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p-Modular TQFT's, Milnor torsion and the Casson-Lescop invariant

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Abstract We derive formulae which lend themselves to TQFT interpretations of the Milnor torsion, the Lescop invariant, the Casson invariant, and the Casson-Morita cocyle of a 3-manifold, and, furthermore, relate them to the Reshetikhin-Turaev theory.

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0 Introduction and summary

Invariants of 3-manifolds that admit extensions to topological quantum field theories (TQFT) are structurally highly organized. Consequently, their evaluations permit an equally deeper insight into the topological structure of the underlying 3-manifolds beyond the mere distinction of their homeomorphism types. Although the notion and many examples of TQFT's have been around for more than a decade there are still surprisingly large gaps in the understanding of the explicit TQFT content of the "classical" invariants such as the Milnor-Turaev torsion or the Casson-Walker-Lescop invariant.

In this article we derive formulae which lend themselves to TQFT interpretations of the Milnor torsion, the Lescop invariant, the Casson invariant, and the Casson-Morita cocyle of a 3-manifold. Specifically, these invariants are expressed in (6) of Theorem 3, in (7) of Theorem 4, in (15) of Theorem 6, and in (18) of Theorem 7 as traces and matrix elements of operators acting on $\bigwedge^* H_1(\Sigma)$ for a surface Σ. We relate these formulae to previous results in [10], [11], [12] and [5] on the Frohman-Nicas and Reshetikhin-Turaev theories. In the course, we develop the general notion of a *q/l-solvable TQFT* and consider reductions to the p-modular cases, as needed for the quantum theories. As an example for the additional structural depth that TQFT interpretations provide we describe results from [5] that allow us to read the cut-numbers of 3-manifolds from the coefficients of the Alexander polynomial, using their relations with the Reshetikhin-Turaev invariants.

After a review of the Alexander polynomial and the Milnor-Turaev torsion in Section 1 we introduce in Section 2 the Frohman-Nicas TQFT modeled over the $\mathbb{Z}$-cohomology of the surface Jacobians, as well as its Lefschetz components. The latter lead us to define the fundamental torsion weights $\Delta_{\varphi}^{(j)}(M) \in \mathbb{Z}$ for a 3-manifold, M, together with a 1-cocycle, φ, which turn out to be representation theoretically more adequate recombinations of the coefficients of the Alexander polynomial. We derive an expression of the Lescop invariant λ_L in terms of these fundamental weights. In Section 3 we construct the Johnson-Morita extensions $\mathcal{U}^{(j)}$ of pairs of Lefschetz components, which have degrees differing by 3. We review elements of Morita's theory for the Casson invariant λ_C and the Casson-Morita cocyle δ_C, and derive formulae that express λ_C and δ_C in terms of matrix elements of operators acting via exterior multiplication on the cohomology of the Jacobian of a surface. We further introduce in Section 3 the notion of a 1/1-solvable TQFT over $\mathbb{M}[\mathrm{y}]/\mathrm{y}^2$ for a commutative ring $\mathbb{M}$, which has an obvious generalization to q/l-TQFT's and can be viewed as a lowest order deformation theory for the $\mathcal{U}^{(j)}$. We derive some immediate functorial properties, and use these and the results for λ_C and δ_C to infer criteria for when a such a TQFT realizes the Casson invariant. In Section 4 we discuss the modular structure and characters of reductions of the TQFT's $\mathcal{U}^{(j)}$ from Section 2 into $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. Finally, in Section 5, we give an example of a 1/1-solvable TQFT over $\mathbb{F}_5[\mathrm{y}]/\mathrm{y}^2$ using the Reshetikhin Turaev theory at a 5-th root of unity. As applications we show that for a 3-manifold with $b_1(M) \geq 1$ the quantum invariant is given in lowest order by the Lescop invariant, and describe how to obtain from the structure theory of these TQFT methods for computing the cut-number of a 3-manifold.

1 The Alexander polynomial, Reidemeister-Milnor-Turaev torsion and the Lescop invariant

We start with a short review of the Alexander Polynomial $\Delta_{\varphi}(M) \in \mathbb{Z}[\Pi]/\pm\Pi$, which is defined for a compact, connected, oriented 3-manifold M with $b_1(M) \geq 1$, and an epimorphism $\varphi : H_1(M,\mathbb{Z}) \twoheadrightarrow \Pi$, where Π is a free abelian group of rank $k \leq b_1(M)$.

The map φ defines a covering space $\widetilde{M}_\varphi \to M$, such that we have a short exact sequence $1 \to \pi_1(\widetilde{M}_\varphi) \to \pi_1(M) \to \Pi \to 1$ and $\widetilde{M}_\varphi$ admits a Π-action by Deck-transformations. In particular, $H_1(\widetilde{M}_\varphi, \mathbb{Z})$ admits a Π-action and thus becomes a $\mathbb{Z}[\Pi] \cong \mathbb{Z}[t_1^{\pm 1}, \ldots, t_k^{\pm 1}]$ module. The Alexander polynomial is then defined as the generator of the smallest principal ideal containing the first elementary ideal of $H_1(\widetilde{M}_\varphi, \mathbb{Z})$. In the case where $k = b_1(M)$, which means $\Pi = H_1^{fr}(M) = H_1(M)/\mathrm{Tors}(H_1(M))$, we simply write $\Delta(M)$ without subscript.

Let us briefly explore this definition in the rank $k = 1$ case, meaning $\Pi = \mathbb{Z}$. Quite often the $\mathbb{Z}[t^{\pm 1}]$-module admits a presentation of the form $H_1(\widetilde{M}_\varphi, \mathbb{Z}) \cong \mathbb{Z}[t^{\pm 1}]^m / A(\mathbb{Z}[t^{\pm 1}]^m)$ obtained from an $m \times m$ Alexander Matrix A with coefficients in $\mathbb{Z}[t^{\pm 1}]$. The Alexander polynomial is thus represented by $\Delta_\varphi(M) = \pm t^l \cdot \det(A)$, with $l \in \mathbb{Z}$. One way of computing such an Alexander Matrix A is as follows. For $\varphi : H_1(M) \twoheadrightarrow \mathbb{Z}$ we find a dual, two-sided, embedded surface $\Sigma \subset M$ of some genus g. If we remove a collar neighborhood of Σ we obtain $C_\Sigma = M - N(\Sigma)$, which we view as a cobordism from Σ to itself. Let $i^\pm : \Sigma \hookrightarrow C_\Sigma$ be the inclusion maps onto the upper and the lower boundary component of C_Σ. Denote by $A^{\pm fr} : H_1(\Sigma_g) \to H_1^{fr}(C_\Sigma)$ the maps induced by $H_1(i^\pm)$ onto the free parts of the homology groups. The sign convention should be such that $A^{\pm fr} = id$ if $M = S^1 \times \Sigma$ and $C_\Sigma = [0,1] \times \Sigma$. The definitions imply the formula

$$\Delta_\varphi(M) \quad = \quad \pm t^{-g} \cdot |\mathrm{Tors}(C_\Sigma)| \cdot \det(A^{+fr} - tA^{-fr}) \quad . \tag{1}$$

We assume in (1) that $b_1(C_\Sigma) = 2g$ so that the $A^{\pm fr}$ are indeed $(2g) \times (2g)$-matrices. For example, if M is the mapping torus with gluing map ψ and φ is canonical then $\Delta_\varphi(M)$ is the characteristic polynomial of $H_1(\psi)$. The case $b_1(C_\Sigma) > 2g$ is equivalent to saying that φ factors through an epimorphism $\pi_1(M) \twoheadrightarrow \mathbb{Z} * \mathbb{Z} \twoheadrightarrow \mathbb{Z}$ onto the (non-abelian) free group in two generators, and implies that $\Delta_\varphi(M) = 0$. The additional factor in (1) and Poincaré duality [16] yield the symmetrized version of the Alexander polynomial. Moreover, we choose the sign such that $\Delta_\varphi(1) \geq 0$. A straight forward homological computation shows $\Delta_\varphi(M)(1) = |\mathrm{Tors}(H_1(M))|$ if $b_1(M) = 1$ and $\Delta_\varphi(M)(1) = 0$ if $b_1(M) \geq 2$. If $\mathcal{K} \subset S^3$ is a knot we obtain the usual Alexander polynomial $\Delta_\mathcal{K}$ of knot theory by applying the above either to the knot complement or to the 3-manifold obtained by doing 0-surgery along $\mathcal{K}$.

A closely related invariant is the Reidemeister-Milnor-Turaev torsion $\tau_\varphi(M) \in \mathbb{Z}[\Pi]$. It is defined as the Reidemeister torsion of the acyclic chain complex associated to the local system defined by φ, see [16], [22] for details. Again, for

$\Pi = H_1^{fr}(M)$ we simply write $\tau(M)$. In Theorems A and B, and 4.1.II. of [22] Turaev shows the following relations.

Theorem 1 [22] *Let $\varphi : H_1(M) \twoheadrightarrow \Pi$ with $k = \mathrm{rank}(\Pi)$:*

(1) *If $k \geq 2$ then $\tau_\varphi(M) = \Delta_\varphi(M)$ (so if $b_1(M) \geq 2$ then $\tau(M) = \Delta(M)$).*
(2) *If $k = 1$ and $\partial M = \emptyset$ then $\tau_\varphi(M) = (t-1)^{-2}\Delta_\varphi(M)$.*
(3) *If $b_1(M) \geq 2$, $k = 1$, and $\partial M = \emptyset$ then $\Delta_\varphi(M) = \overline{\varphi}(\Delta(M))(t-1)^2 t^{-1}$.*
(4) *If $k = 1$ and $\partial M \neq \emptyset$ then $\tau_\varphi(M) = (t-1)^{-1}\Delta_\varphi(M)$.*
(5) *If $b_1(M) \geq 2$, $k = 1$, and $\partial M \neq \emptyset$ then $\Delta_\varphi(M) = \overline{\varphi}(\Delta(M))(t-1)$.*

We next recall the relation between the Alexander polynomial of a link and that of the corresponding closed 3-manifold obtained by surgery. Let $\mathcal{L} \subset N$ be a framed link in a connected rational homology sphere N with $n \geq 2$ number of components, all of which have 0 framings and 0 linking numbers. Denote by $N_{\mathcal{L}}$ the 3-manifold obtained by surgery along $\mathcal{L}$ so that $b_1(N_{\mathcal{L}}) = n$. Hence, $\Delta(N_{\mathcal{L}}) \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, where the generator t_j is given by the meridian l_j of the j-th component of $\mathcal{L}$. By Theorem E of [23] and (1) of Theorem 1 we have $\Delta(N_{\mathcal{L}} - \bigcup_j U(l_j)) = \Delta(N_{\mathcal{L}})\prod_{j=1}^n (t_j - 1)$. Now, it is easily seen that $N_{\mathcal{L}} - \bigcup_j U(l_j) \cong N - U(\mathcal{L})$, where $U(\mathcal{L})$ denotes a tubular neighborhood the link. Moreover, the Alexander polynomial of this is the ordinary one, $\Delta(N - U(\mathcal{L})) = \Delta(\mathcal{L})$, which yields the following relation.

$$\Delta(N_{\mathcal{L}}) \quad = \quad \frac{\Delta(\mathcal{L})}{(t_1 - 1)\ldots(t_n - 1)} \tag{2}$$

We use (2) now to relate the $k = 1$ torsion invariants Δ_φ to the Lescop invariant λ_L for 3-manifolds with $b_1(M) \geq 1$. Recall from [14] that λ_L is an extension of the Casson-Walker invariant.

Lemma 2 *For a closed, compact, oriented manifold M with $b_1(M) \geq 1$ and for <u>any</u> epimorphism $\varphi : H_1(M) \twoheadrightarrow \mathbb{Z}$ we have*

$$\lambda_L(M) \quad = \quad \frac{1}{2}\Delta''_\varphi(M)(1) - \frac{1}{12}\Delta_\varphi(M)(1)\ . \tag{3}$$

Proof For $b_1(M) = 1$ Lescop gives in T5.1 in §1.5 of [14] the formula $\lambda_L(M) = \frac{1}{2}\Delta''(M)(1) - \frac{1}{12}|\mathrm{Tors}(M)|$. In this case φ is canonical, and it follows from (1) that $\Delta_\varphi(M)(1) = \Delta(M)(1) = |\mathrm{Tors}(M)|$.

For $b_1(M) \geq 2$ define the function $\zeta(\mathcal{L}) = \frac{\partial^n \Delta(\mathcal{L})}{\partial t_1 \ldots \partial t_n}(1, \ldots, 1)$ for a link $\mathcal{L} \subset N$ in a rational homology sphere with null homologous components and trivial linking matrix. In 2.1.2 of [14] Lescop then writes $\lambda_L(N_{\mathcal{L}}) = \zeta(\mathcal{L})$. Combining this with (2) and the fact that every 3-manifold is of the form $M = N_{\mathcal{L}}$, with N and $\mathcal{L}$ as in (2), we thus have $\lambda_L(M) = \Delta(M)(1,\dots,1)$.

Let now $\varphi : H_1(M) \twoheadrightarrow \mathbb{Z}$ be an epimorphism for a closed 3-manifold, M, with $b_1(M) \geq 2$ so that we are in the situation (3) of Theorem 1. We know from our previous discussion that $\Delta_\varphi(M)(1) = 0$ and that $\Delta_\varphi(M)$ is invariant under the substitution $t \leftrightarrow t^{-1}$. This implies for the expansion in $(t-1)$ that $\Delta_\varphi(M)(t) = \frac{1}{2}(t-1)^2\Delta''_\varphi(M)(1) + \mathcal{O}((t-1)^3)$. By (3) of Theorem 1 we therefore have $\overline{\varphi}(\Delta(M))(t) = \frac{1}{2}t\Delta''_\varphi(M)(1) + \mathcal{O}((t-1))$. Now, if $\varepsilon : \mathbb{Z}[\Pi] \twoheadrightarrow \mathbb{Z}$ is the augmentation map then $\lambda_L(M) = \Delta(M)(1,\dots,1) = \varepsilon(\Delta(M)) = \varepsilon(\overline{\varphi}(\Delta(M))) = \varepsilon(\frac{1}{2}t\Delta''_\varphi(M)(1) + \mathcal{O}((t-1))) = \frac{1}{2}\Delta''_\varphi(M)(1)$. $\square$

The reason we find (3) useful lies in the fact that it does not distinguish between the cases $b_1(M) = 1$ and $b_1(M) \geq 2$. Moreover, any arbitrary φ can be used to evaluate λ_L, which is, by construction, independent of the choice of φ. This will be essential when we prove in Theorem 18, asserting that for a manifold M with $\lambda_L(M) \neq 0$ we cannot remove more than one surface from M without disconnecting M.

It is now well known that for $b_1(M) \geq 1$ the Milnor torsion $\tau(M)$ also equals the Seiberg-Witten invariant as shown in [15], [24]. In the case $b_1(M) = 0$ the Seiberg-Witten invariant is identified with a combination of the Casson-Walker invariant and $\tau(M) \in \mathbb{Z}[H_1(M)]$, see [21]. We will discuss the Casson invariant for integral homology spheres further in Section 3 below.

2 The Frohman-Nicas TQFT – construction, characters and hard-Lefschetz decompositions

In [1] Frohman and Nicas introduce a ($\mathbb{Z}/2\mathbb{Z}$-projective) TQFT $\mathcal{V}^{FN}_{\mathbb{Z}}$ based on the intersection homology of the Jacobians $J(X) = \mathrm{Hom}(\pi_1(X), U(1))$. Specifically, the functor $\mathcal{V}^{FN}_{\mathbb{Z}}$ associates to every surface Σ the lattice $\mathcal{V}^{FN}_{\mathbb{Z}}(\Sigma) = H^*(J(\Sigma),\mathbb{Z}) \cong \bigwedge^* H_1(\Sigma,\mathbb{Z})$. Furthermore, it assigns to a cobordism $C : \Sigma_A \to \Sigma_B$ a linear (lattice) map $\mathcal{V}^{FN}_{\mathbb{Z}}(C) : \mathcal{V}^{FN}_{\mathbb{Z}}(\Sigma_A) \to \mathcal{V}^{FN}_{\mathbb{Z}}(\Sigma_B)$, which is computed (up to a sign) from intersection numbers of the surface Jacobians with respect to a Heegaard splitting of C. The mapping class group acts canonically on $\mathcal{V}^{FN}_{\mathbb{Z}}(\Sigma_g)$, factoring through the symplectic quotient $\Gamma_g \twoheadrightarrow \mathrm{Sp}(H_1(\Sigma_g,\mathbb{Z})) = \mathrm{Sp}(2g,\mathbb{Z})$.

As in [1] we introduce an inner product $\langle\_,\_\rangle$ on $\bigwedge^* H_1(\Sigma_g)$ with respect to a complex structure $J \in \mathrm{Sp}(2g,\mathbb{Z})$, related to the symplectic skew form $(\_,\_)$ by $(x, Jy) = \langle x, y\rangle$ and $J^2 = -1$. We fix a standard homology basis $a_1, \dots, a_g, b_1, \dots, b_g$, which is an orthonormal as well as a symplectic basis with $b_i = Ja_i$. Denote by $\omega_g = \sum_i a_i \wedge b_i$ the standard symplectic form. In this setting $\mathcal{V}^{FN}_{\mathbb{Z}}$ assigns the map $\alpha \mapsto \alpha \wedge a_{g+1}$ to the cobordism that attaches the $(g+1)$-st 1-handle to a surface Σ_g. (Here the a_{g+1}-cycle is contractible into the interior of the cobordism.) The linear map for the cobordism that attaches the dual 2-handle is the respective conjugate. Hence, for the standard handle body of genus g in S^3 the assigned state is $\Omega_g = a_1 \wedge \ldots \wedge a_g$, the volume form of the corresponding Lagrangian of the handle body. The complementary handle body is mapped to $\langle \Omega_g, \_\rangle$.

In [10] we introduce the Lefschetz-$\mathfrak{sl}_2$-action on $\bigwedge^* H_1(\Sigma_g)$ defined for the standard generators as $E.\alpha = \omega_g \wedge \alpha$, $F = E^*$, and $\hat{H}.\alpha = (\deg(\alpha) - g)\alpha$. It turns out that $\mathcal{V}^{FN}_{\mathbb{Z}}$ is equivariant with respect to this action so that we have the Lefschetz Decomposition,

$$\mathcal{V}^{FN}_{\mathbb{Z}} \quad = \quad \bigoplus_{j\geq 1} V_j \otimes \mathcal{V}^{(j)}_{\mathbb{Z}} \quad , \tag{4}$$

where V_j is the j-dimensional, simple representation of $\mathfrak{sl}_2$. Each $\mathcal{V}^{(j)}_{\mathbb{Z}}$ is an irreducible, lattice TQFT, and can be defined as the restriction to $ker(F) \cap ker(\hat{H} + j - 1)$. For a 3-manifold, M, and a given, $\varphi : H_1(M) \twoheadrightarrow \mathbb{Z}$, let Σ and C_Σ be a dual surface and a covering cobordism as in Section 1. As with any TQFT, it follows from S-equivalence, functoriality, and cyclicity of traces that the expressions

$$\Delta^{(j)}_\varphi(M) \; = \; \mathrm{trace}\Big(\mathcal{V}^{(j)}_{\mathbb{Z}}(C_\Sigma)\Big) \quad \in \mathbb{Z} \tag{5}$$

do not depend on the choice Σ but only on the pair (M, φ). We will call the $\Delta^{(j)}_\varphi$ the *fundamental* torsion coefficients or fundamental torsion weights since they can be understood as characters of fundamental $\mathrm{Sp}(2g,\mathbb{Z})$-representations. In Theorem 4.4 of [1] Frohman and Nicas establish the relation of $\mathcal{V}^{FN}_{\mathbb{Z}}$ with the Alexander polynomial via a Lefschetz trace. Combining their result with (4) and (5) we find the following expression:

Theorem 3 ([1], [10]) *Let $\varphi : H_1(M) \twoheadrightarrow \mathbb{Z}$, Δ_φ, $\Delta^{(j)}_\varphi$, Σ, and $C_\Sigma : \Sigma \to \Sigma$ as above.*

$$\Delta_\varphi(M) \; = \; \mathrm{trace}\Big((-t)^{\hat{H}}\mathcal{V}^{FN}_{\mathbb{Z}}(C_\Sigma)\Big) \; = \; \sum_{j\geq 1} [j]_{-t} \Delta^{(j)}_\varphi(M) \; . \tag{6}$$

We denote as usual $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \in \mathbb{Z}[q^{\pm 1}]$. For example, (6) implies that the fundamental coefficients $(\Delta^{(1)}, \Delta^{(2)}, \dots)$ for the 5_1 and the 5_2 knot are $(-3, -2, 0, \dots)$ and $(0, 1, 1, 0, \dots)$ respectively. If we combine Theorem 3 with Lemma 2 we find the following expression for the Lescop invariant in terms of the $\Delta_\varphi^{(j)}$'s.

Theorem 4 *For a closed 3-manifold and any epimorphism $\varphi : H_1(M) \twoheadrightarrow \mathbb{Z}$ the Lescop invariant is related to the fundamental torsion weights by*

$$\lambda_L(M) \;=\; \sum_{j \geq 1} L^{(j)} \Delta_\varphi^{(j)}(M) \ , \tag{7}$$

where $L^{(j)} = (-1)^{j-1} \dfrac{j(2j^2 - 3)}{12}$.

Clearly, the coefficients $L^{(j)}$ all lie in $\frac{1}{12}\mathbb{Z}$, and the first few of them are given by $L^{(j)} = -\frac{1}{12}, -\frac{5}{6}, \frac{15}{4}, -\frac{29}{3}, \frac{235}{12}, -\frac{69}{2}, \dots$. It is an interesting question whether there are further choices of coefficients, other that the ones used in (7), which would make the sum independent of the choice of φ. In [10] we also identify $\mathcal{V}_{\mathbb{Z}}^{FN}$ with the Hennings TQFT for the quasitriangular Hopf algebra $\mathbb{Z}/2\mathbb{Z} \ltimes \bigwedge^* \mathbb{R}^2$. This entails a calculus for determining $\Delta_\varphi(M)$, and hence also λ_L, from a surgery diagram, extending the traditional Alexander-Conway Calculus for knots and links. It also allows us to remove the sign-ambiguity of $\mathcal{V}_{\mathbb{Z}}^{FN}$ using the 2-framings on the cobordisms that are the standard additional data in the quantum constructions.

3 The Johnson-Morita homomorphisms and the Casson invariant

The mapping class group representations of the TQFT's $\mathcal{V}_{\mathbb{Z}}^{(j)}$ from the previous section all had the Torelli groups $\mathcal{I}_g = \ker(\Gamma_g \to \mathrm{Sp}(2g, \mathbb{Z}))$ in their kernel. The goal of this section is to seek representations and TQFT's with smaller kernels, and determine the information needed to describe the Casson invariant.

Let us first recall the construction of representations of the mapping class groups Γ_g from Section 7 of [11], obtained from the $\mathcal{V}_{\mathbb{Z}}^{(j)}$. They are indeed nontrivial on $\mathcal{I}_g$, but they do vanish on the subgroup $\mathcal{K}_g \subset \mathcal{I}_g$, which is generated by Dehn twists along bounding cycles on the surface Σ_g. Denote also $\mathcal{K}_{g,1}$ and $\mathcal{I}_{g,1}$ the respective subgroups of $\Gamma_{g,1}$ for a surface with one boundary component.

In a series of papers Johnson and Morita have studied these subgroups and their abelian quotients extensively. For $H = H_1(\Sigma_g, \mathbb{Z})$ let $U = \bigwedge^3 H/_{(\omega_g \wedge H)}$. Johnson [7] constructs a homomorphism τ_1 giving rise to the following Sp-equivariant short exact sequence.

$$0 \to \mathcal{K}_g \longrightarrow \mathcal{I}_g \xrightarrow{\tau_1} U \to 0 \ . \tag{8}$$

Here, U is thought of as a free abelian group. In [19] Morita extends this to a homomorphism, $\tilde{k}$, on the entire mapping class group.

$$0 \to \mathcal{K}_g \longrightarrow \Gamma_g \xrightarrow{\tilde{k}} \frac{1}{2}U \rtimes \mathrm{Sp}(2g, \mathbb{Z}) \to 0 \ . \tag{9}$$

As in [11] we introduce for any $x \in \bigwedge^m H$ the maps $\nu(x) : \bigwedge^* H \to \bigwedge^{*+m} H$ and $\mu(x) : \bigwedge^* H \to \bigwedge^{*-m} H$, given by $\nu(x).y = x \wedge y$ and $\mu(x) = \nu(Jx)^*$. It follows from basic relations that $\mu(x)$ maps the $ker(F)$'s to each other, decreasing the $\hat{H}$-weight by m, and that $\mu(\omega_g \wedge z)\big|_{ker(F)} = 0$ for any $(m-2)$-form z. Thus, in the case $m = 3$, μ factors for every $j \geq 1$ into an Sp-equivariant map,

$$\mu^\flat \ : \ U \longrightarrow \mathrm{Hom}(\mathcal{V}_{\mathbb{Z}}^{(j)}(\Sigma_g), \mathcal{V}_{\mathbb{Z}}^{(j+3)}(\Sigma_g)) \ . \tag{10}$$

Any such map serves as an extension map for a representation of $U \rtimes \mathrm{Sp}(2g, \mathbb{Z})$ on $\mathcal{V}_{\mathbb{Z}}^{(j)} \oplus \mathcal{V}_{\mathbb{Z}}^{(j+3)}$, in which U acts non-trivially. We thus have extended Γ_g-modules $\mathcal{U}_{\mathbb{Z}}^{(j)}(\Sigma_g)$, which fit into a short exact sequence as follows.

$$0 \to \mathcal{V}_{\mathbb{Z}}^{(j+3)}(\Sigma_g) \hookrightarrow \mathcal{U}_{\mathbb{Z}}^{(j)}(\Sigma_g) \twoheadrightarrow \mathcal{V}_{\mathbb{Z}}^{(j)}(\Sigma_g) \to 0 \ . \tag{11}$$

The Casson invariant λ_C for integral homology spheres is closely related to the subgroups $\mathcal{K}_g \subset \mathcal{I}_g$. Specifically, let us denote by M_ψ the 3-manifold obtained via a Heegaard construction by cutting S^3 along a standard embedded surface Σ_g and repasting it with an element $\psi \in \mathcal{I}_g$. This yields the following assignment

$$\lambda^* : \mathcal{I}_g \longrightarrow \mathbb{Z} : \quad \psi \mapsto \lambda^*(\psi) := \lambda_C(M_\psi) \,. \tag{12}$$

In [17], [18] Morita studies this map thoroughly. Two important observations of his are that any $\mathbb{Z}$-homology sphere is of the form M_ψ with $\psi \in \mathcal{K}_g$, and that λ^* restricted to $\mathcal{K}_g$ is a homomorphism. Consequently, λ^* is uniquely determined by its values on the generators $\mathsf{D}_\mathcal{C} \in \mathcal{K}_{g,1}$, given by Dehn twists along bounding curves $\mathcal{C} \subset \Sigma_{g,1}$. The difficulty of this description of λ^* lies in the fact that it is often not easy to find a presentation of a specific homology sphere by a product of $\mathsf{D}_\mathcal{C}$'s.

The value of λ^* on one of these generators of $\mathcal{K}_g$ is obtained from the Alexander polynomial via $\lambda^*(\mathsf{D}_\mathcal{C}) = \frac{1}{2}\Delta_\mathcal{C}''(1)$ considering $\mathcal{C}$ as a knot in S^3. If $\Sigma_{h,\mathcal{C}} \subset \Sigma_{g,1}$

is a surface of genus h bounded by $\mathcal{C}$, let $u_1, \ldots, u_h, v_1, \ldots, v_h$ be a symplectic basis of $H_1(\Sigma_{h,\mathcal{C}})$ and $\omega_{\mathcal{C}} = \sum_i u_i \wedge v_i$ the respective symplectic form of this surface. Moreover, for two homology cycles a and b in Σ_g let $l_0(a,b) = lk(a,b^+)$ denote their linking number, where b^+ denotes the "push-off" of the cycle b in positive normal direction. Morita introduces a homomorphism

$$\theta_0 : \bigwedge^2 H \otimes \bigwedge^2 H : a \wedge b \otimes c \wedge d \mapsto l_0(a,c)l_0(b,d) - l_0(a,d)l_0(b,c) \qquad (13)$$

and finds that $\lambda^*(\mathsf{D}_{\mathcal{C}}) = \theta_0(t_{\mathcal{C}})$, where $t_{\mathcal{C}} = -\omega_{\mathcal{C}} \otimes \omega_{\mathcal{C}}$.

In the context of TQFT interpretations it is now remarkable that θ_0 can be reexpressed by matrix elements of operators on $\bigwedge^* H$ using the multiplication and contraction maps ν and μ as before. More precisely, define a map $\Psi : \bigwedge^* H \otimes \bigwedge^* H \longrightarrow \mathrm{End}(\bigwedge^* H)$ by $\alpha \otimes \beta \mapsto \Psi(\alpha \otimes \beta) = \nu(\alpha) \circ \mu(\beta)$. Also, let $\Omega_g = a_1 \wedge \ldots \wedge a_g \in \bigwedge^g H$ be the handle body state of the Frohman-Nicas theory as in Section 2. The next identity follows now from an exercise in multilinear algebra using the relations in [11].

Lemma 5 *For any* $A \in \bigwedge^2 H \otimes \bigwedge^2 H$ *we have*

$$\theta_0(A) \ = \ \langle \Omega_g, \Psi(A)\Omega_g \rangle \ . \qquad (14)$$

We introduce a restricted Lefschetz $\mathfrak{sl}_2$-actions for the surface $\Sigma_{h,\mathcal{C}}$. Specifically, we have a subalgebra $\mathfrak{sl}_2^{\mathcal{C}}$, generated by $E_{\mathcal{C}} = \nu(\omega_{\mathcal{C}})$, $F_{\mathcal{C}} = J \circ E_{\mathcal{C}}^* \circ J^{-1} = \mu(\omega_{\mathcal{C}})$, and $\hat{H}_{\mathcal{C}} = [E_{\mathcal{C}}, F_{\mathcal{C}}] = -h + \sum_{i=1}^h \nu(u_i)\nu(u_i)^* + \nu(v_i)\nu(v_i)^*$. Moreover, we introduce the standard quadratic Casimir Operator $Q_{\mathcal{C}} = E_{\mathcal{C}}F_{\mathcal{C}} + \frac{1}{4}\hat{H}_{\mathcal{C}}(\hat{H}_{\mathcal{C}} - 2)$ of $\mathfrak{sl}_2^{\mathcal{C}}$, as well as $D_{\mathcal{C}} := \frac{1}{4}\hat{H}_{\mathcal{C}}(\hat{H}_{\mathcal{C}} - 2)$.

Theorem 6 *Let* λ^* *be as in (12),* $\Omega_g \in \bigwedge^g L$ *the standard handle body state,* $\mathsf{D}_{\mathcal{C}} \in \mathcal{K}_g$ *the Dehn twist along a bounding curve* $\mathcal{C}$*, and* $E_{\mathcal{C}}, F_{\mathcal{C}}, \hat{H}_{\mathcal{C}}, D_{\mathcal{C}}, Q_{\mathcal{C}} \in U(\mathfrak{sl}_2^{\mathcal{C}})$ *as above. Then*

$$\lambda^*(\mathsf{D}_{\mathcal{C}}) \ = \ -\langle \Omega_g, E_{\mathcal{C}}F_{\mathcal{C}}.\Omega_g \rangle \ = \ \langle \Omega_g, D_{\mathcal{C}}.\Omega_g \rangle \ - \ \langle \Omega_g, Q_{\mathcal{C}}.\Omega_g \rangle \ . \qquad (15)$$

Moreover, if $\psi \in \Gamma_g$ *either preserves or reverses the Lagrangian decomposition* $H = L \oplus L^{\perp}$ *given by the standard handle body, then*

$$\lambda^*(\mathsf{D}_{\psi(\mathcal{C})}) \ = \ \lambda^*(\mathsf{D}_{\mathcal{C}}) \ . \qquad (16)$$

Proof The identity in (15) is readily obtained by combining Lemma 5 and Morita's expression. The second assertion is obvious in the cases when ψ preserves L and $L^{\perp}$, since the same is true for ψ^{-1} and ψ^* and we have $\bar{\psi} \circ X_{\mathcal{C}} \circ \bar{\psi}^{-1} = X_{\psi(\mathcal{C})}$ for every generator $X \in \mathfrak{sl}_2^{\mathcal{C}}$, where $\bar{\psi}$ the symplectic action on $\bigwedge^* H$. For the reversing case it now suffices to consider only a representative $\sigma \in \Gamma_g$ of the complex structure $\bar{\sigma} = J \in \mathrm{Sp}(2g, \mathbb{Z})$. We observe that for all of the operators $Y_{\mathcal{C}} \in \{E_{\mathcal{C}}F_{\mathcal{C}}, Q_{\mathcal{C}}, D_{\mathcal{C}}\}$ we have $Y_{\mathcal{C}}^* = JY_{\mathcal{C}}J^{-1} = Y_{\sigma(\mathcal{C})}$, which implies (16). □

The operator form of (15) suggests several decompositions of λ^*. Both $Q_{\mathcal{C}}$ and $D_{\mathcal{C}}$ have spectrum $\{\frac{j^2-1}{4} : j = 1, 2, \ldots, h'+1\}$, where $h' = \min(h, g-h)$. To describe the eigenspaces more precisely, note that $H_1(\Sigma_g) \cong H_1(\Sigma_{h,\mathcal{C}}) \oplus H_1(\Sigma_{g-h,\mathcal{C}})$, where $\Sigma_{g-h,\mathcal{C}}$ is the complementary surface. Since $Q_{\mathcal{C}}$ also commutes with the total $\mathfrak{sl}_2$-action it preserves $\mathcal{V}^{(0)}_{\mathbb{Z}}(\Sigma_g)$, which contains Ω_g and, hence, also $Q_{\mathcal{C}}.\Omega_g$. The j-th eigensubspace in this restriction is thus $\mathbb{E}_j(Q_{\mathcal{C}}) = \mathcal{V}^{(j)}(\Sigma_{h,\mathcal{C}}) \otimes \mathcal{V}^{(j)}(\Sigma_{g-h,\mathcal{C}})$. Now, $D_{\mathcal{C}}$ does not commute with $\mathfrak{sl}_2$ but still preserves the total degree and hence maps $\bigwedge^g H_1(\Sigma_g)$ to itself. Correspondingly, we find in this restriction $\mathbb{E}_j^{\pm}(D_{\mathcal{C}}) = \bigwedge^{h+1\pm j} H_1(\Sigma_{h,\mathcal{C}}) \otimes \bigwedge^{g-h-1\mp j} H_1(\Sigma_{g-h,\mathcal{C}})$.

The Casson invariant is thus expressible as a sum of terms $\frac{j^2-1}{4}\langle \Omega_g, \mathbb{P}^{(j)}_{\mathcal{C}}.\Omega_g\rangle$, where the $\mathbb{P}^{(j)}_{\mathcal{C}}$ are projectors onto the eigenspaces of $Q_{\mathcal{C}}$ or $D_{\mathcal{C}}$. Since $\bar{\psi} \circ \mathbb{P}^{(j)}_{\mathcal{C}} \circ \bar{\psi}^{-1} = \mathbb{P}^{(j)}_{\psi(\mathcal{C})}$ and Γ_g acts transitively on the set of bounding curves, these operators and spaces can be determined from just the standard curve $\mathcal{C}_0^{(h)}$ for which $H_1(\Sigma_{\mathcal{C}_0^{(h)}})$ has $a_1, \ldots, a_h, b_1, \ldots, b_h$ as a symplectic basis.

One well known decomposition of the Casson invariant is given by its rôle in Floer cohomology. It is interesting to understand whether the eigenspace decompositions discussed here provide similar splittings of λ^* or λ_C as group morphisms or 3-manifold invariants and are, possibly, even related to the Floer group decomposition.

Further investigations of Morita in [17], [18] are devoted to understanding the failure of λ^* to extend as a homomorphism to $\mathcal{I}_g$. More precisely, he considers the integral cocycle (and rational coboundary) on $\mathcal{I}_g$, given as

$$\delta_C(\phi, \psi) \;=\; \frac{1}{2}\delta\lambda^*(\phi, \psi) \;=\; \frac{\lambda^*(\phi\psi) - \lambda^*(\psi) - \lambda^*(\phi)}{2}\,. \tag{17}$$

In Theorem 4.3 of [18] finds the expression $\delta_C(\phi, \psi) = \tilde{s}(\tau_1(\phi), \tau_1(\psi))$, where τ_1 is the Johnson homomorphism onto $U = \bigwedge^3 H/H$. The bilinear form $\tilde{s}(\ ,\ )$ descends from the bilinear form s on $\bigwedge^3 H$ defined by $s(\alpha, \beta) = (\alpha, \Pi_L\beta)$, where Π_L is the canonical projection onto $\bigwedge^3 L \subset \bigwedge^3 H$ and $(\ ,\ )$ is the extension of the standard symplectic form. It is easy to see that $\omega \wedge H$ lies in the left and right null space of s. From (17), Lemma 5, and further multilinear computations we obtain the following.

Theorem 7 *Let $\delta_{\mathcal{C}}$ be as in (17), τ_1 as in (8), and μ and ν as before. Then*

$$\delta_C(\phi,\psi) \;=\; -\langle \Omega_g, \nu(\tau_1(\phi))\mu(\tau_1(\psi)).\Omega_g \rangle \tag{18}$$

Next, we describe a general form of a TQFT, motivated by the structure of the Reshetikhin-Turaev Theories (see Section 4). This will provide a useful framework for finding TQFT interpretations of the formulae (15) and (18) for the Casson invariant.

We start with a commutative ring $\mathbb{M}$ with unit, and denote the ring $\breve{\mathbb{M}} = \mathbb{M}[\mathsf{y}]/\mathsf{y}^2$ (of rank 2 as an $\mathbb{M}$-module). Moreover, we assume two $\mathrm{Sp}(2g,\mathbb{Z})$-representations, $W_0(\Sigma_g)$ and $W_1(\Sigma_g)$, which are free as modules over $\mathbb{M}$. We write $\boldsymbol{\rho}_i : \Gamma_g \to \mathrm{GL}_{\mathbb{M}}(W_i(\Sigma_g))$ for the homomorphism with $\mathcal{I}_g$ in its kernel. Furthermore, we assume that each W_i admits an inner product $\langle\ ,\ \rangle$, and a special unit vector $\vec{\boldsymbol{o}}_g \in W_0(\Sigma_g)$. We also denote $\widetilde{W}_i(\Sigma_g) = W_i(\Sigma_g) \otimes_{\mathbb{M}} \breve{\mathbb{M}}$ and $\widetilde{W}(\Sigma_g) = \widetilde{W}_1(\Sigma_g) \oplus \widetilde{W}_0(\Sigma_g)$ to which we extend $\langle\ ,\ \rangle$ with $W_1 \perp W_0$.

Definition 1 A $1/1$-*solvable TQFT* is a TQFT $\breve{\mathcal{V}}$ over a ring $\breve{\mathbb{M}}$ such that the $\breve{\mathbb{M}}$-modules are of the form $\breve{\mathcal{V}}(\Sigma_g) = \widetilde{W}(\Sigma_g)$ as above. An element $\psi \in \Gamma_g$ is represented by $\breve{\mathcal{V}}$ in the form

$$\breve{\mathcal{V}}(\psi) \;=\; \begin{bmatrix} \boldsymbol{\rho}_1(\psi) & \boldsymbol{\mu}(\psi) \\ 0 & \boldsymbol{\rho}_0(\psi) \end{bmatrix} \;+\; \mathsf{y}\cdot \begin{bmatrix} \boldsymbol{\lambda}_1(\psi) & \boldsymbol{\kappa}(\psi) \\ \boldsymbol{\nu}(\psi) & \boldsymbol{\lambda}_0(\psi) \end{bmatrix} \;. \tag{19}$$

More generally, we require that the space $\widetilde{W}_1 \oplus \mathsf{y}\cdot W_0$ is preserved (hence giving rise to a sub-TQFT over $\mathbb{M}$), and, furthermore, that $\breve{\mathcal{V}}$ assigns to the standard handle bodies the vectors $\vec{\boldsymbol{o}}$ and $\langle \vec{\boldsymbol{o}},\ \rangle$. Finally, the TQFT is *half-projective* with parameter 0 or y.

Clearly, such a TQFT implies two invariants, $\tau^{\mathcal{V}}$ and $\lambda^{\mathcal{V}}$, of closed 3-manifolds into $\mathbb{M}$ defined as the polynomial coefficients of the element in $\breve{\mathbb{M}}$ assigned by the TQFT as follows.

$$\breve{\mathcal{V}}(M) \;=\; \tau^{\mathcal{V}}(M) \,+\, \mathsf{y}\cdot\lambda^{\mathcal{V}}(M)\;. \tag{20}$$

It also produces $\mathbb{M}$-valued invariants, $\Delta^{\mathcal{V}}_{\varphi}(M)$ and $\Xi^{\mathcal{V}}_{\varphi}(M)$, of pairs (M,φ), where $\varphi : H_1(M) \twoheadrightarrow \mathbb{Z}$, Σ, and C_Σ are as in Sections 1 and 2, by the following generalization of (5).

$$\mathrm{trace}\Big(\breve{\mathcal{V}}(C_\Sigma)\Big) \;=\; \Delta^{\mathcal{V}}_{\varphi}(M) \,+\, \mathsf{y}\cdot\Xi^{\mathcal{V}}_{\varphi}(M)\;. \tag{21}$$

Next, let us record a number of immediate consequences of the above definitions.

Lemma 8 *Let $\check{\mathcal{V}}$ be a 1/1-solvable TQFT over a ring $\mathbb{M}$ with unit for which $2 \in \mathbb{M}$ is not a zero-divisor.*

(1) *With the boundary operator $\delta\boldsymbol{\xi}(\psi,\phi) = \boldsymbol{\rho}_j(\psi)\boldsymbol{\xi}(\phi) - \boldsymbol{\xi}(\psi\phi) + \boldsymbol{\xi}(\psi)\boldsymbol{\rho}_i(\phi)$ for $\boldsymbol{\xi} : \Gamma_g \to \mathrm{Hom}(W_i, W_j)$ we have the relations*

$$\delta\boldsymbol{\nu} = \delta\boldsymbol{\mu} = 0 \tag{22}$$

$$-\delta\boldsymbol{\lambda}_1(\psi,\phi) = \boldsymbol{\mu}(\psi)\boldsymbol{\nu}(\phi) \tag{23}$$

$$-\delta\boldsymbol{\lambda}_0(\psi,\phi) = \boldsymbol{\nu}(\psi)\boldsymbol{\mu}(\phi) \tag{24}$$

$$-\delta\boldsymbol{\kappa}(\psi,\phi) = \boldsymbol{\lambda}_1(\psi)\boldsymbol{\mu}(\phi) + \boldsymbol{\mu}(\psi)\boldsymbol{\lambda}_0(\phi) \tag{25}$$

(2) *The restrictions of the maps $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ to $\mathcal{I}_g$ vanish on $\mathcal{I}_g' = [\mathcal{I}_g, \mathcal{I}_g]$.*

(3) *$\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ also factor through $\mathrm{Sp}(2g,\mathbb{Z})$-equivariant, linear maps $U \to \mathrm{Hom}_{\mathbb{M}}(W_0, W_1)$ and $U \to \mathrm{Hom}_{\mathbb{M}}(W_1, W_0)$ respectively.*

(4) *The restrictions of the $\boldsymbol{\lambda}_i$ to $\mathcal{K}_g$ vanish on $\mathcal{K}_g' = [\mathcal{K}_g, \mathcal{K}_g]$, and thus define $\mathrm{Sp}(2g,\mathbb{Z})$-equivariant, linear maps $H_1(\mathcal{K}_g) \to \mathrm{End}_{\mathbb{M}}(W_i)$.*

(5) *For a Heegaard presentation M_ψ we have*

$$\tau^{\mathcal{V}}(M_\psi) = \langle \vec{\boldsymbol{o}}, \boldsymbol{\rho}_0(\psi)\vec{\boldsymbol{o}} \rangle \qquad \text{and} \qquad \lambda^{\mathcal{V}}(M_\psi) = \langle \vec{\boldsymbol{o}}, \boldsymbol{\lambda}_0(\psi)\vec{\boldsymbol{o}} \rangle . \tag{26}$$

(6) *If M is a $\mathbb{Z}$-homology sphere then $\tau^{\mathcal{V}}(M) = 1$.*

(7) *The map $\lambda^{*\mathcal{V}} : \Gamma_g \to \mathbb{M} : \psi \mapsto \lambda^{*\mathcal{V}}(\psi) := \lambda^{\mathcal{V}}(M_\psi)$ restricts to a homomorphism on $\mathcal{K}_g$.*

(8) *The cocycle $\delta\lambda^{*\mathcal{V}}(\psi,\phi) = -\langle \vec{\boldsymbol{o}}, \boldsymbol{\nu}(\psi)\boldsymbol{\mu}(\phi)\vec{\boldsymbol{o}} \rangle$ restricted to $\mathcal{I}_g$ factors through a bilinear form on U.*

Proof The cohomological relations are an immediate consequence of the fact that the map $\check{\mathcal{V}} : \Gamma_g \to \mathrm{GL}_{\check{\mathbb{M}}}(\widetilde{W}(\Sigma_g))$ is a homomorphism. For example, (22) implies $\boldsymbol{\mu}(\mathcal{I}_g') = 0 = \boldsymbol{\nu}(\mathcal{I}_g')$. ¿From Johnson's results, see Theorems 3 and 6 in [8], we have that $\mathcal{K}_g/\mathcal{I}_g' \cong (\mathbb{Z}/2\mathbb{Z})^N$ so that also $\boldsymbol{\mu}(\mathcal{K}_g) = 0 = \boldsymbol{\nu}(\mathcal{K}_g)$ given that $(\mathbb{M})_2 = 0$. Each of the remaining assertions follows now easily from previous assertions and relations (22) and (25). □

The functorial properties of $\lambda^{\mathcal{V}}$ are strikingly similar to those of the Casson invariant λ_C. It is thus plausible to expect a TQFT-interpretation of λ_C to come about in this form. We thus add the notion that a 1/1-solvable TQFT is of *Casson-type* if $\overline{\lambda_C(M)} = \lambda^{\mathcal{V}}(M)$ for any $\mathbb{Z}$-homology sphere. Here, we denote by $n \mapsto \overline{n}$ the canonical map $\mathbb{Z} \to \mathbb{M}$. The similarities of formulae is reflected in the following observation.

For $0 < h < g$ and the standard separating curve $\mathcal{C}_0^{(h)}$ as before let us write $\boldsymbol{\lambda}^{(h)} := \boldsymbol{\lambda}_0(\mathsf{D}_{\mathcal{C}_0^{(h)}})$ and $L^{(h)} := E_{\mathcal{C}_0^{(h)}} F_{\mathcal{C}_0^{(h)}}$. Note, that both operators act on $\mathrm{Sp}(2g,\mathbb{Z})$-modules W_0 and $\bigwedge^* H$ respectively, and that they commute with the action of the standard subgroup $\mathrm{Sp}(2h,\mathbb{Z}) \times \mathrm{Sp}(2(g-h),\mathbb{Z}) \subset \mathrm{Sp}(2g,\mathbb{Z})$. The comparison of formula (26) in Lemma 8 with (15) and (18) is summarized in the next lemma.

Lemma 9 *A 1/1-solvable TQFT over $\mathbb{Z}$ is of Casson-type if and only if*

$$\langle \Omega_g, G \cdot L^{(h)} \cdot G^{-1}.\Omega_g \rangle \;=\; \langle \vec{\boldsymbol{o}}_g, G \cdot \boldsymbol{\lambda}^{(h)} \cdot G^{-1}.\vec{\boldsymbol{o}}_g \rangle \qquad (27)$$

for all $g, h \in \mathbb{N}$ with $0 < h < g$, and for all $G \in {}^{\mathrm{Sp}(2g,\mathbb{Z})}\big/_{\mathrm{Sp}(2h,\mathbb{Z})\times\mathrm{Sp}(2(g-h),\mathbb{Z})}$. In this case we also have for all $a, b \in U$ the relation

$$-2\langle \Omega_g, \nu(a)\mu(b).\Omega_g \rangle \;=\; \langle \vec{\boldsymbol{o}}_g, \boldsymbol{\nu}(a)\boldsymbol{\mu}(b).\vec{\boldsymbol{o}}_g \rangle \;. \qquad (28)$$

Finally, let us point out some subtleties associated with the second Johnson homomorphism. We write $T \subset \bigwedge^2 H \otimes \bigwedge^2 H$ for the symmetric subspace generated by $x \otimes x$ and $x \leftrightarrow y := x \otimes y + y \otimes x$ for all $x, y \in \bigwedge^2 H$, and, further, denote by $\mathfrak{h}_{g,1}(2) = T/T_0$ the quotient of T by the subspace T_0 generated by elements $a \wedge b \leftrightarrow c \wedge d - a \wedge c \leftrightarrow b \wedge d + a \wedge d \leftrightarrow b \wedge c$, see [18]. The second Johnson homomorphism τ_2 is now a map as follows.

$$\tau_2 \;:\; \mathcal{K}_{g,1} \longrightarrow \mathfrak{h}_{g,1}(2) \;:\; \mathsf{D}_{\mathcal{C}} \mapsto \overline{t_{\mathcal{C}}} \qquad \text{with} \quad t_{\mathcal{C}} := -\omega_{\mathcal{C}} \otimes \omega_{\mathcal{C}} \;. \qquad (29)$$

Here, $r \mapsto \overline{r}$ stands for the map $T \to T/T_0$, and $\mathsf{D}_{\mathcal{C}}$ is as in (13). Following (13), the Casson invariant on $\mathsf{D}_{\mathcal{C}}$ is also given by the homomorphism θ_0 evaluated on the element $t_{\mathcal{C}} \in T$. However, θ_0 does not vanish on T_0 and thus does not factor through $\mathfrak{h}_{g,1}(2)$. Consequently, $\lambda^*\big|_{\mathcal{K}_g}$ also does not factor through τ_2. Yet, in [18] Morita is able to define a homomorphism $\eta : \Gamma_{g,1} \longrightarrow \mathbb{Q}$ such that $\eta(\mathsf{D}_{\mathcal{C}}) = \frac{1}{6}h(h-1)$, as well as a homomorphism $\bar{q}_0 : \mathfrak{h}_{g,1}(2) \longrightarrow \mathbb{Q}$, such that $\lambda^*(\psi) = \eta(\psi) + \bar{q}_0(\tau_2(\psi))$. This raises the question what the relation is between this decomposition and the splitting of λ^* entailed by Theorem 7. Moreover, it is interesting to understand the rôle of τ_2 in the general framework of 1/1-solvable TQFT's.

4 p-Modular, Homological TQFT's – their Relation to S_n resolutions, extensions and characters

There are two ways to produce interesting TQFT's over the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, for a prime number $p \geq 3$. One is to consider the $\mathbb{F}_p$-reductions $\mathcal{V}_p^{(j)}$

of the Frohman-Nicas lattice TQFT's $\mathcal{V}_{\mathbb{Z}}^{(j)}$. The second is obtained from the constant order reduction of the cyclotomic integer expansion of the Reshetikhin-Turaev Theories. We will explore some relations between these two theories later. As a preparation let us first discuss the properties of the $\mathcal{V}_p^{(j)}$'s and the p-modular versions of Theorems 3 and 4.

The ring reduction alone from $\mathbb{Z}$ to $\mathbb{F}_p$ turns the irreducible TQFT $\mathcal{V}_{\mathbb{Z}}^{(j)}$ into a generally highly reducible TQFT $\mathcal{V}_p^{(j)}$. Specifically, the inner product on $\bigwedge^* H_1(\Sigma_g)$ induces a pairing $\langle\_,\_\rangle_p : \mathcal{V}_p^{(j)}(\Sigma_g) \otimes \mathcal{V}_p^{(j)}(\Sigma_g) \to \mathbb{F}_p$. It is clear that the null-space of this pairing yields a well-defined sub-TQFT.

Definition 2 [11] Let $\overline{\overline{\mathcal{V}}}_p^{(j)}$ be the quotient-TQFT obtained from $\mathcal{V}_p^{(j)}$ by dividing the vector space of each surface by the null-space of $\langle\_,\_\rangle_p$.

Next, we illustrate explicitly that this is a nontrivial operation.

Example 1 The map $\mathcal{V}_p^{(p-1)}(\Sigma_p) \twoheadrightarrow \overline{\overline{\mathcal{V}}}_p^{(p-1)}(\Sigma_p)$ has nontrivial kernel, given by $\mathbb{F}_p\overline{\omega_p} = im(E)$.

Proof Over $\mathbb{Z}$ the symplectic form ω_p is not in $\mathcal{V}_{\mathbb{Z}}^{(p-1)}(\Sigma_p)$. However, we can pick another representative of $\overline{\omega_p}$, namely, $v = E.1 - p(a_1 \wedge b_1) = \omega_p - p(a_1 \wedge b_1) \in \bigwedge^2 H_1(\Sigma_p)$. Since $F.a_i \wedge b_i = 1$ we find $F.v = 0$ so that indeed $v \in \mathcal{V}_{\mathbb{Z}}^{(p-1)}(\Sigma_p) = ker(F) \cap \ker(\hat{H} + p - 2)$. Now, if $w \in \mathcal{V}_{\mathbb{Z}}^{(p-1)}(\Sigma_p)$ is any other such vector we find $\langle v, w\rangle = \langle E1, w\rangle - \langle p(a_1 \wedge b_1), w\rangle = \langle 1, Fw\rangle - p\langle (a_1 \wedge b_1), w\rangle = -p\langle (a_1 \wedge b_1), w\rangle \in p\mathbb{Z}$. Thus, if $\overline{v} = \overline{\omega_p}$ and $\overline{w}$ are the respective vectors in $\mathcal{V}_p^{(p-1)}(\Sigma_p)$ we see that $\langle \overline{\omega_p}, \overline{w}\rangle_p = 0$ so that $\overline{\omega_p} \neq 0$ lies in the null space of pairing $\langle\_,\_\rangle_p$ and, hence, in the kernel of the above projection. The fact that the kernel is not bigger than this is implied by Theorem 10 below. □

The general relation between the $\mathcal{V}_p$ and $\overline{\overline{\mathcal{V}}}_p$ has the following description.

Theorem 10 [11] *The TQFT's $\overline{\overline{\mathcal{V}}}_p^{(j)}$ are irreducible for any $j \in \mathbb{N}$ and any prime $p \geq 3$. Each $\overline{\overline{\mathcal{V}}}_p^{(j)}(\Sigma_p)$ carries a nondegenerate inner form, with a compatible, irreducible $\mathrm{Sp}(2g, \mathbb{Z})$-representation (i.e., $\psi^* = J\psi^{-1}J^{-1}$).*

Moreover, for any $k \in \mathbb{N}$ with $0 < k < p$ we have a resolution of the quotient-TQFT given by an exact sequence as follows.

$$\ldots \to \mathcal{V}_p^{(c_{i+1})} \to \mathcal{V}_p^{(c_i)} \to \ldots \to \mathcal{V}_p^{(2p+k)} \to \mathcal{V}_p^{(2p-k)} \to \mathcal{V}_p^{(k)} \to \overline{\overline{\mathcal{V}}}_p^{(k)} \to 0\,, \tag{30}$$

where $c_i = ip + k$ if i is even, and $c_i = (i+1)p - k$ if i is odd.

The arrows in the sequence are to be understood as natural transformations between TQFT functors. Particularly, this means that we obtain an $\mathrm{Sp}(2g,\mathbb{Z})$-equivariant resolution of $\overline{\overline{\mathcal{V}}}_p^{(k)}(\Sigma_g)$ for every $g \geq 0$, whose form (30) is, remarkably, independent of g. Quite curiously, the maps in (30) are given by the Lefschetz operators from Section 2. More precisely, we prove in [11] that for $j \equiv k \mod p$ the operator E^k maps $\mathcal{V}_{\mathbb{Z}}^{(j)}(\Sigma_g)$ to $\mathcal{V}_{\mathbb{Z}}^{(j-2k)}(\Sigma_g) + p\bigwedge^{g-j+2k+1}H_1(\Sigma_g)$. Hence, we obtain well defined maps $E^{k_i} : \mathcal{V}_p^{(c_i)} \longrightarrow \mathcal{V}_p^{(c_i-1)}$ in the $\mathbb{F}_p$-reduction.

The rank reduction from $\mathcal{V}_p$ to $\overline{\overline{\mathcal{V}}}_p$ makes the representation theory more challenging as well. The $\mathrm{Sp}(2g,\mathbb{Z})$-representations for the integral TQFT's $\mathcal{V}_{\mathbb{Z}}$ obviously lift to representations of $\mathrm{Sp}(2g,\mathbb{R})$, and are, therefore, highest weight representations in the sense of standard Lie theory. It follows from simple dimension counting that most of the $\overline{\overline{\mathcal{V}}}_p$ cannot be of such a form.

In [11] we prove exactness of (30) by breaking the sequence down into the $\mathfrak{sp}_{2g}$-weight spaces $\mathcal{W}_p^{(c)}(\varpi)$ for a weight ϖ, which are evidently preserved by the E^{k_i}-maps. Each $\mathcal{W}^{(c)}(\varpi)$ carries a natural, equivariant action of the symmetric group S_n, where n is the number of zero components of ϖ, see [11]. The $\mathcal{W}_p^{(c)}(\varpi)$ turn out to be isomorphic to the standard Specht modules $\mathcal{S}_p^{(c)}$ over $\mathbb{F}_p$ associated to the Young diagram $[\frac{n+c-1}{2}, \frac{n-c+1}{2}]$. The respective weight spaces of the $\overline{\overline{\mathcal{V}}}_p^{(k)}$ are easily identified with the irreducible S_n-modules $\mathcal{D}_p^{(k)}$ over $\mathbb{F}_p$, obtained, similarly, by an inner form reduction as in [6]. Exactness in (30), is thus a consequence of the following result in the representation theory of the symmetric groups.

Theorem 11 [11] *Let $\mathcal{S}_p^{(c)}$ and $\mathcal{D}_p^{(k)}$ be S_n-modules as above, and denote by $\chi^{(c)}$ and $\phi_p^{(k)}$ their characters, respectively. We have a resolution as follows.*

$$\ldots \to \mathcal{S}_p^{(c_{i+1})} \to \mathcal{S}_p^{(c_i)} \to \ldots \to \mathcal{S}_p^{(2p-k)} \to \mathcal{S}_p^{(k)} \to \mathcal{D}_p^{(k)} \to 0 \tag{31}$$

Here, k and the c_i are as in Theorem 10. We obtain the relation

$$\phi_p^{(k)} = \sum_{i\geq 0}(-1)^i\chi^{(c_i)} . \tag{32}$$

The proof uses the precise ordered modular structure of the $\mathcal{S}_p^{(c)}$ given by Kleshchev and Sheth in [13], which turns out to be sufficiently rigid to prohibit any homology. The character expansion of irreducible p-modular S_n-characters into p-reductions of the ordinary characters in (32) is a direct consequence of (31), and appears to be new in the modular representation theory of S_n.

In order to extend the results from Section 2 to $\mathbb{F}_p$ we introduce, in analogy to (5), the *p-modular*, fundamental torsion weights, given for a pair $(M, \varphi : H_1(M) \twoheadrightarrow \mathbb{Z})$ by

$$\overline{\overline{\Delta}}^{(j)}_{\varphi,p}(M) \;=\; \mathrm{trace}(\overline{\overline{\mathcal{V}}}^{(j)}_p(C_\Sigma)) \;\in\; \mathbb{F}_p\,. \tag{33}$$

The images of the Alexander polynomial and the Lescop invariant in the cyclotomic integers are next expressed in the weights from (33).

Theorem 12 *Let $f_p : \mathbb{Z}[t, t^{-1}] \to \mathbb{F}_p[\zeta_p]$ be the canonical ring homomorphism, and denote, for $p \geq 5$, by $\overline{L}^{(j)}_p \in \mathbb{F}_p$ of the coefficients $L^{(j)} \in \frac{1}{12}\mathbb{Z}$ from (7). Then*

$$f_p\big(\Delta_\varphi(M)\big) \;=\; \sum_{k=1}^{p-1}[k]_{-\zeta_p} \cdot \overline{\overline{\Delta}}^{(k)}_{\varphi,p}(M) \;\in\; \mathbb{F}_p[\zeta_p]\,. \tag{34}$$

$$\lambda_L(M) \;=\; \sum_{k=1}^{p-1} \overline{L}^{(k)}_p \overline{\overline{\Delta}}^{(k)}_{\varphi,p}(M) \qquad \mathrm{mod}\; p\,. \tag{35}$$

Proof The resolution from (30) implies, analogous to (32), the alternating series

$$\overline{\overline{\Delta}}^{(k)}_{\varphi,p}(M) = \sum_{i\geq 0}(-1)^i \Delta^{(c_i)}_\varphi(M) \quad \mathrm{mod}\; p \qquad 0 < k < p\,. \tag{36}$$

We note also that $f_p([c_i]_{-t}) = [c_i]_{-\zeta_p} = (-1)^i[k]_{-\zeta_p} = (-1)^{i+k-1}[k]_{\zeta_p}$. Combining (36) with (6) we obtain the expansion (34) of the Alexander polynomial in $\mathbb{F}_p[\zeta_p]$ in terms of the irreducible, p-modular weights from (33). For the p-reduction of the Lescop invariant in (35) note that for $p \geq 5$ we have $\frac{1}{12} \in \mathbb{F}_p$ so that the p-reductions $\overline{L}^{(j)}_p$ of the $L^{(j)}$ from (7) are well defined. For example, $\overline{L}^{(j)}_5 = 2, 0, 0, 2, 0, \ldots$ and $\overline{L}^{(j)}_7 = 4, 5, 2, 2, 5, 4, 0, \ldots$. In general, we have $\overline{L}^{(p\pm j)}_p = \overline{L}^{(j)}_p$ so that $\overline{L}^{(c_i)}_p = (-1)^i \overline{L}^{(k)}_p$. From this, (36), and (7) we thus infer (35). □

Finally, let us note that the Johnson-Morita extension we constructed in (11) factors into the irreducible, p-modular quotients so that we have for $0 < k < p-3$ representations $\overline{\overline{\mathcal{U}}}^{(k)}_p$ of $U \rtimes \mathrm{Sp}(2g, \mathbb{Z})$ over $\mathbb{F}_p$, which represent U nontrivially and admit short exact sequences as follows.

$$0 \;\to\; \overline{\overline{\mathcal{V}}}^{(k+3)}_{\mathbb{Z}}(\Sigma_g) \;\hookrightarrow\; \overline{\overline{\mathcal{U}}}^{(k)}_p(\Sigma_g) \;\twoheadrightarrow\; \overline{\overline{\mathcal{V}}}^{(k)}_p(\Sigma_g) \;\to\; 0\,. \tag{37}$$

5 The $\mathbb{F}_p[\zeta_p]$-Expansion of the Reshetikhin-Turaev TQFT, the structure of the Fibonacci case and cut-numbers

Recall that the Reshetikhin-Turaev invariants for $U_{\zeta_p}(\mathfrak{so}_3)$, at a p-th root of unity ζ_p, lie in the cyclotomic integers $\mathbb{Z}[\zeta_p]$ if $p \geq 3$ is a prime. Their expansions in $\mathsf{y} = (\zeta_p - 1)$ yield the Ohtsuki-Habiro invariants, which, in lowest order, are related to the previously discussed torsion and Casson invariants. Gilmer [4] gives an abstract proof that the TQFT's $\mathcal{V}^{RT}_{\zeta_p}$ associated to $U_{\zeta_p}(\mathfrak{so}_3)$ can be properly defined as TQFT's $\mathcal{V}^{I}_{\zeta_p}$ defined over the cyclotomic integers $\mathbb{Z}[\zeta_p]$ for a certain restricted set of cobordisms. Consider the ring epimorphism $\mathbb{Z}[\zeta_p] \twoheadrightarrow \mathbb{F}_p[\mathsf{y}]/\mathsf{y}^{p-1} \,:\, \zeta_p \mapsto 1+\mathsf{y}$ as well as the TQFT $\mathcal{V}^{I}_{p,\mathsf{y}}$ over $\mathbb{F}_p[\mathsf{y}]/\mathsf{y}^{p-1}$ induced by it from $\mathcal{V}^{I}_{\zeta_p}$. For given bases we can, therefore, consider the expansions in y of the linear map assigned to a cobordism C. They give rise to further reduced TQFT's $\mathcal{V}^{[\leq j]}_p$ over $\mathbb{F}_p[\mathsf{y}]/\mathsf{y}^{j+1}$ as follows.

$$\mathcal{V}^{I}_{p,\mathsf{y}}(C) \;=\; \sum_{k=0}^{p-2} \mathsf{y}^k \cdot \mathcal{V}^{I}_{p,[k]}(C) \quad \text{and} \quad \mathcal{V}^{[\leq j]}_p \;:=\; \sum_{k=0}^{j} \mathsf{y}^k \cdot \mathcal{V}^{I}_{p,[k]}(C)\,. \tag{38}$$

Here, each $\mathcal{V}^{I}_{p,[k]}(C)$ is a matrix with entries in $\mathbb{F}_p$. We will focus below on the structure of the TQFT's $\mathcal{V}^{[\leq 0]}_p$ and $\mathcal{V}^{[\leq 1]}_p$ over $\mathbb{F}_p$ and $\check{\mathbb{F}}_p = \mathbb{F}_p[\mathsf{y}]/\mathsf{y}^2$, respectively.

Conjecture 13 *Let $p \geq 5$ be a prime and $q_p = \dfrac{p-3}{2}$.*

A) *There are TQFT's, $\overline{\overline{\mathcal{U}}}{}^{(k)}_p$, which extend the Γ_g representations from (37).*

B) *The TQFT $\mathcal{V}^{[\leq 0]}_p$ over $\mathbb{F}_p$ is a quotient of sub-TQFT's of the q_p-fold symmetric product $S^{q_p}\overline{\overline{\mathcal{U}}}{}^{(1)}_p$.*

C) *The TQFT $\mathcal{V}^{I}_{\zeta_p}$ is* half-projective *with parameter $\mathsf{x} = (\zeta_p - 1)^{q_p}$, and, as such, has a "block-structured" $\mathbb{Z}[\zeta_p]$-basis.*

The statements are not independent. Obviously B) only makes sense if A) is true. Moreover, what we call a "block-structure" in C), meaning roughly that integral bases can be obtained by sewing surfaces together, implies the conditions for half-projectivity. Recall from [9] that a half-projective TQFT $\mathcal{V}$ over a ring R with respect to some $\mathsf{x} \in \mathsf{R}$ fulfills all of the usual TQFT axioms except for the following modification of the functoriality with respect to compositions. For two cobordisms, C_1 and C_2, with well defined composite $C_2 \circ C_1$ we have $\mathcal{V}(C_2 \circ C_1) = \mathsf{x}^{\mu(C_2,C_1)} \cdot \mathcal{V}(C_2)\mathcal{V}(C_1)$, where $\mu(C_2, C_1) \in \mathbb{N} \cup \{0\}$

is the rank of the connecting map $H_1(C_2 \cup C_1) \to H_0(C_2 \cap C_1)$ in the respective Mayer-Vietoris sequence. The tensor product axiom remains the usual. It is immediate that $\mathcal{V}$ satisfies the ordinary functoriality of TQFT's if we either restrict ourselves to connected surfaces or if x is invertible in R and we rescale $\mathcal{V}$. Non-semisimple Hennings TQFT's and homological gauge theories as in Section 2 are the first examples for $\mathsf{x} = 0$ theories, see [10]. Following [9], a consequence of Conjecture 13 C) is the following conjecture raised first by Gilmer.

Conjecture 14 [4] *For a closed, connected 3-manifold we have*

$$cut(M) \;\leq\; \frac{1}{q_p}\mathfrak{o}_p(M)\;. \tag{39}$$

Recall that the cut-number $cut(M)$ of a closed, connected 3-manifold, M, is defined, alternatively, as the maximal number of components that a surface $\Sigma \subset M$ can have for which $M - \Sigma$ is still connected, or as the maximal rank of a (non-abelian) free group F such that there is an epimorphism $\pi_1(M) \twoheadrightarrow F$. The quantum-order $\mathfrak{o}_p(M)$ of a closed 3-manifold is the maximal k such that $\mathcal{V}^{RT}_{\zeta_p}(M) \in \mathsf{y}^k\mathbb{Z}[\zeta_p]$, where $\mathsf{y} = \zeta_p - 1$ as before, assuming that we have a normalization for which $\mathcal{V}^{RT}_{\zeta_p}(S^3)$ is a unit in $\mathbb{Z}[\zeta_p]$.

The evidence for B) of Conjecture 13 comes, e.g, from explicitly matching dimensions for $g = 1, 2$ and general p, from comparison of the asymptotic behavior of the dimensions as $g \to \infty$ (see [11]), and from further consistencies with the cyclotomic integer expansions. The form given in B) also implies that the power $(I\mathcal{I}_g)^{q_p+1} \subset \mathbb{Z}[\Gamma_g]$ of the augmentation ideal of the Torelli group is in the kernel of $\mathcal{V}_p^{[\leq 0]}$. In analogy to Definition 1, we will thus call $\mathcal{V}_p^{[\leq l]}$ a q_p/l-solvable theory. Murakami's result [20] together with B) would hence imply that $\mathcal{V}_p^{[\leq 1]}$ is a $q_p/1$-solvable *Casson*-TQFT in the generalized sense of Definition 1.

The motivation and another strong piece of evidence for Conjecture 13 is the following example.

Theorem 15 [12] *Conjecture 13 holds true for $p = 5$. More precisely, we have the following isomorphism for the constant order TQFT.*

$$\overline{\overline{\mathcal{U}}}{}_5^{(1)} \;\cong\; \mathcal{V}_5^{[\leq 0]}\;. \tag{40}$$

Note first, that, with $q_5 = 1$, the product-TQFT in part B) of Conjecture 13 is simply $\overline{\overline{\mathcal{U}}}{}_5^{(1)}$ itself, which is, clearly, consistent with (40).

Moreover, the projective parameter of $\mathcal{V}^{RT}_{\zeta_5}$ simply becomes $\mathsf{x} = \mathsf{y} = \zeta_5 - 1$. We sometimes call the Reshetikhin-Turaev TQFT at a fifth root of unity the *Fibonacci TQFT's* since the dimension of $\mathcal{V}^{RT}_{\zeta_5}(\Sigma_g)$ is given, e.g., for even g by $5^{\frac{g}{2}} f_{g-1}$, where $f_0 = 0, f_1 = 1, f_2 = 1, \ldots$ are the Fibonacci numbers. (A similar formula holds for odd g, see [11].) Note, that the Kauffman bracket skein theory associated to $U_{\zeta_p}(\mathfrak{so}_3)$ has only two colors, namely 1 and ρ, subject to $\rho \otimes \rho = 1 \oplus \rho$. Despite the seeming simplicity of the Fibonacci TQFT, it is shown in [2, 3] to be of fundamentally greater complexity than the TQFT's for $p = 3, 4, 6$, which are already interesting. The topological content of the $p = 4$-TQFT, for example, has been identified with the Rochlin invariant and the Birman-Craggs-Johnson homomorphisms [25].

We state next the consequences of Theorem 15 and the identification in (40) that concern the explicit relations between the Casson-Walker-Lescop invariant λ_{CWL} and the Fibonacci TQFT.

Corollary 1 *The TQFT $\mathcal{V}_5^{[\leq 1]}$ over $\breve{\mathbb{F}}_5 = \mathbb{F}_5[\mathsf{y}]/\mathsf{y}^2$ is a 1/1-solvable TQFT over $\mathbb{F}_5$ of Casson-type in the sense of Definition 1, and, thus, defines the $\mathbb{F}_5$-valued invariants for closed 3-manifolds τ_5 and λ_5 as in (20), as well as the $\mathbb{F}_5$-invariants $\Delta_{\varphi,5}$ and $\Xi_{\varphi,5}$ for pairs (M,φ) as in (21).*

1. $\tau_5(M) = |H_1(M,\mathbb{Z})| \mod 5$ *if $b_1(M) = 0$ and 0 else wise.*
2. $\lambda_5(M) = \lambda_{CWL}(M) \mod 5$.
3. $\Delta_{\varphi,5}(M) = -2 \cdot \lambda_5(M) \mod 5$.
4. $\Xi_{\varphi,5}(M) = \Xi_5(M)$ *is independent of φ.*

Proof Recall that for closed manifolds $\mathcal{V}^{(j)}_{\mathbb{Z}}(M) = 0$ if $j \geq 2$, and $\mathcal{V}^{(1)}_{\mathbb{Z}}(M) = \mathcal{V}^{FN}(M)$ yields the order of the first homology group of M. The first claim thus follows from the fact that $\mathcal{V}^{(1)}$ occurs precisely once in the resolution of $\mathcal{V}_5^{[\leq 0]}$.

For the case $b_1(M) = 0$ the identification with λ_{CWL} follows from Murakami's work [20]. In the Lescop case, $b_1(M) \geq 1$, we find from (35) and (40) that $\lambda_L(M) = 2(\overline{\overline{\Delta}}{}^{(1)}_{\varphi,5}(M) + \overline{\overline{\Delta}}{}^{(4)}_{\varphi,5}(M)) = 2 \cdot (\mathrm{trace}(\overline{\overline{\mathcal{V}}}{}^{(1)}_5(C_\Sigma) \oplus \overline{\overline{\mathcal{V}}}{}^{(4)}_5(C_\Sigma))) = 2 \cdot \mathrm{trace}(\overline{\overline{\mathcal{U}}}{}^{(1)}_5(C_\Sigma)) = 2 \cdot \mathrm{trace}(\mathcal{V}_5^{[\leq 0]}(C_\Sigma))$. Now, it also follows from TQFT axioms that $\mathcal{V}^I_{\zeta_5}(M) = (\zeta_5 - \zeta_5^{-1}) \cdot \mathrm{trace}(\mathcal{V}^I_{\zeta_5}(C_\Sigma)) = 2\mathsf{y} \cdot \mathrm{trace}(\mathcal{V}_5^{[\leq 0]}(C_\Sigma)) + \mathcal{O}(\mathsf{y}^2)$. Comparison yields the assertion. The identity for $\Delta_{\varphi,5}(M)$ can be read from this calculation as well. The claim for $\Xi_{\varphi,5}(M)$ follows from the analogous identification with the next order Ohtsuki invariant that appears as the coefficient of y^2. $\square$

Let us, moreover, comment on the consequences of Lemma 9 for the Fibonacci case. In [12] we show that $\boldsymbol{\lambda}^{(h)}$ is an orthogonal projector, whose kernel is naturally isomorphic to $\overline{\overline{\mathcal{V}}}^{(1)}(\Sigma_h)\otimes\overline{\overline{\mathcal{V}}}^{(1)}(\Sigma_{g-h})\oplus\overline{\overline{\mathcal{V}}}^{(4)}(\Sigma_h)\otimes\overline{\overline{\mathcal{V}}}^{(4)}(\Sigma_{g-h})$. For small genera this space can be related to the known eigenspaces of the summands of the operator $L^{(h)} = Q_{\mathcal{C}_0^{(h)}} - D_{\mathcal{C}_0^{(h)}}$, and, thereby, yields an alternative proof of Murakami's result [20] in this rather special case. Therefore, it seems likely that with a better understanding of the structure for general genera and primes it is possible to give an entirely independent proof of the result in [20] based on purely representation theoretic and TQFT methods.

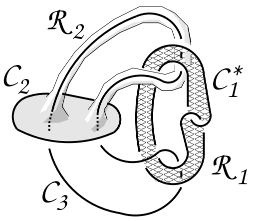
Finally, let us illustrate some concrete topological applications of the structural theory presented in this article. Given Theorem 15 also Conjecture 14 becomes a theorem in the case $p=5$ as follows.

Theorem 16 [5]

$$cut(M) \;\leq\; \mathfrak{o}_5(M)\ . \tag{41}$$

Let us mention here two examples in which (41) allows us to determine the cut number $cut(M)$. Computations of this kind by classical means, generally, entail quite complicated and difficult problems in topology or group theory.

Example C1 Consider the manifold W obtained by 0-surgery along the link with linking numbers 0 shown in the following figure.



We see that we have two disjoint surfaces that do not disconnect W. They consist of the depicted Seifert surfaces $\mathcal{R}_i$ of $\mathcal{C}_1^*$ and $\mathcal{C}_2$ and the discs glued in along the $\mathcal{C}_i$'s by surgery. Also, we know $b_1(W)=3$ so that $cut(W)$ can still be either 2 or 3. It now follows from a short skein theoretic calculation that $\mathfrak{o}_5(W)=2$ and hence $cut(W)=2$.

Example C2 Let $\overline{\psi}\in \mathrm{Sp}(H_1(\Sigma_g,\mathbb{Z}))$ be the symplectic, linear map associated to a mapping class $\psi\in\Gamma_g$. We denote by $a_j(\psi)\in\mathbb{Z}$ the coefficients of the symmetrized, characteristic polynomial $t^{-g}\det(t\mathbb{I}_{2g}-\overline{\psi})=\sum_j a_j(\psi)t^j$ so that $a_{-j}(\psi)=a_j(\psi)$ and $a_j(\psi)=0$ for $|j|>g$. Define next $\delta_5:\Gamma_g\to\mathbb{F}_5$ by

$\delta_5(\psi) = \sum_k a_{5k+2}(\psi) + a_{5k-2}(\psi) - a_{5k}(\psi) \mod 5$. Also, let $T_\psi = \Sigma \times [0,1]/\psi$ be the mapping torus for ψ. The combination of (1), (34) from Theorem 12, (40), (41), and general TQFT properties now yield the following criterion.

Lemma 17 *If $\delta_5(\psi) \neq 0$ then $cut(T_\psi) = 1$.*

Note here that the left hand condition only depends on the action $\overline{\psi}$ on homology, and, e.g., in the case $g = 2$ reduces to $\mathrm{trace}(\overline{\psi}^2) + 1 \not\equiv \mathrm{trace}(\overline{\psi})^2 \mod 5$. (For $g = 1$ we always have $cut(T_\psi) = 1$.) The precise knowledge of the higher order structure of $\mathcal{V}^I_{\zeta_5}$ allows for finer theorems of this type, and we, obviously, expect similar results to hold for general p.

The last example is in fact a special case of a more general relation between the Lescop invariant and cut-numbers, which is independent of the Reshetikhin-Turaev Theory.

Theorem 18 *Let M be a 3-manifold as before with $b_1(M) \geq 1$. Then, if $cut(M) \geq 2$, $\lambda_L(M) = 0$.*

Proof The condition $cut(M) \geq 2$ means that $M - \Sigma - \Sigma'$ is connected for two embedded, oriented, two-sided surfaces, which means that $C_\Sigma - \Sigma'$ is connected. This implies that $C_\Sigma - \Sigma' - \Sigma''$ splits into exactly two connected components for a surface $\Sigma'' \neq \emptyset$. Thus $C_\Sigma = A \circ B$ with connected cobordisms $A : \Sigma' \sqcup \Sigma'' \to \Sigma$ and $B : \Sigma \to \Sigma' \sqcup \Sigma''$ so that $\mu(A, B) \geq 1$. As a result of $\mathsf{x} = 0$ half-projectivity of the Frohman-Nicas TQFT we thus obtain $\mathcal{V}^{FN}(C_\Sigma) = 0$ and hence $\Delta_\varphi(M) = 0$ for φ dual to Σ. By (3) this now implies $\lambda_L(M) = 0$. $\square$

In [5] we will give examples of T_ψ, with $\psi \in \mathcal{I}_g$, for which $cut(T_\psi)$ can no longer be determined from λ_L or Alexander polynomials, but where we have to employ Theorem 16 to determine cut-numbers greater or equal to 2.

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