; A)$ of the dual quandle $\overline{X}$ of a quandle X is isomorphic to $H^2_Q(X; A)$ by an isomorphism taking a 2-cocycle $\overline{\phi}$ to ϕ where $\phi(x, y) = \overline{\phi}(x * y, y)$. It follows that $\Phi_{\overline{\alpha}}(K) = \Phi_\alpha(\overline{K})$, where $\overline{K}$ denotes the mirror image of K. Therefore, the computations of quandle cocycle invariants for $\overline{X}$ can be reduced to those for X.

Remark When $\alpha = 0$, by definition $\Phi_\alpha(K)$ is equal to the number of representations $Q(K) \to X$. In particular, when X is an Alexander quandle, it can be presented by using the ith Alexander polynomials, as mentioned in a remark of Problem 5.2.

Remark [81] When $X = \mathbb{Z}[t^{\pm 1}]/(2, t^2 + t + 1)$, $H^2_Q(X; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. For its non-trivial cohomology class α,

$$\Phi_\alpha(K) = \begin{cases} 4(1+3u) & \text{for } K = 3_1, 4_1, 7_2, 7_3, 8_1, 8_4, 8_{11}, 8_{13}, \text{ and} \\ & \qquad \text{9 certain knots with 9 crossings,} \\ 16(1+3u) & \text{for } K = 8_{18}, 9_{40}, \\ 16 & \text{for } K = 8_5, 8_{10}, 8_{15},\ 8_{19}\text{–}8_{21}, \text{ and} \\ & \qquad \text{16 certain knots with 9 crossings,} \\ 4 & \text{for the other knots } K \text{ with at most 9 crossings,} \end{cases}$$

where u denotes the generator of $\mathbb{Z}/2\mathbb{Z}$. See [81] for details.

When $X = \mathbb{Z}[t^{\pm 1}]/(3, t^2 + 1)$, $H^2_Q(X; \mathbb{Z}/3\mathbb{Z}) = \mathbb{Z}/3\mathbb{Z}$. For a non-trivial cohomology class α of it,

$$\Phi_\alpha(K) = \begin{cases} 9(1+4u+4u^2) & \text{if } K = 4_1, 5_2, 8_3, 8_{17}, 8_{18}, 8_{21}, \text{ and} \\ & \qquad \text{9 certain knots with 9 crossings,} \\ 27(11+8u+8u^2) & \text{if } K = 9_{40}, \\ 81 & \text{if } K = 6_3, 8_2, 8_{19}, 8_{24}, 9_{12}, 9_{13}, 9_{46}, \\ 9 & \text{for the other knots } K \text{ with at most 9 crossings,} \end{cases}$$

where u denotes a generator of $\mathbb{Z}/3\mathbb{Z}$. See [81] for details.

[22] This argument is not available for the link case, since a link quandle is not connected.

Remark It is known, see [78], that each $\alpha \in H^2_Q(X;A)$ gives an *abelian extension* $A \to Y \xrightarrow{p} X$, where $Y = A \times X$, which forms a quandle with the binary operation given by $(a_1, x_1) * (a_2, x_2) = \big(a_1 + \phi(x_1, x_2), x_1 * x_2\big)$ using a 2-cocycle ϕ representing α, and p denotes the natural projection.

Let $a_1, a_2, \cdots, a_N$ be a sequence of generators of $Q(K)$ associated with over paths of a diagram of K which are chosen as going around K. Adjacent generators a_1 and a_2 are related by $a_1 * b = a_2$ (or $a_1 = a_2 * b$) for some generator b. Let $\widetilde{\rho(b)}$ be a pre-image of $\rho(b)$ under the projection p. Then, $S_{\widetilde{\rho(b)}}$ (resp. $S^{-1}_{\widetilde{\rho(b)}}$) induces a map $p^{-1}(a_1) \to p^{-1}(a_2)$, which does not depend on the choice of a pre-image of $\rho(b)$. Composing such maps, we have a sequence of maps $p^{-1}(a_1) \to p^{-1}(a_2) \to \cdots \to p^{-1}(a_N) \to p^{-1}(a_1)$. The composite map of these maps can be expressed $a \mapsto a + m(\rho)$ $(a \in A)$ for some $m(\rho) \in A$. Then, the quandle cocycle invariant can be presented by $\Phi_\alpha(K) = \sum_\rho m(\rho) \in \mathbb{Z}[A]$, where the sum runs over all representations ρ of $Q(K)$ to X.

In particular, as shown in [78], the number of representations $Q(K) \to X$ that can lift to representations $Q(K) \to Y$ is equal to the coefficient of the unit of A in $\Phi_\alpha(K)$. For example, when $A = \mathbb{Z}/2\mathbb{Z}$, it follows that, writing $\Phi_\alpha(K) = a + bt$ (where t is the generator of $\mathbb{Z}/2\mathbb{Z}$), a is equal to the number of representations $Q(K) \to X$ that can lift to representations $Q(K) \to Y$, and b is equal to the number of those that do not.

In this way we can compute $\Phi_\alpha(K)$ in terms of the abelian extension associated to α.

Problem 5.7 *Find relations between quandle cocycle invariants and knot invariants known so far, such as quantum invariants.*

Remark When $\alpha = 0$ and X is an Alexander quandle, $\Phi_\alpha(K)$ can be presented by using the ith Alexander polynomial, as mentioned in a remark of Problem 5.6.

Remark (M. Graña) The quandle cocycle invariants can be presented by knot invariants derived from certain ribbon categories [155].

A central extension of a group G gives an abelian extension of the conjugation quandle of G. It is known that an abelian extension of a group G can be characterized by the cohomology group $H^2(G;A)$ for a G-module A. Motivated by this cohomology group we introduce $H^2_Q(X;A)$ of a quandle X for an "X-module" A as follows. We call an abelian group A an *X-module* of a quandle

X if there is a map $\rho : X \to \mathrm{Aut}(A)$ satisfying that $\rho(x * y) = \rho(y)^{-1}\rho(x)\rho(y)$ for any $x, y \in X$. For simplicity, we often write $\rho(x)^{\pm 1}a$ as $x^{\pm 1}a$ omitting ρ. Let $C_Q^i(X; A)$ be as before. We give the coboundary operators by

$$\begin{aligned} d_1 f(x,y) &= y^{-1}\big(f(x) + xf(y) - f(y)\big) - f(x*y), \\ d_2 g(x,y,z) &= (y*z)^{-1}g(x,z) - z^{-1}g(x,y) + (y*z)^{-1}\big((x*z)-1\big)g(y,z) \\ &\qquad - g(x*y,z) + g(x*z, y*z), \end{aligned}$$

for $f \in C_Q^1(X; A)$ and $g \in C_Q^2(X; A)$. We define the second quandle cohomology group by $H_Q^2(X; A) = (\mathrm{kernel}\, d_2)/(\mathrm{image}\, d_1)$.

Problem 5.8 *Compute $H_Q^2(X; A)$ for each X-module A.*

Remark This cohomology group might be isomorphic to the cohomology group of a quandle space of X (see a remark on Problem 5.4) with coefficients in the local system corresponding to the X-module A.

For an X-module A, each $\alpha \in H_Q^2(X; A)$ gives an extension $A \to Y \to X$, where $Y = A \times X$, which forms a quandle with the binary operation given by $(a_1, x_1) * (a_2, x_2) = \big(x_2^{-1}(a_1 + x_1 a_2 - a_2) + \phi(x_1, x_2), x_1 * x_2\big)$ using a 2-cocycle ϕ representing α.

Problem 5.9 *Let the notation be as above. Then, extending the definition of the quandle cocycle invariant, define a knot invariant associated with α, which is, roughly speaking, an invariant obtained by counting representations of a knot quandle $Q(K)$ to X with information whether each representation can lift to a representation $Q(K) \to Y$.*

5.5 Quantum quandles

Problem 5.10 (M. Polyak) *Define a quantum quandle.*

Remark A quantum group is a quantization of a group, in the sense that it can be regarded as a non-commutative perturbation of a (certain) function algebra on a group. It is a problem to formulate an appropriate quantization of a quandle.

5.6 Rack (co)homology

(C. Rourke, B. Sanderson) A *rack* has the same definition as a quandle, except that axiom (1) is omitted. Quandles are thus a special class of racks. There is a naturally defined classifying space for a rack (in fact a semi-cubical complex), the *rack space*, and hence both homology and cohomology groups are defined. These can be used to define invariants of (framed) links [122, 123, 124]. Regarding a quandle as a rack we have two definitions of cohomology which are closely related. Indeed the cochain group for quandle cohomology is a subgroup of that for rack cohomology with the definition of coboundary unchanged.

Problem 5.11 (C. Rourke, B. Sanderson) *Is there a natural quandle space whose cohomology groups are the quandle cohomology groups?*

Remark The quandle cochain groups do not correspond to setting the rack cochains to be zero on a subcomplex of the rack space. Thus the first guess that the quandle space is obtained by quotienting a certain subcomplex is false.

There is an anologous problem of classification and computation of (co)homology groups for racks as for quandles. One particular interesting question is the following:

Conjecture 5.12 (R. Fenn, C. Rourke, B. Sanderson) $H_3(R_p) \cong \mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ *for* p *prime.*

Remark The conjecture is equivalent to $H_3^Q(R_p) \cong \mathbb{Z}/p\mathbb{Z}$ for p prime. This has been verified for $p \leq 7$ (see table 5) and the rack version has been verified (again for $p \leq 7$) by Maple calculation (Rourke and Sanderson). A direct proof (without using a computer calculation) has been found for $p = 3$. The conjecture is consistent (indeed suggested by) the calculation of T. Mochizuki [284] that $H_3^Q(R_p; \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$.

6 Braid group representations

For $n = 1, 2, \dots$, the braid group B_n is the group generated by $\sigma_1, \dots, \sigma_{n-1}$ modulo the relations:

- $\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| > 1,$
- $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \quad \text{if } |i-j| = 1.$

6.1 The Temperley-Lieb algebra

For $\tau \in \mathbb{C}$, the *Temperley-Lieb algebra* $\mathrm{TL}_n(\tau)$ is the associative $\mathbb{C}$-algebra generated by $1, e_1, \dots, e_{n-1}$ modulo the relations:

- $e_i e_j = e_j e_i \quad \text{if } |i-j| > 1,$
- $e_i e_j e_i = e_i \quad \text{if } |i-j| = 1,$
- $e_i e_i = \tau e_i.$

We will simply write TL_n, where τ is understood. There is a map from B_n to TL_n given by

$$\begin{aligned} \sigma_i &\longmapsto A + A^{-1} e_i, \\ \sigma_i^{-1} &\longmapsto A^{-1} + A e_i, \end{aligned}$$

where $A \in \mathbb{C}$ is such that $\tau = -A^2 - A^{-2}$.

These definitions can be motivated in terms of *tangle diagrams* in $\mathbb{R} \times I$. These are similar to knot diagrams, except that they can include arcs with endpoints on $\mathbb{R} \times \{0, 1\}$. Two tangles are considered the same if they are related by a sequence of isotopies and Reidemeister moves of the second and third type. The generators of B_n and TL_n can be defined to be the tangle diagrams suggested by Figure 17. The arcs of these diagrams have endpoints

$$\{1, 2, \dots, n\} \times \{0, 1\}.$$

The product ab of two such diagrams a and b is obtained by placing a on top of b and then shrinking the result vertically to the required height. The third relation in the Temperley-Lieb algebra allows one to delete a closed loop at the expense of multiplying by τ. Using these definitions, the map from B_n to TL_n is given by resolving all crossings using the Kauffman skein relation.

Chapter 6 was written by S.J. Bigelow.

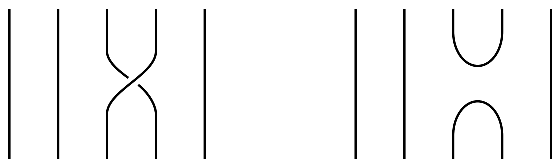


Figure 17: The generator σ_3 of B_5 (the left picture) and the generator e_3 of TL_5 (the right picture).

Problem 6.1 ([188, Problem 3]) *Is the representation of the braid group inside the Temperley-Lieb algebra faithful?*

Remark We are mostly interested in the case τ is a transcendental. The answer is yes for $n \leq 3$, and unknown for all other values of n.

The Jones polynomial of the closure of a braid β is a certain trace function of the image of β inside the Temperley-Lieb algebra. If $\beta \in B_n \setminus \{1\}$ maps to the identity in TL_n, and $\gamma \in B_n$ is any braid whose closure is the unknot, then the closure of $\beta\gamma$ would have Jones polynomial one. It should be easy to arrange for this to be a non-trivial knot. Thus a negative answer to Problem 6.1 would almost certainly lead to a solution to Problem 1.1.

6.2 The Burau representation

For $k = 0, 1, \dots, \lfloor \frac{n}{2} \rfloor$, let V^n_{n-2k} be the vector space spanned by tangle diagrams in $\mathbb{R} \times I$ with no crossings and endpoints

$$\{(1,0),(2,0),\dots,(n-2k,0)\} \cup \{(1,1),(2,1),\dots,(n,1)\}$$

modulo the relations:

- a tangle is zero if it contains an edge with both endpoints on $\mathbb{R} \times \{0\}$,
- a closed loop may be removed at the expense of multiplying by τ.

Let TL_n act on V^n_{n-2k} by stacking tangle diagrams in the usual way. For generic values of τ, TL_n is semisimple and these are its irreducible representations.

We obtain irreducible representations of B_n by taking its induced action on V^n_{n-2k}. By a result of Long [268], the representation of B_n inside TL_n is faithful if and only if each of these irreducible representations is faithful. Note that the action of B_n on the one-dimensional space V^n_n is never faithful for $n > 2$. Also if $n > 2$ is even then the action of B_n on V^n_0 is easily shown to be unfaithful. The action of B_n on V^n_{n-2} is the famous *Burau representation.*

Problem 6.2 *Is the Burau representation of B_4 faithful?*

Remark The Burau representation of B_n is known to be faithful for $n \leq 3$ and not faithful for $n \geq 5$ [51].

The representation of B_4 in TL_4 is faithful if and only if the Burau representation of B_4 is faithful.

Remark The Burau representation of B_4 is faithful if and only if a certain pair of three-by-three matrices generate a free group. The matrices given in [56] contain a misprint, but their description as words in the generators is correct.

The Burau representation of B_4 is faithful if and only if a certain intersection pairing detects intersection of arcs in the four-times punctured disk [51].

Cooper and Long have explicitly calculated the kernel of the Burau representation modulo the primes 2, 3 and 5 [95].

Problem 6.3 (S.J. Bigelow) *Is the action of B_6 on V_2^6 faithful?*

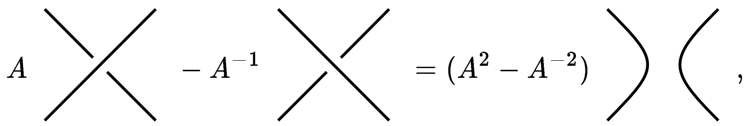
Remark The Burau representation of B_6 is unfaithful [269]. Thus the representation of B_6 in TL_6 is faithful if and only if the action of B_6 on V_2^6 is faithful.

No approach to this problem is known except for a brute force computer search. However such a search might find an example more easily than any of the more subtle approaches to the Burau representation of B_4.

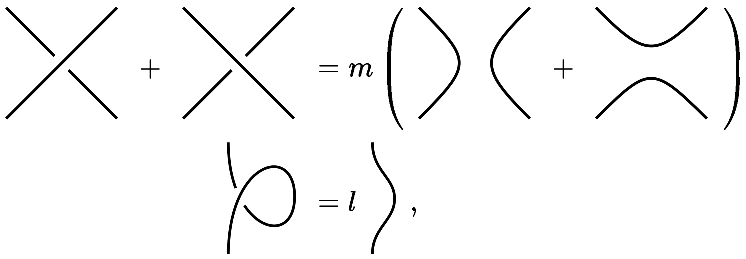
Remark We could also ask whether the action of B_5 on V_1^5 is faithful. A computer search of this representation would be easier because the matrices involved are smaller (five-by-five instead of nine-by-nine). On the other hand, this representation is more likely to be faithful, since if the representation of B_6 in TL_6 is faithful then so is the representation of B_5 in TL_5.

6.3 The Hecke and BMW algebras

We now introduce two algebras which can be defined in a similar way to the Temperley-Lieb algebra. The *Hecke algebra* is the set of formal linear combinations of braids modulo the relation:



where $A \in \mathbb{C}$. The *BMW algebra* is the set of formal linear combinations of tangles whose edges have endpoints $\{1,2,\ldots,n\}\times\{0,1\}$, modulo the relations:



where $m, l \in \mathbb{C}$. See [300] and [58] for an analysis of the BMW algebra. The Temperley-Lieb algebra can be embedded into the Hecke algebra, which in turn can be embedded into the BMW algebra.

These algebras are semisimple for generic values of their parameters. The irreducible representations of the BMW algebra correspond to partitions of $n-2k$ for $k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$. The irreducible representations of the Hecke algebra correspond to partitions of n. The irreducible representations of the Temperley-Lieb algebra correspond to partitions of n into two parts.

Lawrence [240] has used a topological construction to obtain the irreducible representations of the Hecke algebra. The construction uses the definition of the braid group as the mapping class group of a punctured disk to obtain an action on the homology of a related space.

Problem 6.4 (S.J. Bigelow) *Generalise Lawrence's construction to obtain the irreducible representations of the BMW algebra.*

Remark Zinno [413] has shown how to obtain the representation of the BMW algebra corresponding to the partition of $n-2$ into one part.

Problem 6.5 (S.J. Bigelow) *Find a larger family of irreducible representations of B_n which includes those coming from the BMW algebra.*

Remark This might be defined using tangles and some more complicated relations, or by generalising Lawrence's approach.

6.4 Other problems

Problem 6.6 *Classify all irreducible representations of B_n.*

Remark This is probably impossibly hard. However it seems that interesting partial results are possible. Formanek [126] has classified all irreducible complex representations of B_n having degree at most $n-1$.

Problem 6.7 (S.J. Bigelow) *Is there a faithful representation of B_n into a group of matrices over $\bar{\mathbb{Q}}$?*

Remark There is a faithful representation of B_3 into $\mathrm{GL}(2,\mathbb{Z})$. The problem is open for all $n \geq 4$.

There is a faithful representation of B_n into a group of matrices over $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$. Krammer's proof of this fact [229] works when t is assigned any value between 0 and 1. However it is not known whether there is an algebraic value of q for which the representation remains faithful.

7 Quantum and perturbative invariants of 3-manifolds

7.1 Witten-Reshetikhin-Turaev invariants and quantum invariants

Witten [403] proposed that, for a semi-simple compact Lie group G and a positive integer k, a topological invariant of a closed oriented 3-manifold M is given by the path integral

$$Z_k^G(M) = \int e^{2\pi\sqrt{-1}k\mathrm{CS}(A)}\mathcal{D}A, \tag{36}$$

which is a formal integral over gauge equivalence classes of connections A on the trivial G bundle on M. Here, the *Chern-Simons functional* $\mathrm{CS} : \mathcal{A} \to \mathbb{R}$ is defined by

$$\mathrm{CS}(A) = \frac{1}{8\pi^2}\int_M \mathrm{trace}(A \wedge dA + \frac{2}{3}A \wedge A \wedge A), \tag{37}$$

for a connection A, regarding it as a $\mathfrak{g}$-valued 1-form on M, where $\mathfrak{g}$ denotes the Lie algebra of G.

Motivated by Witten's proposal, the *quantum G invariant* $\tau_r^G(M)$ has been defined and studied, first by Reshetikhin and Turaev [346] and later by other researchers, where we put $r = k + h^\vee$ with the dual Coxeter number $h^\vee$ of $\mathfrak{g}$. The quantum invariant is also called the *Witten-Reshetikhin-Turaev invariant.* For example, when M is obtained from S^3 by integral surgery along a framed knot K with a positive framing, $\tau_r^{SU(2)}(M)$ for $r \geq 3$ and $\tau_r^{SO(3)}(M)$ for odd $r \geq 3$ are given by

$$\tau_r^{SU(2)}(M) = \Big(\sum_{n=1}^{r-1}[n]Q^{sl_2;V_n}(U_+)\Big)^{-1}\sum_{n=1}^{r-1}[n]Q^{sl_2;V_n}(K)\Big|_{q=\exp(2\pi\sqrt{-1}/r)},$$

$$\tau_r^{SO(3)}(M) = \Big(\sum_{\substack{0<n<r\\ r \text{ is odd}}}[n]Q^{sl_2;V_n}(U_+)\Big)^{-1}\sum_{\substack{0<n<r\\ r \text{ is odd}}}[n]Q^{sl_2;V_n}(K)\Big|_{q=\exp(2\pi\sqrt{-1}/r)},$$

where $[n] = (q^{n/2} - q^{-n/2})/(q^{1/2} - q^{-1/2})$, and U_+ denotes the trivial knot with $+1$ framing, and $Q^{sl_2;V_n}(K)$ denotes the quantum invariant of K associated with the irreducible n-dimensional representation of the quantum group $U_q(sl_2)$; for details see [221] (see also [321] for the notation). It is known [221] that

$$\tau_r^{SU(2)}(M) = \begin{cases} \tau_3^{SU(2)}(M)\tau_r^{SO(3)}(M) & \text{if } r \equiv 3 \bmod 4, \\ \overline{\tau_3^{SU(2)}(M)}\tau_r^{SO(3)}(M) & \text{if } r \equiv 1 \bmod 4, \end{cases}$$

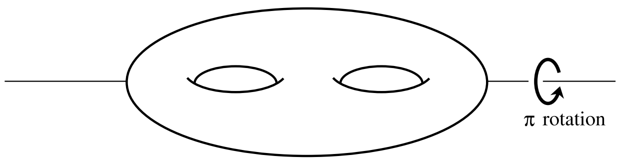
where $\tau_3^{SU(2)}(M)$ is an invariant determined by the cohomology ring and the linking pairing of M, which is equal to zero for some M (see (38)). For details on quantum G invariants, see e.g. [321] and references therein.

Problem 7.1 (see [220, Problem 3.108]) *Does there exist a closed 3-manifold M, other than S^3, such that $\tau_r^{SO(3)}(M) = \tau_r^{SO(3)}(S^3)$ for all odd $r \geq 3$?*

Remark (see [220, Remark on Problem 3.108]) Suppose that $\tau_r^{SO(3)}(M) = \tau_r^{SO(3)}(S^3)$ for a closed 3-manifold M and all odd $r \geq 3$. If the Betti number of M was positive, $\tau_r^{SO(3)}(M)$ is divisible by $q-1$. Hence, M is a rational homology 3-sphere. We have that $\tau^{SO(3)}(M) = \tau^{SO(3)}(S^3)$. Since the leading two coefficients of $\tau^{SO(3)}(M)$ are given by the order of the first homology group and Casson invariant of M, M is an integral homology 3-sphere with Casson invariant zero.

Note that $\tau_r^{SO(3)}\big(L(65,8)\big) = \tau_r^{SO(3)}\big(L(65,18)\big)$ for all odd $r \geq 3$; see [406].

Remark There is a center in the mapping class group of the closed surface of genus 2, shown below.



A *mutation* of a 3-manifold M is defined to be a 3-manifold obtained from M by cutting along a separating closed surface of genus 2 in M and by gluing again after twisting by the above map. It is shown in [205] that $\tau_r^{SO(3)}(M)$ does not depend on a change by any mutation of M.

Problem 7.2 (S.K. Hansen, T. Takata) *Find pairs of non-homeomorphic rational homology 3-spheres that can be distinguished by their quantum G invariants τ_r^G or their quantum PG invariants τ_r^{PG} for some level r and some simply connected compact simple Lie group G but not by their LMO invariants.*

Remark (S.K. Hansen, T. Takata) For example, the LMO invariants of the lens spaces $L(25,4)$ and $L(25,9)$ are equal [37], but their quantum $SU(2)$ invariants for $r=5$ are not equal.

Problem 7.3 (S.K. Hansen, T. Takata) *Do the family of quantum G invariants τ_r^G or the family of quantum PG invariants τ_r^{PG}, G running through all simply connected compact simple Lie groups and r running through all allowed levels, separate rational homology 3-spheres? How well do these families of invariants separate closed oriented 3-manifolds?*

Remark (S.K. Hansen, T. Takata) It is well known that the LMO invariant is a weak invariant outside the class of rational homology 3-spheres; see the last remark on Problem 11.1. On the contrary there are 3-manifolds with arbitrary high first Betti number and non-trivial quantum $SU(2)$ invariants as the example of Seifert manifolds shows. We note that the non-triviality of the invariants of Seifert manifolds e.g. follows from the fact that these invariants have non-trivial asymptotic expansion in the limit of large quantum level; see [354], [168], and Section 7.2. It is likely to believe, e.g. from the asymptotic expansion conjecture of Andersen, see Conjecture 7.7, that the quantum G invariants are quite strong invariants also outside the class of rational homology 3-spheres. It is known, however, that the family of quantum $SU(n)$ invariants, n running through all integers > 1, is not a complete invariant, that is to say that this family of invariants can not separate all closed oriented 3-manifolds, cf. [256]. It is still an open question if this is also the case if we include the quantum invariants for all the other simply connected compact simple Lie groups.

Problem 7.4 *Find a 3-dimensional topological interpretation of quantum invariants of 3-manifolds.*

Remark Certain special values have some interpretations. For a closed oriented 3-manifold M,

$$\tau_3^{SU(2)}(M) = \begin{cases} 0 & \text{if there exists } \alpha \in H^1(M;\mathbb{Z}/2\mathbb{Z}) \text{ with } \alpha^3 \neq 0, \\ \sqrt{2}^{\operatorname{rank} H^1(M;\mathbb{Z}/2\mathbb{Z})} e^{-\beta(M)\pi\sqrt{-1}/4} & \text{otherwise,} \end{cases} \tag{38}$$

where $\beta(M)$ denotes the Brown invariant. Further, for a closed oriented 3-manifold M,

$$\tau_4^{SU(2)}(M) = \sum_{\sigma} e^{-\mu(M,\sigma)\cdot 3\pi\sqrt{-1}/8},$$

where the sum runs over all spin structures σ of M and $\mu(M,\sigma)$ denotes the Rokhlin invariant of a spin structure σ of M. For details, see [221].

It is known [291] that, for any rational homology 3-sphere M and any prime $p > |H_1(M;\mathbb{Z})|$,

$$|H_1(M;\mathbb{Z})| \cdot \tau_p^{SO(3)}(M) \equiv \left(\frac{|H_1(M;\mathbb{Z})|}{p}\right) \Big(1 + 6\lambda(M)(\zeta - 1)\Big)$$

mod $(\zeta-1)^2$ in $\mathbb{Z}[\zeta]$, putting $\zeta = e^{2\pi\sqrt{-1}/p}$, where $\lambda(M)$ denotes the Casson-Walker invariant of M and $\left(\frac{\cdot}{p}\right)$ denotes the Legendre symbol.

Remark The Chern-Simons path integral (36) by Witten [403] gives a 3-dimensional physical interpretation of a quantum invariant of 3-manifolds. Historically speaking, the quantum invariants of 3-manifolds were introduced, motivated by Witten's Chern-Simons path integral.

Conjecture 7.5 [158] *For non-vanishing $\tau_r^G(M)$, the absolute value $|\tau_r^G(M)|$ only depends on the fundamental group $\pi_1(M)$.*

7.2 The asymptotic expansion conjecture

The perturbative expansion of the Chern-Simons path integral (36) is given by the semi-classical approximation and its higher loop perturbations. Roughly speaking, the semi-classical approximation is obtained from the path integral by ignoring the contribution from the third order term of the Chern-Simons functional, and the higher loop perturbation contributions are the corrections to this semi-classical contribution.

To the best of our knowledge, there is today, no complete perturbative treatment of the Chern-Simons quantum field theory available, even from a mathematical physics point of view. In the following few paragraphs we shall try to outline the main activities seen so far in this direction.

The the first formula for the semi-classical approximation of the Chern-Simons path integral was given by Witten in [403], describing it as a sum of contributions, one for each gauge equivalence classes of flat connection, involving the Chern-Simons value, the Reidemeister torsion and a certain spectral flow for each such gauge equivalence class. To test this prediction, Freed and Gompf [127] made for certain Seifert fibered manifolds some computer studies of the large k behavior of $Z_k^{SU(2)}(M)$ and based on these calculations and further discussion of the semi-classical approximation of the path integral, they proposed the following formula for the semi-classical approximation $(r=k+2)$

$$Z_k^{SU(2)}(M) \underset{r\to\infty}{\sim} e^{-3\pi\sqrt{-1}(1+b^1(M))/4} \times \sum_{[A]} e^{2\pi\sqrt{-1}r\mathrm{CS}(A)} r^{(h_A^1-h_A^0)/2} e^{-2\pi\sqrt{-1}(I_A/4+h_A^0/8)} \tau_M(A)^{1/2},$$

The first version of Section 7.2 was written by T. Ohtsuki, following seminar talks given by J.E. Andersen. Based on it, J.E. Andersen wrote this section.

where the sum is over the gauge equivalence classes of flat connections A. Let us explain the quantities involved in this expression and in which cases one can make sense of this expression as it stands.

For any flat connection A, we have the cohomology groups $H^i(M, d_A)$ of the covariant derivative complex $d_A : \Omega^\star(M;\mathfrak{g}) \to \Omega^{\star+1}(M;\mathfrak{g})$ given by $d_A f = df + [A, f]$, and h^i_A is the dimension of $H^i(M, d_A)$. Further associated to this complex we have the Reidemeister torsion $\tau_M(A) \in \otimes_i \big(\det H^i(M, d_A)\big)^{(-1)^i} \cong \Big(\det H^0(M, d_A) \otimes \big(\det H^1(M, d_A)\big)^*\Big)^2$ (by Poincaré duality). If one now assumes that all the gauge equivalence classes of flat connections A are isolated, in fact Freed and Gompf assumed $H^1(M, d_A) = 0$, so that the above sum is finite and such that the square root of the Reidemeister torsion $\tau_M(A)^{1/2}$ is a well-defined number (once a volume on $H^0(M, d_A)$ has been fixed, but for irreducible connections $H^0(M, d_A) = 0$).

The quantity $I_A \in \mathbb{Z}/8\mathbb{Z}$ denotes the spectral flow of the operator

$$\begin{pmatrix} \star d_{A_t} & -d_{A_t}\star \\ d_{A_t}\star & 0 \end{pmatrix}$$

on $\Omega^1(M;\mathfrak{g}) \otimes \Omega^3(M;\mathfrak{g})$, where A_t is a path of connections running from the trivial connection to A. They also looked at some examples where $H^1(M, d_A) \neq 0$ and checked the overall growth predicted by the above formula.

Following this Jeffrey [184] proposed the following more general interpretation of the square root of Reidemeister torsion in the cases where the connections are not isolated: Assume that the moduli space of flat connections $\mathcal{M}$ on M is smooth and that the tangent space at each equivalence class of flat connection A equals $H^1(M, d_A)$. Since $H^0(M, d_A) \subset \mathfrak{g}$ the invariant inner product we have chosen on $\mathfrak{g}$ induces a volume element on $H^0(M, d_A)$. In total this means that the square root of the Reidemeister torsion induces a measure on the moduli space when we pair it with the induced volume element on $H^0(M, d_A)$ divided by the order of the center of G and one arrives at $(r = k + h^\vee)$

$$Z^G_k(M) \underset{r\to\infty}{\sim} e^{-\pi\sqrt{-1}(\dim G)(1+b^1(M))/4} \times \int_{[A]\in\mathcal{M}} e^{2\pi\sqrt{-1}r\,\mathrm{CS}(A)} r^{(h^1_A - h^0_A)/2} e^{-2\pi\sqrt{-1}(I_A/4 + (h^0_A + h^1_A)/8)} \tau_M(A)^{1/2}.$$

For some mapping tori of genus 1 surfaces and lens spaces, Jeffrey verified this form of the semi-classical approximation. Garoufalidis [134] independently proved the semi-classical approximation for lens spaces and studied in various examples the growth rate predicted by these approximations. Rozansky [352] proposed a further refined version of the above semi-classical approximation, and offered calculations for a very large class of Seifert fibered manifolds as evidence. He proposed to divide the volume element on $H^0(M,d_A)$ by the volume of the stabilizer of A, and to use the resulting quantity paired with the square root of the Reidemeister torsion as the measure on $\mathcal{M}$ (generalizing the division by the order of the center above). This gave a natural explanation of factors not accounted for in both the work of Freed and Gompf and the work of Jeffrey. He also proposed corrections to the formula for the growth rate of the invariant (i.e. the power of r in the above), in cases where not all directions in $H^1(M,d_A)$ are tangent to paths of flat connections (see [352] and [353]).

Axelrod and Singer [21, 22] (see also [226]) considered the higher loop contributions in the perturbation expansion and proposed the following:

$$Z_k^G(M) \underset{k\to\infty}{\sim} \int_{[A]\in\mathcal{M}} \begin{pmatrix} \text{semi-classical} \\ \text{approximation} \end{pmatrix} \times \exp\left(\sum_{l=1}^{\infty} \frac{c^l k^{-l}}{(2l)!(3l)!} \sum_{e(\Gamma)=-l} \frac{Z_\Gamma(M,A)}{|\mathrm{Aut}(\Gamma)|} \right) \tag{39}$$

for some scalar c, where the right sum runs over connected trivalent graphs Γ whose Euler number is equal to $-l$, and $|\mathrm{Aut}(\Gamma)|$ denotes the order of the group of automorphisms of Γ. Further, in the case where A is acyclic or when $A\in\mathcal{M}$ is contained in a smooth component, Axelrod and Singer was able to construct $Z_\Gamma(M,A)$ as a topological invariant of (M,A); roughly speaking, it is given as follows in the acyclic case. We identify the set of connection around A with $\Omega^1(M,\mathfrak{g})$. The second order part of the Chern-Simons functional gives a bilinear form on $\Omega^1(M,\mathfrak{g})^{\otimes 2}$, and it determines a 2-form $L\in\Omega^2(M\times M,\mathfrak{g}\otimes\mathfrak{g})$ and its "inverse". Further, the third order part of the Chern-Simons functional gives a trilinear form T on $\Omega^1(M,\mathfrak{g})^{\otimes 3}$. We obtain $Z_\Gamma(M,A)$ by contracting $L^{\otimes(3l)}$ by $T^{\otimes(2l)}$ "along the trivalent graph Γ" (roughly regarding L as in $\Omega^1(M,\mathfrak{g})^{\otimes 2}$); we determine the action of $T^{\otimes(2l)}$ on $L^{\otimes(3l)}\in\Omega^1(M,\mathfrak{g})^{\otimes(6l)}$ by putting copies of L on $3l$ edges of Γ and putting copies of T on $2l$ vertices of Γ. For a precise (mathematical) construction (and its topological invariance) of $Z_\Gamma(M,A)$, see [21, 22].

From the mathematical viewpoint we regard $Z_k^G(M)$ as

$$Z_k^G(M) = \frac{\tau^G_{k+h^\vee}(M)}{\tau^G_{k+h^\vee}(S^1\times S^2)}$$

for the quantum G invariant $\tau_r^G(M)$. Then, the asymptotic expansion of $Z_k^G(M)$ is predicted by the semi-classical approximation and its higher loop corrections stemming from a perturbative expansion of the Chern-Simons path integral, explained above in some cases. This leads us to the following somewhat vague conjecture.

Conjecture 7.6 (The perturbative expansion conjecture) *The asymptotic expansion of $Z_k^G(M)$ of a closed oriented 3-manifold M is given by*

$$\begin{aligned} Z_k^G(M) \underset{k\to\infty}{\sim} & \, e^{-\pi\sqrt{-1}(\dim G)(1+b^1(M))/4} \\ & \times \int_{[A]\in\mathcal{M}} e^{2\pi\sqrt{-1}r\,CS(A)} r^{(h_A^1-h_A^0)/2} e^{-2\pi\sqrt{-1}(I_A/4+(h_A^0+h_A^1)/8)} \tau_M(A)^{1/2} \\ & \times \exp\left(\sum_{l=1}^{\infty} \frac{c^l k^{-l}}{(2l)!(3l)!} \sum_{e(\Gamma)=-l} \frac{Z_\Gamma(M,A)}{|\mathrm{Aut}(\Gamma)|}\right), \end{aligned}$$

putting $r = k + h^\vee$, where the right hand side can be given in the mathematical viewpoint in certain cases, as mentioned above, but which needs further interpretation in general.

Remark The semi-classical approximation stated above (the upper two lines in the above formula), has been confirmed for lens spaces (first partially [127]) and then by [184, 134], for certain mapping tori of diffeomorphisms of a torus [184], and for all finite order mapping tori of automorphisms of any closed orientable surface of genus at least 2 [6]. For a large class of Seifert fibered manifolds [353] and [354] offered calculations which provided evidence that the phases in the semi-classical approximation is given by the Chern-Simons invariants and the measure is given by the square root of the Reidemeister torsion as explained above. Also, expressions for the higher loop corrections was offered. Later the necessary analytic estimates was provided in [168] so as to confirm this. See also the discussion below. For now, there are no examples of hyperbolic manifolds, where parts of the above conjecture has been confirmed.

For other versions of Conjecture 7.6, see [220, Problem 3.108], [136].

The formula in Conjecture 7.6 might not give an exact description of the asymptotic behavior of $Z_k^G(M)$ even in the semi-classical part, neither is it in all cases well-defined. Moreover, it might be difficult at present to calculate the concrete value of the higher loop corrections in the asymptotic expansion of Conjecture 7.6 for given M, A, and Γ. Nor do we have definitions for these terms, which has been proven to be well defined topological invariants in all cases.

The following conjecture offers a kind of reverse viewpoint on Conjecture 7.6, avoiding such ambiguities and difficulties.

Conjecture 7.7 (The asymptotic expansion conjecture, J.E. Andersen [6]) *Let $\{c_0 = 0, c_1, \cdots, c_m\}$ be the set of values of the Chern-Simons functional of flat G connections on a closed oriented 3-manifold M. There exist $d_j \in \mathbb{Q}$, $\tilde{I}_j \in \mathbb{Q}/\mathbb{Z}$, $v_j \in \mathbb{R}_+$, and $a_j^e \in \mathbb{C}$ for $j = 0, 1, \cdots, m$ and $e = 1, 2, 3, \cdots$ such that ($r = k + h^\vee$)*

$$Z_k^G(M) \underset{r\to\infty}{\sim} \sum_{j=0}^{m} e^{2\pi\sqrt{-1}rc_j} r^{d_j} e^{\pi\sqrt{-1}\tilde{I}_j/4} v_j \big(1 + \sum_{e=1}^{\infty} a_j^e r^{-e}\big),$$

that is, for all $E = 0, 1, 2, \ldots$, there exists a constant c_E such that

$$\Big| Z_k^G(M) - \sum_{j=0}^{m} e^{2\pi\sqrt{-1}rc_j} r^{d_j} e^{\pi\sqrt{-1}\tilde{I}_j/4} v_j \big(1 + \sum_{e=1}^{E} a_j^e r^{-e}\big) \Big| \le c_E r^{d-E-1}$$

for all $r = 2, 3, 4, \cdots$. Here, $d = \max\{d_0, \cdots, d_m\}$.

Remark (J.E. Andersen) If such an expansion in the above conjecture exists, then c_j, d_j, $\tilde{I}_j$, v_j, and a_j^e are uniquely determined by $Z_{k+2}^G(M)$ for $k = 0, 1, 2, 3, 4, \cdots$.

Problem 7.8 (J.E. Andersen) *If such an expansion exists, understand how it is related to the expansion of Ohtsuki and the expansion of Habiro.*

It will of course be important to establish, that an expansion of this type exists, however, of far greater importance will be to give independent topological meaning to the many resulting new invariants, e.g. to prove that the phases are the Chern-Simons values c_j. From the discussion above on the semi-classical approximation we derive the following conjecture:

Conjecture 7.9 (Topological interpretations of the d_j's) *Let $\mathcal{M}_j$ be the union of components of the moduli space of flat connections $\mathcal{M}$ which has Chern-Simons value c_j. Then*

$$d_j = \frac{1}{2} \max_{A\in\mathcal{M}_j} (h_A^1 - h_A^0),$$

where max *here means the maximum value that $(h_A^1 - h_A^0)$ assumes on a Zariski open subset of $\mathcal{M}_j$.*

Note that this conjecture might be rather optimistic, and may only hold in the non-degenerate cases. However, we do not know of any cases where it fails (see [136]).

Remark (J.E. Andersen) The special max proposed in Conjecture 7.9 is certainly needed, as shown by the example of the mapping torus of the diffeomorphism $-Id$ of a torus. The quantum $SU(2)$ invariant of this manifold is easily seen to be $r-1$, since $-Id$ is represented trivially for all levels, however, there are flat $SU(2)$ connections for which $(h^1_A - h^0_A) > 2$.

The Conjecture 7.9 implies the following growth rate.

Conjecture 7.10 (The growth rate conjecture) *Let* $d = \max\{d_0, \dots, d_n\}$. *Then* $|Z^G_r(M)| = O(r^d)$.

It is well known that the quantum invariants only grows like r to some power. The power is bounded from above by some simple function (depending on G) of the Heegaard genus of the manifold.

Remark (J.E. Andersen)

(1) Suppose that M is a closed 3-manifold satisfying that $\tau^G_r(M) = \tau^G_r(S^3)$ for all r. If the growth rate conjecture 7.10 is true for the group G, then there is no non-central representation of $\pi_1(M)$ to G.

(2) Kronheimer and Mrowka have proposed a program using Seiberg-Witten theory and Floer homology to establish that any 3-manifold M obtained from S^3 by $+1$ surgery along a non-trivial knot K has a non-trivial (and therefore non-abelian) representation of $\pi_1(M)$ to $SU(2)$. Suppose that this is the case and the growth rate conjecture 7.10 is true. Then, $J_{K,c} = J_{U,c}$ for all $c = 1, 2, \cdots$ if and only if K is the trivial knot U, where $J_{K,c}$ denotes the colored Jones polynomial of a knot K with a color c.

At this time we do not know of a topological interpretation of the values of $\tilde{I}_j$ and v_j which makes sense in all cases. Let us simply just propose the following

Conjecture 7.11 *There is a construct of the* right measure, *say* $\tau_M(A)^{1/2}$ *for* $A \in \mathcal{M}_i$, *from the square root of the Reidemeister torsion generalizing the non-degenerate case explained above and such that*

$$e^{\pi\sqrt{-1}\tilde{I}_j/4} v_j = \int_{A \in \mathcal{M}_i} e^{\pi\sqrt{-1}(-2I_A + h^0_A + h^1_A)/4} \tau_M(A)^{1/2}.$$

Conjectures 7.7 and 7.9 together with Conjecture 7.11 were first proved for mapping tori of all finite order diffeomorphisms of all surfaces of genus at least two in [6]. Recently, Conjecture 7.7 was proved for all Seifert fibered spaces in [168] by supplementing the calculations in [353] and [354] with the need analytic estimates.

Example Let us illustrate the asymptotic behavior of the quantum $SU(2)$ invariant of the lens space $L(5,1)$ of type $(5,1)$. For simplicity, we let r be an odd prime. Since $\tau_3^{SU(2)}\big(L(5,1)\big)=1$, putting $\zeta=\exp(2\pi\sqrt{-1}/r)$, we have that

$$\tau_r^{SU(2)}\big(L(5,1)\big)=\tau_r^{SO(3)}\big(L(5,1)\big)=\left(\frac{5}{r}\right)\zeta^{-3\cdot5^\star}\frac{\zeta^{10^\star}-\zeta^{-10^\star}}{\zeta^{2^\star}-\zeta^{-2^\star}},$$

where $k^\star$ denotes the inverse of k in $\mathbb{Z}/r\mathbb{Z}$. Since $\tau_r^{SU(2)}(S^1\times S^2)=\sqrt{\frac{r}{2}}/\sin(\frac{\pi}{r})$, we have that

$$Z_{r-2}^{SU(2)}\big(L(5,1)\big)=\frac{\tau_r^{SU(2)}\big(L(5,1)\big)}{\tau_r^{SU(2)}\big(S^1\times S^2\big)}=\sqrt{\frac{2}{r}}\sin\frac{\pi}{r}\left(\frac{5}{r}\right)\zeta^{-3\cdot5^\star}\frac{\zeta^{10^\star}-\zeta^{-10^\star}}{\zeta^{2^\star}-\zeta^{-2^\star}}. \tag{40}$$

On the other hand, as in [184], the semi-classical approximation is given as follows. The lens space $L(5,1)$ has three flat connections A_n (n=0,1,2); each A_n is determined by the representation of $\pi_1\big(L(5,1)\big)\cong\mathbb{Z}/5\mathbb{Z}$ to $SU(2)$ which takes a generator of $\mathbb{Z}/5\mathbb{Z}$ to $\begin{pmatrix}e^{2\pi\sqrt{-1}n/5} & 0\\ 0 & e^{-2\pi\sqrt{-1}n/5}\end{pmatrix}$. As in [184], we have that $\mathrm{CS}(A_n)=n^2/5$, $h^0_{A_n}=1$, $h^1_{A_n}=0$, $\tau_M(A_n)^{1/2}=\frac{4\sqrt{2}}{\sqrt{5}}\sin^2\frac{2\pi n}{5}$, and $I_n(\mathrm{mod}\ 4)=1$ if $n<5/2$, and -1 if $n>5/2$. Hence,

$$Z_{r-2}^{SU(2)}\big(\overline{L(5,1)}\big)\underset{r\to\infty}{\sim}2\sqrt{\frac{-2}{5r}}\sum_{n=0,1,2}e^{2\pi\sqrt{-1}rn^2/5}\sin^2\frac{2\pi n}{5}, \tag{41}$$

noting that the notatin of lens spaces in [184] is equal to the notation of their mirror images in [221, 134].

The sequence of $\tau_r^{SU(2)}\big(L(5,1)\big)$ for odd primes r splits into four subsequences according to $r\equiv\pm1,\pm3$ mod 10, and each subsequence can be approximated by a function of a polynomial order. Let us describe the subsequence, say, with $r\equiv-1$ mod 10, as follows. Since $10^\star=(r+1)/10$, we calculate (40) as

$$\begin{aligned}Z_{r-2}^{SU(2)}\big(L(5,1)\big)&=\sqrt{\frac{2}{r}}\sin\frac{\pi}{r}e^{-6\pi\sqrt{-1}/5r}\frac{e^{\pi\sqrt{-1}/5r}-\omega^{-1}e^{-\pi\sqrt{-1}/5r}}{e^{\pi\sqrt{-1}/r}-e^{-\pi\sqrt{-1}/r}}\\ &\underset{r\to\infty}{\sim}\frac{1-\omega^{-1}}{\sqrt{-2}}r^{-1/2},\end{aligned}$$

putting $\omega=\exp(2\pi\sqrt{-1}/5)$. On the other hand, the right hand side of (41) is calculated as

$$2\sqrt{\frac{-2}{5r}}\Big(e^{-2\pi\sqrt{-1}/5}\sin^2\frac{2\pi}{5}+e^{2\pi\sqrt{-1}/5}\sin^2\frac{4\pi}{5}\Big)=\frac{\omega-1}{\sqrt{-2}}r^{-1/2},$$

noting that $\sqrt{5} = 1 + 2\omega + 2\omega^{-1}$ (Gaussian sum). Therefore, it was verified that the semi-classical approximation is correct for this subsequence.

This is related to the perturbative invariant

$$\tau^{SO(3)}\big(L(5,1)\big) = q^{-3/5}\frac{q^{1/10} - q^{-1/10}}{q^{1/2} - q^{-1/2}} \in \mathbb{Q}[[q-1]]$$

as follows. We regard it as a holomorphic function of q in a suitable domain. The asymptotic behavior of $\tau_r^{SO(3)}\big(L(5,1)\big)$, say, for the above mentioned subsequence, can be presented by using this holomorphic function around $q^{1/5} = \omega$.

Example It is known, see [243, 247], that

$$\tau_r^{SO(3)}\big(\Sigma(2,3,5)\big) = \frac{1}{1-\zeta}\sum_{n=0}^{r-1} \zeta^n (1-\zeta^{n+1})(1-\zeta^{n+2})\cdots(1-\zeta^{2n+1})$$

for Poincare homology 3-sphere $\Sigma(2,3,5)$, where we put $\zeta = \exp(2\pi\sqrt{-1}/r)$. It is an exercise to compute the asymptotic behaviour of $Z_{r-2}^{SU(2)}\big(\Sigma(2,3,5)\big)$ as $r \to \infty$ related to Conjecture 7.7, and to formulate a relation with the perturbative invariant given by

$$\tau^{SO(3)}\big(\Sigma(2,3,5)\big) = \frac{1}{1-q}\sum_{n=0}^{\infty} q^n (1-q^{n+1})(1-q^{n+2})\cdots(1-q^{2n+1}).$$

7.3 The volume conjecture

It is known (see Conjecture 7.10 and its remark) that the asymptotic behaviour of the quantum $SU(2)$ invariant $\tau_N^{SU(2)}(M)$ as $N \to \infty$ is a polynomial growth in N. Nevertheless, this asymptotic behaviour might be regarded as an exponential growth in the sense of the following conjecture, which is a 3-manifold version of the volume conjecture (Conjecture 1.19).

Conjecture 7.12 (H. Murakami [294]) *For any closed 3-manifold M,*

$$2\pi\sqrt{-1}\cdot \underset{N\to\infty}{\text{o-lim}} \frac{\log \tau_N^{SU(2)}(M)}{N} = \mathrm{CS}(M) + \sqrt{-1}\mathrm{vol}(M),$$

where $\mathrm{vol}(M)$ *and* $\mathrm{CS}(M)$ *denote the hyperbolic volume*[23] *and the Chern-Simons invariant*[24] *of M respectively, and o-*lim *denotes the "optimistic limit" introduced in [294].*

[23] When M is not hyperbolic, we define $\mathrm{vol}(M)$ to be $v_3||M||$, where $||M||$ is the simplicial volume and v_3 is the hyperbolic volume of the regular ideal tetrahedron.

[24] It is also conjectured (see Problem 7.16) that there exists an appropriate definition of $\mathrm{CS}(M)$ of any closed 3-manifold M, though $\mathrm{CS}(M)$ is defined only for hyperbolic 3-manifolds M at present.

Remark As mentioned in [294] the "definition" of the optimistic limit is not rigorous yet, because there is some ambiguity in the present definition, where formal approximation, such as (4) and (5), are used. It is a problem to find a rigorous formulation of the optimistic limit.

Remark It is shown [294], by using formal approximations, that Conjecture 7.12 is "true" for closed 3-manifolds obtained from S^3 by surgery along the figure-eight knot.

Remark R. Benedetti gave another formulation of the volume conjecture by using quantum hyperbolic invariants; see Conjecture 7.25.

Remark The statement of Conjecture 7.12 should extend for knot (link) complements M, which should be related to the volume conjecture for knots (Conjecture 1.21).

Remark By formally applying the (infinite dimensional) saddle point method to the Chern-Simons path integral, the value (42) appears at a critical point of the Chern-Simons functional. This might give a physical explanation of Conjecture 7.12. Can we justify it in mathematics? There is an approach, by using knotted trivalent graphs (see Conjecture 12.7), to justify the Chern-Simons path integral mathematically, which might be helpful to apply the saddle point method to it rigorously.

Problem 7.13 (H. Murakami) *Calculate* $o\text{-}\lim \frac{\log \tau_N^{SU(2)}(M)}{N}$ *for Seifert fibered 3-manifolds* M.

Remark When M is a mapping torus of a homeomorphism of a surface, a quantum invariant of M can be presented by the trace of the linear map on the quantum Hilbert space associated to the homeomorphism. Such a presentation might be useful to compute the asymptotic behaviour of $\tau_N^{SU(2)}(M)$.

Remark When we choose a simplicial decomposition of M, (the absolute value of) its quantum invariant can be expressed by using quantum $6j$-symbols. The computation of the asymptotic behaviour of $\tau_N^{SU(2)}(M)$ might be reduced to the computation of limits of quantum $6j$-symbols. J. Roberts [347] showed that a limit of classical $6j$-symbols is given by the Euclidean volume of a tetrahedron. Further, J. Murakami and M. Yano [302] recently showed that a limit of quantum $6j$-symbols is related to the hyperbolic volume of a tetrahedron via formal approximation such as (4) and (5).

Problem 7.14 (D. Thurston) *Find a series of invariants of a 3-manifold (depending on roots of unity) that grows as its hyperbolic volume (or its simplicial volume).*

Problem 7.15 (D. Thurston) *Find a correct generalization of the volume conjecture to other non-compact Lie groups.*

Remark The volume conjecture is related to the $SL(2,\mathbb{C})$ Chern-Simons theory, which (formally) deduces the hyperbolic volume and the Chern-Simons invariant. It is a problem to find (or formulate) such invariants of 3-manifolds for other non-compact Lie groups.

The *Chern-Simons functional* $\mathrm{CS}(A) \in \mathbb{C}$ of a $SL(2;\mathbb{C})$ connection A on a closed 3-manifold M is defined by the formula (37), where we regard A in the formula as a $sl(2;\mathbb{C})$-valued 1-form on M in this case. Since a gauge transformation of A changes $\mathrm{CS}(A)$ by an integer, $\mathrm{CS}([A])$ of the gauge equivalence class of A is defined to be in $\mathbb{C}/\mathbb{Z}$. The *Chern-Simons invariant* $\mathrm{CS}(M) \in \mathbb{R}/\mathbb{Z}$ and the volume $\mathrm{vol}(M) \in \mathbb{R}_{>0}$ of a closed hyperbolic 3-manifold M is given by[25]

$$\mathrm{CS}([A_0]) = \mathrm{CS}(M) + \sqrt{-1}\mathrm{vol}(M), \tag{42}$$

where $[A_0]$ is the gauge equivalence class of a $SL(2;\mathbb{C})$ flat connection A_0 associated to the conjugacy class of a holonomy representation $\pi_1(M) \to SL(2;\mathbb{C})$ of the hyperbolic structure on M. Further, when M is the complement of a hyperbolic knot (link) in a closed 3-manifold, $\mathrm{CS}(M)$ can be defined similarly.

Problem 7.16 (S. Morita [228]) *Define the Chern-Simons invariant* $\mathrm{CS}(M)$ *as a topological invariant of any closed oriented 3-manifold* M*, and of any knot (link) complement* M *in a closed 3-manifold.*

This problem includes two problems: to define $\mathrm{CS}(M)$ (topologically or combinatorially) as a topological invariant, and to define it for non-hyperbolic 3-manifolds.

Remark The hyperbolic volume (which is a counterpart of the Chern-Simons invariant) has a definition as a constant multiple of the simplicial volume, which is combinatorial, and can be applied, not only for hyperbolic 3-manifolds, but also for any other 3-manifolds.

[25] The Chern-Simons invariant was introduced by Chern and Simons [85] as an invariant of compact $(4n-1)$-dimensional Riemannian manifolds. For hyperbolic 3-manifolds, Meyerhoff [282] extended $\mathrm{CS}(M)$ for M with cusps. See also [310, 96] for $\mathrm{CS}(M)$ of hyperbolic 3-manifolds M as a counterpart of $\mathrm{vol}(M)$.

Remark (S. Kojima) The Chern-Simons invariant $\mathrm{CS}(M)$ of non-hyperbolic 3-manifolds M should be defined satisfying the following two requirements. One is that $\mathrm{CS}(-M) = -\mathrm{CS}(M)$, where $-M$ denotes M with the opposite orientation. The other is the requirement explained as follows. Let K be a hyperbolic knot in a 3-manifold N. Then, it is known that $N_{K;(p,q)}$ has a hyperbolic structure except for finitely many (p,q), where $N_{K;(p,q)}$ denotes the 3-manifold obtained from N by Dehn surgery along the slope of type (p,q), and that such hyperbolic structures can be obtained in a deformation space of the hyperbolic structures of $N-K$ parameterized by a natural complex parameter, which can be presented by two real parameters p and q. Moreover, the function

$$\mathrm{CS}(M) + \sqrt{-1}\mathrm{vol}(M) \tag{43}$$

is a holomorphic function of the complex parameter. Note that $\mathrm{vol}(M)$ can extend for non-hyperbolic 3-manifolds M by redefining it to be a constant multiple of the simplicial volume $||M||$. $\mathrm{CS}(M)$ should be defined such that, for appropriate[26] knots K in any closed 3-manifold N, the function (43) on the family $\{N_{K;(p,q)}\}_{p,q}$ can extend to a holomorphic function of a complex parameter presented by p and q appropriately.

Problem 7.17 (T. Ohtsuki) *Give a "complex structure" to the set of 3-manifolds. More precisely, find an embedding (or, an immersion) of the set of 3-manifolds to some complex variety such that its restriction to the set* $\{N_{K;(p,q)} \mid p^2+q^2 \gg 0\}$ *can be extended to a holomorphic map of the above mentioned complex parameter for any (hyperbolic) knot* K *in any 3-manifold* N.

We would expect some structures of the set of 3-manifolds such as mentioned in Problems 7.17 and Problem 10.16. Such structures would yield new viewpoints in the study of (the set of, and invariants of) 3-manifolds.

Remark As mentioned above, the set $\{N_{K;(p,q)} \mid p^2+q^2 \gg 0\}$ can be embedded in $\mathbb{C}$, on which the function (43) is holomorphic. In this sense, the infinite family of $N_{K,(p,q)}$ has a "complex structure" around the infinity point of (p,q). The volume conjecture says that the function (43) would be obtained as a certain limit of some series of quantum invariants. This suggests that the above "complex structure" would extend to the whole set of 3-manifolds.

[26] These knots should include, not only all hyperbolic knots, but also other knots. They might not include the trivial knot.

7.4 Quantum hyperbolic invariants of 3-manifolds

The main references for this section are [42, 43, 44], a review being [46]. In [44] the ideas of sections 7-9 in [43] are developed with some important differences in the way they are concretized.

Let W be a compact closed oriented 3-manifold, $L \subset W$ be a non-empty link, ρ be a flat principal B-bundle on W; B is the upper triangular Borel subgroup of $SL(2, \mathbb{C})$. In [43] one constructs a family of "quantum hyperbolic invariants" (QHI) $K_N(W, L, \rho) \in \mathbb{C}$, where $N > 1$ is any odd integer. This consists of two main steps:

(1) For every triple (W, L, ρ), the construction of so-called $\mathcal{D}$-*triangulations* $\mathcal{T} = (T, H, \mathcal{D})$, where: (T, H) is a (singular) triangulation of (W, L) such that each edge has distinct vertices and H contains all the vertices of T; the "decoration" $\mathcal{D}$ is made of a *full* simplicial B-1-cocycle representing ρ on W, a *branching* (for instance one induced by a total ordering of the vertices of T), and an *integral charge*. For these notions, see [46].

(2) The proof that a suitable state sum $H_N(\mathcal{T})$ does not depend on the choice of the $\mathcal{D}$-triangulation $\mathcal{T}$ up to multiplication by N-th roots of unity, so that $K_N(W, L, \rho) = K_N(\mathcal{T}) = H_N(\mathcal{T})^N$ actually defines an invariant.

The proof of the existence of $\mathcal{D}$-triangulations is difficult essentially due to strong global constraints in $\mathcal{D}$. The main building-blocks of the state sums $H_N(\mathcal{T})$ are the "quantum-dilogarithm" $6j$-symbols of the N-dimensional cyclic representations of a quantum Borel subalgebra of $U_\omega(sl(2, \mathbb{C}))$, where $\omega = \exp(2\pi i/N)$. Kashaev proposed in [195] a conjectural purely topological invariant $K_N(W, L)$ which should have been expressed by a state sum of this kind (although in his proposal there were no flat bundles and no notion of $\mathcal{D}$-triangulation); in fact, $K_N(W, L)$ appears as a special case of $K_N(W, L, \rho)$ when ρ is the trivial flat B-bundle on W. The algebraic properties of the $6j$-symbols ensure the invariance of $K_N(\mathcal{T})$ up to certain elementary moves on $\mathcal{D}$-triangulations. Then, the proof of the full invariance of $K_N(\mathcal{T})$ consists in connecting by such elementary moves any two $\mathcal{D}$-triangulations of (W, L, ρ), which is not so easy to achieve.

Problem 7.18 (S. Baseilhac, R. Benedetti) *Generalize the construction of the QHI for flat principal G-bundles, for Lie groups G different from B.*

Section 7.4 was written by S. Baseilhac and R. Benedetti.

Remark A basic ingredient of the B-QHI is the relationship between the cyclic representation theory of a quantum Borel subalgebra of $U_\omega(sl(2,\mathbb{C}))$, and flat B-bundles encoded by simplicial full 1-cocycles. This relationship relies on the theory of quantum coadjoint action of [105], which holds for other Lie groups such as $G = SL(2,\mathbb{C})$.

Problem 7.19 (S. Baseilhac, R. Benedetti) *Fix (W,L) and vary ρ. Study K_N as a function of the bundle, that is as a function defined on the character variety of W with respect to B: regularity, fibers, and so on.*

Remark Denote by z the B-1-cocycle in $\mathcal{T}$ that represents ρ. The state sum $K_N(\mathcal{T})$ is a rational function of the upper diagonal entries of the whole set of values of z. Moreover, the $6j$-symbols are rational functions of the moduli of the idealized triangulation $\widehat{F}(\mathcal{T})$ defined below.

Every $\alpha \in H^1(W;\mathbb{C})$ leads to two flat B-bundles ρ_α and ρ'_α defined as follows. The first one is obtained via the natural identification of $(\mathbb{C},+)$ with the parabolic subgroup $Par(B)$ of B. The second one is obtained by means of the exponential map of $(\mathbb{C},+)$ onto the multiplicative $\mathbb{C}^*$, and the identification of $\mathbb{C}^*$ with the diagonal Cartan subgroup $C(B)$ of B. Similarly, every class in $H^1(W;\mathbb{Z}/p\mathbb{Z})$ leads to a B-bundle by the natural embedding of $\mathbb{Z}/p\mathbb{Z}$ into the group $S^1 \subset \mathbb{C}^*$.

Problem 7.20 (S. Baseilhac, R. Benedetti) *Specialize Problem 7.19 to bundles coming from the ordinary cohomology as above. For real additive ones, analyze the behaviour of the QHI with respect to Thurston's norm. Are they constant on the faces of the corresponding unit sphere ?*

Remark The "projective invariance" property of the QHI (see [43, 46]) implies in particular that they are constant on the rays of $H^1(W;\mathbb{R})$.

Problem 7.21 (S. Baseilhac, R. Benedetti) *Understand the 'phase factor' (i.e. the ambiguity due to N-th roots of unity) of the state sum $H_N(\mathcal{T})$. Possibly derive from it an invariant for (W,L,ρ) endowed with some extra-structure, thus refining $K_N(W,L,\rho)$.*

Remark The phase factor uniquely depends on the branching and the integral charge in the decoration $\mathcal{D}$. On one hand, it is known that branchings can be used to encode, for instance, combings, framings, spin structures and so on. On another hand, combings induce the extra-structure that allows Turaev's refinement of Reidemeister torsions.

Problem 7.22 (S. Baseilhac, R. Benedetti) *Determine a suitable $(2+1)$ 'decorated' cobordism theory supporting a (non purely topological) QFT containing the already defined QHI. Study in particular the behaviour of the QHI with respect to connected sums.*

Problem 7.23 (S. Baseilhac, R. Benedetti) *Develop a 4-dimensional theory of QHI based on Turaev's shadow theory.*

Remark A first step should be to determine the right notion of $\mathcal{D}$-shadow together with a geometric interpretation. In this direction, F. Costantino is completing his PhD thesis at Pisa, where he shows in particular that 'branched shadows' do encode Spin^c structures.

Problem 7.24 (S. Baseilhac, R. Benedetti) *Determine the actual relationship between $K_N(S^3,\cdot)$ and the coloured Jones polynomial $J_N(\cdot)$ (evaluated at $\omega = \exp(2i\pi/N)$ and normalized by $J_N(\mathrm{unknot}) = 1$), as functions of links.*

Remark (1) In [296] it is shown that J_N may be defined by means of usual $(1,1)$-tangle presentations (as for the Alexander polynomial), using an enhanced Yang-Baxter operator whose R-matrix is derived from the quantum-dilogarithm $6j$-symbols. This suggests that there could be a relationship between $K_N(S^3,\cdot)$ (necessarily associated to the trivial flat B-bundle on S^3) and $J_N(\cdot)^N$. The most immediate guess would be that $K_N(S^3,L) = J_N(L)^N$ for each L. In fact, one can give an R-matrix formulation of $K_N(S^3,\cdot)$ involving R-matrices depending on parameters. These parameters are specified in terms of the decorations of special $\mathcal{D}$-triangulations adapted to planar link diagrams [45]. So $K_N(S^3,\cdot)$ can be computed by using suitably decorated link diagrams, and the decoration must satisfy non trivial global constraints. In this setup, $(1,1)$-tangle presentations do not play any role. On another side, the constant R-matrix used for J_N corresponds to one *fixed* particular choice in the parameters. This is not enough to confirm the above guess.

(2) A motivation of Problem 7.24 is also to make working for J_N a theory of scissors congruence classes, as described below for the QHI.

The so-called *Volume Conjectures* concern the asymptotic behaviour of the invariants constructed on the base of the quantum dilogarithm $6j$-symbols, that is of $K_N(W,L,\rho)$ or $J_N(L)$ (for $L \subset S^3$), when $N \to \infty$. They are originally motivated by the asymptotic behaviour of the quantum dilogarithm $6j$-symbols, whose dominant term involves dilogarithm functions that may be used to compute the volume of oriented ideal hyperbolic tetrahedra. In the case of $J_N(L)$ there are also some numerical computations (sometimes using formal manipulations) - see for instance the first section of the present volume for details. In the case of QHI, we develop in [43, 44] (see also [46]) a theory of scissors congruence classes for triples (W, L, ρ) which gives a natural framework for a formulation of a volume conjecture.

This goes roughly as follows. One constructs a 'Bloch-like' group $\mathcal{P}(\mathcal{D})$ based on $\mathcal{D}$-decorated tetrahedra, which maps via an explicit *idealization* map $\widehat{F}$ onto an enriched version $\mathcal{P}(\mathcal{I})$ of the classical Bloch group, built on hyperbolic ideal tetrahedra. Any $\mathcal{D}$-triangulation $\mathcal{T}$ of (W, L, ρ) leads to elements $\mathfrak{c}_{\mathcal{D}}(W, L, \rho) \in \mathcal{P}(\mathcal{D})$ and $\mathfrak{c}_{\mathcal{I}}(W, L, \rho) = \widehat{F}(\mathfrak{c}_{\mathcal{D}}(W, L, \rho)) \in \mathcal{P}(\mathcal{I})$. They are respectively called the $\mathcal{D}$- and $\mathcal{I}$-scissors congruence classes of (W, L, ρ). The QHI essentially depend on the $\mathcal{D}$-class, and for any given $\mathcal{D}$-triangulation $\mathcal{T}$ the $6j$-symbols occurring in $H_N(\mathcal{T})$ depend on the moduli of the hyperbolic tetrahedra of the idealization $\widehat{F}(\mathcal{T})$ of $\mathcal{T}$. By using the classical Rogers dilogarithm one can also define a *dilogarithmic invariant* $R(W, L, \rho)$ which only depends on the $\mathcal{I}$-class.

Conjecture 7.25 (S. Baseilhac, R. Benedetti) *(Real Volume Conjecture for QHI) For any triple (W, L, ρ) one has:*

$$\lim_{N \to \infty} (2\pi/N^2) \log(|K_N(W, L, \rho)|) = \operatorname{Im} R\big(\mathfrak{c}_{\mathcal{I}}(W, L, \rho)\big) .$$

Remark From the explicit formula of $H_N(\mathcal{T})$ one easily shows that the left-hand side of Conjecture 7.25, if it exists, only depends on the moduli of the hyperbolic tetrahedra of $\widehat{F}(\mathcal{T})$. A natural problem is to find a geometric interpretation of the dilogarithmic invariant. Indeed, for scissors congruence classes built with ideal triangulations of genuine (non-compact finite volume) hyperbolic 3-manifolds M, a similar dilogarithmic invariant gives $i(\mathrm{Vol}(M) + i\mathrm{CS}(M))$, where Vol is the Volume and CS is the Chern-Simons invariant (see [309]).

In [46] one proposes a complex version of Conjecture 7.25, for the whole K_N (not only its modulus).

7.5 Perturbative invariants

The *perturbative $SO(3)$ invariant* (or the *Ohtsuki series*) $\tau^{SO(3)}(M) = \sum_{d=0}^{\infty} \lambda_d (q-1)^d \in \mathbb{Q}[[q-1]]$ of a rational homology 3-sphere M is the invariant characterized by the property that $\sum_{d=0}^{k} \lambda_d (e^{2\pi\sqrt{-1}/r} - 1)^d$ for any k is congruent to

$\left(\frac{|H_1(M;\mathbb{Z})|}{r}\right) \tau_r^{SO(3)}(M)$ modulo r for infinitely many primes r; for a detailed definition see [319, 321]. (It is known, see [356, 166], that $\tau^{SO(3)}(M) \in \mathbb{Z}[[q-1]]$ for any integral homology 3-sphere M.) The perturbative PG invariant $\tau^{PG}(M)$ of a rational homology 3-sphere M, say, for $G = SU(N)$, is defined in $\mathbb{Q}[[q-1]]$ similarly, related to the quantum invariant $\tau_r^{PG}(M)$; see [245, 247].

Problem 7.26 *For each rational homology 3-sphere M, calculate $\tau^{SO(3)}(M)$ and $\tau^{PSU(N)}(M)$ for all degrees.*

Remark The value of $\tau_r^{SO(3)}\big(L(a,b)\big)$ of the lens space $L(a,b)$ is concretely calculated in [184, 134]. It follows from those values that

$$\tau^{SO(3)}\big(L(a,b)\big) = q^{-3s(b,a)} \frac{q^{1/2a} - q^{-1/2a}}{q^{1/2} - q^{-1/2}},$$

where we regard it as in $\mathbb{Q}[[q-1]]$ and $s(b,a)$ denotes the Dedekind sum.

Concrete presentations of $\tau^{SO(3)}(M)$ for Seifert fibered 3-manifolds M are given in [242].

Lawrence [241] has given holomorphic expression for the perturbative $SO(3)$ invariants of rational homology 3-spheres obtained by integral surgery along $(2,n)$ torus knot.

Habiro's expansion (45) gives a presentation of $\tau^{SO(3)}(M)$. See examples of Problem 7.31, for presentations of $\tau^{SO(3)}\big(\Sigma(2,3,5)\big)$ and $\tau^{SO(3)}\big(\Sigma(2,3,7)\big)$, which are due to [247]. See also [243] for a computation of $\tau^{SO(3)}\big(\Sigma(2,3,5)\big)$.

Remark From the value of $\tau_r^{PSU(N)}\big(L(a,b)\big)$ of the lens space $L(a,b)$ calculated in [377], we obtain

$$\tau^{PSU(N)}\big(L(a,b)\big) = q^{-N(N^2-1)s(b,a)/2} \frac{[1/a]^{N-1}[2/a]^{N-2}\cdots[(N-1)/a]}{[1]^{N-1}[2]^{N-2}\cdots[N-1]},$$

where we regard it as in $\mathbb{Q}[[q-1]]$ putting $[\alpha] = (q^{\alpha/2} - q^{-\alpha/2})/(q^{1/2} - q^{-1/2})$.

Takata [378] computed the quantum $PSU(N)$ invariant of Seifert fibered manifolds M. Concrete presentations of $\tau^{PSU(N)}(M)$ might follow from the computation.

Remark $\tau^{PSU(N)}(M)$ is recovered from the LMO invariant by

$$\tau^{PSU(N)}(M) = |H_1(M;\mathbb{Z})|^{-n(n-1)/2} \hat{W}_{sl_n}\big(\hat{Z}^{\mathrm{LMO}}(M)\big).$$

In particular, noting $PSU(2) = SO(3)$,

$$\tau^{SO(3)}(M) = |H_1(M;\mathbb{Z})|^{-1}\hat{W}_{sl_2}\big(\hat{Z}^{\mathrm{LMO}}(M)\big).$$

For details see [321]. In this sense Problem 7.26 is related to Problem 11.1.

Problem 7.27 (J. Roberts) *Explain the appearance of modular forms in the Witten invariants.*

Remark (J. Roberts) Lawrence and Zagier discovered in [243] that the perturbative series for the Poincaré homology sphere was close to a modular form. Is this a random coincidence, or is there a more systematic explanation? Does such a relation ever hold for a *hyperbolic* 3-manifold?

Problem 7.28 *Characterize those elements of $\mathbb{Z}[[q-1]]$ of the form $\tau^{SO(3)}(M)$ of integral homology 3-spheres M.*

Remark The degree $\leq d$ part of $\tau^{SO(3)}(M)$ can have any value in the degree $\leq d$ part of $\mathbb{Z}[[q-1]]$. Hence, it is meaningful to consider this problem for the form $\tau^{SO(3)}(M)$ for all degrees.

Remark Problem 7.28 is related to Problem 7.31, which is on the characterization of Habiro's expansion (45). See examples of Problem 7.31, for calculations of Habiro's expansions of $\tau^{SO(3)}\big(\Sigma(2,3,5)\big)$ and $\tau^{SO(3)}\big(\Sigma(2,3,7)\big)$.

Let q be an indeterminate, and let ζ be an r-th root of unity. Set

$$R_1 = \varprojlim{}_n \mathbb{Z}[q,q^{-1}]/\big((q-1)(q^2-1)\cdots(q^n-1)\big).$$

For an integral homology 3-sphere M, relations between $\tau_r^{SU(2)}(M)$ (which equals $\tau_r^{SO(3)}(M)$ for odd r, in this case) and $\tau^{SO(3)}(M)$ can be described in the following commutative diagram.

$$\begin{array}{ccccc} I^{sl_2}(M) \in R_1 & \xrightarrow{\text{injection}} & \mathbb{Z}[[q-1]] \subset \mathbb{Q}[[q-1]] \ni \tau^{SO(3)}(M) \\ \Big\downarrow{\scriptstyle \text{put } q=\zeta} & & \Big\downarrow{\scriptstyle \text{put } q=\zeta} \\ \tau_r^{SU(2)}(M) = \tau_r^{SO(3)}(M) \in \mathbb{Z}[\zeta] & \xrightarrow{\text{injection}} & \mathbb{Z}_r[\zeta] \end{array} \tag{44}$$

Here, the two horizontal maps are defined to be natural injections, and the two vertical maps are defined by substituting $q = \zeta$.

It was conjectured by Lawrence [239], and proved by Rozansky [356], that $\tau^{SO(3)}(M) \in \mathbb{Z}[[q-1]]$ for any integral homology 3-sphere M, and that the images of $\tau^{SO(3)}(M)$ and $\tau_r^{SO(3)}(M)$ coincide in $\mathbb{Z}_r[\zeta]$ in the above diagram for any odd prime power r. See [355] for their numerical examples.

Habiro [166] showed[27] that there exists an R_1-valued invariant $I^{sl_2}(M)$ of an integral homology 3-sphere M whose images in $\mathbb{Q}[[q-1]]$ and $\mathbb{Z}[\zeta]$ in the above diagram are equal to $\tau^{SO(3)}(M)$ and $\tau_r^{SU(2)}(M)$ respectively for any positive integer r. (Here we set $\tau_r^{SU(2)}(M) = 1$ for $r = 1, 2$.) This gives another proof of the above mentioned conjecture of Lawrence for integral homology 3-spheres. This also implies that $\tau^{SO(3)}(M)$ can be presented by

$$\tau^{SO(3)}(M) = \sum_{n=0}^{\infty} \lambda'_n (q-1)(q^2-1)\cdots(q^n-1) \tag{45}$$

with some $\lambda'_n \in \mathbb{Z}[q, q^{-1}]$ (in the above sense) such that

$$\tau_r^{SU(2)}(M) = \sum_{0 \le n < r} \lambda'_n (\zeta-1)(\zeta^2-1)\cdots(\zeta^n-1).$$

Note that the presentation (45) is not unique.

(K. Habiro) Let $\mathfrak{g}$ be a finite dimensional simple complex Lie algebra. Let $d \in \{1,2,3\}$ be such that $d = 1$ in the ADE cases, $d = 2$ in the BCF cases and $d = 3$ in the G_2 case. If M is a closed 3-manifold and if ζ is a root of unity of order r divisible by d, then the quantum $\mathfrak{g}$ invariant $\tau_\zeta^{\mathfrak{g}}(M) \in \mathbb{Q}[\zeta]$ of M at ζ is defined.

Conjecture 7.29 (K. Habiro, T. Le) *For each $\mathfrak{g}$ as above, there is a (unique) invariant $I^{\mathfrak{g}}(M) \in R_1$ of an integral homology 3-sphere M such that for each root of unity ζ of order r divisible by d we have*

$$I^{\mathfrak{g}}(M)\big|_{q=\zeta} = \tau_\zeta^{\mathfrak{g}}(M).$$

Remark When $(r, \det(a_{ij})) = 1$, where (a_{ij}) is the Cartan matrix of the Lie algebra $\mathfrak{g}$, the projective $\mathfrak{g}$-invariant $\tau_\zeta^{P\mathfrak{g}}(M)$ can be defined [247]. Then Habiro and Le also conjecture that $I^{\mathfrak{g}}(M)|_{q=\zeta} = \tau_\zeta^{P\mathfrak{g}}(M)$, if $(r, \det(a_{ij})) = 1$. Note that for an integral homology 3-sphere, $\tau_\zeta^{P\mathfrak{g}}(M) = \tau_\zeta^{\mathfrak{g}}(M)$ when both are defined (i.e. when r is divisible by d and $(r, \det(a_{ij})) = 1$). If this is the case, then we have

$$i\big(I^{\mathfrak{g}}(M)\big) = \tau^{\mathfrak{g}}(M)$$

[27] Hence, $\tau^{SO(3)}(M)$ is as powerful as the set of $\tau_r^{SU(2)}(M)$ for any integer $r \ge 3$, and as powerful as the set of $\tau_r^{SO(3)}(M)$ for any odd $r \ge 3$, for any integral homology 3-sphere M. Further, the LMO invariant dominates $\tau_r^{SU(2)}(M)$ for any integer $r \ge 3$.

where $\tau^{\mathfrak{g}}(M) \in \mathbb{Q}[[q-1]]$ is the perturbative $\mathfrak{g}$ invariant of M [247], and $i : R_1 \to \mathbb{Z}[[q-1]]$ is the upper injection in (44).

Remark The above conjecture implies that the quantum $\mathfrak{g}$ invariant $\tau^{\mathfrak{g}}_{\zeta}(M)$ of an integral homology sphere M takes values in the ring of cyclotomic integers $\mathbb{Z}[\zeta]$, and also that the perturbative invariant $\tau^{\mathfrak{g}}(M)$ takes values in $\mathbb{Z}[[q-1]]$.

Update Habiro and Le [167] proved Conjecture 7.29.

Conjecture 7.30 (K. Habiro) *Suppose that Conjecture 7.29 would hold. For a new indeterminate t, set*

$$R_1' = \varprojlim{}_n R_1[t]/((t-q)(t-q^2)\cdots(t-q^n))$$

Then there exists an invariant $I^{sl}(M) \in R_1'$ of an integral homology 3-sphere M such that $I^{sl}(M)|_{t=q^n} = I^{sl_n}(M)$ for any $n \geq 1$, where we set $I^{sl_1}(M) = 1$.

Problem 7.31 *Characterize those elements of Habiro's expansion (45) of $\tau^{SO(3)}(M)$ of integral homology 3-spheres M.*

Example For the Poincare homology 3-sphere $\Sigma(2,3,5)$ (obtained by surgery on a left-hand trefoil with framing -1) and the Brieskorn sphere $\Sigma(2,3,7)$ (obtained by surgery on a right-hand trefoil with framing -1), it is computed in [247] that

$$\tau^{SO(3)}\big(\Sigma(2,3,5)\big) = \frac{1}{1-q}\sum_{n=0}^{\infty} q^n(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{2n+1}),$$
$$\tau^{SO(3)}\big(\Sigma(2,3,7)\big) = \frac{1}{1-q}\sum_{n=0}^{\infty} q^{-n(n+2)}(1-q^{n+1})(1-q^{n+2})\cdots(1-q^{2n+1}).$$

See also [243] for a computation of $\tau^{SO(3)}\big(\Sigma(2,3,5)\big)$.

Remark Such an infinite sum as (45) would be interesting from the number theoretical viewpoint. For example,

$$1 + \sum_{n=1}^{\infty} q^n(q-1)(q^2-1)\cdots(q^n-1) = \sum_{\substack{k\in\mathbb{Z}\\ k\neq 0}} (-1)^{k+1} q^{\frac{3}{2}k^2-\frac{1}{2}k-1}.$$

A similar infinite sum appears in (12); see also [368].

8 Topological quantum field theory

The notion of topological quantum field theory (TQFT) was introduced in [16, 20], motivated by the operator formalism of a partition function in a quantum field theory which does not depend on the metric of the space. In the mathematical viewpoint, any quantum invariant of 3-manifolds can be formulated by a TQFT, which enables us to compute the invariant by the cut-and-paste method.

A *TQFT* is a functor which takes an oriented closed surface Σ to a finite dimensional complex vector space $V(\Sigma)$, and takes an oriented compact 3-manifold M with boundary Σ to a vector $Z(M) \in V(\Sigma)$, satisfying the following 5 axioms.

(1) $V(-\Sigma) = V(\Sigma)^\star$, where $-\Sigma$ denotes Σ with the opposite orientation and $V(\Sigma)^\star$ denotes the dual vector space of $V(\Sigma)$.

(2) $V(\Sigma_1 \sqcup \Sigma_2) = V(\Sigma_1) \otimes V(\Sigma_2)$, where $\Sigma_1 \sqcup \Sigma_2$ denotes the disjoint union of two surfaces Σ_1 and Σ_2.

(3) $V(\emptyset) = \mathbb{C}$, where $\emptyset$ denotes the empty surface.

(4) For 3-cobordisms M_1 and M_2 with $\partial M_1 = (-\Sigma_1) \sqcup \Sigma_2$ and $\partial M_2 = (-\Sigma_2) \sqcup \Sigma_3$ we have that $Z(M_1 \underset{\Sigma_2}{\cup} M_2) = Z(M_2) \circ Z(M_1)$ as linear maps[28] $V(\Sigma_1) \to V(\Sigma_3)$.

(5) $Z(\Sigma \times I)$ is equal to the identity map of $V(\Sigma)$.

To be precise, in many (but not in all) examples we need "extended 3-manifolds" instead of 3-manifolds to formulate a TQFT, where an *extended 3-manifold* is a 3-manifold M equipped with some kind of framing, e.g. a p_1-structure α on M (see [60])[29]. Namely, we extend the above definition of TQFT to a functor from the category of extended 3-cobordisms in an appropriate way (see [60]). Then, each quantum invariant can be formulated as a TQFT of the category of extended 3-cobordisms. In the remaining part of this section we call such a TQFT simply a TQFT.

The first version of the introductory part of Chapter 8 and Sections 8.1–8.4 was written by T. Ohtsuki, following seminar talks given by G. Masbaum. Based on it, G. Masbaum wrote this introductory part and these sections. Section 8.5 was written by T. Kerler.

[28] For a 3-cobordism M with $\partial M = (-\Sigma_1) \sqcup \Sigma_2$ the vector $Z(M)$ belongs to $V(-\Sigma_1 \sqcup \Sigma_2) = V(-\Sigma_1) \otimes V(\Sigma_2) = V(\Sigma_1)^\star \otimes V(\Sigma_2)$ by the axioms (1) and (2). Hence, $Z(M)$ can be regarded as a linear map $V(\Sigma_1) \to V(\Sigma_2)$.

[29] There is another formulation of a "framing" of a 3-manifold using signature cocycle; see [388].

8.1 Classification and characterization of TQFT's

To understand TQFT's is an important problem in order to investigate the 3-cobordism category, similarly as the representation theory is important in order to investigate groups and algebras.

Problem 8.1 *Find (and classify) all TQFT's.*

Remark The operator formalism of the Chern-Simons path integral suggests the existence of many TQFT's. It is known, see [388, 23], that a modular category is derived from a quantum group at a root of unity and a TQFT is derived from a modular category. The underlying 3-manifold invariant is called the *Reshetikhin-Turaev invariant.* Some other TQFT's might be obtained from quantum groupoids [314]. A TQFT for the LMO invariant is discussed in [301].

Another major construction of TQFT's is derived from sets of $6j$-symbols; for the construction see [392, 41]. When a set of $6j$-symbols arises from a subfactor, the underlying 3-manifold invariant is called the *Turaev-Viro-Ocneanu invariant* (see Section 9.4). Further, when a set of $6j$-symbols comes from a quantum group, such a TQFT is isomorphic to a tensor product of two TQFT's derived from the quantum group [388]. See Problems in Chapter 9 for concrete problems for TQFT's derived from $6j$-symbols.

There are TQFT's derived from finite groups, whose invariants are called the *Dijkgraaf-Witten invariants* [108]. Such TQFT's can alternatively be formulated by using certain sets of $6j$-symbols.

It is known [17] that the vector space $V(\Sigma)$ of a TQFT (V, Z) derived from a quantum group is isomorphic to the space of conformal blocks of a conformal field theory (CFT) of the Wess-Zumino-Witten model. Some other (possibly, "new") TQFT's might be obtained from the orbifold construction of CFT. It is a problem to understand TQFT's derived from the Rozansky-Witten invariant (see [350]); their isomorphism types might be described by known TQFT's, or they might be "new" TQFT's.

The following problem is a part of Problem 8.1 in the sense that some TQFT's are derived from modular categories, as mentioned in a remark after Problem 8.1.

Problem 8.2 *Find (and classify) all modular categories.*

For a TQFT (V, Z), put $P_{(V,Z)}(t) = \sum_{g=0}^{\infty} \big(\dim V(\Sigma_g)\big) t^g$, where Σ_g denotes a closed surface of genus g. The following problem is a refinement of Problem 8.1.

Problem 8.3
(1) Characterize the power series of the form $P_{(V,Z)}(t)$.
(2) For each power series $P(t)$ (satisfying the characterization of (1)), classify all TQFT's (V, Z) such that $P_{(V,Z)}(t) = P(t)$.

Remark A concrete form of such a power series for a TQFT derived from a quantum group is given by Verlinde formula [394]. For example, such a power series of the TQFT derived from $U_q(sl_2)$ at level k is presented by

$$\sum_{g=0}^{\infty} t^g \Big(\frac{k+2}{2}\Big)^{g-1} \sum_{j=1}^{k+1} \Big(\sin \frac{\pi j}{k+2}\Big)^{2-2g}.$$

8.2 Spin TQFT's

There are some refinements of TQFT's.

A *spin TQFT* is a TQFT on the category of spin 3-cobordisms, whose invariants depend on spin structures; such a TQFT can be formulated by extending the definition of a usual TQFT (see [61]). It is shown [61] that a spin TQFT can be obtained from the modular category of $U_q(sl_2)$ at level $k \equiv 2 \pmod 4$.

Problem 8.4 *Find other spin TQFT's.*

Remark Some examples of spin TQFT's can be constructed from the refined quantum invariants of [47, Theorem 6.2].

Remark A spin TQFT is expected to be a refinement of a usual TQFT in the sense that a spin TQFT (V^s, Z^s) should be related to a usual TQFT (V, Z) such that $V(\Sigma)$ for connected Σ can be described by the direct sum of $V^s(\Sigma, \sigma_\Sigma)$ over the spin structures σ_Σ on Σ (see [61]) and $Z(M)$ of a closed manifold M can be described by the sum of $Z^s(M, \sigma_M)$ over the spin structures σ_M on M.

A spin^c TQFT should be a TQFT on the category of spin^c 3-cobordisms, whose invariants depend on spin^c structures.

Problem 8.5 *Formulate and find spin^c TQFT's.*

Remark The Seiberg-Witten invariant (for its exposition see, e.g. [271]) and the torsion invariant τ (see [391]) are defined for closed 3-manifolds with spin^c structures. Are there TQFT's which are related to these invariants?

8.3 Homotopy QFT's

V. Turaev [389, 390] introduced and developed *HQFT* (homotopy QFT) with a target space X in dimension $d+1$.

Problem 8.6 (V. Turaev) (1) *Extend HQFT's to spin and spin^c settings.*
(2) *Find algebra structures behind spin and spin^c HQFT's in dimension* $1+1$.

Problem 8.7 (V. Turaev) *Study (spin and spin^c) HQFT's with the target space $K(H,2)$ in dimensions $1+1, 2+1$, and $3+1$ for $H=\mathbb{Z}^N$.*

Remark It is shown by V. Turaev that HQFT's with the target space $K(\pi,1)$ in dimension $1+1$ can be described by crossed π-algebras, and that any modular G-category gives rise to a HQFT with the target space $K(G,1)$ in dimension $2+1$ [390]. HQFT's with the target space $K(H,2)$ in dimension $1+1$ were studied and classified by M. Brightwell and P. Turner [66].

8.4 Geometric construction of TQFT's

Assume that the surface Σ is equipped with the structure of a smooth algebraic curve over $\mathbb{C}$. We denote by $H^0(\mathcal{M}_\Sigma, \mathcal{L}^{\otimes k})$ the space of sections of $\mathcal{L}^{\otimes k}$ on $\mathcal{M}_\Sigma$, where $\mathcal{M}_\Sigma$ is the moduli space of semi-stable rank N bundles with trivial determinant over Σ, and $\mathcal{L}$ is the determinant line bundle on $\mathcal{M}_\Sigma$. It is known that $H^0(\mathcal{M}_\Sigma, \mathcal{L}^{\otimes k})$ is isomorphic to $V(\Sigma)$ of a TQFT (V,Z) derived from the quantum group $U_q(sl_N)$ at a $(k+N)$-th root of unity. In this sense, $H^0(\mathcal{M}_\Sigma, \mathcal{L}^{\otimes k})$ gives a geometric construction of such a $V(\Sigma)$.

Problem 8.8 *Find a geometric construction of a TQFT using $H^0(\mathcal{M}_\Sigma, \mathcal{L}^{\otimes k})$. Namely, find a geometric way to associate a vector in $H^0(\mathcal{M}_\Sigma, \mathcal{L}^{\otimes k})$ to a 3-manifold M with $\partial M = \Sigma$.*

Remark In physics such a vector is obtained by applying an infinite dimensional formal analogue of the geometric invariant theory and the symplectic quotient to the Chern-Simons path integral; see [17]. It is a problem to justify this argument in some mathematical sense.

Here is a concrete problem which may be of interest in studying the relationship between $V(\Sigma)$ and $H^0(\mathcal{M}_\Sigma, \mathcal{L}^{\otimes k})$. The group $J^{(N)}(\Sigma)$ of N-torsion points on the Jacobian $J(\Sigma)$ acts on $\mathcal{M}_\Sigma$ by tensoring. This gives an action of a central extension $\mathcal{E}(\Sigma)$ of $J^{(N)}(\Sigma)$ on $H^0(\mathcal{M}_\Sigma, \mathcal{L}^{\otimes k})$.

Problem 8.9 (G. Masbaum) *Study this action of the finite group $\mathcal{E}(\Sigma)$ on $H^0(\mathcal{M}_\Sigma, \mathcal{L}^{\otimes k})$, and describe the induced decompositions of this vector space according to the characters of $\mathcal{E}(\Sigma)$. Also relate these decompositions to decompositions of $V(\Sigma)$ for the TQFT (V, Z) derived from the quantum group $U_q(sl_2)$ at a $(k+N)$-th root of unity.*

Remark This was done for $N = 2$ in [7].

Remark The group $J^{(N)}(\Sigma)$ is isomorphic to $H^1(\Sigma; \mathbb{Z}/N)$ and the extension $\mathcal{E}(\Sigma)$ is described using the Weil pairing, which corresponds to the intersection form on $H_1(\Sigma; \mathbb{Z}/N)$. For $N = 2$, an action of $\mathcal{E}(\Sigma)$ on the vector space $V(\Sigma)$ is described in [60, Section 7], and it was shown in [7] that $V(\Sigma)$ and $H^0(\mathcal{M}_\Sigma, \mathcal{L}^{\otimes k})$ are isomorphic as representations of $\mathcal{E}(\Sigma)$; here the torsion points on the Jacobian $J(\Sigma)$ correspond to simple closed curves on the surface Σ. For example, if $k \equiv 2 \mod 4$, one obtains decompositions indexed by spin structures (theta-characteristics) on Σ. For $N \geq 3$, the action of $\mathcal{E}(\Sigma)$ and the spin decompositions of $V(\Sigma)$ were constructed in [59].

Let $\mathfrak{M}_g$ denote the mapping class group of a closed surface Σ_g of genus g, and let $\tilde{\mathfrak{M}}_g$ denote its central extension (see [18, 275]) arising in the category of extended 3-cobordisms.

Problem 8.10 *For a given TQFT (V, Z), determine whether the image of $\tilde{\mathfrak{M}}_g$ in $End\big(V(\Sigma_g)\big)$ is finite.*

Remark Using physical arguments, Bantay [24] (see also references therein) showed that for every CFT the image of $\tilde{\mathfrak{M}}_1$ in $\mathrm{End}\big(V(S^1 \times S^1)\big)$ is finite. This had been rigorously proved by Gilmer [144] for the $SU(2)$ case.

In higher genus, it is known [132, 274] that the image of $\mathfrak{M}_g$ $(g \geq 2)$ is infinite in general.

Problem 8.11 (G. Masbaum) *Is there a relation between the Nielsen-Thurston classification of mapping classes of Σ_g and their images on $V(\Sigma_g)$ for TQFT's (V, Z)?*

Remark The Nielsen-Thurston classification says that any mapping class of a surface is either finite order, reducible, or pseudo-Anosov (see, e.g. [83]). It is known that a Dehn twist is taken to a matrix of finite order by any TQFT derived from a modular category of a quantum group. On the other hand, it is shown in [274] that a certain product of two non-commuting Dehn twists is taken to a matrix of infinite order in the $SU(2)$ TQFT at level k unless $k = 1, 2, 4, 8$.

8.5 Half-projective and homological TQFT's

In [145] it is shown that, for a restricted set of cobordisms, the Reshetikhin-Turaev TQFT at a prime p-th root of unity ζ_p can be defined, at least abstractly, as a functor $\mathcal{V}_p : \boldsymbol{Cob} \to \mathbb{Z}[\zeta_p]$-mod, meaning the category of *free* $\mathbb{Z}[\zeta_p]$-modules. Note that there is a well defined ring epimorphism $\mathbb{Z}[\zeta_p] \twoheadrightarrow \mathbb{F}_p[\mathsf{y}]/\mathsf{y}^{p-1}$, which sends $\zeta_p \mapsto 1 + \mathsf{y}$ and maps integer coefficients canonically onto the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Thus an endomorphism, which for a choice of basis of the free $\mathbb{Z}[\zeta_p]$-modules is given by a matrix with entries in $\mathbb{Z}[\zeta_p]$, will be represented by the same matrix with reduced coefficients now in $\mathbb{F}_p[\mathsf{y}]/\mathsf{y}^{p-1}$. Collecting the coefficients for each degree we can thus reexpress such a matrix as a sum of matrices over $\mathbb{F}_p$ multiplied with powers of y, or, more succinctly, use $\mathrm{Mat}(\mathbb{F}_p[\mathsf{y}]/\mathsf{y}^{p-1}) = \mathrm{Mat}(\mathbb{F}_p)[\mathsf{y}]/\mathsf{y}^{p-1}$. This means that in the ring-reduction the TQFT assigns to cobordisms a polynomial $\overline{\mathcal{V}}_p(M) = \sum_{j=0}^{p-2} \mathsf{y}^j \cdot \mathcal{V}_p^{[j]}(M)$, where each $\mathcal{V}_p^{[j]}(M)$ is a matrix over $\mathbb{F}_p$ and is well defined for given bases.

Recall also the notion of a half-projective TQFT with respect to an element $\mathsf{x} \in \mathsf{R}$ in the base ring, introduced in [210]. It is defined, by perturbing functoriality into $\mathcal{V}(N \circ M) = \mathsf{x}^{\mu(M,N)} \mathcal{V}(N)\mathcal{V}(M)$, where $\mu(M,N) = \mathrm{rank}\big(H_1(N \circ M) \xrightarrow{\delta} H_0(N \cap M)\big)$.

Problem 8.12 (T. Kerler) [Cyclotomic integer TQFT's]

(1) *Find explicit/computable bases for the $\mathcal{V}_p(\Sigma_g)$ as free modules over $\mathbb{Z}[\zeta_p]$.*

(2) *Show that $\mathcal{V}_p$ can be extended to all cobordisms as a half-projective TQFT with $\mathsf{x} = (\zeta_p - 1)^{\frac{p-3}{2}} \in \mathsf{R} = \mathbb{Z}[\zeta_p]$.*

(3) *Determine the structure of the* $\mathcal{V}_p^{[j]}(M)$ *and in how far they have lifts from* $\mathbb{F}_p$ *to* $\mathbb{Z}$, *analogous to the Ohtsuki invariants for closed 3-manifolds.*

(4) *Find a universal TQFT that combines all* $\mathcal{V}_p$, *at least perturbatively, into one.*

In the case of $p = 5$ the program for items (1)–(3) has been mostly carried out in [213], for primes $p \geq 7$ not much is known though. Some explicit bases have been found for genus $g = 1$ by Gilmer, but the situation for higher genera $g \geq 3$ is unknown. An immediate application of item (2) is that the quantum order, as introduced in [92], is also an upper bound for the cut-number of a 3-manifold. A closely related statement for (2) would also yield a very different proof for the fact that the Ohtsuki invariants are of finite type. In item (3) the "lift" must depend on p since the dimensions of the vector spaces do, and must also involve further quotients that arise since the irreducible TQFT's over $\mathbb{Z}$ do not match the required dimensions either, but they become reducible when reduced to $\mathbb{F}_p$. Item (4) is rather vague at this point, indicating for some sort of infinite filtered space with finite graded components.

Any TQFT $\mathcal{V} : \boldsymbol{\mathcal{C}ob} \to \mathsf{R}\text{-mod}$ implies a sequence of representation $\mathcal{V}_{[g]} : \Gamma_g \to \mathrm{GL}_{\mathsf{R}}(\mathcal{V}(\Sigma_g))$ of the mapping class groups. We say that a TQFT is *homological* if each of these representations factors through the quotient $\Gamma_g \twoheadrightarrow \mathrm{Sp}(2g, \mathbb{Z})$ (given by the action on $H_1(\Sigma_g)$), and we say it is *strictly homological* if each of the $\mathrm{Sp}(2g, \mathbb{Z})$-representations is algebraic, i.e. either faithful or zero. A particular example of strictly homological TQFT's over $\mathsf{R} = \mathbb{Z}$ are the Lefschetz components $\mathcal{V}^{(j)}$ of the Frohman-Nicas TQFT, see [128, 214]. From these we can generate a larger family $\boldsymbol{\mathcal{Q}}^0$ of such TQFT's by taking all direct sums of $\mathcal{V}^{(j)}$'s. For example all the TQFT's constructed in [110] lie in $\boldsymbol{\mathcal{Q}}^0$. An even larger family $\boldsymbol{\mathcal{Q}}^*$ is found by taking also tensor products and their irreducible summands.

Problem 8.13 (T. Kerler) [Homological TQFT's]

(1) *Find the irreducible components and ring structure (w.r.t* $\oplus$ *and* $\otimes$*) of* $\boldsymbol{\mathcal{Q}}^*$.

(2) *Determine whether all strictly homological TQFT's lie in* $\boldsymbol{\mathcal{Q}}^*$.

(3) *Identify the homological TQFT's that arise from the gauge theory of higher rank groups (such as* $PSU(n)$ *in [129]) with elements in* $\boldsymbol{\mathcal{Q}}^*$.

(4) *Identify the irreducible factors of the constant orders* $\mathcal{V}_p^{[0]}$ *of the cyclotomic integer expansion of the Reshetikhin-Turaev theory with elements in* $\boldsymbol{\mathcal{Q}}^*$.

The first item is in some sense about finding the representation ring of $\mathrm{Sp}(2,\mathbb{Z})\times \mathrm{Sp}(4,\mathbb{Z})\times\ldots\times\mathrm{Sp}(2g,\mathbb{Z})\times\ldots$ equipped with further generators and relations given by the standard handle attachments. The constraints given by the latter may be just good enough to ensure that the answer to item (2) is positive. The application of (3) is a better understanding and possibly a closed form for the polynomials from [129] that express the $PSU(n)$-invariants in terms of the coefficients of the Alexander polynomial. Evidence seems to suggest that the TQFT's from (4) stem from $\frac{p-3}{2}$-fold symmetric products of elements in $\mathcal{Q}^0$. A plausible corollary would be that for a closed manifold with $b_1(M)\geq 1$ we have

$$\mathcal{V}_p(M) \;=\; (\zeta_p-1)^{\frac{p-3}{2}}P_{\frac{p-3}{2}}(\lambda_{CWL}(M)) \;+\; \mathcal{O}((\zeta_p-1)^{\frac{p-1}{2}})\,, \qquad (46)$$

where λ_{CWL} is the Casson-Walker-Lescop invariant, and P_j is a polynomial of degree j with integer coefficients. (Note our normalization $\mathcal{V}_p(S^3)=1$). As remarked in [212] the identity in (46) is true for $p=5$ and general M with $b_1(M)\geq 1$. Moreover, work in progress shows that (46) holds also for general p if M is a torus-bundle over a circle.

The homological TQFT's are the starting point for a more general, perturbative view point on TQFT's that should parallel and extend that of the finite type theory of homology-3-spheres. At least for fixed p one can understand, for example, the Reshetikhin-Turaev theory as deformation of the $\mathcal{Q}^*$-theories. The notion that is somewhat parallel to that of finite type for closed 3-manifolds is what we shall call *finite length*. More precisely, the representations $\mathcal{V}_{[g]}:\Gamma_g\to \mathrm{GL}_{\mathsf{R}}(\mathcal{V}(\Sigma_g))$ of the mapping class groups extend linearly to homomorphisms $\mathcal{V}_{[g]}:\mathbb{Z}[\Gamma_g]\to\mathrm{End}_{\mathsf{R}}(\mathcal{V}(\Sigma_g))$. Denote by $I\mathcal{I}_g\subset\mathbb{Z}[\Gamma_g]$ the augmentation ideal of the Torelli group. The *length* of $\mathcal{V}$ is the maximal $L\in\mathbb{N}$ such that $\mathcal{V}_{[g]}((I\mathcal{I}_g)^{L+1})=0$. Clearly, the $L=0$-theories are just the homological ones. The $L=1$-theories can be thought of as elements of some $\mathrm{Ext}(\mathcal{V},\mathcal{W})$ with $\mathcal{V},\,\mathcal{W}\in\mathcal{Q}^*$. Restricted to representations of the Γ_g's they factor (in *char* $\neq 2$) through the Johnson-Morita-homomorphism $\Gamma_g\to\bigwedge^3 H_1(\Sigma_g)\rtimes\mathrm{Sp}(2g,\mathbb{Z})$, for which such extension are explicitly constructible [211].

Problem 8.14 (T. Kerler) [Length = 1 TQFT's]

(1) *Describe and construct algebraic $L=1$-extensions of Γ_g-representations to TQFT's, preferably as "simple" generalizations of the Frohman-Nicas-$U(1)$-theory.*

(2) *Produce a classification of $L=1$-TQFT's in the sense of an extension theory of $\mathcal{Q}^*$.*

(3) *Identify the Γ_g-representations on relative $SU(2)$-moduli space from [77] with these TQFT's, and find similar, higher rank theories.*

(4) *Identify the $\mathcal{V}_p^{[0]}$ as $L = 1$-theories, if possible.*

The conceivable generalizations of the TQFT construction of Frohman and Nicas described in (1) include using different, possibly non-compact gauge groups instead of $U(1)$ and using more refined versions of intersection homologies for stratified moduli spaces. Given the theory for $\mathcal{Q}^*$ the solution to item (2) will lead to well defined problems in $\mathfrak{sp}$-invariant theory. Constructions of $L = 1$-theories follow the schemes from (1) and (3). The identification in (4) is carried out for $p = 5$ in [211].

The notion of *finite length* can be refined into the notion of *q/l-solvable* introduced in [212], indicating a TQFT over $\mathsf{R} = \mathbb{M}[\mathsf{y}]/\mathsf{y}^{l+1}$ such that the constant order TQFT over the ground ring $\mathbb{M}$ is of length q. This, clearly, defines a special case of a TQFT of length $\leq (q \cdot l + q + l)$. Murakami's result [291] can be restated as saying that the Reshetikhin-Turaev theory gives rise to a 1/1-solvable TQFT $\mathcal{V}_p^{[\leq 1]}$ with ground ring $\mathbb{F}_p$ (i.e. a TQFT of length 3 over $\mathbb{F}_p[\mathsf{y}]/\mathsf{y}^2$) such that

$$\mathcal{V}_p^{[\leq 1]}(M) = 1 + \mathsf{y}\frac{1}{6}\lambda_{CWL}(M) \tag{47}$$

for any closed homology sphere M. Following Ohtsuki's work Murakami's identity (with some extra renormalizations by the order of $H_1(M)$) extends also to rational homology spheres. Let us call a theory with this property a TQFT of *Casson type*.

Recall, that the similar relation (46) for λ_{CWL} for manifolds with $b_1(M) \geq 1$ is already contained in the information of a homological ($L = 0$) TQFT, and is indeed a special evaluation of the Turaev-Milnor Torsion, see [212]. Given the richer structure of a 1/1-solvable TQFT we will expect new invariants Ξ that are refinements of λ_{CWL} and the torsion invariants.

To be more precise, note that for a pair (M, φ), where $\varphi : \pi_1(M) \twoheadrightarrow \mathbb{Z}$ defines a cyclic cover, any TQFT $\mathcal{V}$ yields an invariant $\mathcal{V}(M, \varphi) = trace(\mathcal{V}(C_\Sigma))$ where $C_\Sigma = \overline{M - \Sigma} : \Sigma \to \Sigma$ and $\Sigma \subset M$ is any surface dual to φ. In this way the Frohman Nicas theories $\mathcal{V}^{(j)}$ yields the coefficients of the Alexander Polynomial, and, as shown in [212], thus also λ_{CWL}.

A more refined invariant, which, roughly speaking, generalizes the Alexander module, is the Turaev-Viro module $\mathcal{M}_{TV}(M, \varphi)$. It is described by Gilmer in [143]. $\mathcal{M}_{TV}(M, \varphi)$ is given, up to conjugacy, by $\mathcal{V}(\Sigma)/\ker(\mathcal{V}(C_\Sigma)^N)$ (with

N large enough) together with the action of $\mathcal{V}(C_\Sigma)$ on it. The traces of $\mathcal{V}(C_\Sigma)$ or its powers are the most obvious well defined numerical invariants of $\mathcal{M}_{TV}(M,\varphi)$. The dimension of the module is yet another such invariant.

For a 1/1-solvable theory $\mathcal{V}$ the invariant $\mathcal{V}(M,\varphi)$ takes values in $\mathbb{M}[\mathrm{y}]/\mathrm{y}^2$ and can hence be written as $\mathcal{V}(M,\varphi) = \lambda^{\mathcal{V}}_{\varphi}(M) + \mathrm{y}\cdot\Xi^{\mathcal{V}}_{\varphi}(M)$, where $\lambda^{\mathcal{V}}$ and $\Xi^{\mathcal{V}}$ are now $\mathbb{M}$-valued invariants. If y coincides with the half projective parameter $\lambda^{\mathcal{V}}$ does not depend on φ, and we expect it to be some function of λ_{CWL}. Moreover, if $\mathcal{V}$ descends from a 1/2-solvable TQFT with the same property also $\Xi^{\mathcal{V}}$ would be independent of φ.

For the modular TQFT over $\mathbb{F}_5[\mathrm{y}]/\mathrm{y}^2$ obtained from the Reshetikhin Turaev theory this invariant has already been defined in [212], and we may expect it to lift, similarly, to an invariant $\Xi_{\mathbb{Z}}$ over $\mathbb{Z}$. For $p>5$ we expect, as in the case of λ_{CWL}, the next order terms in the expansions (46) of the Reshetikhin Turaev theories to be polynomial expressions in λ_{CWL} and $\Xi_{\mathbb{Z}}$.

Problem 8.15 (T. Kerler) [q/l-solvable and Casson TQFT's]

(1) *Lift the 1/1-solvable TQFT's of Casson type over $\mathbb{F}_p$ to a universal 1/1-solvable TQFT's of Casson type over $\mathbb{Z}$.*

(2) *Describe the* resulting *invariant $\Xi_{\mathbb{Z}}$ for 3-manifolds with $b_1(M)\geq 1$.*

(3) *Develop a perturbation theory for general q/l-solvable TQFT's.*

(4) *Relate those with the various, standard resolutions of Γ_g.*

(5) *Relate them also to the traditional finite type theory for closed 3-manifolds.*

(6) *Describe the Reshetikhin-Turaev theories in this pattern.*

Preparations for item (1) can be found in [212] in which formulae for the Casson invariant over $\mathbb{Z}$ are derived that have the same form as general TQFT formulae. Item (2) is immediate from the preceding discussion. The remaining items are logical continuations.

The category of 3-dim cobordisms $\mathcal{C}ob^\bullet$ between compact, oriented surfaces with one boundary component has a natural structure of a braided tensor category. Another, category $\mathcal{A}lg$ can be defined entirely algebraically in terms of generators and relations with respect to a tensor product and a composition product. On the level of objects it has exactly one generator, say A, so that all other objects are of the form $A^{\otimes g}$ with $1=A^{\otimes 0}$. The morphisms are given by all words that can be generated by taking composition and tensor products of elementary morphisms $m: A\otimes A\to A$, $\Delta: A\to A\otimes A$, $e: 1\to A$,

$\varepsilon : A \to 1, \ldots$, that appear in the definition of a braided, ribbon Hopf algebra with integrals and a non-degenerate pairing. For example, in [215] a surjective functor $\mathcal{A}lg \twoheadrightarrow \boldsymbol{\mathcal{C}ob}^{\bullet}$ is constructed, which, in the genus one restriction in fact an isomorphism.

Problem 8.16 (T. Kerler) [3-dim cobordisms from Hopf algebras]

(1) *Find further relations on $\mathcal{A}lg$, besides the ones arising from the axiomatics of Hopf algebras, that would make $\mathcal{A}lg \to \boldsymbol{\mathcal{C}ob}^{\bullet}$ an isomorphism.*

(2) *Find relations on $\mathcal{A}lg$ such that the maps*

$$\mathrm{Aut}_{\mathcal{A}lg}(A^{\otimes g}) \to \Gamma_g \cong \mathrm{Aut}_{\boldsymbol{\mathcal{C}ob}^{\bullet}}(\Sigma_{g,1})$$

are isomorphisms.

(3) *Relate this to obstructions, such as Steinberg and Whitehead groups, via stratified function spaces.*

(4) *What are the analogous algebraic structures in higher dimensions.*

The first problem is easily stated, but presumably very difficult as it implies a faithful translation of 3-dimensional topology into an algebraic gadget. In this respect it is vaguely parallel to the geometrization and Poincaré conjectures. The easier problem stated in item (2) can, in theory, be attacked head-on, given the known presentations of the mapping class groups. The third point hints to the fact that the generators in $\mathcal{A}lg$ correspond to Morse-theoretically elementary cobordisms, and the relations can be interpreted, similarly, in terms of handle slides and cancellation. This is, thus, reminiscent of the definitions of, e.g. Steinberg groups of 3-manifolds. The problem in item (4) is, again, easily stated but even in 4 dimensions lingers in almost complete total darkness. It is not hard to understand that higher category theory has to be invoked and not just one "object" A suffices as a "generator". Any partial answers may open the possibility of constructing functorial 4-manifold invariants by "linear representation" of such structures.

In [216] ETQFT's $\boldsymbol{\mathcal{V}}$ are defined as double functors from the double category of relative, 2-framed 1+1+1-dim cobordisms $\boldsymbol{\mathcal{C}ob}^*$ to the double category of linear, abelian categories over a perfect field. (The "E" stands for "extended to surfaces with boundaries"). Applied to a single circle, thought of as a 0-object in $\boldsymbol{\mathcal{C}ob}^*$, it yields an abelian category $\mathcal{C}_{\boldsymbol{\mathcal{V}}} = \boldsymbol{\mathcal{V}}(S^1)$, which we call the associated *circle category*. The main result of [216] is a construction of a $\boldsymbol{\mathcal{V}}_{\mathcal{C}}$, for each given *modular* tensor category $\mathcal{C}$ (meaning a bounded, ribbon, braided tensor category with some additional properties) such that $\mathcal{C}_{\boldsymbol{\mathcal{V}}_{\mathcal{C}}} = \mathcal{C}$.

The construction is made for all semisimple $\mathcal{C}$, and is extended, in the case of non-semisimple $\mathcal{C}$, to both to the situation of connected surfaces with boundary as well as disconnected, closed surfaces using the previously mentioned notion of half-projective TQFT's.

Problem 8.17 (T. Kerler) [Extended and half-projective TQFT's]

(1) *Describe in how far an ETQFT $\boldsymbol{\mathcal{V}}$ with circle category $\mathcal{C}$ can differ from $\boldsymbol{\mathcal{V}}_{\mathcal{C}}$, thus introducing a equivalence notion that would establish a bijective correspondence between the class of ETQFT's and the class of modular tensor categories.*

(2) *Find an extended notion of half-projectivity that includes also surfaces that are both disconnected and have boundary.*

(3) *Find constructions and axioms of ETQFT's that apply to more relaxed notions of boundedness or modularity.*

The functor $\mathcal{A}lg \to \boldsymbol{\mathcal{C}ob}^{\bullet}$ already imposes that a circle category $\mathcal{C}_{\boldsymbol{\mathcal{V}}}$ must fulfill about all axioms of a modular tensor category, and contain a Hopf algebra object with properties. Given some rigidity assumption it actually must be the same chosen in the construction of $\boldsymbol{\mathcal{V}}_{\mathcal{C}}$. What may still differ is the choice of algebra structures of the same object in the same category, which is thus the main source of possible ambiguities. Already in [216] it is clear that there are several choices. The correct axiomatics for item (2) should follow from a careful analysis of the double composition laws for surgery tangles from [216] and generalization of [210]. Item (3) is relevant to include more general notions of TQFT's as they would be of interest in the theory of finite type invariants.

The Reshetikhin-Turaev theory typically starts with non-semisimple modular category $\mathcal{C}$, typically the representation category of a non-semisimple quantum groups $U_q(\mathfrak{g})$, and then considers a canonical semisimple sub-quotient $\overline{\mathcal{C}}$, see [208]. Thus $\mathcal{V}_{\overline{\mathcal{C}}}$ yields a semisimple TQFT. It is known that this is different from the non-semisimple TQFT $\mathcal{V}_{\mathcal{C}}$, which in the case of a quantum group is obtained via the Hennings algorithm.

TQFT's can also be generated from a rigid, monoidal category $\mathcal{B}$ without any braiding. One way is to take the Drinfel'd double $D(\mathcal{B})$, which is then a modular category for some choice of ribbon element, and use $\mathcal{V}_{D(\mathcal{B})}$. For semisimple $\mathcal{B}$ one can also extract the 6j-symbol data and follow the Turaev-Viro construction to obtain a TQFT $\mathcal{W}_{\mathcal{B}}$.

Problem 8.18 (T. Kerler) [Non-semisimple vs. semisimple TQFT's, the double conjecture]

(1) *Clarify the difference in the content of $\mathcal{V}_{\overline{\mathcal{C}}}$ and $\mathcal{V}_{\mathcal{C}}$! Are there homological TQFT's $\mathcal{H}$ such that $\mathcal{V}_{\mathcal{C}}$ is in some essential way equivalent to $\mathcal{V}_{\overline{\mathcal{C}}} \otimes \mathcal{H}$?*

(2) *Find a construction of $\mathcal{W}_{\mathcal{B}}$ that generalizes the Turaev-Viro TQFT's to non-semisimple $\mathcal{B}$'s, similar to the way [216] generalized the Reshetikhin-Turaev construction. In the case of quantum groups and closed 3-manifolds this should reproduce a version of the Kuperberg invariant.*

(3) *What is the relation between $\mathcal{W}_{\mathcal{B}}$ and $\mathcal{V}_{D(\mathcal{B})}$? Are they in some sense isomorphic TQFT's?*

For the case of $U_q(\mathfrak{sl}_2)$ there is evidence from the genus=1 case that such an $\mathcal{H}$ is indeed given by the Frohman-Nicas-$U(1)$-theory. Item (2) is rather natural as a problem. As is apparent in [236] one may expect technical challenges requiring "minimal" cell decompositions of cobordisms, as opposed to general triangulations, as well as "combings" instead of framings.

The last conjecture appears also as Question 5 in [209] which was motivated by works of and discussions with D. Kazhdan and S. Gelfand in 1994. Since it is a rather nearby conjecture from a formal point of view it may have been posed already earlier. For categories arising from subfactors and closed manifolds results answering this conjecture have been obtained in [204]. As outlined in [209] further, more general results in this direction should yield a deeper understanding of both TQFT constructions involved as well as entail a topological picture for the Drinfel'd double construction.

9 The state-sum invariants of 3-manifolds derived from $6j$-symbols

Turaev and Viro [392] introduced a formulation of a state-sum invariant of 3-manifolds as a state-sum on triangulations of 3-manifolds derived from certain $6j$-symbols. After that, Ocneanu gave a general formulation of this state-sum for general $6j$-symbols and constructed 3-manifold invariants from subfactors based on this formulation. This general formulation was also given by Barrett and Westbury [41].

9.1 Monoidal categories, $6j$-symbols, and subfactors

Consider a collection, $\{V_i\}_{i\in I}$, of (irreducible) modules over $\mathbb{C}$ (of a quantum group or a subfactor) which is closed under tensor product, i.e. for any $i,j\in I$, $V_i\otimes V_j\cong\oplus_{k\in I}\mathcal{H}^k_{i,j}\otimes V_k$ for some $N^k_{i,j}$ dimensional vector space $\mathcal{H}^k_{i,j}$, which expresses the multiplicity of V_k in $V_i\otimes V_j$. Such a collection (with a certain property) is called a *monoidal category*, where each V_i is called a *simple object* of the category (for details see [23]). A monoidal category is provided by a certain set of representations of a quantum group (see, e.g. [201]), and also by a certain set of N-N bimodules arising from a subfactor $N\subset M$ (as explained below). The algebra spanned by I with the multiplication given by $a\cdot b=\sum_{c\in I}N^c_{a,b}c$ for $a,b\in I$ is called the *fusion rule algebra.*

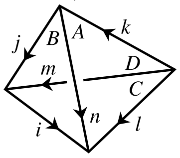
Let $\{V_i\}_{i\in I}$ be a monoidal category (with a finite set I) provided by a quantum group (at a root of unity) or a subfactor (of finite depth). Fix the above mentioned isomorphism $V_i\otimes V_j\cong\oplus_{k\in I}\mathcal{H}^k_{i,j}\otimes V_k$ for each i,j. Then, we have two bases of the vector space $\mathrm{Hom}(V_l,V_i\otimes V_j\otimes V_k)$ for each i,j,k,l as follows. Consider the maps

$$V_l\longrightarrow V_n\otimes V_k\longrightarrow(V_i\otimes V_j)\otimes V_k$$

determined by basis vectors $A\in\mathcal{H}^l_{n,k}$ and $B\in\mathcal{H}^n_{i,j}$. The composition of these maps gives a vector of $\mathrm{Hom}(V_l,V_i\otimes V_j\otimes V_k)$. Thus, we obtain a basis of this vector space consisting of vectors labeled by triples (n,A,B). Moreover, we obtain another basis consisting of vectors labeled by triples (m,C,D), where $C\in\mathcal{H}^l_{i,m}$, $D\in\mathcal{H}^m_{j,k}$, by considering the following maps,

$$V_l\longrightarrow V_i\otimes V_m\longrightarrow V_i\otimes(V_j\otimes V_k).$$

The introductory part of each section of Chapter 9 was written by T. Ohtsuki, following suggestions given by Y. Kawahigashi and J. Roberts.

The collection of the entries of the matrix which relates these two bases is a typical example of a set of $6j$-symbols, where a *set of $6j$-symbols* is defined to be a solution of certain polynomial equations: the tetrahedral symmetry, the unitarity, and the pentagon relation. Each $6j$-symbol is labeled by $i,j,k,l,m,n \in I$, and $A \in \mathcal{H}^l_{n,k}$, $B \in \mathcal{H}^n_{i,j}$, $C \in \mathcal{H}^l_{i,m}$, and $D \in \mathcal{H}^m_{j,k}$. This $6j$-symbol will be associated to a tetrahedron 

labeled by them.

A *subfactor* is a pair of infinite dimensional algebras N and M with an inclusion relation $N \subset M$ satisfying some property. A major class of subfactors is a class of *WZW model subfactors* of level $k = 1, 2, \cdots$, which are related to quantum groups. Another well-known class is a class of subfactors of the Jones index < 4; they are classified to be of types A_n, D_{2n}, E_6, or E_8. A left X right Y module Z is called a X-Y *bimodule*, and is written ${}_XZ_Y$. For a subfactor $N \subset M$, consider irreducible N-N bimodules appearing as direct summands of N-N bimodules in the following sequence,

$$ {}_NN_N, \qquad {}_NM_M \underset{M}{\otimes} {}_MM_N, \qquad {}_NM_M \underset{M}{\otimes} {}_MM_N \underset{N}{\otimes} {}_NM_M \underset{M}{\otimes} {}_MM_N, \qquad \cdots. $$

The collection of (isomorphism classes of) such irreducible modules provides a monoidal category $\{V_i\}_{i\in I}$. It is known that I is a finite set when the subfactor is of finite depth (this always holds when its index < 4). For a fusion rule algebra with a set of $6j$-symbols there exists a subfactor (if quantum dimensions are positive) such that the diagram in Figure 18 commutes. For details of this paragraph see [148, 118].

Thus, the following classification problems are almost equivalent. Each of them is fundamental, but probably impossibly hard. (See also Problem 8.2.)

Problem 9.1

(1) *Find (and classify) all semi-simple monoidal categories (with finitely many isomorphism classes of simple objects).*

(2) *Find (and classify) (finite dimensional) fusion rule algebras and sets of $6j$-symbols.*

(3) *Find (and classify) all subfactors (of finite depth).*

Remark Major sets of $6j$-symbols, what we call *quantum $6j$-symbols*, are the sets of $6j$-symbols derived from quantum groups, resp. WZW model subfactors.

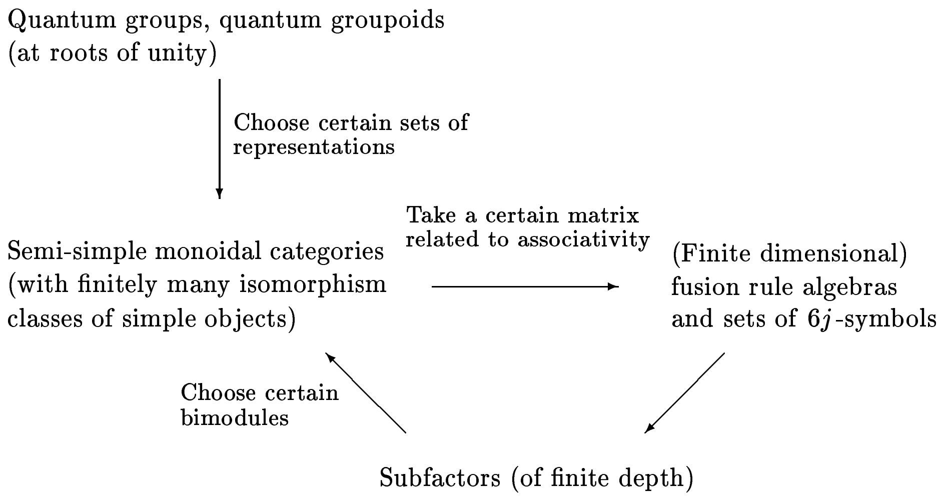
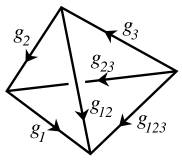


Figure 18: $6j$-symbols and related objects

Another class of $6j$-symbols is derived from finite groups; for a 3-cocycle α of a finite group G, a set of $6j$-symbols is given by

$W\Big($  $\Big) = \begin{cases} \alpha(g_1, g_2, g_3) & \text{if } g_{12}{=}g_1g_2,\ g_{23}{=}g_2g_3, \text{ and } g_{123}{=}g_1g_2g_3, \\ 1 & \text{otherwise,} \end{cases}$

where the tetrahedra is given a trivial face coloring. There are still other infinitely many sets of $6j$-symbols arising from subfactors; see Table 6. These $6j$-symbols might have a universal presentation given by a tetrahedron in the theory of knotted trivalent graphs (see Section 12.4).

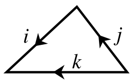
9.2 Turaev-Viro invariants and the state-sum invariants derived from monoidal categories

A state-sum invariant of 3-manifolds is defined by using such a set of $6j$-symbols with a monoidal category $\{V_i\}_{i\in I}$, as follows. Choose a simplicial decomposition of a closed 3-manifold M, and fix a total order of its vertices, which induces

[30] To be precise, the even part of the subfactor of type D_{2n} is braided, and its S-matrix is non-degenerate.
[31] This is trivially braided.

subfactor		monoidal category	S-matrix
WZW model subfactors of level $k = 1, 2, \cdots$ $SU(N)_k$, $SO(N)_k$, $Sp(N)_k$, $\cdots$		braided	non-degenerate
subfactors of index < 4	type A_n $(= SU(2)_n)$	braided	non-degenerate
	type D_{2n}	braided[30]	non-degenerate[30]
	type E_6, E_8	not braided	none
subfactors of index > 4: exotic subfactors, $\cdots$	(generalized) Haagerup, Asaeda-Haagerup, $\cdots$	not braided	none
	quantum doubles of Haagerup subfactor, $\cdots$	braided	non-degenerate
subfactors from —	3-cocycles of finite groups	not braided	none
	representations of finite groups	braided[31]	degenerate

Table 6: Subfactors, their monoidal categories, and S-matrices

orientations of edges. Further, choose an *edge coloring* λ, which is a map of the set of edges to I, and choose a *face coloring* φ, which is a collection of such assignments that a basis vector of $\mathcal{H}_{i,j}^k$ is assigned to a triangle 

with an edge coloring. To a tetrahedron σ with an edge coloring λ and a face coloring φ, we associate the above mentioned $6j$-symbol, which we denote by $W(\sigma; \lambda, \varphi)$. Then, a *state-sum invariant* of M is defined by

$$Z(M) = w^{-v} \sum_{\lambda} \Big(\prod_{E} \mu_{\lambda(E)} \Big) \sum_{\varphi} \prod_{\sigma} W(\sigma; \lambda, \varphi), \tag{48}$$

where the sums of λ and φ run over all edge colorings and all face colorings, and the products of E and σ run over all edges and all tetrahedra of the simplicial decomposition of M, and μ_i is a constant, which corresponds to a "quantum dimension", and $w = \sum_{i \in I} \mu_i^2$, and v is the number of vertices of the simplicial decomposition. It is known (see [41], [118, Chapter 12]) that the invariant (48) is a topological invariant of M. The definition of the invariant (48) can naturally be extended to an invariant of 3-manifolds with boundaries, and a TQFT can be formulated based on it.

In particular, for the set of $6j$-symbols arising from representations of the quantum group $U_q(Sl_2)$ at a root of unity, the invariant (48) is called *the Turaev-Viro invariant* [392]. In its definition it is not necessary to introduce face colorings (because $N_{i,j}^k$ is always equal to 0 or 1 for any i, j, k in this case) and orientations of edges (because each representation of $U_q(sl_2)$ is self-dual).

The monoidal category of a set of quantum $6j$-symbols is a modular category, and we can construct the Reshetikhin-Turaev invariant from it (see Section 9.3). The square of the absolute value of the invariant is equal to the value of the state-sum invariant derived from these $6j$-symbols.

The state-sum invariant derived from the set of $6j$-symbols given by a 3-cocycle α of a finite group G is called the *Dijkgraaf-Witten invariant* [108]. In particular, when $\alpha = 1$, it is equal to the number of conjugacy classes of representations $\pi_1(M) \to G$. It is further equal to the state-sum invariant derived from the set of $6j$-symbols obtained from the representations of the finite group G.

When a set of $6j$-symbols arises from a subfactor, the state-sum invariant derived from these $6j$-symbols is called the *Turaev-Viro-Ocneanu invariant.* There are infinitely many subfactors other than the above cases as shown in Table 6. The Turaev-Viro-Ocneanu invariants derived from such subfactors might be new invariants of 3-manifolds.

Problem 9.2 (Y. Kawahigashi) *Suppose we have a three-dimensional TQFT. Can we determine whether it arises from a fusion rule algebra and $6j$-symbols? If yes, can we describe all fusion rule algebras with $6j$-symbols producing the TQFT?*

Remark (Y. Kawahigashi) By a result of Ocneanu, we have at most only finitely many such fusion rule algebras with $6j$-symbols, up to equivalence of $6j$-symbols.

Problem 9.3 (Y. Kawahigashi) *Suppose we have two fusion rule algebras with $6j$-symbols and that two TQFT's arising from them are isomorphic. What relation do we have for the two sets of $6j$-symbols?*

Remark (Y. Kawahigashi) Are they equivalent in the sense of [362]?

Problem 9.4 (Y. Kawahigashi) *Suppose we have a TQFT arising from a fusion rule algebra with $6j$-symbols. Using a fusion rule subalgebra and $6j$-symbols restricted on it, we can construct another TQFT. What relation do we have for these TQFT's?*

Remark (Y. Kawahigashi) How about the case where the fusion rule subalgebra arises from α-induction? The α-induction produces a fusion rule algebra with $6j$-symbols from a semisimple ribbon category with finitely many isomorphism classes of simple objects and a specific choice of an object satisfying certain axioms. See [62], [222] and their references. If the original ribbon category is modular, we have some answer in [62], so it is particularly interesting when the S-matrix is not invertible.

9.3 The state-sum invariants derived from ribbon categories

A *ribbon category* is a monoidal category $\{V_i\}_{i\in I}$ equipped with a *braiding* $V\otimes W \to W\otimes V$ and a *twist* $V\to V$ for any objects V and W which are maps satisfying certain properties. We obtain an invariant of framed links from a ribbon category by associating a braiding to a crossing of a link diagram and a twist to a full-twist of a framing of a link. A monoidal category is called *semi-simple* if any object is isomorphic to a direct sum of simple ones. The *S-matrix* $S=(S_{ij})_{i,j\in I}$ of a semi-simple ribbon category $\{V_i\}_{i\in I}$ is defined by putting S_{ij} to be the invariant of the Hopf link whose components are associated with V_i and V_j. A *modular category* is a semi-simple ribbon category with finitely many isomorphism classes of simple objects whose S-matrix is invertible. We obtain the Reshetikhin-Turaev invariant of 3-manifolds and its TQFT from a modular category by using surgery presentations of the 3-manifolds. See [23] for details of this paragraph.

$$
\begin{tikzcd}
\text{Monoidal category} \arrow[r, Rightarrow] \arrow[d, "\begin{array}{l}+\text{ braiding}\\ +\text{ twist}\end{array}"] & \text{State-sum invariant of 3-manifolds and its TQFT} \\
\text{Ribbon category} \arrow[r, Rightarrow] \arrow[d, "\begin{array}{l}+\text{ semi-simple}\\ +\text{ finiteness of } I\\ +\text{ invertibility of } S\end{array}"] & \text{Invariant of framed links} \\
\text{Modular category} \arrow[r, Rightarrow] & \text{Reshetikhin-Turaev invariant of 3-manifolds and its TQFT}
\end{tikzcd}
$$

Figure 19: Monoidal, ribbon, modular categories and their consequences

The quantum $6j$-symbols are typical $6j$-symbols which induce modular categories. The square of the absolute value of the Reshetikhin-Turaev invariant derived from a modular category is equal to the value of the state-sum invariant derived from the category. It is suggested by Ocneanu that the monoidal category of the quantum double of each of such subfactors would be braided, and that the Reshetikhin-Turaev invariant derived from this quantum double would be equal to the Turaev-Viro-Ocneanu invariant derived from the original subfactor.

Problem 9.5 (Y. Kawahigashi) *Suppose we have a semisimple ribbon category C with finitely many isomorphism classes of simple objects. If the S-*

matrix is invertible, we can construct the Reshetikhin-Turaev invariant and the state-sum invariant from C and the latter is the square of the absolute value of the former. If the S-matrix is not invertible, do we still have a similar description of the state-sum invariant?

Remark See also Problem 9.11 for a similar problem for the Turaev-Viro-Ocneanu invariants.

Problem 9.6 (Y. Kawahigashi) *Suppose we have a semisimple ribbon category C_1 with finitely many isomorphism classes of simple objects, but the S-matrix is not invertible. Then we can construct a new modular category C_2 containing C_1 as a full subcategory by the "quantum double" construction [315, 316, 182], but there may be another extension of C_1 to a modular category. Theorem 2.13 in [315] claims that we have a "minimal" extension in an "essentially unique" way. Do we indeed have existence and certain uniqueness of such an extension? If so, what is the relation between the two TQFT's arising from C_1 and its minimal extension?*

Problem 9.7 (Y. Kawahigashi) *Suppose we have a semisimple ribbon category C_1 with a degenerate S-matrix as in Problem 9.6. By the method in [288], we can also make a modular tensor category C_2 from C_1. What is the relation between the two TQFT's arising from C_1 and C_2?*

Problem 9.8 (Y. Kawahigashi) *There are some fusion rule algebras with $6j$-symbols that do not seem to arise from quantum groups in [14] and more conjectured candidates of such examples in [160]. What are the corresponding TQFT's? Especially if the series conjectured in [160] does exist, it would give a parametrized family of TQFT's. Does a differentiation by a parameter (after a certain reparametrization) give a more interesting invariant, possibly of Vassiliev type?*

9.4 Turaev-Viro-Ocneanu invariants

The state-sum invariant of 3-manifolds derived from $6j$-symbols is called the *Turaev-Viro-Ocneanu invariant* when the set of $6j$-symbols arises from a subfactor. There are infinitely many subfactors other than those derived from quantum groups or finite groups. The Turaev-Viro-Ocneanu invariants derived from such subfactors might be new invariants of 3-manifolds.

(N. Sato) The Haagerup subfactor of Jones index $\frac{5+\sqrt{13}}{2}$ has the smallest index among finite depth subfactors with Jones index bigger than 4 and it is expected to have some "exotic" properties from the subfactor theoretical viewpoint. However, it does not seem so sensitive to classify 3-manifolds. The Turaev-Viro-Ocneanu invariant constructed from the Haagerup subfactor cannot distinguish lens spaces $L(5,1)$ and $L(5,2)$, as well as $L(7,1)$ and $L(7,2)$. On the other hand, generalized E_6-subfactors with the group symmetries $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$ can distinguish $L(3,1)$ and $L(3,2)$, $L(5,1)$ and $L(5,2)$, respectively.

Problem 9.9 (N. Sato) *Find a subfactor which can distinguish lens spaces $L(7,1)$ and $L(7,2)$. Moreover, find a subfactor to classify 3-manifolds as well as possible.*

In the lattice field theory, Ponzano and Regge [332] constructed a state sum model for $SU(2)$ and investigated an asymptotic behavior of the model.

Some infinite depth subfactors are manageable in the sense of growth rate (amenability). Such subfactors are called *strongly amenable.* The strong amenability condition might be enough to control the asymptotic behavior of the state sum model constructed from a strongly amenable subfactor.

Problem 9.10 (N. Sato) *Construct a well-defined state sum type invariant from a strongly amenable subfactor.*

Note that, unlike the Ponzano-Regge model, we do not have an asymptotic description of the quantum $6j$-symbols in general. (Recall that $6j$-symbols of $SU(2)$ have an asymptotic description.)

Let us consider the Turaev-Viro-Ocneanu invariant for a closed 3-manifold constructed from a subfactor. Then, this invariant can be considered as a Reshetikhin-Turaev type invariant constructed from a subfactor by passing the initial subfactor through the Longo-Rehren construction. If we start with a subfactor which has a non-degenerate braiding in particular, then this Turaev-Viro-Ocneanu invariant splits into a Reshetikhin-Turaev invariant and its complex conjugate. The following question will open a way to establish a theory of the minimal non-degenerate extension of a degenerate braiding.

Problem 9.11 (N. Sato) *Let us consider the Turaev-Viro-Ocneanu invariant from a subfactor with a degenerate braiding. Then, find a description of this invariant as a Reshetikhin-Turaev invariant.*

Remark See also Problem 9.5 for a similar problem for the state-sum invariants derived from ribbon categories.

10 Casson invariant and finite type invariants of 3-manifolds

10.1 Casson and Rokhlin invariants

It is known as Rokhlin theorem that the signature of a spin smooth closed 4-manifold is divisible by 16, which deduce the following definition of the Rokhlin invariant. For a closed 3-manifold M and a spin structure σ on M, the *Rokhlin invariant* $\mu(M,\sigma) \in \mathbb{Z}/16\mathbb{Z}$ is defined to be the signature of any smooth compact spin 4-manifold with spin boundary (M,σ). In particular, for a $\mathbb{Z}/2\mathbb{Z}$ homology 3-sphere M, the *Rokhlin invariant* $\mu(M) \in \mathbb{Z}/16\mathbb{Z}$ is defined to be the signature of any smooth compact spin 4-manifold with boundary M, noting that there exists a unique spin structure on such a M. The Casson invariant is a $\mathbb{Z}$-valued lift of the Rokhlin invariant of integral homology 3-spheres. Further, it is known [398] that

$$\mu(M) \equiv 4|H_1(M;\mathbb{Z})|^2\lambda_{\mathrm{CW}}(M) \equiv 8|H_1(M;\mathbb{Z})|\lambda_{\mathrm{CWL}}(M) \qquad (\mathrm{mod}\ 16)$$

for any $\mathbb{Z}/2\mathbb{Z}$ homology 3-sphere, where λ_{CW} denotes the Casson-Walker invariant[32] [398] and λ_{CWL} denotes the Casson-Walker-Lescop invariant[33] [251]. For an exposition of the Casson and Rokhlin invariants, see [221, 251, 363].

Problem 10.1 *Can the Casson invariant of an integral homology 3-sphere M be characterized by the signature of a certain 4-manifold bounded by M?*

Remark It is shown in [131] that the Casson invariant of the Seifert fibered homology 3-sphere $\Sigma(\alpha_1,\cdots,\alpha_n)$ is equal to $1/8$ times the signature of its Milnor fiber.

The Casson-Walker-Lescop invariant of closed 3-manifolds with positive Betti number can be computed from the torsion invariant τ of V. Turaev. He [391] gave a surgery formula for τ, which implies a surgery formula for the Casson-Walker-Lescop invariant.

Problem 10.2 (V. Turaev) *Relate this surgery formula for the Casson-Walker-Lescop invariant with that of Lescop [251].*

[32] The normalization here is that $\lambda_{\mathrm{CW}}(M) = 2\lambda_{\mathrm{C}}(M)$ for an integral homology 3-sphere M.

[33] The normalization here is that $\lambda_{\mathrm{CWL}}(M) = \big(|H_1(M;\mathbb{Z})|/2\big)\lambda_{\mathrm{CW}}(M)$ for a rational homology 3-sphere M.

(C. Lescop) In 1984, Casson defined his invariant of integral homology 3-spheres as an integer that "counts" the $SU(2)$-representations of their fundamental group in an appropriate way (see [3, 159]). Cappell, Lee and Miller [76] showed that the Casson way of counting $SU(2)$-representations of the π_1 works for any compact Lie group and provides other invariants of integral homology spheres.

Question 10.3 (C. Lescop) *Are the Cappell-Lee-Miller Casson-type $SU(n)$-invariants of finite type? If so, what are their degrees and their weight systems?*

Problem 10.4 (M. Polyak) *Define an invariant λ of a pair (M,σ) of a closed 3-manifold M and a spin structure σ on M such that*

$$\lambda_{\mathrm{CWL}}(M) = \sum_{\sigma} \lambda(M,\sigma)$$

for any closed 3-manifold M, where the sum runs over all spin structures σ on M.

Note that the set of spin structures on M is a torsor over $H^1(M;\mathbb{Z}/2\mathbb{Z})$ in the sense that differences of spin structures can be detected by cohomology classes in $H^1(M;\mathbb{Z}/2\mathbb{Z})$, while the set of spin^c structures on M is a torsor over $H^1(M;\mathbb{Z})$ in a similar sense.

Remark It is shown [326] that there exists an invariant $\hat{\theta}$ of a rational homology 3-sphere M associated with a spin^c structure α on M such that

$$\frac{1}{2}|H_1(M,\mathbb{Z})|\lambda_{\mathrm{CW}}(M) = \sum_{\alpha} \hat{\theta}(M,\alpha)$$

for any rational homology 3-sphere M, where the sum runs over all spin^c structures α on M. It is conjectured [326] that $\hat{\theta}$ is equal to Seiberg-Witten invariant for all rational homology 3-spheres.

Remark (M. Polyak) The Casson invariant is a lift of The Rokhlin invariant. We expect that $\lambda(M,\sigma)$ of Problem 10.4 should be a lift of $\mu(M,\sigma)$. How is $\sum_{\sigma}\mu(M,\sigma) \in \mathbb{Z}/16\mathbb{Z}$ related to $\lambda_{\mathrm{CWL}}(M)$?

It is known that this sum vanishes in $\mathbb{Z}/16\mathbb{Z}$ when $b_1(M) > 3$, while it is known [251] that $\lambda_{\mathrm{CWL}}(M) = 0$ when $b_1(M) > 3$.

Remark (C. Lescop) Let M be the 3-manifold obtained by surgery along a framed link L, and let W be the 4-manifold associated to the surgery presentation. Then, $24\lambda_{\mathrm{CWL}}(M) - 3|H_1(M;\mathbb{Z})|\mathrm{sign}W$ can be presented by a formula of Alexander polynomial coefficients and linking numbers of L [251, Formula 6.3.1], which might be helpful.

Note that the list of $\mu(M,\sigma)$ for a given M is richer than their sum $\sum_\sigma \mu(M,\sigma)$. For example, $\mu(\mathbb{R}P^3,\sigma) = 1, -1$ and $\mu(\mathbb{R}P^3\#(\text{Poincare sphere}),\sigma) = 7, 9$, while their sums are equal in $\mathbb{Z}/16\mathbb{Z}$.

Remark The invariant of Problem 10.4 should be related to the Goussarov-Habiro theory for spin 3-manifolds [276]. Recall that the Rokhlin and Casson invariants can be characterized as invariants under Y_2-equivalence and Y_3-equivalence among $\mathbb{Z}HS$'s respectively. It was shown [276] that Rokhlin invariant of spin closed 3-manifolds is the invariant under spin Y_2-equivalence among spin closed 3-manifolds. What is the invariant under spin Y_3-equivalence?

Remark The Casson-Walker invariant can be characterized as the first coefficient of the perturbative expansion of the quantum $SO(3)$ invariant $\tau^{SO(3)}(M)$ [291]. We have a spin refinement $\tau_r^{SU(2)}(M,\sigma)$ of the quantum $SU(2)$ invariant $\tau_r^{SU(2)}(M)$ for $r \equiv 0 \bmod 4$ such that

$$\tau_r^{SU(2)}(M) = \sum_\sigma \tau_r^{SU(2)}(M,\sigma),$$

where the sum runs over all spin structures σ on M [221]. We expect that $\lambda(M,\sigma)$ of Problem 10.4 should be related to the first coefficient of the perturbative expansion of $\tau_r^{SU(2)}(M,\sigma)$.

For $r \equiv 2 \bmod 4$, we have another refinement $\tau_r^{SU(2)}(M,\xi)$ for $\xi \in H_1(M;\mathbb{Z}/2\mathbb{Z})$ such that

$$\tau_r^{SU(2)}(M) = \sum_\xi \tau_r^{SU(2)}(M,\xi),$$

where the sum runs over all cohomology classes in $H^1(M;\mathbb{Z}/2\mathbb{Z})$. The first coefficient of the perturbative expansion of $\tau_r^{SU(2)}(M,\xi)$ was discussed in [292, 293]. It might be a problem to find a refinement $\lambda(M,\xi)$ of $\lambda_{\mathrm{CW}}(M)$ for some cohomology class ξ.

Remark Problem 10.4 is related to Problem 11.7, which is a problem to find a spin refinement of the LMO invariant, noting that the first coefficient of the LMO invariant is given by the Casson-Walker-Lescop invariant.

Question 10.5 (M. Polyak) *Is there a "Rokhlin invariant" of a pair (M, α) of a closed 3-manifold M and a spin^c structure α on M? (See Question 10.21.)*

Problem 10.6 (M. Polyak) *By presenting 3-manifolds by surgery along framed links in S^3, we can regard an invariant of 3-manifolds as an invariant of framed links. Establish a Gauss diagram formula for the link invariant derived from each finite type invariant of 3-manifolds.*

Remark (M. Polyak) The first step is to find a Gauss diagram formula for the Casson invariant. The Casson-Walker invariant as an invariant of 2-component links is studied in [223].

If we would obtain a Gauss diagram formula for the Casson-Walker-Lescop invariant, then a spin refinement of it (of Problem 10.4) would be obtained by decorating the Gauss diagram formula by characteristic sublinks, noting that the spin structures on the 3-manifold obtained by surgery along a framed link L can be presented by characteristic sublinks of L (see [221]).

10.2 Finite type invariants

A link in an integral homology 3-sphere is called *algebraically-split* if the linking number of any pair of its components vanishes, and is called *boundary* if all its components bound disjoint surfaces. A framed link is called *unit-framed* if the framings of its components are ± 1. Let $\mathbb{M}$ be the set of (homeomorphism classes of) oriented integral homology 3-spheres, and let R be a commutative ring with 1. For an algebraically-split unit-framed link L in an integral homology 3-sphere M, we put

$$[M, L] = \sum_{L' \subset L} (-1)^{\#L'} M_{L'} \in R\mathbb{M},$$

where the sum runs over all sublinks L' of L, and $\#L'$ denotes the number of components of L', and $M_{L'}$ denotes the 3-manifold obtained from M by surgery along L'. Let $\mathcal{F}_d^{\mathrm{as}}(R\mathbb{M})$ [318] (resp. $\mathcal{F}_d^{\mathrm{b}}(R\mathbb{M})$ [135]) denote the submodule of $R\mathbb{M}$ spanned by $[M, L]$ such that M is an integral homology 3-sphere and L is a unit-framed algebraically-split link L with d components in M (resp. a unit-framed boundary link L in M). Let $\mathcal{F}_d^{\mathrm{Y}}(R\mathbb{M})$ [137] denote the submodule of $R\mathbb{M}$ spanned by $[M, G]$ such that M is an integral homology 3-sphere and G is a collection of d disjoint Y-graphs (see Figure 11) in M, where $[M, G]$ is defined similarly as $[M, L]$ (see [137]).[34] A homomorphism $v : R\mathbb{M} \to R$ is

[34] $\mathcal{F}_d^{\mathrm{Y}}(R\mathbb{M})$ can alternatively be defined by using blinks [140]; see [137].

called a *finite type invariant* of $\mathcal{F}^{\mathrm{as}}_\star$-degree d (resp. $\mathcal{F}^{\mathrm{b}}_\star$-degree d, or $\mathcal{F}^{\mathrm{Y}}_\star$-degree d) if v vanishes on $\mathcal{F}^{\mathrm{as}}_{d+1}(R\mathbb{M})$ (resp. $\mathcal{F}^{\mathrm{b}}_{d+1}(R\mathbb{M})$, or $\mathcal{F}^{\mathrm{Y}}_{d+1}(R\mathbb{M})$). It is known [141] that

$$\mathcal{F}^{\mathrm{as}}_{3d}(\mathbb{QM}) = \mathcal{F}^{\mathrm{as}}_{3d-1}(\mathbb{QM}) = \mathcal{F}^{\mathrm{as}}_{3d-2}(\mathbb{QM})$$

and that there is an isomorphism

$$\mathcal{A}(\emptyset;\mathbb{Q})^{(d)} \longrightarrow \mathcal{F}^{\mathrm{as}}_{3d}(\mathbb{QM})/\mathcal{F}^{\mathrm{as}}_{3d+3}(\mathbb{QM})$$

between vector spaces [141, 244]. It is known [137] that

$$\begin{aligned}
&\mathcal{F}^{\mathrm{b}}_{d}(\mathbb{ZM}) \supset \mathcal{F}^{\mathrm{Y}}_{2d}(\mathbb{ZM}), \qquad \mathcal{F}^{\mathrm{as}}_{3d}(\mathbb{ZM}) \supset \mathcal{F}^{\mathrm{Y}}_{2d}(\mathbb{ZM}),\\
&\mathcal{F}^{\mathrm{as}}_{3d}(R\mathbb{M}) = \mathcal{F}^{\mathrm{b}}_{d}(R\mathbb{M}) = \mathcal{F}^{\mathrm{Y}}_{2d}(R\mathbb{M}),\\
&\mathcal{F}^{\mathrm{Y}}_{2d-1}(R\mathbb{M}) = \mathcal{F}^{\mathrm{Y}}_{2d}(R\mathbb{M})
\end{aligned}$$

if $1/2 \in R$.

10.2.1 Torsion and finite type invariants

Conjecture 10.7 *$\mathcal{F}^{\mathrm{as}}_{d}(\mathbb{ZM})/\mathcal{F}^{\mathrm{as}}_{d+1}(\mathbb{ZM})$ (resp. $\mathcal{F}^{\mathrm{b}}_{d}(\mathbb{ZM})/\mathcal{F}^{\mathrm{b}}_{d+1}(\mathbb{ZM})$) is torsion free for each d.*

Remark (K. Habiro) The group $\mathcal{F}^{\mathrm{Y}}_{d}(\mathbb{ZM})/\mathcal{F}^{\mathrm{Y}}_{d+1}(\mathbb{ZM})$ has 2-torsion for each $d > 0$.

Conjecture 10.8 *$\mathcal{A}(\emptyset;\mathbb{Z})$ is torsion free.*

10.2.2 Do finite type invariants distinguish homology 3-spheres?

Conjecture 10.9 *Finite type invariants distinguish integral homology 3-spheres. (See Conjecture 11.2.)*

10.2.3 Dimensions of spaces of finite type invariants

A finite type invariant v is called *primitive* if $v(M_1 \# M_2) = v(M_1) + v(M_2)$ for any integral homology 3-spheres M_1 and M_2. We denote by $\mathcal{A}(\emptyset;R)_{\mathrm{conn}}$ the submodule of $\mathcal{A}(\emptyset;R)$ spanned by Jacobi diagrams with connected trivalent graphs. As a graded vector space $\mathcal{A}(\emptyset;\mathbb{Q})$ is isomorphic to the symmetric tensor algebra of $\mathcal{A}(\emptyset;\mathbb{Q})_{\mathrm{conn}}$.

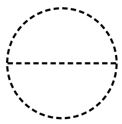
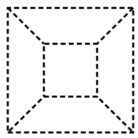
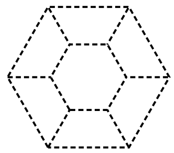
Problem 10.10 *Determine the dimension of the space of primitive finite type invariants of integral homology 3-spheres of each degree d. Equivalently, determine the dimension of the space $\mathcal{A}(\emptyset;\mathbb{Q})^{(d)}_{conn}$ for each d.*

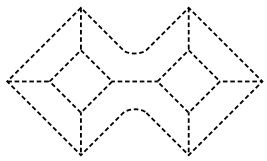
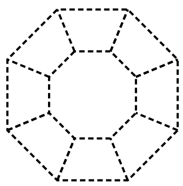
d	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
prime diag.	0	1	0	0	1	0	1	1	1	1	2	1			
dim $\mathcal{A}(\emptyset)^{(d)}_{\text{conn}}$	0	1	1	1	2	2	3	4	5	6	8	9	≥ 11	≥ 13	≥ 15
dim $\mathcal{A}(\emptyset)^{(d)}$	1	1	2	3	6	9	16	25	42	65	105	161	≥ 254	≥ 386	≥ 595

Table 7: Some dimensions for Problem 10.10

Remark $\mathcal{A}(\emptyset;\mathbb{Q})^{(d)}_{\text{conn}}$ is isomorphic to $\mathcal{B}^{(d+1,2)}_{\text{conn}}$ mentioned in a remark of Problem 2.12, by the isomorphism taking a trivalent graph to a uni-trivalent graph obtained from the trivalent graph by cutting a middle point of an edge. Hence, the dimension of $\mathcal{A}(\emptyset;\mathbb{Q})^{(d)}_{\text{conn}}$ is equal to the dimension $\beta_{d+1,2}$ of $\mathcal{B}^{(d+1,2)}_{\text{conn}}$. Therefore, we obtain the row of $\mathcal{A}(\emptyset;\mathbb{Q})^{(d)}_{\text{conn}}$ in Table 7 from a column of Table 2.

Remark $\mathcal{A}(\emptyset)_{\text{conn}}$ is an algebra with the product given by connected sum of Jacobi diagrams. Let us look for prime diagrams with respect to the connected sum; they generate the algebra $\mathcal{A}(\emptyset)_{\text{conn}}$. By the AS and IHX relations, we can remove a triangle, and we can break a polygon with odd edges. Hence, prime diagrams are given by

$p_1 =$ , $p_4 =$ , $p_6 =$ ,

$p_7 =$ , $p_8 =$ , $\cdots$.

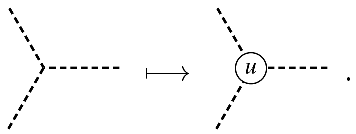
They have the relation $p_1 p_7 = p_4^2$, since $p_1 p_7 = p_1(x_3 p_4) = (x_3 p_1) p_4 = p_4^2$, where x_3 is the element of Vogel's algebra Λ given in (49) below, which acts on $\mathcal{A}(\emptyset)_{\text{conn}}$. It is a problem to find a complete list of generators and relations of the algebra $\mathcal{A}(\emptyset)_{\text{conn}}$.

Remark $\mathcal{A}(\emptyset;\mathbb{Q})_{\text{conn}}$ is a Λ-algebra, where Λ is Vogel's algebra given below, whose generators and relations have been known in degree ≤ 10; see a remark on Problem 10.11. It is a problem to find generators and relations of $\mathcal{A}(\emptyset;\mathbb{Q})_{\mathrm{conn}}$ as a Λ-algebra.

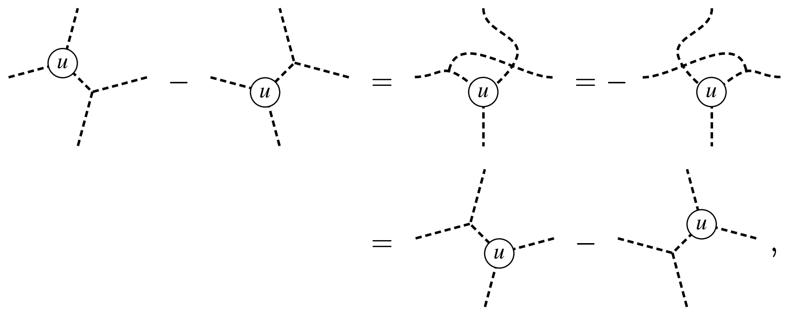
Update The prime diagrams of degree ≤ 11 are given in [88].

10.2.4 Vogel's algebra

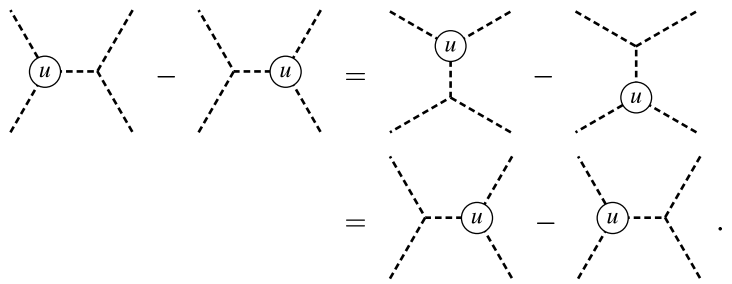
Vogel's algebra [395] is defined as follows. For fixed 3 points, we denote by $\mathcal{A}(3\text{ points})_{\mathrm{conn}}$ the module over $\mathbb{Q}$ spanned by vertex-oriented connected uni-trivalent graphs whose univalent vertices are the fixed 3 points subject to the AS and IHX relations. The symmetric group $\mathfrak{S}_3$ acts on $\mathcal{A}(3\text{ points})_{\mathrm{conn}}$ by permutation of 3 points. The module Λ is defined to be the submodule of $\mathcal{A}(3\text{ points})_{\mathrm{conn}}$ consisting of all elements u satisfying that $\sigma(u) = \mathrm{sgn}(\sigma)\cdot u$ for any $\sigma\in\mathfrak{S}_3$. It is well defined to insert $u\in\Lambda$ in a vertex-oriented trivalent vertex as



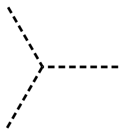
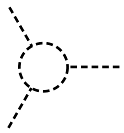
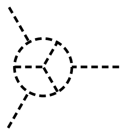
Moreover, this insertion is independent, modulo the AS and IHX relations, of a choice of a trivalent vertex as follows. By the AS and IHX relations,



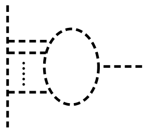
where the middle equality is derived from the anti-symmetry of u. By the $\pi/4$ and $-\pi/4$ rotations of the above formula, we have that



Hence, the left hand side of the above formula is equal to 0. This implies that the insertion of u is independent of a choice of a trivalent vertex. The module Λ is an algebra, called *Vogel's algebra*, whose product of $x, y \in \Lambda$ is defined to be the element of Λ obtained by inserting x in a trivalent vertex of y. It is a commutative algebra. Some generators of Λ in low degrees are given by

$1 =$  , $t =$  , $x_3 =$  , (49)

and further,

$x_n =$  ,

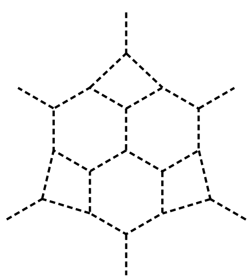
having n horizontal lines between the vertical line and the circle. It is known that the even x_n's can be presented by odd x_n's.

Problem 10.11 *Describe Vogel's algebra Λ, say, by giving complete sets of generators and relations of Λ.*

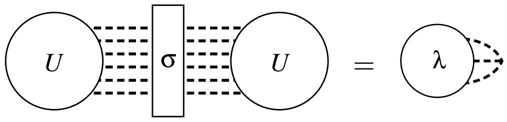
Remark Vogel [396] conjectured that the homomorphism $\varphi : R_0 \to \Lambda$ given in [396] was bijective, where R_0 is the subalgebra of a polynomial algebra in 3 variables, generated by elements given in [396]. As mentioned in [396], φ has been known to be bijective in degree ≤ 10, and injective in degree ≤ 15.

Recently (in June, 2001), Vogel found a polynomial in R_0 whose image in Λ vanishes; this implies that φ is not injective. Surjectivity of φ (which implies that Λ is generated by $t, x_3, x_5, x_7, \cdots$) is still an open problem.

Vogel [397] further found a divisor of zero in Λ. It is given as follows. Putting

$U =$  , we define $W \in \mathcal{A}(\emptyset)$ and $\lambda \in \Lambda$ by

$$W = \sum_{\sigma} \mathrm{sign}(\sigma)$$



,

where the sum of σ runs over all permutations $\sigma \in \mathfrak{S}_6$, and λ is obtained from W by removing a neighborhood of a trivalent vertex. Vogel showed that $t\lambda = 0 \in \Lambda$ and $\lambda \neq 0 \in \Lambda$.

10.2.5 Other problems

Problem 10.12 *Find a constructive combinatorial presentation of each finite type invariant of integral homology 3-spheres, and, in particular, of the Casson invariant, by localizing configuration space integrals.*

Remark The perturbative expansion of the path integral of the Chern-Simons field theory suggests that each Vassiliev invariant of knots can be obtained as a mapping degree of a certain map on a configuration space, whose localization deduces a Gauss diagram formula of this Vassiliev invariant; see comments before Problem 3.11. In the 3-manifold case G. Kuperberg and D. Thurston [237] gave a presentation of each finite type invariant by using configuration space integrals, whose localization might deduce a combinatorial formula, similarly as a Gauss diagram formula. It would be a difficult point of such localization to deal with "hidden strata" (anomaly faces).

Problem 10.13 (J. Roberts) *What is the space of* 3*-manifolds?*

Remark (J. Roberts) Vassiliev invariants are usually characterised in purely combinatorial terms, but it is worth remembering that Vassiliev was led to this definition by considering the natural stratification of the space of smooth maps $S^1 \to \mathbb{R}^3$. The combinatorial theory of finite type invariants of homology spheres is now equally well-developed but there remains no natural justification for considering the relations introduced by Ohtsuki, other than that these turn out to interact very well with the perturbative expansion of the Witten invariants. One would like to find a stratified space of integer homology spheres, in which crossing a codimension 1 stratum corresponds to doing ± 1 surgery on a knot. Now the space of smooth maps $f : S^{n+3} \to S^n$ is a natural choice for a "space of framed 3-manifolds", via the Pontrjagin-Thom construction (take the preimage of a fixed point in S^n). But this space gives the wrong filtration, and it's not clear how to alter it to implement (for example) constraints on the homology of the preimages. See Shirokova [365].

10.3 Goussarov-Habiro theory

10.3.1 Goussarov-Habiro theory for 3-manifolds

Related to finite type invariants of 3-manifolds, equivalence relations among 3-manifolds have been studied by Goussarov [152, 153] and Habiro [165], which is called the Goussarov-Habiro theory for 3-manifolds. These equivalence relations are helpful for us to study structures of the set of 3-manifolds.

The Y_d*-equivalence*[35] among oriented 3-manifolds is the equivalence relation generated by either of the following relations,

(1) surgery on a tree clasper with d trivalent vertices [165],

(2) Goussarov's d-variation (which generates Goussarov's notion of $(d-1)$-equivalence) [152, 153],

(3) surgery by an element in the dth lower central series subgroup of the Torelli group of a compact connected surface.

It is known [165] that these relations generate the same equivalence relation among $\mathbb{Z}HS$'s. Two closed 3-manifolds M and M' are Y_1-equivalent if and only if there is an isomorphism $H_1(M;\mathbb{Z}) \to H_1(M';\mathbb{Z})$ which induces an isomorphism between their linking pairings [278].

It is known [165] that {integral homology 3-spheres ($\mathbb{Z}HS$'s)}$/\underset{Y_2}{\sim} \cong \mathbb{Z}/2\mathbb{Z}$ and that $\{\mathbb{Z}HS\text{'s}\}/\underset{Y_3}{\sim} \cong \mathbb{Z}$, which deduce the Rokhlin and Casson invariants respectively. Further, it is known [165] that $\{M \underset{Y_{2d-1}}{\sim} S^3\}/\underset{Y_{2d}}{\sim} = 0$ for $d>1$ and that there exists a natural surjective homomorphism

$$\mathcal{A}(\emptyset;\mathbb{Z})^{(d)}_{\mathrm{conn}} \longrightarrow \{M \underset{Y_{2d}}{\sim} S^3\}/\underset{Y_{2d+1}}{\sim} \tag{50}$$

such that the tensor product of this map and $\mathbb{Q}$ is an isomorphism. In particular, $\{M \underset{Y_{2d}}{\sim} S^3\}/\underset{Y_{2d+1}}{\sim}$ forms an abelian group with respect to the connected sum of $\mathbb{Z}HS$'s, and hence, so does $\{\mathbb{Z}HS\text{'s}\}/\underset{Y_{2d+1}}{\sim}$.

Conjecture 10.14 *The map (50) is an isomorphism.*

[35] The Y_d-equivalence is also called the $(d-1)$*-equivalence* (due to Goussarov) in some literatures.

This conjecture might be reduced to Conjecture 10.8 and the following conjecture.

Conjecture 10.15 *$\{M \underset{Y_{2d}}{\sim} S^3\}/_{\underset{Y_{2d+1}}{\sim}}$ is torsion free for each d.*

Remark Conjecture 10.8 implies this conjecture, since the surjective homomorphism (50) gives a $\mathbb{Q}$-isomorphism.

Remark (K. Habiro) It is also a problem to describe the graded set $\{M \underset{Y_d}{\sim} M_0\}/_{\underset{Y_{d+1}}{\sim}}$ for an arbitrarily given 3-manifold M_0. For $d = 0$, the quotient set $\{\text{3-manifolds}\}/_{\underset{Y_1}{\sim}}$ can be identified with the set of isomorphism classes of $H_1(M;\mathbb{Z})$ and their linking pairings (as mentioned above). For $d > 0$, there is a surjective map to this graded set from a certain module of Jacobi diagrams (subject to the AS and IHX relations).

Problem 10.16 (T. Ohtsuki) *Define a product $M_1 \circ M_2$ of integral homology 3-spheres M_1 and M_2 which is related, by (50), to the product of Jacobi diagrams given by their connected sum.*

Remark $\mathcal{A}(\emptyset)_{\rm conn}$ is an algebra with the product given by connected sum of Jacobi diagrams. The connected sum of Jacobi diagrams on $\emptyset$ is well defined by the AS and IHX relations. The sum of $\mathcal{A}(\emptyset)_{\rm conn}$ corresponds, by (50), to the connected sum of integral homology 3-spheres. The problem is to define a product among integral homology 3-spheres corresponding to the product of $\mathcal{A}(\emptyset)_{\rm conn}$ by (50).

It is known [152, 165] that two integral homology 3-spheres M and M' are Y_d-equivalent if and only if $v(M) = v(M')$ for any A-valued finite type invariant[36] v of $\mathcal{F}^{\rm Y}_{\star}$-degree $< d$ for any abelian group A. In fact, a natural quotient map $\{\mathbb{Z}HS\text{'s}\} \to \{\mathbb{Z}HS\text{'s}\}/_{\underset{Y_d}{\sim}}$ is a finite type invariant of $\mathcal{F}^{\rm Y}_{\star}$-degree $< d$, which classifies Y_d-equivalence classes of integral homology 3-spheres.

For an oriented compact surface F, a *homology cylinder* over F is a homology $F \times I$ whose boundary is parameterized by $\partial(F \times I)$.

[36] For an abelian group A, a homomorphism $v : \mathbb{Z}\mathbb{M} \to A$ is called a *finite type invariant* of $\mathcal{F}^{\rm Y}_{\star}$-degree d if v vanishes on $\mathcal{F}^{\rm Y}_{d+1}(\mathbb{Z}\mathbb{M})$.

Conjecture 10.17 (M. Polyak, see [153, "Theorem 4"]) *Let F be an oriented compact surface. Two homology cylinders C and C' over F are Y_d-equivalent if and only if $v(C) = v(C')$ for any A-valued finite type invariant v of $\mathcal{F}^{\mathrm{Y}}_{\star}$-degree $< d$ for any abelian group A.*

Remark (M. Polyak) The corresponding assertion for closed 3-manifolds does not hold; note that $\{\text{closed 3-manifolds}\}/\underset{Y_d}{\sim}$ does not (naturally) form a group. Recall that $\{\mathbb{Z}HS\text{'s}\}/\underset{Y_d}{\sim}$ forms an abelian group, which guarantees the corresponding assertion for $\mathbb{Z}HS$'s, as mentioned above. The set $\{\text{homology cylinders on } F\}/\underset{Y_d}{\sim}$ forms a group with respect to the composition of homology cylinders, though it is not abelian.

10.3.2 Goussarov-Habiro theory for spin and spin^c 3-manifolds

As shown in [276], we have a natural spin (resp. spin^c structure) on the 3-manifold obtained from a spin (resp. spin^c) 3-manifold by surgery along a Y graph (or a tree clasper). We define the *Y^s_d-equivalence* (*spin Y_d-equivalence*) (resp. *Y^c_d-equivalence* (*spin^c Y_d-equivalence*)) to be the equivalence relation among spin (resp. spin^c) 3-manifolds given by the Y_d-equivalence. It is known [276] that the quotient set $\{\text{spin closed 3-manifolds}\}/\underset{Y^s_1}{\sim}$ can be identified with the isomorphism classes of pairs of $H_1(M;\mathbb{Z})$ and certain quadratic forms $\phi_{M,\sigma} : \mathrm{Tor}H_1(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$, or equivalently, the isomorphism classes of triples of $H_1(M;\mathbb{Z})$ and linking pairings $\lambda_M : \big(\mathrm{Tor}H_1(M;\mathbb{Z})\big)^{\otimes 2} \to \mathbb{Q}/\mathbb{Z}$ and the mod 8 reduction of the Rokhlin invariant $\mu(M,\sigma)$. Further, it is known [107] the quotient set $\{\text{spin}^c \text{ closed 3-manifolds}\}/\underset{Y^c_1}{\sim}$ can be identified with the set of the isomorphism classes of pairs of $H_1(M;\mathbb{Z})$ and certain quadratic forms q_σ. This set would be well described by the classification of the following problem.

Problem 10.18 (F. Deloup) *Classify the monoid (for orthogonal sum) of isomorphism classes of quadratic forms q_σ.*

Remark The quotient set $\{\text{closed 3-manifolds}\}/\underset{Y_1}{\sim}$ can be identified with the set of the isomorphism classes of pairs of $H_1(M;\mathbb{Z})$ and linking pairings. This set can be well described by the classification of linking pairings given in [207].

The first version of Section 10.3.2 was written by T. Ohtsuki, following a report of F. Deloup. Based on it, F. Deloup wrote this section.

Problem 10.19 (G. Massuyeau) *Describe the quotient set* $\{\textit{spin closed 3-manifolds}\}/\underset{Y_d^s}{\sim}$, *in particular, for* $d=2,3$.

Problem 10.20 (F. Deloup, G. Massuyeau) *Describe the quotient set* $\{\textit{spin}^c \textit{ closed 3-manifolds}\}/\underset{Y_d^c}{\sim}$, *in particular, for* $d=2,3$.

Remark There is a unique spin (resp spin^c) structure on a $\mathbb{Z}HS$. Hence, $\{\text{spin } \mathbb{Z}HS\text{'s}\}/\underset{Y_d^s}{\sim}$ (resp. $\{\text{spin}^c\ \mathbb{Z}HS\text{'s}\}/\underset{Y_d^s}{\sim}$) is equal to $\{\mathbb{Z}HS\text{'s}\}/\underset{Y_d}{\sim}$. This quotient set can be described by Jacobi diagrams (see Conjecture 10.15).

Remark The above two problems are related to spin and spin^c refinements of the Casson-Walker-Lescop invariant; see Problem 10.4.

Deloup and Massuyeau [107] obtained a complete system of invariants for quadratic functions on finite abelian groups which involves the Gauss-Brown invariant $\gamma(q)=\sum_{x\in G} e^{2\pi\sqrt{-1}q(x)}$ of a quadratic form q. In the case q_σ comes from a usual spin structure, q_σ is homogeneous[37] and the argument of $\gamma(q_\sigma)$ is just the mod 8 reduction of the Rokhlin invariant. (Here we take the classical Rokhlin invariant of a spin structure on M to be the signature mod 16 of an oriented smooth simply-connected 4-manifold bounded by M.) Thus, in general, $\arg\gamma(q_\sigma)\in\mathbb{Q}/\mathbb{Z}$ may be viewed as mod 8 generalization of Rokhlin invariant for spin^c structures. In the context of spin Goussarov-Habiro theory, Massuyeau proved that the Rokhlin invariant is a finite type invariant of degree 1. This suggests the following question.

Question 10.21 (F. Deloup) *Is there a lift of* $\arg\gamma(q_\sigma)$ *to a mod 16 invariant? This would give a finite type invariant of degree 1 in the spin*^{c} *Goussarov-Habiro theory.*

[37] A quadratic function q is a a map such that $q(x+y)-q(x)-q(y)$ is bilinear in x and y. It is called *homogeneous* if $q(nx)=n^2q(x)$ for any $n\in\mathbb{Z}$ and $x\in G$. In fact, there is a canonical map $\sigma\mapsto q_\sigma$ from spin^c structures to quadratic functions and q_σ is homogeneous if and only if σ actually comes from a spin structure. Note that not all spin^c structures come from spin structures.

11 The LMO invariant

The LMO invariant $Z^{\mathrm{LMO}}(M) \in \mathcal{A}(\emptyset)$ of closed oriented 3-manifolds was introduced in [249]. The LMO invariant of rational homology 3-spheres was reformulated by Aarhus integral [36]. The LMO invariant is a universal perturbative invariant of rational homology 3-spheres (see [320, 36, 321]), and a universal finite type invariant of integral homology 3-spheres [244].

11.1 Calculation of the LMO invariant

Problem 11.1 *For each rational homology 3-sphere M, calculate $Z^{\mathrm{LMO}}(M)$ for all degrees.*

Remark Bar-Natan and Lawrence [37] showed a rational surgery formula for the LMO invariant. By using it, they obtained

$$\hat{Z}^{\mathrm{LMO}}\big(L(p,q)\big) = \langle \Omega_x, \Omega_x^{-1} \Omega_{x/p} \rangle_x \exp \frac{-s(q,p)}{48} \theta \tag{51}$$

for the lens space $L(p,q)$ of type (p,q), where $s(q,p)$ denotes the Dedekind sum. For the notation $\langle \Omega_x, \Omega_x^{-1} \Omega_{x/p} \rangle_x$ see [37].

Remark The degree 1 part of $Z^{\mathrm{LMO}}(M)$ is given by the Casson-Walker invariant of M [249]. Further, the degree $\leq d$ part of $Z^{\mathrm{LMO}}(M)$ of integral homology 3-spheres are given by finite type invariants of degree $\leq d$. Hence, it is algorithmically possible to calculate the degree $\leq d$ part of $Z^{\mathrm{LMO}}(M)$ of an integral homology 3-sphere for each d. It is meaningful to calculate $Z^{\mathrm{LMO}}(M)$ for all degrees.

Remark It is meaningful to calculate $Z^{\mathrm{LMO}}(M)$ when M is a rational homology 3-sphere. Otherwise, it is known that $Z^{\mathrm{LMO}}(M)$ can be given by some "classical" invariants. When $b_1(M) = 1$, the value of $Z^{\mathrm{LMO}}(M)$ can be presented by using the Alexander polynomial of M [138, 259]. When $b_1(M) = 2$, the value of $Z^{\mathrm{LMO}}(M)$ can be presented by using the Casson-Walker-Lescop invariant of M [162]. When $b_1(M) = 3$, the value of $Z^{\mathrm{LMO}}(M)$ can be presented by using the cohomology ring of M [161]. When $b_1(M) > 3$, we always have that $Z^{\mathrm{LMO}}(M) = 1$ [161].

11.2 Does the LMO invariant distinguish integral homology 3-spheres?

Conjecture 11.2 *The LMO invariant distinguishes integral homology 3-spheres. (See Conjecture 10.9.)*

Remark Bar-Natan and Lawrence [37] showed (as a corollary of their calculation (51)) that the LMO invariant does not separate lens spaces. They also showed in [37] that the LMO invariant separates integral homology Seifert fibered spaces.

Problem 11.3 *Does there exist an integral/rational homology 3-sphere M such that $Z^{\text{LMO}}(M) = Z^{\text{LMO}}(S^3)$?*

11.3 Characterization of the image of the LMO invariant

Problem 11.4 *Characterize those elements of $\hat{\mathcal{A}}(\emptyset)_{conn}$ which are of the form $\log Z^{\text{LMO}}(M)$ for integral/rational homology 3-spheres.*

Remark Since $\tau^{SO(3)}(M)$ can be obtained from $Z^{\text{LMO}}(M)$ by applying the weight system W_{sl_2}, some characterization of this problem might be obtained from the characterization of the form $\tau^{SO(3)}(M)$ (Problem 7.28), say, from the integrality of the coefficients of $\tau^{SO(3)}(M)$ for integral/rational homology 3-spheres M. Some other characterization of this problem might by obtained from the loop expansion of the Kontsevich invariant.

11.4 Variations of the LMO invariant

Problem 11.5 *Construct the LMO invariant with coefficients in a finite field.*

Remark If the Kontsevich invariant with coefficients in a finite field would be constructed (see Problem 3.7), then it would be helpful for this problem.

Problem 11.6 *Construct the LMO invariant (or the theory of finite type invariants) in arrow diagrams.*

11.5 Refinements of the LMO invariant

(T. Le) As mentioned in a remark in Problem 11.1, the LMO invariant is a weak invariant when $b_1(M) > 0$; in particular, $Z^{\text{LMO}}(M) = 1$ when $b_1(M) > 3$. The following two problems might give refinements of $Z^{\text{LMO}}(M)$ which would be stronger than $Z^{\text{LMO}}(M)$, in particular, when $b_1(M) > 0$.

Problem 11.7 (T. Le, V. Turaev) *Define the LMO invariant $Z^{\text{LMO}}(M,\sigma)$ of the pair of a closed 3-manifold M and a spin structure σ of M such that $Z^{\text{LMO}}(M) = \sum_\sigma Z^{\text{LMO}}(M,\sigma)$, where the sum runs over all spin structures on M. There is also a similar problem for spin^c structures.*

Remark The quantum $SU(2)$ invariant of (M,σ) satisfies that $\tau_r^{SU(2)}(M) = \sum_\sigma \tau_r^{SU(2)}(M,\sigma)$ for r divisible by 4 (see [221]). The $Z^{\mathrm{LMO}}(M,\sigma)$ should be defined such that $\tau_r^{SU(2)}(M,\sigma)$ can be recovered from $Z^{\mathrm{LMO}}(M,\sigma)$ in an appropriate sense, and such that the coefficients of $Z^{\mathrm{LMO}}(M,\sigma)$ are "finite type invariants" of (M,σ) under an appropriate definition of finite type invariants of (M,σ).

The set of spin structure is a torsor over $H^1(M;\mathbb{Z}/2\mathbb{Z})$ in the sense that the difference of two spin structures is an element in $H^1(M;\mathbb{Z}/2Z)$, and every element of $H^1(M;\mathbb{Z}/2Z)$ is the difference of some spin structure and a fixed one. Similarly, the set of all spin^c structure is a torsor over $H^1(M,\mathbb{Z})$. In this sense the previous problem might be related to the following problem.

Problem 11.8 (T. Le, V. Turaev) *For every element $\xi \in H^1(M,\mathbb{Z})$ construct an extension of $Z^{\mathrm{LMO}}(M,\xi)$ of the LMO invariant such that when $\xi = 0$ one recovers the usual LMO invariant.*

The idea is that the usual LMO invariant corresponds only to the trivial cohomology class, and for manifolds with high Betti number, it is equal to 0. K. Habiro has an extension of the LMO invariant that might be a solution to this problem.

Remark For a finite abelian group A and $\xi \in H^1(M,A)$, let $\tau(M,\xi)$ be the invariant of (M,ξ), defined from a modular A-category, and let $\tau(M)$ be the invariant of M derived from a modular category forgetting A-grading. Then, $\tau(M) = \sum_\xi \tau(M,\xi)$. (For details, see [250].) The $Z^{\mathrm{LMO}}(M,\xi)$ should be defined such that a suitable $\tau(M,\xi)$ can be recovered from $Z^{\mathrm{LMO}}(M,\xi)$ in an appropriate sense, and such that the coefficients of $Z^{\mathrm{LMO}}(M,\xi)$ are "finite type invariants" of (M,ξ) under an appropriate definition of finite type invariants of (M,ξ).

11.6 Other problems

Question 11.9 (1) *Find a surgery formula for the Kuperberg-Thurston invariant [237] in terms of the Chern-Simons series of Question 3.12*

(2) *Compare the Kuperberg-Thurston invariant to the LMO invariant.*

Problem 11.10 (D. Thurston) *Do configuration spaces of [237] have torsion in $\mathbb{Z}$-homology? Does such torsion deduce a torsion invariant of homology 3-spheres?*

12 Other problems

12.1 (Pseudo) Legendrian knot invariants

Let W be a compact closed oriented 3-manifold. (K,v) is said a *pseudo Legendrian pair* in W if $K \subset W$ is a knot, v is a non singular vector field on W and K is transverse to v. K is simply said a (pL)-knot. (K_t, v_t), $t \in [0,1]$, is a *pseudo Legendrian isotopy* if K_t is an ambient isotopy of knots, v_t is a homotopy of fields and (K_t, v_t) is a (pL)-pair for every $t \in [0,1]$. Every (pL)-knot is naturally a *framed* knot, and every (pL)-isotopy is in particular a framed knots isotopy. If ξ is a transversely oriented contact structure on and K is ξ-Legendrian in the classical sense, then K is a (pL)-knot w.r.t. any field v which is positively transverse to ξ. Every Legendrian isotopy between ξ-Legendrian knots induces a (pL)-isotopy. So we have 3 categories of knots, related by natural forgetting maps:

$$\{\text{Legendrian knots}\} \xrightarrow{f_1} \{\text{(pL)-knots}\} \xrightarrow{f_2} \{\text{framed knots}\}.$$

Note that, for each one of these categories, $\mathcal{C}$ say, also the *$\mathcal{C}$-homotopy immersion class* of any $\mathcal{C}$-knot is naturally defined, this contains the $\mathcal{C}$-isotopy class and is preserved by the forgetting maps.

In [49] one has introduced the *Reidemeister-Turaev torsions* of (pL)-knots; one has realized that torsions include a correct lifting to the (pL)-category of the classical Alexander invariant; moreover, in many cases (for instance when W is a $\mathbb{Z}$-homology sphere), they can distinguish (pL)-knots which are isotopic as framed knots.

Question 12.1 (R. Benedetti) *Are torsions actually sensitive only to the (pL)-*homotopy immersion classes *of (pL)-knots?*

If one fix a $\mathcal{C}$- homotopy immersion class of knots, say α , then one can define the set of *finite type invariants* $\mathcal{F}(\alpha)$ of the $\mathcal{C}$-isotopy classes contained in α. If α_0 is a class of Legendrian knots, one can take $\alpha_1 = f_1(\alpha_0)$ and $\alpha_2 = f_2(\alpha_1)$; a finite type invariant for α_i lifts to a finite type invariant for α_{i-1}. So one has natural maps

$$\mathcal{F}(\alpha_2) \xrightarrow{f_2^*} \mathcal{F}(\alpha_1) \xrightarrow{f_1^*} \mathcal{F}(\alpha_0).$$

It is known [130] that, under certain hypotheses on W (for instance when W is a $\mathbb{Z}$-homology sphere), $f_1^* \circ f_2^*$ is a bijection. On the oder hand, one can

Section 12.1 was written by R. Benedetti.

find in [380] examples where $f_1^* \circ f_2^*$ is not surjective and Legendrian finite type invariants can eventually distinguish some Legendrian knots which are isotopic as framed knots. In fact one can realize that for these examples f_2^* is already not surjective and that (pL)-finite type invariants can eventually distinguish some (pL)-knots which are isotopic as framed knots. The following conjecture is not in contradiction with all these known results on the subject.

Conjecture 12.2 (R. Benedetti) *For every W, for every (pL)-class α_1 as above, f_1^* is an isomorphism. This means, in particular, that finite type invariants of Legendrian knots should be definitely not sensitive to geometric (rigid) properties of the contact structures like "tightness".*

See also [50] for a more detailed discussion and related questions.

12.2 Knots and finite groups

Knot groups are known to be residually finite, that is, any non-trivial element can be detected by a homomorphism to some finite group.

Now by Dehn's lemma and the loop theorem a knot is trivial if and only if its longitude represents the trivial element of the knot group. Consequently for each non-trivial knot there is a homomorphism to *some* finite group which carries the longitude to a non-trivial element.

Problem 12.3 (H.R. Morton) *From a knot diagram find an explicit such homomorphism to some permutation group or establish that the knot is trivial.*

Refinements.

(1) Give an upper bound in terms of the diagram for the order of the permutation groups which need to be considered.

(2) See what happens if the meridians (which are all conjugate) are restricted to map to permutations of some specified cycle type, for example, single transpositions.

Remark Every finite group is the subgroup of a permutation group, so no restrictions are implied here.

The language of quandles could be adopted for 2 when referring to the chosen meridian conjugacy class.

Section 12.2 was written by H.R. Morton.

It is possible to represent some knot groups onto a finite non-cyclic group with the longitude mapping trivially. This always happens when n-colouring a knot, as the knot group is mapped onto the dihedral group D_n, and the longitude goes into its commutator subgroup. The problem here focusses on the stronger question of representing the longitude non-trivially.

12.3 The numbers of 3-, 5-colorings and some local moves

A *p-coloring* of a link L is a homomorphism of the link quandle of L to the dihedral quandle R_p of order p (or, alternatively, a homomorphism of $\pi_1(S^3 - L)$ to the dihedral group of order $2p$ which takes each meridian to a reflection).[38] Let $Col_p(L)$ denote the number of p-colorings of L (see the remark of Problem 4.16). The following conjecture implies that the 3-move (see Figure 20) would topologically characterize the partition of the set of links given by $Col_3(L)$; note that $Col_3(L)$ is unchanged under the 3-move.

Conjecture 12.4 (3-move conjecture, Y. Nakanishi [305]) *Any link can be related to a trivial link by a sequence of 3-moves.*

Remark $Col_3(L)$ is equal to 3^{n+1}, where n is the rank of $H_1(M_2(L); \mathbb{Z}/3\mathbb{Z})$ and $M_2(L)$ denotes the double cover of S^3 branched along L. Further, Col_3 of the trivial link with n components is equal to 3^n. Hence, if a link L is related to a trivial link by 3-moves, then such a trivial link has $\log_3 Col_3(L)$ components.

Remark [220, Remark on Conjecture 1.59 (1)] Since $B_n/\langle \sigma_i^3 \rangle$ is finite for $n \leq 5$, the proof of this conjecture for closures of braids of at most 5 strands is reduced to verifying finitely many cases. According to Y. Nakanishi, the smallest known obstruction of this conjecture is the 2-parallel of a set of Borromean rings.

Remark [376] This conjecture is true for weak genus two knots.

Update Dabkowski and Przytycki [99] showed that some links cannot be reduced to trivial links by 3-moves, which are counterexamples to this conjecture.

[38] The original definition of a 3-coloring by Fox (see [98, Chapter VI, Exercise 6]) is (an equivalent notion of) a *non-trivial* homomorphism of the link quandle of L to the dihedral quandle R_3. Przytycki [337] studied the number of 3-colorings. His definition allows trivial homomorphisms.

It is shown in [169] that $Col_5(L)$ is invariant under the (2,2)-move (see Figure 20). The following conjecture implies that the (2,2)-move would topologically characterize the partition of the set of links given by $Col_5(L)$.

Conjecture 12.5 (Y. Nakanishi, T. Harikae [220, Conjecture 1.59 (6)]) *Any link can be related to a trivial link by a sequence of (2,2)-moves.*

Remark This conjecture holds for algebraic links; see [220, Conjecture 1.59 (6)], [337], and references therein.

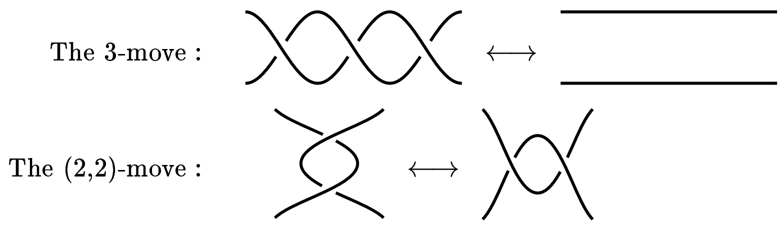


Figure 20: The 3-move and the (2,2)-move

12.4 Knotted trivalent graphs

D. Bar-Natan and D. Thurston [39, 40, 384] developed the theory of knotted trivalent graphs and their algebra, related to shadow surfaces of V. Turaev [388] and Lie groups/algebras.

A *knotted trivalent graph* (*KTG*) is a (framed) embedding of a (ribbon) trivalent graph Γ into S^3, where framing is an integer of a half integer (hence, the ribbon of a trivalent graph is not necessarily orientable). There are four operations of KTG's: connected sum, unzip, bubbling and unknot; see Figure 21. Any KTG (in particular, any link) can be obtained from copies of tetrahedron and Möbius strip with $\pm 1/2$ framing by applying KTG operations. Further, two sequences of KTG operations give the same KTG, if and only if they are related by certain (finitely many) relations including the pentagon and hexagon relations (see [40]). Thus, the theory of KTG's is finitely presented in this sense.

The Kontsevich invariant of framed links have an extension for KTG's (see [301]) and the extended Kontsevich invariant is well-behaved under the KTG

Section 12.4 was written by T. Ohtsuki, following seminar talks given by D. Thurston.

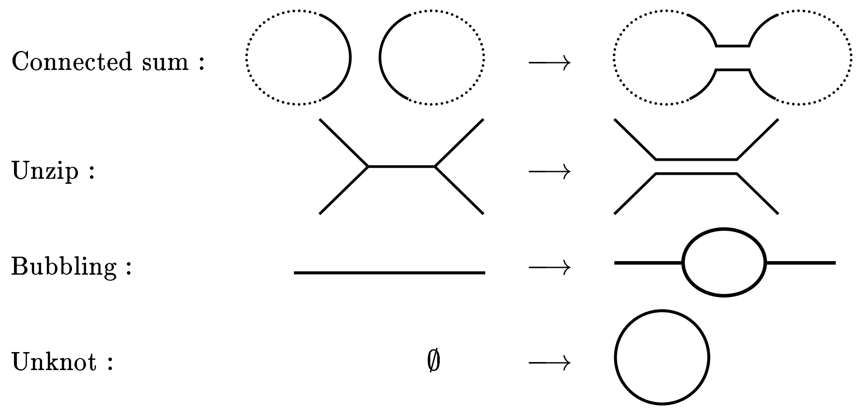


Figure 21: Four operations of KTG's [40]. The left hand side of the connected sum denotes a disjoint union of two separate graphs.

operations such that they give another construction of the Kontsevich invariant starting from the invariants of tetrahedron and Möbius strip.

Problem 12.6 *Find a new proof of the existence of a universal Vassiliev invariant of knots, presenting them by KTG's and their operations.*

Conjecture 12.7 (D. Bar-Natan, D. Thurston) *For each compact Lie group G, level k, and every KTG $K : \Gamma \to \mathbb{R}^3$, there exists a collection of measures μ_K on the space of gauge equivalence classes of G-connections on Γ satisfying the following conditions.*

- *It is well-behaved under KTG operations.*
- *It is "localized" near connections that extend to $S^3 - K$.*
- *A half-twist framing change acts by $e^{\sqrt{-1}H\hbar/2}$, where H is the Schrödinger operator on G.*
- *It recovers quantum invariants by*

$$I_R(K) = \int h_R(A) d\mu_K(A),$$

where $h_R(A)$ denotes the holonomy of A in R. Here, R is a set of representations of G associated to edges of Γ and appropriate intertwiners associated to vertices of Γ.

Remark [31, 39] The physical presentation of the quantum invariant of a knot K associated with a representation R of G is given by the Chern-Simons path integral,

$$Z_k(S^3, K) = \int h_R(A) e^{2\pi\sqrt{-1}k\,\mathrm{CS}(A)} \mathcal{D}A,$$

where $\mathrm{CS}(A)$ denotes the Chern-Simons functional of A and the integral is a formal integral over the infinite dimensional space of all G connections on S^3. It is a motivation of Conjecture 12.7 that a collection of μ_K should play a role of $e^{2\pi\sqrt{-1}k\,\mathrm{CS}(A)}\mathcal{D}A$. It is expected [31, 39] that the collection of measures μ_K of Conjecture 12.7 would prove the asymptotic expansion conjecture (Conjecture 7.6).

Problem 12.8 *Construct an invariant of KTG's from configuration space integrals in a natural way.*

Turaev [388] introduced a presentation of 3-manifolds as S^1-bundles over "shadow surfaces", as follows (for details see [388, 40, 384]). A *fake surface* is a singular surface such that a neighborhood of each point is homeomorphic to an open subset of the cone over a tetrahedron. A S^1-bundle over a fake surface can appropriately be defined and its isomorphism class is determined by the Chern number, which is an integer or half-integer associated to each face; we call the Chern number the *gleam*. A *shadow surface* is a fake surface with gleams associated to the faces. Every (closed) 3-manifold can be presented by a S^1-bundle over a (closed) shadow surface. The pentagon and hexagon relations (see [388, Figure 1.1 of Chapter VIII]) are moves among shadow surfaces which present a homeomorphic 3-manifold, though they are not enough to characterize a homeomorphism class of 3-manifolds.

Exercise 12.9 *Find a complete set of moves among shadow surfaces which present a homeomorphic 3-manifold.*

We obtain a shadow surface as a time evolution of a sequence of KTG's given by KTG operations. Thus, we have relations among links, 3-manifolds, KTG's and shadow surfaces as in the commutative diagram in Figure 22; for detailed statements see [40, 384].

Motivated by a complexity of 3-manifolds discussed in [279, 272, 273], D. Thurston introduced the shadow number of 3-manifolds. The *shadow number* is defined to be the minimal number of vertices of a shadow surface. All graph manifolds have shadow number 0 and all surgeries on the Borromean rings have shadow number 1. The volume conjecture might be related to the following conjecture.

$$
\begin{tikzcd}
\text{Framed links} \arrow[r, "\text{exterior}"] & \text{Framed link exteriors} \arrow[r, "\text{surgery}"] & \text{Closed 3-manifolds} \\
\begin{array}{l}\text{Certain sequences}\\ \text{of KTG's}\end{array} \arrow[u, "\text{presentation}"] \arrow[r, "\begin{array}{c}\text{time}\\ \text{evolution}\end{array}"'] & \begin{array}{l}\text{Collapsible}\\ \text{shadow surfaces}\end{array} \arrow[u, "\begin{array}{l}\text{by making}\\ S^1\text{-bundle}\end{array}"'] \arrow[r, "\text{cap off } \partial"'] & \text{Closed shadow surfaces} \arrow[u, "\begin{array}{l}\text{by making}\\ S^1\text{-bundle}\end{array}"']
\end{tikzcd}
$$

Figure 22: Links, 3-manifolds, KTG's, and shadow surfaces

Conjecture 12.10 (D. Thurston) *The shadow number of a 3-manifold is quasi-linear in its Gromov norm. That is, there exist constants c_1 and c_2 such that*

$$c_1||M|| \leq \textit{(shadow number of } M\textit{)} \leq c_2||M||$$

for any 3-manifold M, where $||M||$ denotes the Gromov norm of M.

Remark (D. Thurston) It is easy to bound the Gromov norm in terms of the shadow number (i.e. to prove the left inequality for some c_1).

Remark (D. Thurston) It is shown by W. Thurston that the hyperbolic volume of a hyperbolic 3-manifold is quasi-linear in the minimal number of ideal tetrahedra in a "spun triangulation" (i.e. the minimal number of ideal tetrahedra in some link complement in the 3-manifold). It is shown by J. Brock [68] that the volume of a mapping torus is quasi-linear in the pants translation distance (for fixed genus).

Lackenby [238] showed that alternating knot diagrams give good information about the hyperbolic volume. Knot diagrams are a special case of shadow diagrams, but shadow diagrams can be much more efficient. This suggests the following problem:

Problem 12.11 (D. Thurston) *Find a condition on shadow diagrams which is satisfied by shadow diagrams from alternating knots; and gives a lower bound on the hyperbolic volume.*

The Reshetikhin-Turaev invariant and the Turaev-Viro-Ocneanu invariant can be described in terms of the KTG algebra, via I-bundles and S^1-bundles over shadow surfaces respectively. The relation between the two invariants is derived from the relation between the two construction of 3-manifolds shown in Figure 23.

$$
\begin{tikzcd}[column sep=small]
& \text{Closed shadow surfaces with 0 gleams} \arrow[dl, "\text{by making } I\text{-bundles and cap off boundary}"'] \arrow[dr, "\text{by making } S^1\text{-bundles}"] & \\
\text{Closed 3-manifolds} \arrow[rr] & & \text{Certain closed 3-manifolds} \\
M \arrow[rr, mapsto] & & M\#\overline{M}\#(S^2 \times S^1)\text{'s}
\end{tikzcd}
$$

Figure 23: Two ways to obtain 3-manifolds from shadow surfaces

Problem 12.12 *Construct a universal Reshetikhin-Turaev invariant and a universal Turaev-Viro-Ocneanu invariant of closed 3-manifolds, in terms of the KTG algebra.*

Remark The LMO invariant and the even degree part of it might be a universal Reshetikhin-Turaev invariant and a universal Turaev-Viro-Ocneanu invariant of rational homology 3-spheres, respectively.

12.5 Quantum groups

Problem 12.13 (J. Roberts) *What are quantum groups?*

Remark (J. Roberts) A naive answer is to simply define them by means of generators and relations, but this is appallingly unsatisfying. Better is Drinfel'd's original construction [111], which begins with the geometric construction of *quasi-quantum groups* using the monodromy of the KZ equation. He then uses completely algebraic results about uniqueness of deformations to obtain from each one a quantum group, whose category of representations is equivalent to that of the quasi-quantum group, though the first has a trivial associator and a complicated R-matrix, the second vice versa. (In particular, the braid group representation associated to a quantum group is *local* in the sense that the R-matrix implementing the action of a braid generator on a tensor product of representations of the quantum group involves only the tensor factors associated to the two strings concerned. This is certainly not true for the KZ equation. Is there any way to understand this using geometry?)

These constructions are very subtle and complicated. What really is a quantum group, in fact? I believe that algebraists have some reasonably geometric descriptions of pieces of them in terms of perverse sheaves, etc., but I do not pretend to understand these. Atiyah made the very interesting suggestion that quantum groups might be in some sense the "quaternionifications" of compact Lie groups. Literal quaternionification does not make sense, but substitutes might be available, in the sense that hyperkähler geometry provides a working substitute for the non-existent quaternionic version of complex manifold theory. Some evidence for this point of view is presented in Atiyah and Bielawski [19].

12.6 Other problems

Problem 12.14 (N. Askitas) *Can a knot of 4-genus g_s always be sliced (made into a slice knot) by g_s crossing switches?*

Remark (A. Stoimenow) Clearly (at least) g_s crossing switches are needed, but sometimes more are needed to *unknot* the knot.

Update Livingston [266] showed that the knot 7_4 provides a counterexample to this problem; $g_s(7_4) = 1$ but no crossing change results in a slice knot.

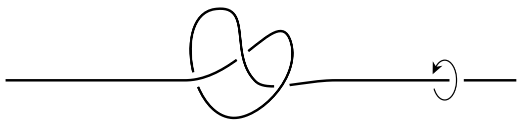
Problem 12.15 (M. Boileau [220, Problem 1.69 (C)]) *Are there mutants of distinct unknotting numbers?*

Remark (A. Stoimenow) There are mutants of distinct genera (Gabai [133]) and slice genera (Livingston [265]).

Let G be the graph such that its vertices are isotopy classes of unoriented knots, and two vertices are adjacent if the corresponding knots differ by a single crossing change.

Conjecture 12.16 (X.-S. Lin [262]) *Any automorphism of G is either the identity or the mirror map, that is, any automorphism of G is induced by a diffeomorphism of the ambient space.*

Problem 12.17 (X.-S. Lin [262]) *What is the homotopy type of the space $L(K)$ of long ropes (as shown in the picture below) with the fixed knot type K?*



Remark [262] A conjecture would be that, if K is a prime knot, $L(K)$ is homotopy equivalent to the circle if and only if K is non-trivial, with the fundamental group generated by the obvious loop in $L(K)$ shown in the above picture. This question is motivated by the paper [287]. If the conjecture holds, the homotopy type of the space of short ropes studied by Mostovoy would be clear. A paper of Hatcher [171] seems to be related with this problem.

Problem 12.18 (J. Roberts) *Extend Kuperberg's work on webs.*

Remark (J. Roberts) Kuperberg posed in [234] the question of giving a presentation, as a tensor category, of the representation category of a compact Lie group or quantum group. The generators should be (roughly) the fundamental modules and their bilinear and trilinear invariants; more complicated morphisms in the category can be built out of these according to a graphical calculus (essentially Penrose's tensor calculus) of "webs". The first main problem is to describe a set of elementary linear relations (skein relations) among such pictures which generates all the relations among morphisms in the category. The second is to describe a canonical basis of any invariant space in terms of canonical pictures in the disc. Kuperberg solved both these problems for groups of ranks one (in which case the pictures are just Temperley-Lieb diagrams) and two and, with Khovanov in [219], made tantalising but imprecise conjectures about how in the higher-rank case the pictures might be related to the geometry of the weight lattice. These ideas are closely related to the work of Vaughan Jones [187] on planar algebra, which is a similar kind of calculus describing the category of bimodules over a subfactor. (Aside: Is it possible to find a bimodule category whose intertwining rules are described by quasiperiodic Penrose tiles?)

Problem 12.19 (J. Roberts) *Extend the theory of measured laminations to higher rank groups.*

Remark (J. Roberts) Let Σ be a closed oriented surface of genus g, and let $C(\Sigma)$ be its set of multicurves (isotopy classes of collections of disjoint simple closed curves). Let $T(\Sigma)$ be its Teichmüller space; that is, the space of hyperbolic structures, considered up to diffeomorphisms isotopic to the identity. Topologically, $T(\Sigma)$ is an open ball of dimension $6g-6$.

Each of $C(\Sigma), T(\Sigma)$ has a natural embedding in the space of functions $C(\Sigma) \to \mathbb{R}_{\geq 0}$: one sends a multicurve to its associated minimal geometric intersection number function, and a metric to its associated geodesic length function. It is a remarkable fact that the $\mathbb{R}_+$-projective boundaries of these sets coincide. They define the space of *measured laminations*, which compactifies $T(\Sigma)$ into a closed ball and is of great importance in Thurston's theory of surface automorphisms. For further details see for example Penner and Harer [327].

Now $T(\Sigma)$ may also be described algebraically as a certain component of the space of flat $SL(2,\mathbb{R})$ connections on Σ (that is, homomorphisms $\pi_1(\Sigma) \to SL(2,\mathbb{R})$), and in this context the geodesic length function is replaced by a trace-of-holonomy function. Is there a generalisation of the above picture to a higher rank group such as $SL(n,\mathbb{R})$?

Hitchin [172] proves that in fact the space of flat $SL(n,\mathbb{R})$ connections has a special "Teichmüller component", which is topologically an open ball, so we have a candidate for $T(\Sigma)$.(Aside: he asks whether there is an interpretation of the points of the Teichmüller component in terms of some kind of geometric structures on Σ. Choi and Goldman showed that for $n = 3$ they parametrise convex real projective structures, but no general answer is known.)

A candidate for $C(\Sigma)$ might be the set of Kuperberg-style (closed) webs drawn on the surface, for there is then a natural holonomy-type map $T(\Sigma)\times C(\Sigma) \to \mathbb{R}$ which is a substitute for the geodesic length function. (In the $SL(2)$ case, this $C(\Sigma)$ is just the set of multicurves, as it should be.) What might replace the geometric intersection number, and lead to some notion of "measured lamination" for higher-rank groups, is unclear.

Problem 12.20 (J. Roberts) *What is the generating function for q-spin net evaluations?*

Remark (J. Roberts) A q-spin net is a trivalent planar graph whose edges are labelled by irreducible representations of $SU(2)$. By placing idempotents from the Temperley-Lieb algebra on its edges and joining up their external strings in a planar fashion at the vertices, one forms an evaluation in $\mathbb{Z}[q^{\pm 1}]$. The goal is to find a power series in variables associated to the edges which serves as a generating function for the evaluations corresponding to all possible labellings of a given graph. Such a formula is known for any graph at the classical value $q = 1$, and Westbury [400] found a generating function for the tetrahedral graph (the quantum $6j$-symbol). A general formula is, however, unknown, and Westbury also shows that the naive guess (simply replacing factorials in the $q = 1$ formula by quantum factorials) is wrong.

Problem 12.21 (Y. Shinohara [364]) *If $n = 4k + 1$ with $k > 0$, is there a knot with determinant n and signature 4?*

Remark (A. Stoimenow) The form $4k+1$ follows from Murasugi [303], and the condition $k \neq 0$ from a signature theorem for even unimodular quadratic forms over $\mathbb{Z}$. If a counterexample for $n > 1$ exists, then all prime divisors of n are of the form $24k+1$ and not smaller than 2857. If $\sigma_{4+8l,8l+5}$ is the elementary symmetric polynomial of degree $4+8l$ in $8l+5$ variables, then all values of $\sigma_{4+8l,8l+5}$ on *positive odd* arguments are no counterexamples, so the problem could "reduce" to showing that some of the $\sigma_{4+8l,8l+5}$ realizes almost all n on positive odd arguments. This appears number theoretically hard, however.

The set of concordance classes of 2-strand string links forms a group $\mathcal{C}_2$. Stanford showed that $\mathcal{C}_2$ is not nilpotent, in particular not abelian.

Problem 12.22 (T. Stanford) *Is $\mathcal{C}_2$ solvable? Does $\mathcal{C}_2$ contain a free group?*

Problem 12.23 (A. Stoimenow) *Do positive links of given signature σ have bounded (below) maximal Euler characteristic χ?*

Remark (A. Stoimenow) So far for general positive links only $\sigma > 0$ is known [361, 90], and for positive knots $\sigma \geq 4$ if $2g = 1-\chi \geq 4$ (it follows from [379]). For positive braid links the answer is positive, and also for special alternating links by Murasugi [303].

Problem 12.24 (A. Stoimenow) *If a prime knot K can be transformed into its mirror image by one crossing change, is K achiral or (algebraically?) slice?*

Remark (A. Stoimenow) Smoothing out this crossing gives a link of zero Tristram-Levine-signatures [385, 254] and zero Alexander polynomial. Many such links are slice, and then K would be slice also. But unlikely.

Problem 12.25 (A. Stoimenow) *Let n be an odd natural number, different from 1, 9, and 49, such that n is the sum of two squares. Is there a prime alternating achiral knot of determinant n?*

Remark (A. Stoimenow) If there is an achiral knot of determinant n, then n is the odd sum of two squares [170]. The converse is also true, and the achiral knot of determinant n can be chosen to be alternating *or* prime, but *not* always both. For $n = 1$, 9, and 49, there is no prime alternating achiral knot of determinant n. If there is another such n, then $n > 2000$ and n is not a square. See [374].

Conjecture 12.26 (V. Turaev) *A pair (a finitely generated abelian group H of rank 1, an element $\Delta(t) \in \mathbb{Z}[H/\mathrm{Tors}\,H] = \mathbb{Z}[t^{\pm 1}]$) (where t is a generator of $H/\mathrm{Tors}\,H$) can be realized as the pair ($H_1(M)$, the Alexander polynomial Δ_M of M) for a closed connected oriented 3-manifold M if and only if $\Delta(t) = t^k \Delta(t^{-1})$ with even $k \in \mathbb{Z}$ and $\Delta(1) = \pm|\mathrm{Tors}\,H|$.*

Remark (V. Turaev) Both conditions are known to be necessary. They are presumably sufficient. This is known for $H = \mathbb{Z}$ and for $H = \mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$ with $n \geq 2$. When M is obtained from S^3 by 0-surgery along a knot K, $H_1(M) = \mathbb{Z}$ and $\Delta_M(t) = \Delta_K(t)$. It is known that a Laurent polynomial $f(t) \in \mathbb{Z}[t^{\pm 1}]$ is realized as the Alexander polynomial of a knot if and only if $f(t) = t^k f(t^{-1})$ with even k and $f(1) = 1$. Using surgery on a 2-component link in S^3 with linking number 0 and framing numbers $0, n$, respectively, one can prove (cf. [253]) the conjecture for $H = \mathbb{Z} \times (\mathbb{Z}/n\mathbb{Z})$.

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