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Periodicity of Goussarov-Vassiliev knot invariants

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Abstract The paper is a survey of known periodicity properties of finite type invariants of knots, and their applications.

AMS Classification 57N10; 57M25

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Dedicated to the memory of M. Goussarov

1 Introduction

1.1 History

According to the story communicated to us by M. Polyak and O. Viro, in the fall of 1988 M. Goussarov gave two talks about some invariants of knots that behave in a certain sense polynomially with respect to modifications of a knot. Goussarov's talk was neglected, and a few years later with entirely different motivation V. Vassiliev axiomatically introduced the same invariants of knots. These Goussarov-Vassiliev invariants have come to be known as *finite type invariants.*

This expository paper is concerned with periodicity properties of the finite type invariants of knots. Our aim is to focus on the major ideas and open problems; we will avoid proofs which the reader may find in the existing literature.

1.2 What is periodicity for the colored Jones function?

In order to motivate and explain these properties, we need to reverse our steps a few years earlier. In 1985 Jones discovered his famous knot polynomial, [21]. To a knot Jones associated a polynomial, i.e., an element of $\mathbb{Z}[q, q^{-1}]$ (here and below we normalize our invariants to equal to 1 for the unknot). In one of its

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views, the Jones polynomial comes from a 2-dimensional irreducible representation of the Lie algebra $\mathfrak{sl}_2$. By considering for each natural number d the $(d+1)$-dimensional irreducible representation V_d of $\mathfrak{sl}_2$, one obtains a sequence of polynomials associated to a given knot. This sequence is best organized in the *colored Jones function* $J(K)$ of a knot K, which is a 2-parameter formal power series

$$J(K)(h,\lambda) = \sum_{n,m=0}^{\infty} a_{n,m}(K)h^n\lambda^m$$

(see [2]) with remarkable periodicity properties. By its definition, if $\lambda = d$ is a natural number, $J(K)(h,d)$ coincides up to a change of variables with the Jones polynomial using V_d, thus for each fixed d we have:

$$J(K)(h,d) \in \mathbb{Z}[e^h, e^{-h}].$$

We can think of this as a *periodicity property* (that is, a *recursion relation*) for the coefficients $a_{n,m}$ of J. By this we mean the following: we call a sequence (b_n) *periodic* if the generating function $\sum_n b_n t^n$ is a rational function of e^t. Thus, for each fixed d, the sequence $(\sum_m a_{n,m}d^m)$ is periodic. From our point of view, this is an obvious periodicity property of J.

We now discuss a *hidden periodicity* property of J. Each coefficient $a_{n,m}$ is a finite type invariant of degree n and vanishes if $m > n$, [2]. Thus, we can rewrite the colored Jones function as a sum of subdiagonal terms $Q_{J,k}$

$$J(K)(h,\lambda) = \sum_{k=0}^{\infty} h^k Q_{J,k}(K)(h\lambda)$$

where

$$Q_{J,k}(K)(h) = \sum_{m=0}^{\infty} a_{k+m,m}(K)h^m.$$

It was conjectured by Melvin-Morton and Rozansky, and proven in [2], that $Q_{J,0}$ equals (up to a logarithm and a change of variables) to the *Alexander polynomial* Δ of the knot. More precisely, we have

$$Q_{J,0}(K)(h) = -\frac{1}{2}\log\Delta(K)(e^h).$$

This translates to a hidden periodicity of the coefficients $a_{n,n}$ of J.

Rozansky, in his study of the colored Jones function conjectured that for every $n \geq 1$, $Q_{J,n}$ is (up to a change of variable $t = e^h$) a rational function of t, whose denominator is a $(2n+1)$st power of the Alexander polynomial. Using a variety of quantum field theory techniques and an appropriate expansion of the R-martix, Rozansky proved this Rationality Conjecture for the colored Jones fucntion, see [28].

1.3 What is periodicity for the Kontsevich integral?

It follows from work of Drinfeld [4] that the colored Jones function is the image of a universal finite type invariant, the so-called *Kontsevich integral*

$$Z : \text{Knots} \longrightarrow \mathcal{A}(\star)$$

where $\mathcal{A}(\star)$ is a vector space over $\mathbb{Q}$ spanned by graphs, modulo the AS and IHX relations, [1]. The graphs in question are uni-trivalent graphs (with vertex orientations) and have two notions of complexity: the degree (i.e., the number of vertices) and the number of trivalent vertices. The Kontsevich integral has a subdiagonal expansion (in terms of graphs with a fixed number of trivalent vertices). Each term of that expansion is a series of uni-trivalent graphs with fixed number of trivalent vertices and an arbitrary number of univalent ones. Rozansky conjectured that each term of the expansion of the Kontsevich integral should be given by a series of trivalent graphs with rational functions attached to their edges. This is often called the *Rationality Conjecture* for the Kontsevich integral, which we now describe.

A weak form of the Rationality Conjecture was proven by Kricker [22]. A stong form followed by joint work with Kricker [9], where a *rational form* Z^{rat} of the Kontsevich integral Z was constructed.

The construction of the Z^{rat} invariant is rather involved. A key ingredient is the *surgery view of knots*, that is the presentation of knots by surgery on framed links in a solid torus, modulo some kirby-type relations, [11]. Another key ingredient is that the values of the Z^{rat} invariant are trivalent graphs *with beads*, that is rational functions attached on their edges. More precisely,

$$Z^{\mathrm{rat}} : \text{Knots} \longrightarrow \mathcal{B}(\Lambda \to \mathbb{Z}) \times \mathcal{A}(\Lambda_{\mathrm{loc}})$$

where

- $\mathcal{B}(\Lambda \to \mathbb{Z})$ is a quotient of the set of Hermitian matrices over $\Lambda = \mathbb{Z}[t, t^{-1}]$ which are invertible over $\mathbb{Z}$, modulo the equivalence $A \sim B$ iff $A \oplus D = P(B \oplus E)P^\star$ for diagonal matrices D, E with monomials in t on the diagonal and for P invertible over Λ
- $\mathcal{A}(\Lambda_{\mathrm{loc}})$ is a vector space spanned by trivalent graphs with rational functions (that is elements of $\Lambda_{\mathrm{loc}} = \{f/g \,|\, f, g \in \Lambda,\, g(1) = 1\}$), attached on their edges, modulo some relations explained in [9].

There is a *hair* map

$$\text{Hair} : \mathcal{B}(\Lambda \to \mathbb{Z}) \times \mathcal{A}(\Lambda_{\mathrm{loc}}) \longrightarrow \mathcal{A}(\star)$$

defined by

$$\operatorname{Hair}(A, s) = \exp\left(-\frac{1}{2}\operatorname{tr}\log(A)(e^h)|_{h^n \to w_n}\right) \operatorname{Hair}(s)$$

for $A \in \mathcal{B}(\Lambda \to \mathbb{Z})$ and $s \in \mathcal{A}(\Lambda_{\mathrm{loc}})$, where w_n is the wheel with n legs, multiplication is given by the disjoint union and $\operatorname{Hair}(s)$ replaces t in each bead of s by an exponential of legs as follows:

$$t \to \sum_{n=0}^{\infty} \frac{1}{n!} \quad n \text{ legs}$$

In other words, the Hair map sends the matrix part to wheels and replaces the variable t of a bead in terms of the exponential of legs. The content of [9, Theorem 1.3] is the following commutative diagram

$$\begin{tikzcd}
\text{Knot} \arrow[r, "Z^{\mathrm{rat}}"] \arrow[dr, "Z"'] & \mathcal{B}(\Lambda \to \mathbb{Z}) \times \mathcal{A}(\Lambda_{\mathrm{loc}}) \arrow[d, "\text{Hair}"] \\
& \mathcal{A}(\star)
\end{tikzcd}$$

which is in a strong form the Rationality Conjecture for the Kontsevich integral (a weak form of the conjecture only states that the image of Z is in the image of the Hair map). It was recently shown by Patureau-Mirand that the Hair map is not 1-1, [27], thus the strong form of the Rationality Conjecture is potentially stronger than the weak form.

2 Properties and applications of the Z^{rat} invariant

Although the Z^{rat} invariant is a rather complicated object, certain parts of it can be explained using classical topology.

2.1 The matrix part of Z^{rat} and the Blanchfield pairing

In [9] it was shown that the matrix part of $Z^{\mathrm{rat}}(M, K)$ determines the *Blanchfield pairing* of (M, K), that is the intersection form

$$H_1(\widetilde{M - K}, \mathbb{Z}) \times H_1(\widetilde{M - K}, \mathbb{Z}) \longrightarrow \Lambda_{\mathrm{loc}}/\Lambda.$$

The converse is also true, and will be postponed to a future publication.

2.2 The loop move, finite type invariants and 0-equivalence

One of the beautiful ideas originating in the work of Goussarov (and Habiro) is that a *geometric move* on a set S leads, upon iteration, to the dual notions of *finite type invariants* (i.e., certain numerical invariants $f : S \to A$ with values in an abelian group A (eg. $A = \mathbb{Z}, \mathbb{Q}$)) and n*-equivalence* (i.e., certain quotients of the set S that are often abelian groups).

A main example of this idea is to consider the set of knots in S^3 and the move to be an I*-modification* (this is a special case of surgery on a clasper, and has the same result as a crossing change of a knot). This leads to the theory of finite type invariants of knots in S^3. 0-equivalence (that is knots, modulo I-modifications) is trivial since every knot can be unknotted by a sequence of I-modifications.

Another example of this idea is to consider the set of knots K in integral homology 3-spheres M, and the *loop move* introduced at [12]. The loop move replaces a pair (M, K) by the result of surgery on a *clasper* $G \subset M - K$ whose leaves have linking number zero with K.

The graph-part of Z^{rat} takes values in a graded vector space (where the degree of a trivalent graph is the number of trivalent vertices), thus we can talk about the degree $2n$ part Z^{rat}_{2n} of Z^{rat}. Although Z^{rat}_{2n} is not a finite type invariant (since it determines a power series of finite type invariants under the Hair map), it is a finite type invariant of type $2n$ with respect to the loop move, as was shown in [9].

In [12] it was shown that for two pairs (M, K) and (M', K') the following are equivalent:

- They can be obtained one from the other via a sequence of loop moves (this is often called 0*-equivalence* with respect to the loop move).
- They are S*-equivalent*, i.e., their Seifert matrices are S-equivalent.
- They have isometric Blanchfield pairings.

In conjunction with the above, it follows that the matrix part of the Z^{rat} invariant precisely describes 0-equivalence with respect to the loop move.

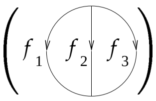
We will now sample some applications of the Z^{rat} invariant.

2.3 The Casson-Walker invariant of cyclic branched coverings of a knot

This application involves the computation of the Casson-Walker invariant of cyclic branched covers of a knot. Given a knot K in an integral homology 3-sphere M and a positive integer p, one can construct a closed 3-manifold, the p-fold branched covering $\Sigma^p_{(M,K)}$ of K in M. The sequence of 3-manifolds $\Sigma^p_{(M,K)}$ is closely related to the *signature function* $\sigma(M,K): S^1 \to \mathbb{Z}$ of a knot, which properly normalized is a key concordance invariant. The first nontrivial finite type invariant of 3-manifolds is the Casson-Walker invariant λ. Suppose that $\Sigma^p_{(M,K)}$ is a rational homology 3-sphere (or equivalently, that the Alexander polynomial of (M,K) has no complex pth roots of unity). In [10, Corollary 1.4] we showed that for all (M,K) and p as above, we have

$$\lambda(\Sigma^p_{(M,K)}) = \frac{1}{3}\mathrm{Res}_p Z_2^{\mathrm{rat}}(M,K) + \frac{1}{8}\sigma_p(M,K)$$

where $\sigma_p(M,K) = \sum_{\omega^p=1} \sigma(M,K)(\omega)$ is the *total p-signature* and

$\mathrm{Res}_p\Bigg($ 

$\Bigg) = \frac{1}{p}\sum f_1(\omega_1) f_2(\omega_2) f_3(\omega_3)$

where the sum is over all triples $(\omega_1, \omega_2, \omega_3)$ that satisfy $\omega_1^p = \omega_2^p = \omega_3^p = \omega_1\omega_2\omega_3 = 1$. In particular, the growth rate of the Casson invariant of cyclic branched covers is given by:

$$\lim_{p\to\infty} \frac{\lambda(\Sigma^p_{(M,K)})}{p} = \frac{1}{3}\int_{S^1\times S^1} Q(M,K)(s)d\mu(s) + \frac{1}{8}\int \sigma_s(M,K)d\mu(s)$$

where $d\mu$ is the Haar measure. The above formulas are part of the computation of the full LMO invariant of cyclic branched covers in terms of the signature function and residues of the Z^{rat} invariant, [10, Theorem 1].

2.4 On knots with trivial Alexander polynomial

Topological surgery in dimension 4 is a list of surgery problems that sometimes one can be reduced to another. A list of *atomic surgery problems* (that is problems that every surgery problem reduces to, in dimension 4) were compiled by Casson and Freedman. Such a list is not unique but various versions of it involve the *slicing of boundary links* whose free covers are acyclic. In the case of knots, this condition is equivalent to the triviality of the Alexander polynomial.

Freedman showed that Alexander polynomial 1 knots are indeed topologically slice, [6].

However, for over 15 years there was a confusion: it was thought that every Alexander polynomial 1 knot bounds a Seifert surface whose Seifert form has minimal rank (that is, rank equal to the genus of the surface), [5]. This turnt out to be false, [15]. The required invariant turnt out to be the 2-loop part Z_2^{rat} of the rational invariant Z^{rat}. No classical knot invariant (such as signature, Blanchfield pairing, Casson-Gordon, and Cochran-Orr-Teichner L^2 signature invariants) would have worked, as all these vanish on knots with trivial Alexander polynomial. Thus, the Z_2^{rat} invariant is an obstruction to a knot bounding a Seifert surface of a prescribed type.

In subsequent work with P. Teichner, we introduce a decreasing filtration on the set of Alexander polynomial 1 knots [16], which is strictly decreasing every other step, and which in degree 1 equals to the set of knots which bound minimal rank Seifert surfaces. The *Hyperbolic Volume Conjecture* of knots (in its Simplicial Volume formulation) implies that the intersection of this filtration is the unknot, [25].

Whether this descreasing filtration on good knots is related to a tower of smooth slicing obstructions of good knots is an unknown problem.

2.5 Lifting from the Lie algebra to the Lie group

Given a Lie algebra $\mathfrak{g}$ with invariant inner product, there is a map (often called a *weight system*)

$$W_{\mathfrak{g}}^h : \mathcal{A}(\star) \longrightarrow S(\mathfrak{g})^{\mathfrak{g}}[[h]]$$

where $S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$. The image of the Kontsevich integral under this map coincides with the colored Jones function. This weight system replaces unitrivalent graphs by Lie invariant tensors.

Similarly, given a compact connected Lie group G with (complexified) Lie algebra $\mathfrak{g}$, there is a map

$$W_G^h : \mathcal{A}(\Lambda_{\mathrm{loc}}) \longrightarrow C_{\mathrm{alg}}(G)^G[[h]]$$

where $C_{\mathrm{alg}}(G)$ is the algebra of *almost invariant functions* $f : G \to \mathbb{C}$ (that is those continuous functions that generate a finite dimensional subspace of the algebra of continuous functions $C(G)$ under the action of G on itself by conjugation). $C_{\mathrm{alg}}(G)^G$ is a finitely generated algebra, as follows from the Peter-Weyl theorem. For a discussion, see [14].

Using this, it is fairly easy to show that the Rationality Conjecture for the Kontsevich integral implies the Rationality Conjecture for the colored Jones function, see [14, Theorem 2] and also [29].

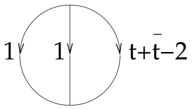
2.6 Sample calculations

Can we compute the Z^{rat} of any knot? This is a hard question, since such a computation would in particular compute the Kontsevich integral of that knot. Computations can be done for torus knots, generalizing work of Bar-Natan and Lawrence, [3]. On the other hand, the author does not know how to compute the Z^{rat} invariant of the figure eight knot.

One can ask for less, however. Rozansky has written a computer program that can compute the Z_2^{rat} invariant of knots which are presented as closures of braids, [29]

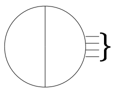
A different algorithm for computing Z_2^{rat} for knots has been given in [12], where knots are presented via surgery on claspers. For more computations, see also [15] and [23]. We mention a sample computation here for an untwisted Whitehead double $\mathrm{Wh}(K)$ of a knot K, taken from [13, Corollary 1.1]:

$$Z_2^{\mathrm{rat}}(\mathrm{Wh}(K)) = a(K) \cdot$$



where $a(K) = \frac{1}{4}\frac{d^2}{dh^2}\Delta(K)(e^h)_{h=0} \in \mathbb{Z}$. This implies that the Kontsevich integral of $\mathrm{Wh}(K)$ is given by

$$Z(\mathrm{Wh}(K)) = \exp\left(2\sum_{n=1}^{\infty}\frac{a(K)}{(2n)!}\right.$$



$$\left.\}_{2n\,\mathrm{legs}} + \dots\right)$$

where the dots indicate connected unitrivalent graphs with at least three loops and an arbitrary number of legs. This illustrates the periodicity of the 2-loop part of the Kontsevich integral.

2.7 Integrality properties of the colored Jones function

This is a topic complementary to periodicity, that we briefly discuss. Recall that $J(h,d) \in \mathbb{Z}[e^h, e^{-h}]$, which is an *integrality property* of coefficients $a_{n,m}$ of J, in the sense that certain linear combinations of rational numbers are actually integers. Rozansky conjectured and further proved in [28] that each of the subdiagonal terms $Q_{J,k}$ $(k \geq 1)$ of the colored Jones polynomial can be written in the form

$$Q_{J,k}(h) = \frac{P_{J,k}(e^h)}{\Delta^{2n+1}(e^h)}$$

for polynomials $P_{J,k}(t) \in \mathbb{Z}[t, t^{-1}]$. This is a hidden integrality property of the coefficients of J, and can be used, as Rozanky showed, to show that the image of the LMO invariant under the $\mathfrak{sl}_2$ weight system coincides with the p-adic expansion of the *Reshetikhin-Turaev* invariant of manifolds obtained by nonzero surgery on a knot, [30].

3 What next?

3.1 Boundary links

The Rationality Conjecture (RC) for the Kontsevich integral of knots has been extended to links in two different directions. In one extension, one considers boundary links. If we are to have a RC for the Kontsevich integral of links, we need to restrict to links whose Kontsevich integral has vanishing tree part. This is equivalent to the vanishing of all Milnor invariants, as was shown by Habegger-Masbaum, [19]. The class of links with vanishing Milnor invariants contains (and perhaps coincides with) the class of *homology boundary links*, [24]. For simplicity, we can restrict our attention to boundary links.

An extra point to keep in mind is that the Kontsevich integral of a link takes value in uni-trivalent graphs whose legs are colored by the components of the link. Further, legs with different colored do not commute (i.e., the order by which they are attached on an edge is important). In this context, the RC for the Kontsevich integral of boundary links is a statement about *rational functions in noncommuting variables*. Luckily, there is well-developed algebra to deal with this, that comes under the name of *noncommutative localization* (i.e., Cohn localization) of the group-ring of the free group. Adapting this, it was possible to define a rational noncommutative invariant Z^{rat} of boundary links that determines the Kontsevich integral of the boundary link.

3.2 Algebraically connected links

In a direction perpendicular to that of boundary links, Rozansky considers links L with nonvanishing (multivariable) Alexander polynomial. The simplest example of such a link is the Hopf link.

Rozansky's RC is a *relative* conjecture, formulated for the colored Jones function. Whether it can be lifted to a relative RC for the Kontsevich integral of alg. connected links remains to be seen.

3.3 More periodicity properties?

The best that one could hope for is that after a change of variables, the two parameter series $J(h, \lambda)$ is given by a single rational function. This is probably false, but can we prove it?

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