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Polynomial invariants and Vassiliev invariants

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Abstract We give a criterion to detect whether the derivatives of the HOMFLY polynomial at a point is a Vassiliev invariant or not. In particular, for a complex number b we show that the derivative $P_K^{(m,n)}(b,0) = \frac{\partial^m}{\partial a^m}\frac{\partial^n}{\partial x^n}P_K(a,x)|_{(a,x)=(b,0)}$ of the HOMFLY polynomial of a knot K at $(b,0)$ is a Vassiliev invariant if and only if $b=\pm 1$. Also we analyze the space V_n of Vassiliev invariants of degree $\leq n$ for $n=1,2,3,4,5$ by using the $^-$–operation and the *–operation in [5]. These two operations are unified to the $\hat{}$–operation. For each Vassiliev invariant v of degree $\leq n$, $\hat{v}$ is a Vassiliev invariant of degree $\leq n$ and the value $\hat{v}(K)$ of a knot K is a polynomial with multi–variables of degree $\leq n$ and we give some questions on polynomial invariants and the Vassiliev invariants.

AMS Classification 57M25

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1 Introduction

In 1990, V. A. Vassiliev introduced the concept of a finite type invariant of knots, called Vassiliev invariants [13]. There are some analogies between Vassiliev invariants and polynomials. For example, in 1996 D. Bar–Natan showed that when a Vassiliev invariant of degree m is evaluated on a knot diagram having n crossings, the result is approximately bounded by a constant times of n^m [2] and S. Willerton [15] showed that for any Vassiliev invariant v of degree n, the function $p_v(i,j):=v(T_{i,j})$ is a polynomial of degree $\leq n$ for each variable i and j. Recently, we [4] defined a sequence of knots or links induced from a double dating tangle and showed that any Vassiliev invariant has a polynomial growth on this sequence.

J. S. Birman and X.–S. Lin [3] showed that each coefficient in the Maclaurin series of the Jones, Kauffman, and HOMFLY polynomial, after a suitable

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change of variables, is a Vassiliev invariant, and T. Kanenobu [7, 8] showed that some derivatives of the HOMFLY and the Kauffman polynomial are Vassiliev invariants. For the question whether the n–th derivatives of knot polynomials are Vassiliev invariants or not, we [5] gave complete solutions for the Jones, Alexander, Conway polynomial and a partial solution for the Q–polynomial. Also we introduced the $^-$–operation and the *–operation to obtain polynomial invariants from a Vassiliev invariant of degree n. From each of these new polynomial invariants, we may get at most $(n+1)$ linearly independent numerical Vassiliev invariants.

In this paper, we find a line and two points in the complex plane where the derivatives of the HOMFLY polynomial can possibly be Vassiliev invariants and analyze the space V_n of Vassiliev invariants for $n \leq 5$ by using the $^-$–operation and the *–operation.

Throughout this paper all knots or links are assumed to be oriented unless otherwise stated. For a knot K and $i \in \mathbb{N}$, K^i denotes the i–times self–connected sum of K and $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ denote the sets of nonnegative integers, integers, rational numbers, real numbers and complex numbers, respectively.

A knot or link invariant v taking values in an abelian group can be extended to a singular knot or link invariant by taking the difference between the positive and negative resolutions of the singularity. A knot or link invariant v is called a *Vassiliev invariant of degree* n if n is the smallest nonnegative integer such that v vanishes on singular knots or links with more than n double points. A knot or link invariant v is called a *Vassiliev invariant* if v is a Vassiliev invariant of degree n for some nonnegative integer n.

Definition 1.1 [4] Let $\mathbf{J}$ be a closed interval $[a,b]$ and k a positive integer. Fix k points in the upper plane $\mathbf{J}^2 \times \{b\}$ of the cube $\mathbf{J}^3$ and their corresponding k points in the lower plane $\mathbf{J}^2 \times \{a\}$ of the cube $\mathbf{J}^3$. A (k,k)*–tangle* is obtained by attaching, within $\mathbf{J}^3$, to these $2k$ points k curves, none of which should intersect each other. A (k,k)–tangle is said to be *oriented* if each of its k curves is oriented. Given two (k,k)–tangles S and T, roughly the *tangle product* ST is defined to be the tangle obtained by gluing the lower plane of the cube containing S to the upper plane of the cube containing T. The *closure* $\overline{T}$ of a tangle T is the unoriented knot or link obtained by attaching k parallel strands connecting the k points and their corresponding k points in the exterior of the cube containing T. When the tangles S and T are oriented, the oriented tangle ST is defined only when it respects the orientations of S and T and the closure $\overline{S}$ has the orientation inherited from that of S and $\overline{ST}$ is the oriented knot or link obtained by closing the (k,k)–tangle ST.

Definition 1.2 [4] An oriented (k,k)–tangle T is called a *double dating tangle* (*DD–tangle* for short) if there exist some ordered pairs of crossings of the form $(*)$ in Figure 1, so that T becomes the trivial (k,k)–tangle when we change all the crossings in the ordered pairs, where i and j in Figure 1, denote components of the tangle. Note that a DD–tangle is always an oriented tangle.

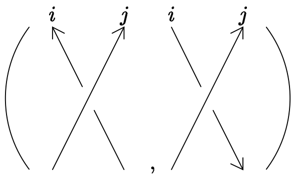

Figure 1: $(*)$

Since every $(1,1)$–tangle is a double dating tangle, every knot is a closure of a double dating $(1,1)$–tangle. But there is a link which is not the closure of any DD–tangle since the linking number of two components of the closure of a DD–tangle must be 0.

Definition 1.3 [4] Given an oriented (k,k)–tangle S and a double dating (k,k)–tangle T such that the product ST is well–defined, we have a sequence of links $\{L_i(S,T)\}_{i=0}^{\infty}$ obtained by setting $L_i(S,T)=\overline{ST^i}$ where $T^i=TT\cdots T$ is the i–times self–product of T and T^0 is the trivial (k,k)–tangle. We call $\{L_i(S,T)\}_{i=0}^{\infty}$ ($\{L_i\}_{i=0}^{\infty}$ for short) the *sequence induced from the (k,k)–tangle S and the double dating (k,k)–tangle T* or simply a *sequence induced from the double dating tangle T*.

In particular, if $\overline{S}$ is a knot for a (k,k)–tangle S, then $L_i(S,T)=\overline{ST^i}$ is a knot for each $i\in\mathbb{N}$ since T^i can be trivialized by changing some crossings.

Theorem 1.4 [5] *Let $\{L_i\}_{i=0}^{\infty}$ be a sequence of knots induced from a DD–tangle. Then any Vassiliev knot invariant v of degree n has a polynomial growth on $\{L_i\}_{i=0}^{\infty}$ of degree $\leq n$.*

Corollary 1.5 [5] *Let L and K be two knots. For each $i\in\mathbb{N}$, let $K_i=K\sharp L\sharp\cdots\sharp L$ be the connected sum of K to the i–times self–connected sum of L. If v is a Vassiliev invariant of degree n, then $v|_{\{K_i\}_{i=0}^{\infty}}$ is a polynomial function in i of degree $\leq n$.*

The converse of Corollary 1.5 is not true. In fact, the maximal degree $u(K)$ of the Conway polynomial $\nabla_K(z)$ for a knot K is a counterexample.

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2 The derivatives of the HOMFLY polynomial and Vassiliev invariants.

From now on, the notations 3_1, 4_1, 5_1 and 6_1 will mean the knots in the Rolfsen's knot table [11]. For the definitions of the HOMFLY polynomial $P_L(a,z)$ and the Kauffman polynomial $F_L(a,x)$ of a knot or link L, see [10].

Note that the *Jones polynomial* $J_L(t)$, the *Conway polynomial* $\nabla_L(z)$, and the *Alexander polynomial* $\Delta_L(t)$ of a knot or link L can be defined from the HOMFLY polynomial $P_L(a,z) \in \mathbb{Z}[a,a^{-1},z,z^{-1}]$ via the equations $J_L(t) = P_L(t, t^{1/2} - t^{-1/2})$, $\nabla_L(z) = P_L(1,z)$ and $\Delta_L(t) = P_L(1, t^{1/2} - t^{-1/2})$ respectively and that the *Q-polynomial* $Q_L(x)$ can be defined from the Kauffman polynomial $F_L(a,x)$ via the equation $Q_L(x) = F_L(1,x)$.

By using the skein relations, we can see that $P_L(a,z)$ and $F_L(a,x)$ are *multiplicative under the connected sum.* i.e. $P_{L_1\sharp L_2}(a,z) = P_{L_1}(a,z)P_{L_2}(a,z)$ and $F_{L_1\sharp L_2}(a,x) = F_{L_1}(a,x)F_{L_2}(a,x)$ for all knots or links L_1 and L_2. So the Jones, Conway, Alexander and Q–polynomials are also multiplicative under the connected sum.

It is well known that $P_K(a,z) \in \mathbb{Z}[a^2, a^{-2}, z^2]$ and $F_K(a,x) \in \mathbb{Z}[a,a^{-1},x]$ for a knot K. For each $i \in \mathbb{N}$ and each knot K, we denote by $F_i(K;a)$ and $P_{2i}(K;a)$ the coefficient of x^i in $F_K(a,x)$ and the coefficient of z^{2i} in $P_K(a,z)$, respectively, which are polynomials in a.

Throughout this section, knot polynomials are always assumed to be multiplicative under the connected sum.

We consider 1–variable knot polynomials first and then 2–variable knot polynomials.

Lemma 2.1 [5] *Let $f_K(x)$ be a knot polynomial of a knot K such that $f_K(x)$ is infinitely differentiable in a neighborhood of a point a and assume that $f_K^{(1)}(a) \neq 0$. Then there exists a unique polynomial $p(x)$ of degree m such that $f_{K^i}^{(m)}(a) = (f_K(a))^i p(i)$ for $i > m$.*

Theorem 2.2 [5] *For each $n \in \mathbb{N}$, we have*

(1) $J_K^{(n)}(a)$ *is a Vassiliev invariant if and only if* $a = 1$.

(2) $\nabla_K^{(n)}(a)$ *is a Vassiliev invariant if and only if* $a = 0$.

(3) $\Delta_K^{(n)}(a)$ *is a Vassiliev invariant if and only if* $a = 1$.

(4) $Q_K^{(n)}(a)$ *is not a Vassiliev invariant if* $a \neq -2, 1$.

Theorem 2.3 *Let $g\colon \mathbb{R} \to \mathbb{R}$ be infinitely differentiable function at $x = a$ with $g^{(1)}(a) \neq 0$. Assume that $f_K(x)$ is a knot polynomial which is infinitely differentiable in a neighborhood of $g(a)$ for all knots K and that there exists a knot L such that $f_L(g(a)) \neq 0, 1$ and $f_L^{(1)}(g(a)) \neq 0$. Then each coefficient of $(x - a)^n$ in the Taylor expansion of $f_K \circ g(x)$ at $x = a$, is not a Vassiliev invariant.*

Proof Consider a sequence $\{L^i\}_{i=0}^{\infty}$ of knots. By Lemma 2.1, we see that $(f_{L^i}(g(x)))^{(n)}|_{x=a} = (f_L(g(a)))^i p(i)$, where $p(i)$ is a polynomial in i of degree n, and hence the coefficient $\frac{1}{n!}(f_K(g(x)))^{(n)}|_{x=a}$ of $(x - a)^n$ does not have a polynomial growth on $\{L^i\}_{i=0}^{\infty}$.

It follows from Corollary 1.5 that the coefficient of $(x - a)^n$ in the Taylor expansion of $f_K \circ g(x)$ is not a Vassiliev invariant. □

J. S. Birman and X.–S. Lin [3] showed that each coefficient in the Maclaurin series of $J_K(e^x)$ is a Vassiliev invariant. As a generalization of Birman and Lin's type of changing variables, we have

Theorem 2.4 *Let $g\colon \mathbb{R} \to \mathbb{R}$ be an infinitely differentiable function at $x = a$. Assume that $g^{(1)}(a) \neq 0$. Then*

(1) *each coefficient of $(x - a)^n$ in the Taylor expansion of $J_K \circ g(x)$ at $x = a$, is a Vassiliev invariant if and only if $g(a) = 1$,*

(2) *each coefficient of $(x - a)^n$ in the Taylor expansion of $\nabla_K \circ g(x)$ at $x = a$, is a Vassiliev invariant if and only if $g(a) = 0$,*

(3) *each coefficient of $(x - a)^n$ in the Taylor expansion of $\Delta_K \circ g(x)$ at $x = a$, is not a Vassiliev invariant if and only if $g(a) = 1$ and*

(4) *if $g(a) \neq -2, 1$ then each coefficient of $(x - a)^n$ in the Taylor expansion of $Q_K \circ g(x)$ at $x = a$, is not a Vassiliev invariant.*

Proof (1) Let $A_K = \{t|\ J_K(t) = 0, 1\} \bigcup \{t|\ J_K^{(1)}(t) = 0\}$ for a knot K. Then $A_{3_1} \bigcap A_{4_1} = \{1\}$. Thus if $g(a) \neq 1$, then $g(a) \in \mathbb{R} \backslash (A_{3_1} \bigcap A_{4_1})$. Take $L = 3_1$ in Theorem 2.3 if $g(a) \in \mathbb{R} \backslash A_{3_1}$ and $L = 4_1$ in Theorem 2.3 if $g(a) \in \mathbb{R} \backslash A_{4_1}$. Then $J_L(g(a)) \neq 0, 1$ and $J_L^{(1)}(g(a)) \neq 0$. So by Theorem 2.3, each coefficient of $(x - a)^n$ in the Taylor expansion of $J_K \circ g(x)$ is not a Vassiliev invariant. Conversely, assume that $g(a) = 1$ and that $n \in \mathbb{N}$. Since the coefficient of $(x - a)^n$ in the Taylor expansion of $J_K(g(x))$ is a linear combination of $1, J_K^{(1)}(1), \cdots, J_K^{(n)}(1)$, by Theorem 2.2, it is a Vassiliev invariant. The proofs of (2), (3) and (4) are similar. □

Example 2.5 Take $f(x) = \sin(x)$ for $x \in \mathbb{R}$. Then $f(0) \neq 1$ and $f^{(1)}(0) \neq 0$. Thus each coefficient in the Maclaurin series of $J_K(\sin(x)) = J_K(f(x))$ is not a Vassiliev invariant. But each coefficient in the Maclaurin series of $\nabla_K(\sin(x)) = \nabla_K(f(x))$ is a Vassiliev invariant, since it is a finite linear combination of the coefficients of the Conway polynomial $\nabla_K(z)$ of a knot K.

Now we will deal with 2–variable knot polynomials such as the HOMFLY polynomial $P_K(a, z) \in \mathbb{Z}[a, a^{-1}, z]$ and the Kauffman polynomial $F_K(a, x) \in \mathbb{Z}[a, a^{-1}, x]$. For a 2–variable Laurent polynomial $g(x, y)$ which is infinitely differentiable on a neighborhood of (a, b), we denote $\frac{\partial^m}{\partial x^m} \frac{\partial^n}{\partial y^n} g(x, y)|_{(x,y)=(a,b)}$ by $g^{(m,n)}(a, b)$ for each pair $(m, n) \in \mathbb{N}^2$.

Theorem 2.6 [5] *Let $g_K(x, y)$ be a 2–variable knot polynomial which is infinitely differentiable on a neighborhood of (a, b) for all knots K. If there exists a knot L such that $g_L(a, b) \neq 0, 1$, $g_L^{(1,0)}(a, b) \neq 0$ and $g_L^{(0,1)}(a, b) \neq 0$ then $g_K^{(m,n)}(a, b)$ is not a Vassiliev invariant for all $m, n \in \mathbb{N}$.*

Lemma 2.7 *Let $g_K(x, y)$ be a 2–variable knot polynomial which is infinitely differentiable on a neighborhood of $(a, b) \in \mathbb{C}^2$ for all knots K and let $m, n \in \mathbb{N}$. If there exists a knot L such that $g_L(a, b) \neq 0, 1$, $g_L^{(1,0)}(a, b) \neq 0$, $g_L^{(0,1)}(a, b) = 0$ and $g_L^{(0,2)}(a, b) \neq 0$ then there exists a polynomial $p(i)$ of degree $m + n$ such that $g_{L^i}^{(m,2n)}(a, b) = (g_L(a, b))^i p(i)$ for $i > m + 2n$.*

Proof It is similar to that of Theorem 2.12 in [5]. □

Lemma 2.8 *Let $g_K(x, y)$ be a 2–variable knot polynomial which is infinitely differentiable on a neighborhood of $(a, b) \in \mathbb{C}^2$ for all knots K. If there exists a knot L such that $g_L(a, b) \neq 0, 1$, $g_L^{(1,0)}(a, b) \neq 0$, $g_L^{(0,1)}(a, b) = 0$ and $g_L^{(0,2)}(a, b) \neq 0$ then $g_K^{(m,2n)}(a, b)$ is not a Vassiliev invariant for all $m, n \in \mathbb{N}$.*

Proof It follows from Lemma 2.7 and Corollary 1.5. □

Theorem 2.9 *Let $n \in \mathbb{N}$ and $a \in \mathbb{C}$. $P_{2i}^{(n)}(K;a)$ is a Vassiliev invariant if and only if $a = \pm 1$.*

Proof Note that $P_{2i}^{(n)}(K;a) = (2i)!P_K^{(n,2i)}(a,0)$. Since $P_K(a,z) \in \mathbb{Z}[a^2,a^{-2},z^2]$ for all knots K, $P_K^{(n,1)}(a,0) = 0$ for all $a \in \mathbb{C}$ and all knots K. For each knot K, let $A_K^1 = \{a \in \mathbb{C} \mid P_K(a,0) = 0 \text{ or } 1\}$, $A_K^2 = \{a \in \mathbb{C} \mid P_K^{(1,0)}(a,0) = 0\}$, $A_K^3 = \{a \in \mathbb{C} \mid P_K^{(0,2)}(a,0) = 0\}$ and $A_K = A_K^1 \bigcup A_K^2 \bigcup A_K^3$. Since $P_{3_1}(a,z) = (-a^{-4}+2a^{-2})+a^{-2}z^2$ and $P_{4_1}(a,z) = (a^{-2}-1+a^2)-z^2$, we have $A_{3_1} = \{\pm\frac{\sqrt{2}}{2}, \pm 1\}$, $A_{4_1} = \{\pm(\frac{\sqrt{3}+\sqrt{-1}}{2}), \pm(\frac{\sqrt{3}-\sqrt{-1}}{2}), \pm 1, \pm\sqrt{-1}\}$ and hence $A_{3_1} \bigcap A_{4_1} = \{\pm 1\}$. Thus if $a \neq \pm 1$, then, by Lemma 2.8, $P_{2i}^{(n)}(K;a)$ is not a Vassiliev invariant. Conversely, T. Kanenobu [8] showed that $P_{2i}^{(n)}(K;1)$ is a Vassiliev invariant. Since $P_{2i}^{(n)}(K;-1) = (-1)^n P_{2i}^{(n)}(K;1)$, $P_{2i}^{(n)}(K;-1)$ is also a Vassiliev invariant. □

By Theorem 2.9, for $b \in \mathbb{C}$, $P_K^{(m,n)}(b,0)$ is a Vassiliev invariant if and only if n is odd or $b = \pm 1$. For $(b,y) \in \mathbb{C}^2$ with $y \neq 0$, we have the following

Theorem 2.10 *Let m,n be nonnegative integers. If $(b,y) \in \mathbb{C}^2$ with $y \neq 0$ such that $P_K^{(m,n)}(b,y)$ is a Vassiliev invariant, then $(b,y) = (b, \pm(b-b^{-1}))$, $(\pm\sqrt{-1}, \sqrt{-3})$ or $(\pm\sqrt{-1}, -\sqrt{-3})$.*

Proof By direct calculations, $P_{3_1}(a,z) = (-a^{-4}+2a^{-2})+a^{-2}z^2$, $P_{4_1}(a,z) = (a^{-2}-1+a^2)-z^2$ and $P_{6_1}(a,z) = (a^{-4}-a^{-2}+a^2)+z^2(-a^{-2}-1)$. Let $A_K^1 = \{(b,y) \mid P_K(b,y) = 0 \text{ or } 1\}$, $A_K^2 = \{(b,y) \mid P_K^{(1,0)}(b,y) = 0\}$, $A_K^3 = \{(b,y) \mid P_K^{(0,1)}(b,y) = 0\}$ and $A_k = A_K^1 \bigcup A_K^2 \bigcup A_K^3$ for each knot K. Then

$$\begin{aligned}
A_{3_1} \cap A_{4_1} &= (A_{3_1}^1 \cap A_{4_1}^1) \cup (A_{3_1}^1 \cap A_{4_1}^2) \cup \cdots \cup (A_{3_1}^3 \cap A_{4_1}^3) \\
&= \{\ (\pm\sqrt{-1}, 2\sqrt{-1}),\ (\pm\sqrt{-1}, -2\sqrt{-1})\} \\
&\quad \cup\{(\pm\sqrt{-1}, \sqrt{-3}),\ (\pm\sqrt{-1}, -\sqrt{-3}),\ (\pm 1, \sqrt{-1}),\ (\pm 1, -\sqrt{-1})\} \\
&\quad \cup\{(\frac{-1\pm\sqrt{5}}{2}, \sqrt{1\pm\sqrt{5}}),\ (\frac{-1\pm\sqrt{5}}{2}, -\sqrt{1\pm\sqrt{5}})\} \\
&\quad \cup\{(b,y) \mid y = \pm(b-b^{-1})\}.
\end{aligned}$$

So we get

$$\begin{aligned}
& A_{3_1} \cap A_{4_1} \cap A_{6_1} \\
= \ & ((A_{3_1} \cap A_{4_1}) \cap A_{6_1}^1) \cup ((A_{3_1} \cap A_{4_1}) \cap A_{6_1}^2) \cup ((A_{3_1} \cap A_{4_1}) \cap A_{6_1}^3) \\
= \ & \{(b,y) \mid y = \pm(b-b^{-1})\} \cup \{(\pm\sqrt{-1}, \sqrt{-3}),\ (\pm\sqrt{-1}, -\sqrt{-3})\}.
\end{aligned}$$

If $(b, y) \in \mathbb{C}^2 \setminus (A_{3_1} \cap A_{4_1} \cap A_{6_1})$, then, by Theorem 2.6, $P_K^{(m,n)}(b, y)$ is not a Vassiliev invariant. □

Whether a finite product of the derivatives of knot polynomials at some points is a Vassiliev invariant or not can be detected by using Lemma 2.1, Theorem 2.6, Lemma 2.7 and Corollary 1.5. For example if there is a knot L such that $J_L^{(1)}(a) \neq 0, Q_L^{(1)}(b) \neq 0, P_L^{(1,0)}(c, y) \neq 0, P_L^{(0,1)}(c, y) \neq 0$ and $J_L(a)Q_L(b)P_L(c, y)$ $\neq 0, 1$, then the product $J_K^{(k)}(a)Q_K^{(l)}(b)P_K^{(m,n)}(c, y)$ is not a Vassiliev invariant for any $k, l, m, n \in \mathbb{N}$.

Since $Q_K^{(1)}(-2) = J_K^{(2)}(1)$ (T. Kanenobu [6]), $Q_K^{(1)}(-2)$ is a Vassiliev invariant of degree ≤ 2. Note that $Q_K^{(0)}(1) = 1$ for any knot K and hence $Q_K^{(0)}(1)$ is a Vassiliev invariant of degree 0, but $Q_K^{(1)}(1)$ and $Q_K^{(2)}(1)$ are not Vassiliev invariants [5].

Open Problem (A. Stoimenow [12]) Is $Q_K^{(n)}(-2)$ a Vassiliev invariant for $n \geq 2$?

Question 2.11 Is $Q_K^{(n)}(1)$ a Vassiliev invariant for $n \geq 3$?

The above two problems are the only remaining unsolved problems in one variable knot polynomials [5].

Question 2.12 Find all the points at which the derivatives of the Kauffman polynomial are Vassiliev invariants.

Question 2.13 Find all linear combinations of any finite products of derivatives of knot polynomials, which are Vassiliev invariants.

3 New polynomial invariants from Vassiliev invariants

In this section, a Vassiliev invariant v always means a Vassiliev invariant taking values in a numerical number field $\mathbf{F} = \mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. We begin with introducing the constructions of new polynomial invariants from a given Vassiliev invariant (see [4]) and then we will define a new polynomial invariant unifying the polynomial invariants obtained from the constructions in [4]. The new polynomial invariant is also a Vassiliev invariant and so we get various numerical Vassiliev invariants from the coefficients of the new polynomial invariant.

Let K and L be two knots and let $\{L_i\}_{i=0}^{\infty}$ be a sequence of knots induced from a DD–tangle. Since any $(1,1)$–tangle is a DD–tangle, we get two sequences $\{L\sharp K^i\}_{i=0}^{\infty}$ and $\{K\sharp L_i\}_{i=0}^{\infty}$ of knots induced from DD–tangles.

Let v be a Vassiliev invariant of degree n and fix a knot L. Then by Corollary 1.5, for each knot K there exist unique polynomials $p_K(x)$ and $q_K(x)$ in $\mathbf{F}[x]$ with degrees $\leq n$ such that $v(L\sharp K^i) = p_K(i)$ and $v(K\sharp L_i) = q_K(i)$. We define two polynomial invariants $\bar{v}$ and v^* as follows: $\bar{v}\colon \{\text{knots}\} \to \mathbf{F}[x]$ by $\bar{v}(K) = p_K(x)$ and $v^*\colon \{\text{knots}\} \to \mathbf{F}[x]$ by $v^*(K) = q_K(x)$. Then $\bar{v}(K)|_{x=j} = p_K(j) = v(L\sharp K^j)$ and $v^*(K)|_{x=j} = q_K(j) = v(K\sharp L_j)$ for all $j \in \mathbb{N}$.

Then we have the following

Theorem 3.1 [5] *Let v be a Vassiliev invariant of degree n taking values in a numerical field $\mathbf{F}$.*

(1) *For a fixed knot L, $\bar{v}$ is a Vassiliev invariant of degree $\leq n$ and the degree of x in $\bar{v}(K)$ is $\leq n$. In particular if L is the unknot, $\bar{v}$ is a Vassiliev invariant of degree n and $\bar{v}(K)|_{x=1} = v(K)$.*

(2) *For a fixed sequence $\{L_i\}_{i=0}^{\infty}$ of knots induced from a DD–tangle, v^* is a Vassiliev invariant of degree $\leq n$ and the degree of x in $v^*(K)$ is $\leq n$. In particular if L_j is the unknot for some $j \in \mathbb{N}$, then v^* is a Vassiliev invariant of degree n and $v^*(K)|_{x=j} = v(K)$.*

Given a Vassiliev invariant v of degree n, we may get at most $(n+1)$ linearly independent numerical Vassiliev invariants which are the coefficients of the polynomial invariants $\bar{v}$ and v^* respectively and then apply $^-$–operation and *–operation repeatedly on these new Vassiliev invariants to get another new Vassiliev invariants. Inductively we may obtain various Vassiliev invariants.

We note that for a Vassiliev invariant v of degree n, since $\bar{v}(K)$ and $v^*(K)$ are polynomials of degrees $\leq n$ for any knot K, the polynomial invariants $\bar{v}$ and v^* are completely determined by $\{\bar{v}(K)|_{x=i} \mid 0 \leq i \leq n\}$ and $\{v^*(K)|_{x=i} \mid 0 \leq i \leq n\}$ respectively.

Let V_n be the space of Vassiliev invariants of degrees $\leq n$ and let $A_n \subset V_n$. For each nonnegative integer j, define A_n^j as follows. Set $A_n^0 = A_n$ and define inductively A_n^j to be the set of all Vassiliev invariants obtained from the coefficients of the new polynomial invariants $\bar{v}$ and v^* ranging over all $v \in A_n^{j-1}$, all knots L and all sequences $\{L_i\}_{i=0}^{\infty}$ induced from all DD–tangles in Theorem 3.1.

Define $A_n^* = \cup_{j=0}^{\infty} A_n^j$. We ask ourselves the following:

Question [5] Find a minimal finite subset A_n of V_n such that span(A_n^*) $= V_n$.

Let V_n be the space of Vassiliev invariants of degree $\leq n$. Then the dimension of V_n/V_{n-1} is $0, 1, 1, 3, 4, 9, 14$ for $n = 1, 2, 3, 4, 5, 6, 7$ [1].

Proposition 3.2 [7, 8] *For each nonnegative integer k and l,*

(1) $P_{2k}^{(l)}(K; 1)$ *is a Vassiliev invariant of degree* $\leq 2k + l$.

(2) $(\sqrt{-1})^{k+l} F_k^{(l)}(K; \sqrt{-1})$ *is a Vassiliev invariant of degree* $\leq k + l$.

If v_n and v_m are Vassiliev invariants of degrees n and m respectively, then the product $v_n v_m$ is a Vassiliev invariant of degree $\leq n + m$ [1, 14].

We get a base for each V_n $(n \leq 5)$ from the results of J. S. Birman and X.–S. Lin (citeBL, D. Bar–Natan [1] and T. Kanenobu [9].

Theorem 3.3 [9, 3, 1] *Let V_n be the space of Vassiliev invariants of degree $\leq n$. Then*

(1) $\{1\}$ *is a basis for* $V_0 = V_1$, *where* 1 *is the constant map with image* $\{1\}$.

(2) $\{a_2(K)\}$ *is a basis for* V_2/V_1.

(3) $\{J_K^{(3)}(1)\}$ *is a basis for* V_3/V_2.

(4) $\{(a_2(K))^2, a_4(K), J_K^{(4)}(1)\}$ *is a basis for* V_4/V_3.

(5) $\{a_2(K)P_0^{(3)}(K; 1), P_0^{(5)}(K; 1), P_4^{(1)}(K; 1), \sqrt{-1}F_4^{(1)}(K; \sqrt{-1})\}$ *is a basis for* V_5/V_4.

We can easily see that the Vassiliev invariants $a_2(K)$, $\sqrt{-1}F_4^{(1)}(K; \sqrt{-1})$ and $J_K^{(3)}(1)$ are additive. If v is an additive Vassiliev invariant, then, from the coefficients of the polynomial invariants $\overline{v}$ and v^*, we cannot get Vassiliev invariants other than linear combinations of v and the constant Vassiliev invariants.

Let v be a Vassiliev invariant of degree n and L a knot. Define v_L^i to be the Vassiliev invariant defined by $v_L^i(K) = v(L \sharp K^i)$ and define v_L to be the Vassiliev invariant defined by $v_L(K) = v(L \sharp K)$ [5]. Then we can see that the

Vassiliev invariants obtained from the coefficients of $\overline{v}$ and v^* are contained in the spans of the sets $\{v_L^i \mid L \text{ is a knot}, i = 0, 1, 2, \cdots, n\}$ and $\{v_L \mid L \text{ is a knot}\}$ respectively.

Take the trivial knot, 3_1, 4_1 and 5_1 for L and $(3_1)^i$, $(4_1)^i$ and $(5_1)^i$ for L_i in Theorem 3.1. Then all linearly independent Vassiliev invariants obtained by applying the $^-$–operations and the *–operations for the non–additive Vassiliev invariants of degree ≤ 5 in Theorem 3.3 can be found as follows.

$$(a_2(K))^2 \xrightarrow{-} \{a_2(K)\}$$

$$a_4(K) \xrightarrow{-} \{a_2(K), (a_2(K))^2\}$$

$$J_K^{(4)}(1) \xrightarrow{-} \{a_2(K), (a_2(K))^2\}$$

$$a_2(K)P_0^{(3)}(K;1) \xrightarrow{-} \{a_2(K), J_K^{(3)}(1)\}, \quad a_2(K)P_0^{(3)}(K;1) \xrightarrow{*} \{a_2(K)J_K^{(3)}(1)\}$$

$$a_2(K)J_K^{(3)}(1) \xrightarrow{-} \{a_2(K), J_K^{(3)}(1)\}$$

$$P_0^{(5)}(K;1) \xrightarrow{-} \{a_2(K)P_0^{(3)}(K;1)\}, \quad P_0^{(5)}(K;1) \xrightarrow{*} \{a_2(K), J_K^{(3)}(1)\}$$

$$P_4^{(1)}(K;1) \xrightarrow{-} \{a_2(K)P_2^{(1)}(K;1)\}, \quad P_4^{(1)}(K;1) \xrightarrow{*} \{a_2(K), J_K^{(3)}(1)\}$$

$$a_2(K)P_2^{(1)}(K;1) \xrightarrow{-} \{a_2(K), J_K^{(3)}(1)\}$$

For simplicity, for each Vassiliev invariant v, we unlist the Vassiliev invariants obtained from v^* if they can be obtained from $\overline{v}$ and we also exclude the constant map 1 whose image is $\{1\}$ and v itself in the list of Vassiliev invariants obtained from $\overline{v}$ and v^*.

Thus we get the following

Theorem 3.4 *Let A_n be a subset of the space V_n of the Vassiliev invariants of degree $\leq n$ such that span$(A_n^*) = V_n$. Then A_n can be chosen as follows.*

(1) $A_0 = A_1 = \{1\}$, *where* 1 *denotes the constant map with image* $\{1\}$.

(2) $A_2 = \{a_2(K)\}$.

(3) $A_3 = \{a_2(K), J_K^{(3)}(1)\}$.

(4) $A_4 = \{J_K^{(3)}(1), a_4(K), J_K^{(4)}(1)\}$.

(5) $A_5 = \{P_0^{(5)}(K;1), P_4^{(1)}(K;1), \sqrt{-1}F_4^{(1)}(K;\sqrt{-1}), a_4(K), J_K^{(4)}(1)\}$.

Let v be a Vassiliev invariant of degree n. In [5], the authors generalized the one–variable knot polynomial invariants $\bar{v}$ and v^* to two–variable knot polynomial invariants $\bar{v}$ and v^*, respectively with the same notation.

Now we want to generalize the two–variable knot polynomial invariants $\bar{v}$ and v^* in Theorem 3.1 simultaneously to a multi–variable knot polynomial invariant $\hat{v}$ by unifying both $\bar{v}$ and v^* to a multi-variable polynomial invariant $\hat{v}$ whose proof is analogous to that of Theorem 3.1. See [5].

Given sequences $\{L_i^{(1)}\}_{i=0}^{\infty}, \cdots, \{L_i^{(k)}\}_{i=0}^{\infty}$ of knots induced from DD–tangles, for each knot K, there exists a unique polynomial

$$p_K(x_0, x_1, \cdots, x_k) \in \mathbf{F}[x_0, x_1, \cdots, x_k]$$

such that for all $(i_0, i_1, \cdots, i_k) \in \mathbb{N}^{k+1}$, $v(K^{i_0} \sharp L_{i_1}^{(1)} \sharp \cdots \sharp L_{i_k}^{(k)}) = p_k(i_0, i_1, \cdots, i_k)$.

Now we define a new polynomial invariant $\hat{v}$: $\{\text{knots}\} \to \mathbf{F}[x_0, \cdots, x_k]$ by $\hat{v}(K) = p_K(x_0, \cdots, x_k)$.

Then by applying the similar argument to the case of $\bar{v}$ and v^* [5], we can see that $\hat{v}$ is a Vassiliev invariant of degree $\leq n$ and the degree of each variable x_i in $\hat{v}(K)$ is $\leq n$. Thus we get the following

Theorem 3.5 *Let v be a Vassiliev invariant of degree n taking values in a numerical field $\mathbf{F}$ and let $\{L_i^{(1)}\}_{i=0}^{\infty}, \cdots, \{L_i^{(k)}\}_{i=0}^{\infty}$ be sequences of knots induced from DD–tangles. Then $\hat{v}$: $\{knots\} \to \mathbf{F}[x_0, \cdots, x_k]$ is a Vassiliev invariant of degree $\leq n$ and the degree of each variable x_i in $\hat{v}(K)$ is $\leq n$.*

For a Vassiliev invariant v, let $C_v :=$ {the coefficients of the polynomial $\hat{v}(K)$}. Then, in Theorem 3.5, $\hat{v}$ is completely determined by C_v. Since the degree of each variable in $\hat{v}$ is $\leq n$, we see that

$$\operatorname{span}(C_v) = \operatorname{span}(\{\hat{v}(K)|_{(x_0, \cdots, x_k) = (i_0, \cdots, i_k)} \mid 0 \leq i_0, \cdots, i_k \leq n\}).$$

Question 3.6 Let v be a Vassiliev invariant of degree n. Find sequences $\{L_i^{(1)}\}_{i=0}^{\infty}, \cdots, \{L_i^{(k)}\}_{i=0}^{\infty}$ of knots induced from DD–tangles such that $\operatorname{span}(C_v) = \operatorname{span}(\{v\}^*)$ where C_v is the set of coefficients of the polynomial invariant $\hat{v}$ induced from v and $\{L_i^{(1)}\}_{i=0}^{\infty}, \cdots, \{L_i^{(k)}\}_{i=0}^{\infty}$.

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