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A homological definition of the Jones polynomial

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Abstract We give a new definition of the Jones polynomial. Let L be an oriented knot or link obtained as the plat closure of a braid $\beta \in B_{2n}$. We define a covering space $\tilde{C}$ of the space of unordered n-tuples of distinct points in the $2n$-punctured disk. We then describe two n-manifolds $\tilde{S}$ and $\tilde{T}$ in $\tilde{C}$, and show that the Jones polynomial of L can be defined as an intersection pairing between $\tilde{S}$ and $\beta\tilde{T}$. Our construction is similar to one given by Lawrence, but more concrete.

AMS Classification 57M25; 57M27, 20F36

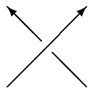
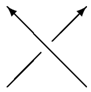
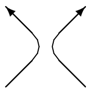
Keywords Jones polynomial, braid group, plat closure, bridge position

1 Introduction

The Jones polynomial is most easily defined using skein relations. Consider the set of formal linear combinations of oriented knots or links in S^3 over the ring $\mathbf{Z}[q^{\pm\frac{1}{2}}]$ modulo the *skein relation*

$$q^{-1}L_+ - qL_- = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})L_0,$$

where L_+, L_- and L_0 are oriented knots or links that are identical except in a ball, where they are as follows.

$L_+ =$  , $L_- =$  , $L_0 =$  .

Using this relation, any oriented knot or link L can be written as some scalar multiple of the unknot. This scalar $V_L \in \mathbf{Z}[q^{\pm\frac{1}{2}}]$ is uniquely determined by the isotopy class of L, and is called the *Jones polynomial* of L.

The original definition in [3] gives the Jones polynomial of the closure of a braid as a trace function of the image of that braid in the Hecke algebra. It is natural to ask whether there is a more topological definition - one which is based not on algebraic properties of a braid, or combinatorial properties of a projection onto

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the plane, but only on topological properties of the way the link is embedded into three-dimensional space.

Despite a great deal of effort, no satisfactory answer to this question is known. A partial answer was provided by the groundbreaking paper of Witten [7], which gives an interpretation of the Jones polynomial in terms of quantum field theory. Reshetikhin and Turaev [6] gave a mathematically rigorous formulation of this theory, using quantum groups instead of the Feynman path integral, which is not yet known to be well-defined.

In this paper, we follow the approach used by Lawrence in [5]. Let L be the *plat closure* of a braid $\beta \in B_{2n}$. Jones [3] showed that V_L appears as an entry of the matrix for β in a certain irreducible representation of B_{2n}. Lawrence [4] gave a topological interpretation of this representation. This leads to an interpretation of V_L as an intersection pairing between a certain element of cohomology and the image under β of a certain element of homology.

The definition of V_L given in this paper is essentially the same as Lawrence's. However our description of the relevant elements of homology and cohomology is more explicit. Indeed we explain how one could use them to directly calculate V_L, which is not clear from [4]. We also give a new and more elementary proof that our invariant is the Jones polynomial.

This interpretation of V_L might represent some progress towards a truly topological definition. However it has a flaw in common with many definitions. Namely, it is first defined for links in a special form, and then shown to be an isotopy invariant by checking it is invariant under certain moves. In our case, the special form is the plat closure of a braid, and the moves are those described by Birman in [2].

This paper is an attempt to fill in some details of a talk given at a workshop in RIMS, Kyoto, in September 2001. I thank the organisers for their kind hospitality. This research was supported by the Australian Research Council.

2 Definitions

Let D be the unit disk centred at 0 in the complex plane. Let $p_1, \dots, p_{2n}$ be points on the real line such that

$$-1 < p_1 < \dots < p_{2n} < 1.$$

Call these *puncture points*. Let

$$D_{2n} = D \setminus \{p_1, \dots, p_{2n}\}.$$

The braid group B_{2n} is the mapping class group of D_{2n}. We will also use other equivalent definitions of B_{2n} as a group of geometric braids, as the fundamental group of a configuration space of points in the plane, and as given by generators and relations. For these and more, see [1]. We use the convention that braids act on D_{2n} on the left, and geometric braids read from top to bottom.

Let $\bar{C}$ be the set of ordered n-tuples of distinct points in D_{2n}. Let C be the quotient of $\bar{C}$ by the symmetric group S_n, that is, the set of unordered n-tuples of distinct points in D_{2n}. We now define a homomorphism

$$\Phi\colon \pi_1 C \to \langle q \rangle \oplus \langle t \rangle.$$

The motivation for our definition is the fact, which we will not prove, that it has a certain universal property. Namely, any map from $\pi_1 C$ to an abelian group which is invariant under the action of B_{2n} must factor uniquely through Φ.

Let $\alpha\colon I \to C$ be a loop in C. By ignoring the puncture points we can consider α as a loop of unordered n-tuples in the disk, and hence as a braid in B_n. Let b be the image of this braid under the usual abelianisation map from B_n to $\mathbf{Z}$, which takes each of the the standard generators to 1. Similarly, the map

$$s \mapsto \{p_1, \dots, p_{2n}\} \cup \alpha(s)$$

determines a braid in B_{3n}. Let b' be the image of this braid under the usual abelianisation map from B_{3n} to $\mathbf{Z}$. Note that b and b' have the same parity, equal to the parity of the image of the braid α in the symmetric group S_n. Let $a = \frac{1}{2}(b' - b)$. We define

$$\Phi(\alpha) = q^a t^b.$$

This definition was intended to be easy to state and clearly well-defined, but it is also somewhat artificial. A more intuitive definition is as follows. A loop $\alpha\colon I \to C$ can be written as

$$\alpha(s) = \{\alpha_1(s), \dots, \alpha_n(s)\}$$

for some arcs $\alpha_1, \dots, \alpha_n$ in D_{2n}. The exponent of q in $\Phi(\alpha)$ records the total winding number of these arcs around the puncture points. The exponent of t records twice the winding number of these arcs around each other. Thus if two arcs switch places by a counterclockwise half twist then this contributes a factor of t.

Let $\tilde{C}$ be the covering space of C corresponding to Φ. The group of covering transformations of $\tilde{C}$ is $\langle q \rangle \oplus \langle t \rangle$. We define the following intersection pairing in $\tilde{C}$.

Definition 2.1 Suppose A and B are immersed n-manifolds in C such that at least one of them is closed, and the other intersects every compact subset of C in a compact set. Suppose $\tilde{A}$ and $\tilde{B}$ are lifts of A and B respectively to $\tilde{C}$. For any $a, b \in \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$ let $q^a t^b \tilde{A}$ be the image of $\tilde{A}$ under the covering transformation $q^a t^b$. There is a well-defined algebraic intersection number $(q^a t^b \tilde{A}, \tilde{B}) \in \mathbf{Z}$. We define

$$\langle \tilde{A}, \tilde{B} \rangle = \sum_{a,b \in \mathbf{Z}} (q^a t^b \tilde{A}, \tilde{B}) q^a t^b \in \mathbf{Z}[q^{\pm 1}, t^{\pm 1}].$$

This is a finite sum, since the number of nonzero terms is at most the geometric intersection number of A and B in C.

To specify an n-manifold $\tilde{C}$, it will help to have a fixed basepoint.

Definition 2.2 Let $d_1, \ldots, d_n$ be distinct points on ∂D. We take them to lie in the lower half plane, ordered from left to right. Let $\mathbf{c} = \{d_1, \ldots, d_n\}$ and fix a choice of $\tilde{\mathbf{c}} \in \tilde{C}$ in the fibre over $\mathbf{c}$.

We now define a certain type of picture in the disk which we will use to represent an immersed n-manifold in $\tilde{C}$.

Definition 2.3 A *fork diagram* in D_{2n} consists of maps

$$E_1, \ldots, E_n \colon I \to D$$

called *tine edges*, and maps

$$E'_1, \ldots, E'_n \colon I \to D$$

called *handles*, subject to the following conditions.

- the tine edges are disjoint embeddings of the interior of I into D_{2n}, and map the endpoints of I to the puncture points in D (not necessarily injectively),
- the handles are disjoint embeddings of I into D_{2n},
- E'_i is a path from d_i to a point in the interior of E_i.

Such a fork diagram determines an immersed open n-ball $\tilde{U}$ in $\tilde{C}$ as follows. Note that $E_1 \times \ldots \times E_n$ maps the interior of $I \times \ldots \times I$ into $\tilde{C}$. Now let U be the projection of this map to C. Let γ be the path in C given by

$$\gamma(s) = \{E'_1(s), \ldots, E'_n(s)\}.$$

Lift this to a path $\tilde{\gamma}$ in $\tilde{C}$ starting at $\tilde{\mathbf{c}}$. We define $\tilde{U}$ to be the lift of U which contains $\tilde{\gamma}(1)$. This is an oriented open n-ball in $\tilde{C}$.

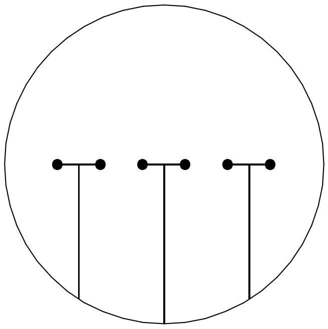


Figure 1: The standard fork diagram for $n = 3$

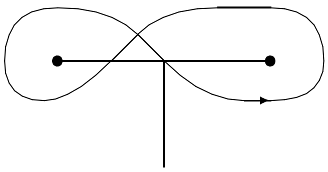


Figure 2: The figure eight F_i corresponding to E_i

Definition 2.4 Let $\tilde{S}$ denote the open n-ball in $\tilde{C}$ corresponding to the *standard fork diagram* shown in Figure 1, where all tine edges are oriented from left to right.

Definition 2.5 For each tine edge E_i in the standard fork diagram, let F_i be the map from S^1 to the figure-eight as shown in Figure 2. Then $F_1 \times \ldots \times F_n$ is an immersion from the n-torus into $\tilde{C}$. Let T be the projection of this map into C. Use the handles to specify a lift $\tilde{T}$ of T to $\tilde{C}$, as in the definition of $\tilde{S}$.

If $[\beta] \in B_{2n}$, let β be a homeomorphism from D_{2n} to itself that represents the mapping class $[\beta]$. Let β' be the induced map from C to itself. This can be lifted to a map from $\tilde{C}$ to itself. Let $\tilde{\beta}'$ be the lift of β' which fixes $\tilde{\mathbf{c}}$. By abuse of notation we will use β to denote $[\beta]$, β' and $\tilde{\beta}'$.

Definition 2.6 An *oriented braid* is a braid $\beta \in B_{2n}$ together with a choice of orientation for each of the $2n$ strands such that the orientations match up correctly when we take the plat closure of β.

We now define our invariant V'_β of an oriented braid $\beta \in B_{2n}$.

Definition 2.7 Suppose $\beta \in B_{2n}$ is an oriented braid. Let w be the writhe of β, that is, the number of right handed crossings minus the number of left handed crossings. Let e be the sum of the exponents of the generators in any word representing β. Let

$$\lambda = (-q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{-1}(-q^{\frac{3}{4}})^{w}(q^{-\frac{1}{4}})^{e+2n}.$$

Then we define

$$V'_\beta = \lambda \langle \tilde{S}, \beta\tilde{T} \rangle|_{(t=-q^{-1})}.$$

The main result of this paper is the following.

Theorem 2.8 *V'_β is the Jones polynomial of the plat closure of β.*

Note that the simpler formula

$$q^{-\frac{1}{4}(e+2n)}\langle \tilde{S}, \beta\tilde{T} \rangle|_{(t=-q^{-1})}$$

therefore gives the Kauffman bracket of the plat closure of β, normalised to equal one for the empty diagram.

3 How to compute the invariant

In this section, we describe of how one could compute $\langle \tilde{S}, \beta\tilde{T} \rangle$, and hence V'_β.

Let $E_1, \dots, E_n$ and $E'_1, \dots, E'_n$ be the tine edges and handles of the standard fork diagram. Let $F_1, \dots, F_n$ be the corresponding figure-eights as in Figure 2. By applying an isotopy, we can assume that each βF_i and E_j intersect transversely and there are no triple points.

The intersection $S \cap \beta T$ consists of points $\mathbf{e} = \{e_1, \dots, e_n\}$ in C such that $e_i \in E_i \cap F_{\pi i}$ for some permutation π of $\{1, \dots, n\}$. Each such $\mathbf{e}$ contributes a monomial $\pm q^a t^b$ to $\langle \tilde{S}, \beta\tilde{T} \rangle$. The sign of this monomial is the sign of the intersection of S with βT at $\mathbf{e}$. Let $\tilde{\mathbf{e}}$ be the lift of $\mathbf{e}$ which lies in $\tilde{S}$. The integers a and b are such that $q^a t^b \tilde{\mathbf{e}}$ lies in $\beta\tilde{T}$. The sum of these terms $\pm q^a t^b$ over all $\mathbf{e}$ in $S \cap \beta T$ is the required polynomial $\langle \tilde{S}, \beta\tilde{T} \rangle$.

We now describe how to compute the monomial for a given $\mathbf{e}$ as above. Let m be the number of points e_i such that the sign of the intersection of E_i with $F_{\pi i}$ at e_i is negative. The sign of our required monomial is then $(-1)^m$ times the parity of the permutation π.

Let h be the path

$$h(s) = \{E'_1(s), \dots, E'_n(s)\}$$

in C. Let ϵ be the path

$$\epsilon(s) = \{\epsilon_1(s), \dots, \epsilon_n(s)\},$$

where ϵ_i is a segment of E_i going from $E'_i(1)$ to e_i. Then the composition $h\epsilon$ lifts to a path from $\tilde{\mathbf{c}}$ to $\tilde{\mathbf{e}}$. Let ϕ be the path

$$\phi(s) = \{\phi_1(s), \dots, \phi_n(s)\},$$

where ϕ_i is a segment of $\beta F_{\pi i}$ going from $\beta E'_{\pi i}(1)$ to e_i. Then the composition $\beta(h)\phi$ lifts to a path from $\tilde{\mathbf{c}}$ to $q^a t^b \tilde{\mathbf{e}}$. Recall that $q^a t^b \tilde{\mathbf{e}} \in \beta\tilde{T}$. We conclude that the path

$$\delta = \beta(h)\phi\epsilon^{-1}h^{-1}$$

lifts to a path from $\tilde{\mathbf{c}}$ to $q^a t^b \tilde{\mathbf{c}}$. Thus

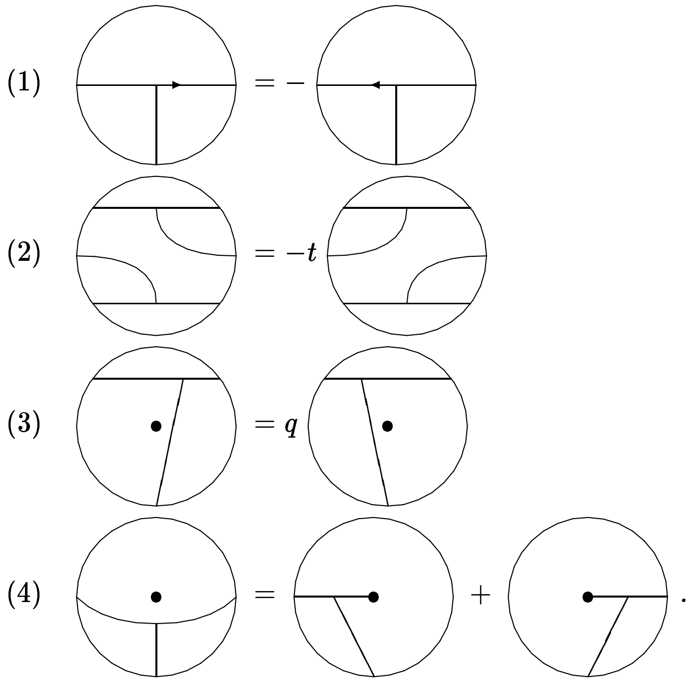
$$q^a t^b = \Phi(\delta).$$

It is possible to calculate the Jones polynomial of a knot or link by these methods. Given a knot diagram, replace every sequence of consecutive undercrossings with a figure-eight. Attach handles in any convenient fashion and proceed as described above. Then either compute the correct factor λ, or simply settle for the value of the Jones polynomial up to sign and multiplication by a power of $q^{\frac{1}{2}}$. I have performed this calculation by hand for the unknot, the Hopf link, and the trefoil knot. Even for the trefoil, it was necessary to correct several mistakes before reaching the correct answer. A computer implementation would be more reliable, but probably no more efficient than existing methods.

4 Lemmas

The pairing $\langle \cdot, \cdot \rangle$ is really a pairing between $H^n(\tilde{C})$ and $H_n(\tilde{C})$. These cohomology and homology groups are modules over $\mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$, where q and t act by covering transformations. The open n-ball corresponding to a fork diagram represents an element of $H^n(\tilde{C})$. We can thus extend the algebraic pairing to take linear combinations of fork diagrams in its first entry. We now prove some relations that hold between fork diagrams considered as elements of $H^n\tilde{C}$.

Lemma 4.1 *The following relations hold.*



Here, the diagrams in each relation are understood to be identical except in the disk shown. The disks shown are also allowed to contain an arbitrary number of additional tine edges, which are required to be identical for all diagrams in the relation. All tine edges are oriented in any consistent fashion except in relation (1).

Proof of Lemma 4.1 In the first three relations, the tine edges are the same in both sides of the equation. Thus the corresponding open n-balls are lifts of the same open n-ball in C, possibly with different orientations.

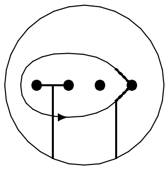
To determine the orientation, recall that the open n-ball in C was defined as the product $T_1 \times \ldots \times T_n$ of tine edges, where T_i is the tine edge whose handle is attached to d_i. Thus the orientation is determined by the orientations of the tine edges and the order in which they occur. In the first relation, the orientation of one tine edge was reversed. In the second, the order in which they occur underwent a transposition. In the third, there was no change to the order or orientation of the tine edges. Thus the signs are as claimed in these relations.

The choice of lift is determined by the handles. In the first relation, the handles are unchanged. In the second and third, the changes in the handles correspond to the scalars t and q as given.

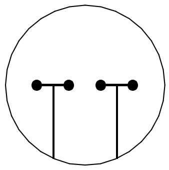
The final relation describes the process of cutting the open n-ball along a hyperplane by pushing that hyperplane into a puncture point. □

One useful consequence of these relations is the following relation. (Thanks to Saul Schleimer for suggesting the name.)

Lemma 4.2 (The fork-spoon relation) *Modulo* $(1+qt)$, *we have*



$= (1-q^{-1})$



,

where the horizontal tine edges are oriented left to right.

5 Proof of the main theorem

Throughout this section we use the conventions that $t = -q^{-1}$, and the horizontal tine edges in any fork diagram are oriented from left to right.

Let β be an oriented braid and let L be the plat closure of β. The aim of this section is to prove Theorem 2.8, that $V'_\beta = V_L$. First we prove that V'_β depends only on the isotopy class of L. We use a result due to Birman which gives the "Markov moves" for plat closures.

Definition 5.1 Let K_{2n} be the subgroup of B_{2n} generated by

- σ_1,
- $\sigma_2\sigma_1^2\sigma_2$,
- $\sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i}$ for $i = 1, \dots, n-1$,

where $\sigma_1, \dots, \sigma_{2n-1}$ are the standard generators of B_{2n}.

Lemma 5.2 (Birman) *Two oriented braids $\beta_1 \in B_{2n_1}$ and $\beta_2 \in B_{2n_2}$ have isotopic plat closures if and only if they are related by a finite sequence of the following moves.*

- $\beta \mapsto g\beta h$, *where* $\beta \in B_{2n}$ *and* $g, h \in K_{2n}$.
- $\beta \leftrightarrow \sigma_{2n}\beta$, *where* $\beta \in B_{2n}$ *and* $\sigma_{2n}\beta \in B_{2n+2}$.

The move $\beta \mapsto \sigma_{2n}\beta \in B_{2n+2}$ is called *stabilisation.*

Our statement of Lemma 5.2 differs from that of [2] in two respects. First, Birman's result applies only to knots. This was necessary in order to show that the two types of move commute. We do not need the moves to commute, so we can apply Birman's proof to the case of links.

Second, Birman did not consider the issue of orientation. There is a small technical point here. Our statement of the lemma should really specify the effect of each move on the orientation of the braid. However since each move corresponds to an isotopy of the plat closure, it is clear what this effect should be. With this in mind, Birman's proof goes through unchanged.

Lemma 5.3 *If $\beta \in B_{2n}$ is an oriented braid and $g \in K_{2n}$ then $V'_{g\beta} = V'_{\beta}$.*

Proof It suffices to prove this in the case g is one of the generators of K_{2n}.

Recall that the coefficient in the definition of V'_{β} is

$$\lambda = (-q^{\frac{1}{2}} - q^{-\frac{1}{2}})^{-1}(-q^{\frac{3}{4}})^{w}(q^{-\frac{1}{4}})^{e+2n},$$

where w and e are respectively the writhe and exponent sum of β. Now σ_1 contributes a left-handed crossing to $\sigma_1\beta$, so the writhe of $\sigma_1\beta$ is $w-1$. For all other generators, $g\beta$ has writhe w. Thus we must show that

$$\delta\langle \tilde{S}, g\beta\tilde{T}\rangle = \langle \tilde{S}, \beta\tilde{T}\rangle,$$

where δ is $-q^{-1}$ for $g = \sigma_1$, and q^{-1} for all other generators. We can rewrite this as

$$\langle \delta^{-1}\tilde{S}, g\beta\tilde{T}\rangle = \langle g\tilde{S}, g\beta\tilde{T}\rangle.$$

Thus it suffices to prove the identities

$$g\tilde{S} = \delta^{-1}\tilde{S}.$$

In the cases $g = \sigma_1$ and $g = \sigma_{2i}\sigma_{2i-1}\sigma_{2i+1}\sigma_{2i}$, this follows easily from Lemma 4.1. In the case $g = \sigma_2\sigma_1^2\sigma_2$, it helps to first use the fork-spoon relation. □

Lemma 5.4 *If $\beta \in B_{2n}$ is an oriented braid and $h \in K_{2n}$ then $V'_{\beta h} = V'_{\beta}$.*

Proof Note that $\tilde{T}$ can be isotoped to be equal to $(1-q)^n\tilde{S}$ except in an arbitrarily small neighbourhood of the puncture points. Thus

$$\begin{aligned}\langle \tilde{S}, \beta\tilde{T}\rangle &= (1-q)^{-n}\langle \tilde{T}, \beta\tilde{T}\rangle \\ &= (1-q)^{-n}\langle \beta^{-1}\tilde{T}, \tilde{T}\rangle \\ &= \langle \beta^{-1}\tilde{T}, \tilde{S}\rangle.\end{aligned}$$

Similarly

$$\langle \tilde{S}, \beta h \tilde{T} \rangle = \langle \beta^{-1} \tilde{T}, h \tilde{S} \rangle.$$

The result now follows from the proof of the previous lemma. □

Before we move on to stabilisation, we prove the following special case of the skein relation.

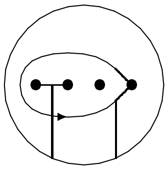
Lemma 5.5 *Let $\beta \in B_{2n}$ be an oriented braid. Suppose the strands second and third from the right at the top have a parallel orientation. Let $\beta_+ = \sigma_{2n-2}\beta$ and $\beta_- = \sigma_{2n-2}^{-1}\beta$. Then*

$$q^{-1} V'_{\beta_+} - q V'_{\beta_-} = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) V'_{\beta}.$$

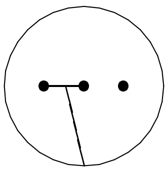
Proof If the writhe of β is w then the writhe of β_+ is $w+1$ and the writhe of β_- is $w-1$. A simple calculation reduces the problem to showing that

$$(\sigma_{2n-2} - 1)(\sigma_{2n-2} + q)\tilde{S} = 0.$$

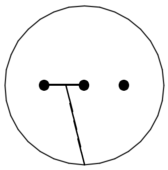
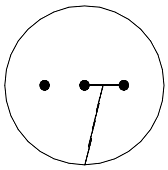
By the fork-spoon lemma, this is equivalent to

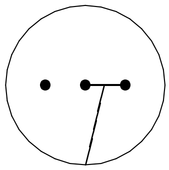
$(\sigma_{2n-2} - 1)(\sigma_{2n-2} + q)$  $= 0,$

where we have shown only a disk containing the puncture points $p_{2n-3}, \dots, p_{2n}$. To simplify notation, assume that $n = 2$ and show only a disk containing p_1, p_2 and p_3. This reduces the problem to showing that

$(\sigma_2 - 1)(\sigma_2 + q)$  $= 0.$

By Lemma 4.1,

$(\sigma_2 - 1)$  $=$  , and

$(\sigma_2 + q)$  $= 0.$

This completes the proof. □

Lemma 5.6 *Suppose $\beta \in B_{2n}$ is an oriented braid. Let $\beta'_+ = \sigma_{2n}\beta \in B_{2n+2}$. Then $V'_{\beta'_+} = V'_\beta$.*

Proof Let β' be the image of β in B_{2n+2}. Note that $\beta'_+ = \sigma_{2n}\beta'$. Let $\beta'_- = \sigma_{2n}^{-1}\beta'$. Then

$$\beta'_+\sigma_{2n+1}^2 = \sigma_{2n}\sigma_{2n+1}^2\sigma_{2n}\beta'_-.$$

One can show that σ_{2n+1} and $\sigma_{2n}\sigma_{2n+1}^2\sigma_{2n}$ lie in K_{2n+1}. By Lemmas 5.3 and 5.4, it follows that

$$V'_{\beta'_+} = V'_{\beta'_-}.$$

By Lemma 5.5,

$$q^{-1}V'_{\beta'_+} - qV_{\beta'_-} = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})V'_{\beta'}.$$

Thus

$$(-q^{\frac{1}{2}} - q^{-\frac{1}{2}})V'_{\beta'_+} = V'_{\beta'}.$$

To show that $V'_{\beta'_+} = V'_\beta$, it therefore suffices to show that

$$V'_{\beta'} = (-q^{\frac{1}{2}} - q^{-\frac{1}{2}})V'_\beta.$$

Note that β and β' have the same writhe w, and the same exponent sum e, but have $2n$ and $2n+2$ strands respectively. Thus coefficient λ used in the computation of $V'_{\beta'}$ is $q^{-\frac{1}{2}}$ times that of V'_β. The problem is therefore reduced to showing that

$$\langle \tilde{S}', \beta\tilde{T}' \rangle = (-1 - q)\langle \tilde{S}, \beta\tilde{T} \rangle,$$

where $\tilde{S}'$ is the open $(n+1)$-ball corresponding to the standard fork diagram in D_{2n+2}, and $\tilde{T}'$ is the corresponding $(n+1)$-torus.

Now $\tilde{S}'$ is the product of $\tilde{S}$ with an edge, and $\tilde{T}'$ is the product of $\tilde{T}$ with a circle. This edge meets this circle at two points in the disk. These two points contribute -1 and $-q$ times $\langle \tilde{S}, \beta\tilde{T} \rangle$. To see this, consider the computation of the pairing described in Section 3. □

We have shown that V'_β is invariant under each of the moves in 5.2. Thus it is an isotopy invariant of the plat closure L of β.

We now prove the skein relation given in Section 1. Let L_+, L_- and L_0 be as defined there. Since V' is an isotopy invariant, we are free to choose any representation of these links as plat closures of a braid. In particular, we can move the ball on which the three links differ to the top right of the diagram. Then we can isotope the rest of the diagram to be a plat closure. Thus we can reduce to the case which was already proved in Lemma 5.5.

Finally, a direct computation as described in Section 3 verifies that our invariant of the unknot is one. This completes the proof that V'_β is the Jones polynomial of the plat closure of β.

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