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Quantum invariants of Seifert 3–manifolds and their asymptotic expansions

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Abstract We report on recent results of the authors concerning calculations of quantum invariants of Seifert 3–manifolds. These results include a derivation of the Reshetikhin–Turaev invariants of all oriented Seifert manifolds associated with an arbitrary complex finite dimensional simple Lie algebra, and a determination of the asymptotic expansions of these invariants for lens spaces. Our results are in agreement with the asymptotic expansion conjecture due to J. E. Andersen [1], [2].

AMS Classification 57M27; 17B37, 18D10, 41A60

Keywords Quantum invariants, Seifert manifolds, modular categories, quantum groups, asymptotic expansions

1 Introduction

In 1988 E. Witten [34] proposed new invariants $Z_k^G(X,L) \in \mathbb{C}$ of an arbitrary closed oriented 3–manifold X with an embedded colored link L by quantizing the Chern–Simons field theory associated to a simple and simply connected compact Lie group G, k being an arbitrary positive integer called the (quantum) level. The invariant $Z_k^G(X,L)$ is given by a Feynman path integral over the (infinite dimensional) space of gauge equivalence classes of connections in a G bundle over X. This integral should be understood in a formal way since, at the moment of writing, it seems that no mathematically rigorous definition is known, cf. [17, Sect. 20.2.A]. The invariants Z_k^G are called the quantum G–invariants or Witten's invariants associated to G.

Shortly afterwards, N. Reshetikhin and V. G. Turaev [27] defined in a mathematically rigorous way invariants $\tau_r^{\mathfrak{sl}_2(\mathbb{C})}(X,L) \in \mathbb{C}$ of the pair (X,L) by combinatorial means using irreducible representations of the quantum deformations of $\mathfrak{sl}_2(\mathbb{C})$ at certain roots of unity, r being an integer ≥ 2 associated to the

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order of the root of unity. Later quantum invariants $\tau_r^{\mathfrak{g}}(X,L) \in \mathbb{C}$ associated to other complex simple Lie algebras $\mathfrak{g}$ were constructed using representations of the quantum deformations of $\mathfrak{g}$ at 'nice' roots of unity, see [32]. We call $\tau_r^{\mathfrak{g}}$ for the quantum $\mathfrak{g}$–invariants or the RT–invariants associated to $\mathfrak{g}$.

Both in Witten's approach and in the approach of Reshetikhin and Turaev the invariants are part of a topological quantum field theory (TQFT) (or more correctly a family of TQFT's). This implies that the invariants are defined for compact oriented 3–dimensional cobordisms (perhaps with some extra structure on the boundary), and satisfy certain cut-and-paste axioms, see [3], [9], [25], [31]. The TQFT's of Reshetikhin and Turaev can from an algebraic point of view be given a more general formulation by using so-called modular (tensor) categories [31]. The representation theory of the quantum deformations of $\mathfrak{g}$ at certain roots of unity, $\mathfrak{g}$ an arbitrary finite dimensional complex simple Lie algebra, induces such modular categories, see e.g. [20], [8], [21].

For an invariant to be powerful one should be able to calculate it. A problem with the quantum invariants of knots and 3–manifolds is that they are rather hard to calculate. In fact people have only been able to calculate these invariants for certain (families of) knots and 3–manifolds. The lens spaces and more generally the Seifert 3–manifolds constitute such a family, and there is a wealth of literature about different calculations of quantum invariants of these spaces, see [11, Introduction] for some references. In [11] the RT–invariant associated to an arbitrary modular category is calculated for any Seifert manifold, cf. [11, Theorem 4.1]. (Here and in the rest of this paper a 3–manifold means a closed oriented 3–manifold. In particular, a Seifert manifold is an oriented Seifert manifold.)

A solution to the above problem and to the general problem of understanding the topological 'meaning' of the quantum invariants could be to determine relationships between the quantum invariants and classical (well understood and calculable) invariants. However, this seems to be a rather hard task. This leads us into one of the themes in this article, namely asymptotic expansions of the invariants. By using stationary phase approximation techniques together with path integral arguments Witten was able [34] to express the leading asymptotics of $Z_k^G(X)$ in the limit $k \to \infty$ as a sum over the set of stationary points for the Chern–Simons functional. The terms in this sum are expressed by such topological/geometric invariants as Chern–Simons invariants, Reidemeister torsions and spectral flows, so here we see a way to extract topological information from the invariants. A full asymptotic expansion of Witten's invariant is expected on the basis of a full perturbative analysis of the Feynman path integral, see [6], [7], [5]. It is generally believed that the family of TQFT's of Reshetikhin and

Turaev is a mathematical realization of Witten's family of TQFT's. This belief has together with known results concerning asymptotics of the RT–invariants lead to a conjecture, the asymptotic expansion conjecture (AEC), which specifies in a rather precise way the asymptotic behaviour of the RT–invariants. The AEC was proposed by Andersen in [1], where he proved it for mapping tori of finite order diffeomorphisms of orientable surfaces of genus at least two using the gauge theoretic definition of the quantum invariants.

In this paper we explain recent results of the authors concerning the RT–invariants of Seifert manifolds. Explicitly we state formulas for the invariants $\tau_r^{\mathfrak{g}}$ of all Seifert manifolds in terms of the Seifert invariants and standard data for $\mathfrak{g}$, $\mathfrak{g}$ being an arbitrary complex finite dimensional simple Lie algebra, cf. Theorem 3.5. Moreover, we analyse more carefully the invariants $\tau_r^{\mathfrak{g}}(X)$ for X any lens space, thereby determining a formula for the large r asymptotics of these invariants, cf. Theorem 4.2 and the remark following this theorem. This formula is in agreement with the AEC.

A part of the paper is concerned with studying a certain family of finite dimensional complex representations $\mathcal{R} = \mathcal{R}_r^{\mathfrak{g}}$ of $SL(2,\mathbb{Z})$. These representations are known from the study of theta functions and modular forms in connection with the study of affine Lie algebras, cf. [19], [18, Sect. 13]. They also play a fundamental role in conformal field theory and (therefore) in the Chern–Simons TQFT's of Witten, see e.g. [10], [33], [34]. In case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, Jeffrey [15], [16] has determined a nice formula for $\mathcal{R}_r^{\mathfrak{g}}(U)$ in terms of the entries in $U \in SL(2,\mathbb{Z})$. Theorem 3.3 is a direct extension of Jeffrey's result to arbitrary $\mathfrak{g}$. The representations $\mathcal{R}_r^{\mathfrak{g}}$ are of interest when calculating the RT–invariants of the Seifert manifolds since certain matrices, which can be expressed through $\mathcal{R}_r^{\mathfrak{g}}$, enter into the formulas of the invariants.

The paper is organized as follows. In Sect. 2 we introduce notation for the Seifert manifolds and recall surgery presentations for these manifolds. In Sect. 3 we explain our calculation of the $\mathfrak{g}$–invariants of the Seifert manifolds. In Sect. 4 we state the asymptotic expansion conjecture and determine the asymptotic expansions of the $\mathfrak{g}$–invariants of the lens spaces. In the appendix we sketch the proof of the formula for the entries in $\mathcal{R}_r^{\mathfrak{g}}(U)$, $U \in SL(2,\mathbb{Z})$, Theorem 3.3. The paper is to some extend expository. Details and most technicalities, in particular in connection to the proof of Theorem 3.3, will be given in [13].

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Mathématique Avancée (IRMA), Université Louis Pasteur and C.N.R.S., Strasbourg, while being a Marie Curie Fellow. A part of this work was done while the second author visited IRMA. She thanks this department for hospitality during here stay. Another part was done while both authors visited the Research Institute for Mathematical Sciences (RIMS), Kyoto University. We would like to thank RIMS for hospitality during the special month on *Invariants of knots and 3–manifolds*, September 2001. We also thank the organisers of this program for letting us present this work at the workshop of the special month. Finally the first author thanks J. E. Andersen for helpful conversations about quantum invariants in general and about asymptotics of these invariants in particular.

2 Seifert manifolds

For Seifert manifolds we will use the notation introduced by Seifert in his classification results for these manifolds, see [29], [30], [11, Sect. 2]. That is, $(\epsilon; g \mid b; (\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$ is the Seifert manifold with orientable base of genus $g \geq 0$ if $\epsilon = \mathrm{o}$ and non-orientable base of genus $g > 0$ if $\epsilon = \mathrm{n}$ (where the genus of the non-orientable connected sum $\#^k \mathbb{RP}^2$ is k). (In [29], [30] $(\epsilon; g \mid b; (\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$ is denoted $(\mathrm{O}, \epsilon; g \mid b; \alpha_1, \beta_1; \dots ; \alpha_n, \beta_n)$, but we leave out the O, since we are only dealing with oriented Seifert manifolds.) The pair (α_j,β_j) of coprime integers is the (oriented) Seifert invariant of the j'th exceptional (or singular) fiber. We have $0 < \beta_j < \alpha_j$. The integer $-b$ is equal to the Euler number of the Seifert fibration $(\epsilon; g \mid b)$ (which is a locally trivial S^1–bundle). More generally, the Seifert Euler number of $(\epsilon; g \mid b; (\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$ is $E = -\left(b + \sum_{j=1}^n \beta_j/\alpha_j\right)$. We note that lens spaces are Seifert manifolds with base S^2 and zero, one or two exceptional fibers. According to [23, Fig. 12 p. 146], the manifold $(\epsilon; g \mid b; (\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$ has a surgery presentation as shown in Fig. 1 if $\epsilon = \mathrm{o}$ and as shown in Fig. 2 if $\epsilon = \mathrm{n}$. The $\underbrace{g}$ indicate g repetitions.

For completeness we will also state the results in terms of the non-normalized Seifert invariants due to W. D. Neumann, see [14]. For a Seifert manifold X with non-normalized Seifert invariants $\{\epsilon; g; (\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n)\}$ the invariants ϵ and g are as above. The (α_j,β_j) are here pairs of coprime integers with $\alpha_j > 0$ but not necessarily with $0 < \beta_j < \alpha_j$. These pairs are not invariants of X, but can be varied according to certain rules. In fact, X has a surgery presentation as shown in Fig. 1 with $b = 0$ if $\epsilon = \mathrm{o}$ and as shown in Fig. 2 with $b = 0$ if $\epsilon = \mathrm{n}$. The Seifert Euler number of X is $-\sum_{j=1}^n \beta_j/\alpha_j$ (which is an invariant of the Seifert fibration X). For more details, see [14, Sect. I.1].

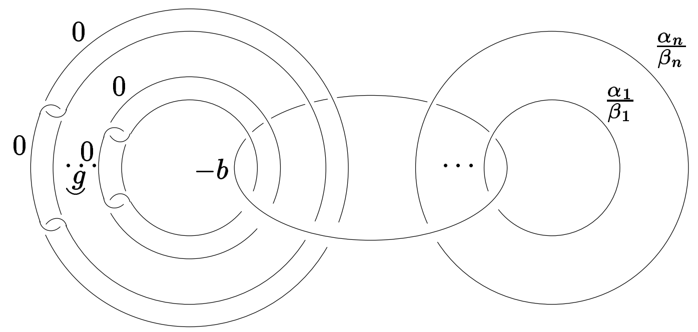

Figure 1: Surgery presentation of $(\mathrm{o}; g \mid b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$

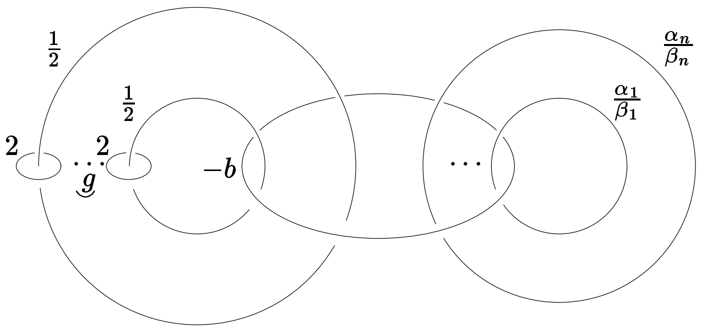

Figure 2: Surgery presentation of $(\mathrm{n}; g \mid b; (\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n))$

3 Quantum invariants of Seifert manifolds

In this section we explain our calculation of the $\mathfrak{g}$–invariants of the Seifert manifolds, $\mathfrak{g}$ being an arbitrary complex finite dimensional simple Lie algebra. Our starting point is a formula for the RT–invariants associated to an arbitrary modular category of the Seifert manifolds, derived in [11].

The RT–invariants of the Seifert manifolds for modular categories Let us first give some preliminary remarks on modular categories. We use notation as in [31]. Let $(\mathcal{V}, \{V_i\}_{i \in I})$ be an arbitrary modular category with braiding c and twist θ. The ground ring is $K = \mathrm{Hom}_{\mathcal{V}}(\mathbb{I}, \mathbb{I})$, where $\mathbb{I}$ is the unit object. Let $i \mapsto i^*$ be the involution in I determined by the condition that V_{i^*} is isomorphic to the dual of V_i. An element $i \in I$ is called self-dual if $i = i^*$.

For such an element we have a K–module isomorphism $\mathrm{Hom}_{\mathcal{V}}(V \otimes V, \mathbb{I}) \cong K$, $V = V_i$. The map $x \mapsto x(\mathrm{id}_V \otimes \theta_V)c_{V,V}$ is a K–module endomorphism of $\mathrm{Hom}_{\mathcal{V}}(V \otimes V, \mathbb{I})$, so is a multiplication by a certain $\varepsilon_i \in K$. By the definition of the braiding and twist we have $(\varepsilon_i)^2 = 1$. In particular $\varepsilon_i \in \{\pm 1\}$ if K is a field. There is a distinguished element in I denoted 0, such that $V_0 = \mathbb{I}$.

The S– and T–matrices of $\mathcal{V}$ are the matrices $S = (S_{ij})_{i,j\in I}$, $T = (T_{ij})_{i,j\in I}$ given by $S_{ij} = \mathrm{tr}(c_{V_j,V_i} \circ c_{V_i,V_j})$ and $T_{ij} = \delta_{ij} v_i$, where tr is the categorical trace of $\mathcal{V}$, δ_{ij} is the Kronecher delta equal to 1 if $i = j$ and zero elsewhere, and $v_i \in K$ such that $\theta_{V_i} = v_i \mathrm{id}_{V_i}$.

Assume that $\mathcal{V}$ has a rank $\mathcal{D}$, i.e. an element of K satisfying

$$\mathcal{D}^2 = \sum_{i\in I} \dim(i)^2,$$

where $\dim(i) = \dim(V_i) = \mathrm{tr}(\mathrm{id}_{V_i})$. We let

$$\Delta = \sum_{i\in I} v_i^{-1} \dim(i)^2.$$

Moreover, let $\tau = \tau_{(\mathcal{V},\mathcal{D})}$ be the RT–invariant associated to $(\mathcal{V}, \{V_i\}_{i\in I}, \mathcal{D})$, cf. [31, Sect. II.2]. For a tuple of integers $\mathcal{C} = (m_1, \ldots, m_t)$, let

$$G^{\mathcal{C}} = T^{m_t} S \cdots T^{m_1} S.$$

The Rademacher Phi function is defined on $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm 1\}$ by

$$\Phi\left(\begin{bmatrix} p & r \\ q & s \end{bmatrix}\right) = \begin{cases} \frac{p+s}{q} - 12(\mathrm{sign}(q))\mathrm{s}(s,|q|) & ,q \neq 0, \\ \frac{r}{s} & ,q = 0, \end{cases} \tag{1}$$

see [26]. Here, for $q \neq 0$, the Dedekind sum $\mathrm{s}(s,q)$ is given by

$$\mathrm{s}(s,q) = \frac{1}{4|q|} \sum_{j=1}^{|q|-1} \cot\frac{\pi j}{q} \cot\frac{\pi s j}{q} \tag{2}$$

for $|q| > 1$ and $\mathrm{s}(s,\pm 1) = 0$. We put $a_{\mathrm{o}} = 2$ and $a_{\mathrm{n}} = 1$. Moreover, let $b_j^{(\mathrm{o})} = 1$ and $b_j^{(\mathrm{n})} = \delta_{j,j^*}$, $j \in I$. Given pairs (α_j, β_j) of coprime integers we let $\mathcal{C}_j = (a_1^{(j)}, \ldots, a_{m_j}^{(j)})$ be a continued fraction expansion of α_j/β_j, $j = 1, 2, \ldots, n$, i.e.

$$\frac{\alpha_j}{\beta_j} = a_{m_j}^{(j)} - \cfrac{1}{a_{m_j-1}^{(j)} - \cfrac{1}{\cdots - \cfrac{1}{a_1^{(j)}}}}.$$

Theorem 3.1 [11] *Let* $M = (\epsilon; g \mid b; (\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$, $\epsilon = \mathrm{o},\mathrm{n}$. *Then*

$$\tau(M) = (\Delta\mathcal{D}^{-1})^{\sigma_\epsilon}\mathcal{D}^{a_\epsilon g-2-\sum_{j=1}^n m_j}$$
$$\times \sum_{j\in I} (\varepsilon_j)^{a_\epsilon g}\, b_j^{(\epsilon)} v_j^{-b}\dim(j)^{2-n-a_\epsilon g}\left(\prod_{i=1}^n (SG^{\mathcal{C}_i})_{j,0}\right),$$

where

$$\sigma_\epsilon = (a_\epsilon - 1)\operatorname{sign}(E) + \sum_{j=1}^n \operatorname{sign}(\alpha_j\beta_j) + \frac{1}{3}\sum_{j=1}^n\left(\sum_{k=1}^{m_j} a_k^{(j)} - \Phi(B^{\mathcal{C}_j})\right).$$

Here $E = -\left(b + \sum_{j=1}^n \frac{\beta_j}{\alpha_j}\right)$ *is the Seifert Euler number.*

The RT–invariant $\tau(M)$ *of the Seifert manifold* M *with non-normalized Seifert invariants* $\{\epsilon; g; (\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n)\}$ *is given by the same expression with the exceptions that the factor* v_j^{-b} *has to be removed and* $E = -\sum_{j=1}^n \frac{\beta_j}{\alpha_j}$. □

The theorem is also valid in case $n = 0$. In this case one just has to put all sums $\sum_{j=1}^n$ equal to zero and all products $\prod_{i=1}^n$ equal to 1. Note that $\epsilon_j^k = 1$ if k is even and $\epsilon_j^k = \epsilon_j$ if k is odd since $\epsilon_j^2 = 1$. In particular, $(\varepsilon_j)^{a_\epsilon g} = 1$ if $\epsilon = \mathrm{o}$. The sum $\sum_{j=1}^n \operatorname{sign}(\alpha_j\beta_j)$ is of course equal to n for normalized Seifert invariants.

Let us next consider the lens spaces. For p,q a pair of coprime integers, recall that $L(p,q)$ is given by surgery on S^3 along the unknot with surgery coefficient $-p/q$. In the following corollary we include the possibilities $L(0,1) = S^1\times S^2$ and $L(1,q) = S^3$, $q\in\mathbb{Z}$.

Corollary 3.2 [11] *Let* p,q *be a pair of coprime integers. If* $q \neq 0$ *we let* $(a_1,\dots,a_{m-1})$ *be a continued fraction expansion of* $-p/q$. *If* $q = 0$, *put* $m = 3$ *and* $a_1 = a_2 = 0$. *Then*

$$\tau(L(p,q)) = (\Delta\mathcal{D}^{-1})^{\sigma}\mathcal{D}^{-m}G^{\mathcal{C}}_{0,0},$$

where $\mathcal{C} = (a_1,\dots,a_{m-1},0)$ *and* $\sigma = \frac{1}{3}\left(\sum_{l=1}^{m-1} a_l - \Phi(B^{\mathcal{C}})\right)$. □

The RT–invariants of the Seifert manifolds for the classical Lie algebras It is a well-known fact that quantum deformations of the classical Lie algebras at roots of unity induce modular categories, see [20], [8], [21]. Let us provide the details needed. For simplicity we will only consider simply laced Lie algebras in this paper, except in Remark 3.6 where we give a few remarks with respect to what have to be adjusted to include the general case. (See also [13] for the general case.) Therefore, let in the following $\mathfrak{g}$ be a fixed complex finite dimensional simple and simply laced Lie algebra.

First let us fix some notation for $\mathfrak{g}$. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and let $\alpha_1, \dots, \alpha_l$ be a set of simple (basis) roots in the dual space of $\mathfrak{h}$. We denote by $\mathfrak{h}_{\mathbb{R}}^*$ the $\mathbb{R}$–vector space spanned by $\alpha_1, \dots, \alpha_l$ and let $\langle\ ,\ \rangle$ be the inner product on $\mathfrak{h}_{\mathbb{R}}^*$ defined by $\langle \alpha_i, \alpha_j \rangle = a_{ij}$, $(a_{ij})_{1\le i,j\le l}$ being the Cartan matrix for $\mathfrak{g}$. In particular, all roots have length $\sqrt{2}$. The root lattice Λ^R is the $\mathbb{Z}$–lattice generated by $\alpha_1, \dots, \alpha_l$, and the weight lattice Λ^W is the $\mathbb{Z}$–lattice generated by the fundamental weights $\lambda_1, \dots, \lambda_l$, i.e. $\lambda_i \in \mathfrak{h}_{\mathbb{R}}^*$ such that $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$ for all $i, j \in \{1, 2, \dots, l\}$. The (open) fundamental Weyl chamber is the set

$$C = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \alpha_i \rangle > 0, i = 1, \dots, l\}.$$

For a positive integer k, the k–alcove is the (closed) set

$$C_k = \{x \in \bar{C} \mid \langle x, \alpha_0 \rangle \le k\},$$

where $\bar{C}$ is the topological closure of C and α_0 is the highest root of $\mathfrak{g}$, i.e. α_0 is the unique root in C. The Weyl group is denoted W.

Let $q = e^{\pi\sqrt{-1}/r}$, where r is an integer $\ge h^\vee$. Here $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ (equal to the Coxeter number h of $\mathfrak{g}$, since $\mathfrak{g}$ is simply laced). By $U_q(\mathfrak{g})$ we denote the quantum group associated to these data as defined by Lusztig, see [22, Part V]. We follow [8, Sect. 1.3 and 3.3] here but will mostly use notation from [31] for modular categories as above. (Note that what we denote $U_q(\mathfrak{g})$ here is denoted $U_q(\mathfrak{g})|_{q=e^{\pi\sqrt{-1}/r}}$ in [8].) Let $\left(\mathcal{V}_r^{\mathfrak{g}}, \{V_i\}_{i\in I}\right)$ be the modular category induced by the representation theory of $U_q(\mathfrak{g})$, cf. [8, Theorem 3.3.20]. In particular, the index set for the simple objects is $I = \mathrm{int}(C_r) \cap \Lambda^W$. We use here the shifted indexes (shifted by ρ) (contrary to [8]). (Normally the irreducible modules of $U_q(\mathfrak{g})$ (of type 1), q a formal variable, are indexed by the cone of dominant integer weights Λ_+^W. Here we denote the irreducible module associated to $\mu \in \Lambda_+^W$ by $V_{\mu+\rho}$.) For q a root of unity as above, V_λ is an irreducible module of $U_q(\mathfrak{g})$ of non-zero dimension if $\lambda \in I$. The involution $I \to I$, $\lambda \mapsto \lambda^*$, is given by $\lambda^* = -w_0(\lambda - \rho) + \rho$, where w_0 is the longest element in W and ρ is half the sum of positive roots. The distinguished element $0 \in I$ is equal to ρ. According to [8, Theorem 3.3.20] we can use

$$\mathcal{D} = r^{l/2} \left| \frac{\mathrm{vol}(\Lambda^R)}{\mathrm{vol}(\Lambda^W)} \right|^{1/2} \left(\prod_{\alpha\in\Delta_+} 2\sin\left(\frac{\pi\langle \alpha, \rho\rangle}{r}\right)\right)^{-1} \qquad (3)$$

as a rank of $\mathcal{V}_r^{\mathfrak{g}}$. Here Δ_+ is the set of positive roots. According to the same theorem we have

$$\Delta\mathcal{D}^{-1} = \omega^{-3}, \tag{4}$$

where

$$\omega = e^{\frac{2\pi\sqrt{-1}c}{24}} = \exp\left(\frac{\pi\sqrt{-1}}{h}|\rho|^2\right)\exp\left(-\frac{\pi\sqrt{-1}}{r}|\rho|^2\right), \tag{5}$$

where $c = \frac{r-h}{r}\dim(\mathfrak{g})$ is the central charge. The last equality in (5) follows from Freudenthal's strange formula $|\rho|^2/h = \dim\mathfrak{g}/12$.

The matrices S and T for $\mathcal{V}_r^{\mathfrak{g}}$ are tightly related to a certain unitary representation $\mathcal{R} = \mathcal{R}_r^{\mathfrak{g}}$ of $SL(2,\mathbb{Z})$. On the standard generators

$$\Xi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{6}$$

of $SL(2,\mathbb{Z})$ we have

$$\begin{aligned} \mathcal{R}(\Xi)_{\lambda\mu} &= \frac{\sqrt{-1}^{|\Delta_+|}}{r^{l/2}}\left|\frac{\mathrm{vol}(\Lambda^W)}{\mathrm{vol}(\Lambda^R)}\right|^{1/2}\sum_{w\in W}\det(w)\exp\left(-\frac{2\pi\sqrt{-1}}{r}\langle w(\lambda),\mu\rangle\right), \\ \mathcal{R}(\Theta)_{\lambda\mu} &= \delta_{\lambda\mu}\exp\left(\frac{\pi\sqrt{-1}}{r}\langle\lambda,\lambda\rangle - \frac{\pi\sqrt{-1}}{h}\langle\rho,\rho\rangle\right) \end{aligned} \tag{7}$$

for $\lambda,\mu\in I$. In the following we also write $\tilde{U}$ for $\mathcal{R}(U)$. By using the results in [8, Sect. 3.3], in particular [8, Theorem 3.3.20], we find

$$S_{\lambda\mu} = \mathcal{D}\tilde{\Xi}_{\lambda\mu}, \qquad T_{\lambda\mu} = \omega\tilde{\Theta}_{\lambda\mu} \tag{8}$$

for $\lambda,\mu\in I$. Let $\mathcal{C} = (a_1,\dots,a_n)\in\mathbb{Z}^n$ and let $m\in\{0,1\}$. By (8) we immediately get

$$(S^mG^{\mathcal{C}})_{\lambda\rho} = \mathcal{D}^{m+n}\omega^{\sum_{j=1}^n a_j}\left(\tilde{\Xi}^m\tilde{\Theta}^{a_n}\tilde{\Xi}\tilde{\Theta}^{a_{n-1}}\cdots\tilde{\Theta}^{a_1}\tilde{\Xi}\right)_{\lambda\rho} \tag{9}$$

for $\lambda\in I$. Finally we have for any $\lambda\in I$ that

$$\dim(\lambda) = S_{\lambda\rho} = \mathcal{D}\tilde{\Xi}_{\lambda\rho} = \mathcal{D}r^{-l/2}\left|\frac{\mathrm{vol}(\Lambda^W)}{\mathrm{vol}(\Lambda^R)}\right|^{1/2}\prod_{\alpha\in\Delta_+}2\sin\left(\frac{\pi\langle\alpha,\lambda\rangle}{r}\right),$$

see also [8, Formulas (3.3.2) and (3.3.5)]. All the above data can now be put into the expression in Theorem 3.1 to give a formula for $\tau_r^{\mathfrak{g}}(X)$, X an arbitrary Seifert manifold, where $\tau_r^{\mathfrak{g}}$ is the RT–invariant associated to $\mathcal{V}_r^{\mathfrak{g}}$. However, the formula to emerge is not detailed enough to be of any use, at least not when it comes to a determination of asymptotics of the invariants. The reason is, that the formula will contain matrix products as in the right-hand side of (9).

A way out of this problem is to determine nice formulas for the entries of $\mathcal{R}(U)$ in terms of the entries of U. This has in fact been done for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ by Jeffrey [15], [16]. Here method is to write the matrix $U \in SL(2,\mathbb{Z})$ as a product in the generators Ξ and Θ and make a certain induction argument. A main ingredient is a reciprocity formula for Gaussian sums. We use a similar argument to extend Jeffrey's results to an arbitrary complex finite dimensional simple Lie algebra. The following theorem generalizes [16, Propositions 2.7 and 2.8].

Theorem 3.3 *Let* $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ *with* $c \neq 0$. *Then there exists an* $\epsilon \in \{\pm 1\}$ *such that*

$$\begin{aligned} \mathcal{R}(\epsilon U)_{\lambda\mu} = {} & \frac{\sqrt{-1}^{|\Delta_+|}\operatorname{sign}(\epsilon c)^{|\Delta_+|}}{(r|c|)^{l/2}\operatorname{vol}(\Lambda^R)} \exp\left(-\frac{\pi\sqrt{-1}}{h}|\rho|^2\Phi(U)\right) \\ & \times \exp\left(\frac{\pi\sqrt{-1}}{r}\frac{d}{c}|\mu|^2\right) \sum_{\nu\in\Lambda^R/c\Lambda^R} \exp\left(\frac{\pi\sqrt{-1}}{r}\frac{a}{c}|\lambda + r\nu|^2\right) \\ & \times \sum_{w\in W} \det(w)\exp\left(-\frac{2\pi\sqrt{-1}}{r\epsilon c}\langle\lambda + r\nu, w(\mu)\rangle\right). \end{aligned}$$

The function Φ is given in (1). If $c = 0$, then $U = \epsilon\Theta^b$ for some $b \in \mathbb{Z}$ and $\epsilon \in \{\pm 1\}$ and the expression for $\mathcal{R}(\epsilon U)_{\lambda\mu}$ follows immediately from (7). At first sight the above theorem looks a little strange because of the undetermined sign ϵ. This sign has to do with the fact that $\mathcal{R}$ is a representation of $SL(2,\mathbb{Z})$ and not of $PSL(2,\mathbb{Z})$ (except for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, where $\mathcal{R}$ is in fact a representation of $PSL(2,\mathbb{Z})$). However, as we shall see now, we will get rid of this sign in the cases we need. In fact, according to Theorem 3.1 and (9), we only need the expression for $\mathcal{R}(U)_{\lambda\mu}$ in case λ or μ is equal to ρ in the calculation of the invariants of the Seifert manifolds. Since $\rho^* = \rho$, $\Xi^2 = -1$, and $\mathcal{R}(\Xi^2)_{\lambda\mu} = \delta_{\lambda\mu^*}$, we have

$$\mathcal{R}(-U)_{\lambda\rho} = \mathcal{R}(U)_{\lambda\rho}, \quad \mathcal{R}(-U)_{\rho\lambda} = \mathcal{R}(U)_{\rho\lambda}$$

for all $\lambda \in I$. By using this fact and the Weyl denominator formula one can show the following corollary to Theorem 3.3. (To show the first formula in Corollary 3.4 one also has to use unitarity of $\mathcal{R}$.)

Corollary 3.4 *Let* $U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ *with* $c \neq 0$. *Then*

$$\begin{aligned} \mathcal{R}(U)_{\lambda\rho} = & \frac{\sqrt{-1}^{|\Delta_+|}\,\mathrm{sign}(c)^{|\Delta_+|}}{(r|c|)^{l/2}\,\mathrm{vol}(\Lambda^R)} \exp\left(-\frac{\pi\sqrt{-1}}{h}|\rho|^2\Phi(U)\right) \\ & \times \exp\left(\frac{\pi\sqrt{-1}}{r}\frac{a}{c}|\lambda|^2\right) \sum_{\nu\in\Lambda^R/c\Lambda^R} \exp\left(\frac{\pi\sqrt{-1}}{r}\frac{d}{c}|\rho+r\nu|^2\right) \\ & \times \sum_{w\in W} \det(w) \exp\left(-\frac{2\pi\sqrt{-1}}{rc}\langle\rho+r\nu, w(\lambda)\rangle\right). \end{aligned}$$

If $a \neq 0$ *we also have*

$$\begin{aligned} \mathcal{R}(U)_{\lambda\rho} = & \frac{\sqrt{-1}^{|\Delta_+|}\,\mathrm{sign}(c)^{|\Delta_+|}}{(r|c|)^{l/2}\,\mathrm{vol}(\Lambda^R)} \exp\left(\frac{\pi\sqrt{-1}}{r}\frac{b}{a}|\rho|^2\right) \\ & \times \exp\left(-\frac{\pi\sqrt{-1}}{h}|\rho|^2\Phi(U)\right) \\ & \times \sum_{w\in W} \det(w) \sum_{\nu\in\Lambda^R/c\Lambda^R} \exp\left(\frac{\pi\sqrt{-1}}{r}\frac{a}{c}|\lambda+r\nu-\frac{w(\rho)}{a}|^2\right). \end{aligned}$$

□

Because of the length and technical nature of the proof of Theorem 3.3, we defer the argument to the Appendix, and will only give the main ideas there. Detailed arguments will appear in [13].

Given a pair of coprime integers (α,β), $\alpha > 0$, we let β^* be the inverse of β in the multiplicative group of units in $\mathbb{Z}/\alpha\mathbb{Z}$. The following theorem is a generalization of [11, Theorem 8.4] (which concerns the case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$). The proof follows closely the proof of [11, Teorem 8.4] and is therefore left out here. (One has to use the first formula in Corollary 3.4.)

Theorem 3.5 *Let* $M = (\epsilon; g \mid b; (\alpha_1,\beta_1), \dots, (\alpha_n,\beta_n))$, $\epsilon \in \{\mathrm{o},\mathrm{n}\}$. *Then*

$$\begin{aligned} \tau_r^{\mathfrak{g}}(M) \;=\; & \exp\left(\frac{\pi\sqrt{-1}}{r}|\rho|^2\left[3(a_\epsilon-1)\,\mathrm{sign}(E) - E - 12\sum_{j=1}^n \mathrm{s}(\beta_j,\alpha_j)\right]\right) \\ & \times \frac{\sqrt{-1}^{n|\Delta_+|}\,r^{l(a_\epsilon g/2-1)}}{2^{|\Delta_+|(n+a_\epsilon g-2)}\,\mathrm{vol}(\Lambda^R)^{2-a_\epsilon g}} \frac{1}{\mathcal{A}^{l/2}} e^{\frac{3\pi\sqrt{-1}}{h}|\rho|^2(1-a_\epsilon)\,\mathrm{sign}(E)} Z_\epsilon^{\mathfrak{g}}(M;r), \end{aligned}$$

where $\mathrm{s}(\beta_j,\alpha_j)$ *is given by* (2), $\mathcal{A}=\prod_{j=1}^n \alpha_j$, *and*

$$\begin{aligned} Z_\epsilon^{\mathfrak{g}}(M;r) &= \sum_{\lambda\in I} b_\lambda^{(\epsilon)} \varepsilon_\lambda^{a_\epsilon g}\left(\prod_{\alpha\in\Delta_+}\sin^{2-n-a_\epsilon g}\left(\frac{\pi\langle\lambda,\alpha\rangle}{r}\right)\right)\exp\left(\frac{\pi\sqrt{-1}}{r}E|\lambda|^2\right) \\ &\quad\times\sum_{w_1,\dots,w_n\in W}\ \sum_{\nu_1\in\Lambda^R/\alpha_1\Lambda^R}\cdots\sum_{\nu_n\in\Lambda^R/\alpha_n\Lambda^R}\left(\prod_{j=1}^n \det(w_j)\right) \\ &\quad\times\exp\left(-\pi\sqrt{-1}\sum_{j=1}^n\frac{\beta_j^*}{\alpha_j}\left(r|\nu_j|^2+2\langle w_j(\rho),\nu_j\rangle\right)\right) \\ &\quad\times\exp\left(-\frac{2\pi\sqrt{-1}}{r}\langle\lambda,\sum_{j=1}^n\frac{r\nu_j+w_j(\rho)}{\alpha_j}\rangle\right). \end{aligned}$$

The RT–invariant $\tau_r^{\mathfrak{g}}(M)$ *of the Seifert manifold* M *with non-normalized Seifert invariants* $\{\epsilon;g;(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n)\}$ *is given by the same expression.* □

The theorem is also valid in case $n=0$. In this case one just has to put the sum $\sum_{w_1,\dots,w_n\in W}\sum_{\nu_1\in\Lambda^R/\alpha_1\Lambda^R}\cdots\sum_{\nu_n\in\Lambda^R/\alpha_n\Lambda^R}$ in $Z_\epsilon(M;r)$ equal to $1,\epsilon=\mathrm{o},\mathrm{n}$, and put $\mathcal{A}=1$ and $\sum_{j=1}^n \mathrm{s}(\beta_j,\alpha_j)=0$.

Let us finally consider the lens space $L(p,q)$. Let b,d be any integers such that $U=\begin{pmatrix} q & b \\ p & d\end{pmatrix}\in SL(2,\mathbb{Z})$. Assume $q\neq 0$, let $V=-\Xi U=\begin{pmatrix} p & d \\ -q & -b\end{pmatrix}$, and let $C'=(a_1,a_2,\dots,a_{m-1})\in\mathbb{Z}^{m-1}$ such that $B^{\mathcal{C}'}=V$. Then $\mathcal{C}'$ is a continued fraction expansion of $-p/q$ and $U=\Xi V=B^{\mathcal{C}}$, where $\mathcal{C}=(a_1,a_2,\dots,a_{m-1},0)$. By Corollary 3.2, (4) and (9) we therefore get

$$\tau_r^{\mathfrak{g}}(L(p,q))=\omega^{\Phi(U)}\tilde{U}_{\rho\rho}, \tag{10}$$

where ω is given by (5). If $q=0$ we have $p=1$ and $L(p,q)=S^3$. In this case we have $\tau_r^{\mathfrak{g}}(L(p,q))=\mathcal{D}^{-1}$. We also have $U=\Xi\Theta^d$, so by using (7), (3) and (5) we find that the right-hand side of (10) is also equal to $\mathcal{D}^{-1}$. The identity (10) coincides with [16, Formula (3.7)] for $\mathfrak{g}=\mathfrak{sl}_2(\mathbb{C})$, see also [11, Formula (49)].

Remark 3.6 Let us briefly mention the adjustments to be done for including the non-simply laced Lie algebras. In the general case the root of unity $q=e^{\pi\sqrt{-1}/(dr)}$, where $d=1$ for $\mathfrak{g}$ simply laced, $d=2$ if $\mathfrak{g}$ belongs to the series BCF and $d=3$ if $\mathfrak{g}$ is of type G_2. Moreover, Λ^R is in general the coroot lattice, which is dual to the weight lattice. The inner product $\langle\ ,\ \rangle$ in $\mathfrak{h}_{\mathbb{R}}^*$ is induced by an invariant bilinear form on $\mathfrak{g}$, and is normalized such that a long root has length $\sqrt{2}$. We stress that α_0 is the long highest root of $\mathfrak{g}$.

4 The asymptotic expansion conjecture and Seifert manifolds

For X a fixed closed oriented 3–manifold we consider $r \mapsto \tau_r^{\mathfrak{g}}(X)$ as a complex valued function on $\{h^\vee, h^\vee+1, h^\vee+2, \dots\}$. We are interested in the behaviour of this function in the limit of large r, i.e. $r \to \infty$.

It is believed that Witten's TQFT's associated to G coincides with the TQFT's of Reshetikhin–Turaev associated to $\mathfrak{g}$, where G is a simply connected compact simple Lie group with complexified Lie algebra $\mathfrak{g}$. In particular it is conjectured that Witten's leading large k asymptotics for $Z_k^G(X)$ should be valid for the function $r \mapsto \tau_r^{\mathfrak{g}}(X)$ in the limit $r \to \infty$, and furthermore, that this function should have a full asymptotic expansion. The precise formulation of this is stated in the following conjecture, called the asymptotic expansion conjecture (AEC).

Conjecture 4.1 (J. E. Andersen [1], [2]) *Let $\{\alpha_1, \dots, \alpha_M\}$ be the set of values of the Chern–Simons functional of flat G connections on a closed oriented 3–manifold X. Then there exist $d_j \in \mathbb{Q}$, $\tilde{I}_j \in \mathbb{Q}/\mathbb{Z}$, $b_j \in \mathbb{R}_+$ and $c_m^j \in \mathbb{C}$ for $j = 1, \dots, M$ and $m = 1, 2, 3, \dots$ such that*

$$\tau_r^{\mathfrak{g}}(X) \sim_{r\to\infty} \sum_{j=1}^{M} b_j e^{2\pi\sqrt{-1}r\alpha_j} r^{d_j} e^{\pi\sqrt{-1}\tilde{I}_j/4} \left(1 + \sum_{m=1}^{\infty} c_m^j r^{-m}\right), \tag{11}$$

that is, for all $N = 0, 1, 2, \dots$

$$\tau_r^{\mathfrak{g}}(X) = \sum_{j=1}^{M} b_j e^{2\pi\sqrt{-1}r\alpha_j} r^{d_j} e^{\pi\sqrt{-1}\tilde{I}_j/4} \left(1 + \sum_{m=1}^{N} c_m^j r^{-m}\right) + o(r^{d-N})$$

in the limit $r \to \infty$, where $d = \max\{d_1, \dots, d_M\}$.

As noticed by Andersen [1], [2], a complex function defined on the positive integers has at most one asymptotic expansion on the form (11) if the α_j's are rational (and mutually different), see also [12]. This means that if the AEC is true and if we futhermore have that the Chern–Simons invariants are rational (as conjectured by e.g. Auckly [4]), then all the quantities c_m^j, α_j, d_j,... are topological invariants. The AEC was first proved for the mapping tori of finite order diffeomorphisms of orientable surfaces of genus at least 2 and for any $\mathfrak{g}$ by Andersen [1] using the gauge theory definition of the invariants. Note that these mapping tori are Seifert manifolds with orientable base and Seifert Euler number equal to zero. Later on, the AEC was proved for all Seifert manifolds with orientable base or non-orientable base with even genus in case $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, cf. [12]. The proof of this result is partly based on calculations of Rozansky [28]. For more details about the AEC and conjectures about the topological interpretation of the different parts of the asymptotic formula (11) we refer to [1], [2], [12]. One can also find a review about the status of the AEC in these references.

By elaborating on the expression (10) along the same lines as in [16, Sect. 3] (using the last formula in Corollary 3.4) we find the following generalization of [16, Theorem 3.4] (valid in case $p \neq 0$).

Theorem 4.2 *The RT–invariant associated to $\mathfrak{g}$ of the lens space $L(p,q)$ is given by*

$$\begin{aligned}\tau_r^{\mathfrak{g}}(L(p,q)) = \frac{\operatorname{sign}(p)^{|\triangle_+|}\sqrt{-1}^{|\triangle_+|}}{(r|p|)^{l/2}\operatorname{vol}(\Lambda^R)} \exp\left(\frac{\pi\sqrt{-1}}{r} 12\operatorname{sign}(p)s(q,|p|)|\rho|^2\right) \\ \times \sum_{w\in W}\det(w)\exp\left(-\frac{2\pi\sqrt{-1}}{pr}\langle\rho, w(\rho)\rangle\right) \\ \times \sum_{\nu\in\Lambda^R/p\Lambda^R}\exp\left(\pi\sqrt{-1}\frac{q}{p}r|\nu|^2\right)\exp\left(2\pi\sqrt{-1}\frac{1}{p}\langle\nu, q\rho - w(\rho)\rangle\right). \quad \square\end{aligned}$$

From this theorem it is obvious, that the large r asymptotics of $\tau_r^{\mathfrak{g}}(L(p,q))$ is on the same form as in (11) (expand the factor $\exp\left(\frac{\pi\sqrt{-1}}{r}12\operatorname{sign}(p)s(q,|p|)|\rho|^2\right)$ $\exp\left(-\frac{2\pi\sqrt{-1}}{pr}\langle\rho, w(\rho)\rangle\right)$ as a power series in r^{-1}). Proving the following conjecture will therefore finalize the proof of the AEC for the invariants $\tau_r^{\mathfrak{g}}(L(p,q))$.

Conjecture 4.3 *The set of values of the Chern–Simons functional of flat G connections on $L(p,q)$ is given by*

$$\left\{ \frac{q}{2p}|\nu|^2 \pmod{\mathbb{Z}} \;\middle|\; \nu \in \Lambda^R/p\Lambda^R \right\}.$$

For $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ $(G = SU(n))$ this conjecture should follow from results in [24].

5 Appendix. The proof of Theorem 3.3

In this section we will explain the ideas behind the proof of Theorem 3.3. The underlying Lie algebra $\mathfrak{g}$ is still assumed to be simply laced for simplicity. However, with some minor adjustments the arguments given are also true for the non-simply laced Lie algebras, see Remark 3.6.

The proof of Theorem 3.3 builds mainly on the key-lemma, Lemma 5.1. Let us introduce some notation. For a tuple of integers $\mathcal{C} = (m_1, \ldots, m_t)$, let

$$B_k^{\mathcal{C}} = \begin{pmatrix} a_k^{\mathcal{C}} & b_k^{\mathcal{C}} \\ c_k^{\mathcal{C}} & d_k^{\mathcal{C}} \end{pmatrix} = \Theta^{m_k}\Xi\Theta^{m_{k-1}}\Xi\ldots\Theta^{m_1}\Xi \tag{12}$$

for $k = 1, 2, \ldots, t$, and let $B^{\mathcal{C}} = B_t^{\mathcal{C}}$, where Ξ and Θ are given by (6). Moreover, we put

$$a_0^{\mathcal{C}} = d_0^{\mathcal{C}} = 1, \qquad b_0^{\mathcal{C}} = c_0^{\mathcal{C}} = 0.$$

We say that $\mathcal{C}$ has length $|\mathcal{C}| = t$. If it is clear from the context what $\mathcal{C}$ is we write a_k for $a_k^{\mathcal{C}}$ etc. From [16, Proposition 2.5], the elements a_i, b_i, c_i, d_i satisfy the recurrence relations

$$\begin{aligned} a_k &= m_k a_{k-1} - c_{k-1}, \quad c_k = a_{k-1}, \\ b_k &= m_k b_{k-1} - d_{k-1}, \quad d_k = b_{k-1} \end{aligned} \tag{13}$$

for $k = 1, 2, \ldots, t$. One should note that the expressions (7) for the entries of $\mathcal{R}(\Xi)$ and $\mathcal{R}(\Theta)$ are well-defined for all $\lambda, \mu \in \Lambda^W$. Note also that if λ or μ are elements of Λ^W belonging to the boundary of C_r then $\mathcal{R}(\Xi)_{\lambda\mu} = 0$. This observation allows us to shift between $I = \text{int}(C_r) \cap \Lambda^W$ and $C_r \cap \Lambda^W$ as summation index set in formulas below. This shift is important in the proof of Lemma 5.1. Following Jeffrey [15, Sect. 2], [16, Sect. 2] we consider

$$\mathcal{T}_{\lambda_0,\lambda_{t+1}}^{\mathcal{C}} = \sum_{\lambda_1,\ldots,\lambda_t \in C_r \cap \Lambda^W} \tilde{\Xi}_{\lambda_{t+1}\lambda_t} \tilde{\Theta}_{\lambda_t}^{m_t} \tilde{\Xi}_{\lambda_t\lambda_{t-1}} \tilde{\Theta}_{\lambda_{t-1}}^{m_{t-1}} \tilde{\Xi}_{\lambda_{t-1}\lambda_{t-2}} \cdots \tilde{\Theta}_{\lambda_1}^{m_1} \tilde{\Xi}_{\lambda_1\lambda_0}$$

for $\lambda_0, \lambda_{t+1} \in C_r \cap \Lambda^W$, where we write $\tilde{\Theta}_\lambda$ for $\tilde{\Theta}_{\lambda\lambda}$. Then we have the following generalization of [16, Lemma 2.6]:

Lemma 5.1 *Assume that $\mathcal{C} = (m_1, \ldots, m_t)$ is a sequence of integers such that a_k is nonzero for $k = 1, \ldots, t$. Then*

$$\mathcal{T}_{\lambda_0,\lambda_{t+1}}^{\mathcal{C}} = K_{\lambda_0}^{\mathcal{C}} \sum_{w \in W} \det(w) \sum_{\mu \in \Lambda^R / a_t \Lambda^R} \exp\left(-\frac{\pi\sqrt{-1}c_t}{a_t r} \left| \lambda_{t+1} + r\mu + \frac{w(\lambda_0)}{c_t} \right|^2 \right),$$

where

$$K_{\lambda_0}^{\mathcal{C}} = \frac{\sqrt{-1}^{(t+1)|\triangle_+|}}{(r|a_t|)^{l/2}\operatorname{vol}(\Lambda^R)}\,\zeta^{l\,D_t}\exp\left(-\frac{\pi\sqrt{-1}}{h}(\sum_{i=1}^{t} m_i)|\rho|^2\right)$$
$$\times\exp\left(-\frac{\pi\sqrt{-1}}{r}(\sum_{i=1}^{t-1}\frac{1}{a_{i-1}a_i})|\lambda_0|^2\right).$$

Here $\zeta = \exp\frac{\pi\sqrt{-1}}{4}$ *and* $D_t = \operatorname{sign}(a_0a_1) + \cdots + \operatorname{sign}(a_{t-1}a_t)$. □

We will not give the proof of this lemma here, since it is long and technical. The lemma is proved by induction on the length of $\mathcal{C}$. The reciprocity formula for Gaussian sums, Proposition 5.2, plays a prominent role in the proof. A proof of this reciprocity formula can be found in [15, Sect. 2].

Let V be a real vector space of dimension l with inner product $\langle\ ,\ \rangle$, Λ a lattice in V and Λ^* the dual lattice. For an integer r, a self-adjoint automorphism $B\colon V\to V$, and an element $\psi\in V$, we assume

$$\frac{1}{2}\langle\lambda, Br\lambda\rangle,\ \ \langle\lambda, B\eta\rangle,\ \ r\langle\lambda,\psi\rangle\in\mathbb{Z},\qquad \forall\lambda,\eta\in\Lambda,$$
$$\frac{1}{2}\langle\mu, Br\mu\rangle,\ \ \langle\mu, r\xi\rangle,\ \ r\langle\mu,\psi\rangle\in\mathbb{Z},\qquad \forall\mu,\xi\in\Lambda^*$$

and $B\Lambda^*\subseteq\Lambda^*$. Then we have the following:

Proposition 5.2 (Reciprocity formula for Gauss sums)

$$\operatorname{vol}(\Lambda^*)\sum_{\lambda\in\Lambda/r\Lambda}\exp\left(\frac{\pi\sqrt{-1}}{r}\langle\lambda,B\lambda\rangle\right)\exp\left(2\pi\sqrt{-1}\langle\lambda,\psi\rangle\right)$$
$$=\left(\det\frac{B}{\sqrt{-1}}\right)^{-1/2} r^{l/2}\sum_{\mu\in\Lambda^*/B\Lambda^*}\exp\left(-\pi r\sqrt{-1}\langle\mu+\psi, B^{-1}(\mu+\psi)\rangle\right). \quad □$$

In the proof of Lemma 5.1 we use Proposition 5.2 with $\Lambda=\Lambda^W$, the dual lattice being the root lattice Λ^R. Basically we use the reciprocity formula (t times, recursively) to change the sums in the expression for $\mathcal{T}^{\mathcal{C}}$ to a sum with a range which does not depend on r. Another main ingredient in the proof of Lemma 5.1 is symmetry considerations along the same lines as the discussion in [16, pp. 584–586], see in particular [16, Proposition 4.4]. In the proof of Lemma 5.1 and Theorem 3.3 we use several results in [16, Sect. 2], in particularly [16, Proposition 2.5].

Lemma 5.1 nearly proves Theorem 3.3. There is, however, a small hurdle to overcome because of the assumption on the a_k's in the lemma. The following small result does the job.

Lemma 5.3 *Let* $U = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2,\mathbb{Z})$ *with* $c \neq 0$. *Then we can write* $U = V\Theta^n$, *where* $n \in \mathbb{Z}$ *and where* V *is given in the following way: If* $a = 0$, *then* $V = \Xi$; *if* $a \neq 0$, *then there exists a sequence of integers* $\mathcal{C}$ *such that* $V = B^{\mathcal{C}}$, *see* (12), *and such that* $a_k^{\mathcal{C}} \neq 0$, $k = 1,2,\dots,|\mathcal{C}|$. □

From this lemma we see the origin of the undetermined sign ϵ. Let us use the above lemmas to sketch the proof of Theorem 3.3.

Proof of Theorem 3.3 According to the previous lemma there exists an integer n, a sign $\epsilon \in \{\pm 1\}$, and a $V \in SL(2,\mathbb{Z})$ as in the Lemma 5.3 such that $U = \epsilon V\Theta^n$. First assume $a \neq 0$ and that $n = 0$, i.e. assume that $\epsilon U = B^{\mathcal{C}}$, where $\mathcal{C} = (m_1,\dots,m_t)$ and $a_k \neq 0$, $k = 1,2,\dots,t$. Let $\mathcal{C}' = (m_1,\dots,m_{t-1})$. Then, by Lemma 5.1, (13), and a small calculation, we get

$$\begin{aligned}\mathcal{R}(\epsilon U)_{\lambda\mu} &= \tilde{\Theta}^{m_t}_{\lambda\lambda}\mathcal{T}^{\mathcal{C}'}_{\mu,\lambda} \\ &= \sum_{w\in W}\det(w)\sum_{\nu\in\Lambda^R/c\Lambda^R}\exp\left(\frac{\pi\sqrt{-1}}{r}\frac{a}{c}\left|\lambda + r\nu - \frac{w(\mu)}{\epsilon a}\right|^2\right),\end{aligned}$$

where

$$K = K^{\mathcal{C}'}_{\mu}\exp\left(-\frac{\pi\sqrt{-1}}{h}m_t|\rho|^2\right)\exp\left(-\frac{\pi\sqrt{-1}}{a_{t-1}a_{t-2}r}|\mu|^2\right)\exp\left(-\frac{\pi\sqrt{-1}}{a_t a_{t-1} r}|\mu|^2\right).$$

The formula for the entries of $\mathcal{R}(\epsilon U)$ now follows by the fact $b/a + 1/(ac) = d/c$ (except for the factor K which we will not rewrite here). Next assume that $\epsilon U = B^{\mathcal{C}}\Theta^n$ with $n \neq 0$, where $\mathcal{C}$ is as above. Then

$$B^{\mathcal{C}} = \epsilon U\Theta^{-n} = \epsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} = \epsilon\begin{pmatrix} a & -na+b \\ c & -nc+d \end{pmatrix},$$

and since Theorem 3.3 is valid for $U = B^{\mathcal{C}}$ (with $\epsilon = 1$) we get the result after a small calculation. Finally one has to consider the case where $a = 0$, in which case $\epsilon U = \Xi\Theta^n = \begin{pmatrix} 0 & -1 \\ 1 & n \end{pmatrix}$. Here the result follows by inserting the expressions in (7) into $\mathcal{R}(\epsilon U)_{\lambda\mu} = \mathcal{R}(\Xi)_{\lambda\mu}\mathcal{R}(\Theta)^n_{\mu\mu}$. □

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