Geometry & Topology Monographs

Volume 5 (2002)

Four-manifolds, geometries and knots

J.A.Hillman

The University of Sydney

jonh@maths.usyd.edu.au

Geometry & Topology Monographs
ISSN 1464-8997 (on-line) 1464-8989 (printed)
Volume 5 (2002)
Four-manifolds, geometries and knots, by J.A.Hillman
Published 9 December 2002

Geometry & Topology is a fully refereed international journal dealing with all aspects of geometry and topology. It is intended to provide free electronic dissemination of high quality research. The *Geometry & Topology Monographs* series is intended to provide a similar forum for conference proceedings and research monographs. Printed copy is available. Full details, including pricing, are on the journal web site (see below).

Submission must be made electronically. For full instructions visit the journal web site at:
`http://www.maths.warwick.ac.uk/gt/`
or choose your nearest mirror site from the EMIS network: `http://www.emis.de/`
or use anonymous ftp to: `ftp.maths.warwick.ac.uk`

Geometry and Topology Publications
Mathematics Institute
University of Warwick
Coventry, CV4 7AL, UK
Email: `gt@maths.warwick.ac.uk` Fax: +44-1203-524182

For printed copy see:
`http://www.maths.warwick.ac.uk/gt/gtp-subscription.html`

To order use the on-line order form:
`http://www.maths.warwick.ac.uk/gt/gtp-online-order-form.html`

Or write to Geometry and Topology Orders at the above address or:
Email: `gtorders@maths.warwick.ac.uk` Fax: +44-1203-524182

Contents

Preface

Every closed surface admits a geometry of constant curvature, and may be classified topologically either by its fundamental group or by its Euler characteristic and orientation character. It is generally expected that all closed 3-manifolds have decompositions into geometric pieces, and are determined up to homeomorphism by invariants associated with the fundamental group (whereas the Euler characteristic is always 0). In dimension 4 the Euler characteristic and fundamental group are largely independent, and the class of closed 4-manifolds which admit a geometric decomposition is rather restricted. For instance, there are only 11 such manifolds with finite fundamental group. On the other hand, many complex surfaces admit geometric structures, as do all the manifolds arising from surgery on twist spun simple knots.

The goal of this book is to characterize algebraically the closed 4-manifolds that fibre nontrivially or admit geometries, or which are obtained by surgery on 2-knots, and to provide a reference for the topology of such manifolds and knots. In many cases the Euler characteristic, fundamental group and Stiefel-Whitney classes together form a complete system of invariants for the homotopy type of such manifolds, and the possible values of the invariants can be described explicitly. If the fundamental group is elementary amenable we may use topological surgery to obtain classifications up to homeomorphism. Surgery techniques also work well "stably" in dimension 4 (i.e., modulo connected sums with copies of $S^2 \times S^2$). However, in our situation the fundamental group may have nonabelian free subgroups and the Euler characteristic is usually the minimal possible for the group, and it is not known whether s-cobordisms between such 4-manifolds are always topologically products. Our strongest results are characterizations of manifolds which fibre homotopically over S^1 or an aspherical surface (up to homotopy equivalence) and infrasolvmanifolds (up to homeomorphism). As a consequence 2-knots whose groups are poly-Z are determined up to Gluck reconstruction and change of orientations by their groups alone.

We shall now outline the chapters in somewhat greater detail. The first chapter is purely algebraic; here we summarize the relevant group theory and present the notions of amenable group, Hirsch length of an elementary amenable group, finiteness conditions, criteria for the vanishing of cohomology of a group with coefficients in a free module, Poincaré duality groups, and Hilbert modules over the von Neumann algebra of a group. The rest of the book may be divided into three parts: general results on homotopy and surgery (Chapters 2-6), geometries and geometric decompositions (Chapters 7-13), and 2-knots (Chapters 14-18).

Some of the later arguments are applied in microcosm to 2-complexes and PD_3-complexes in Chapter 2, which presents equivariant cohomology, L^2-Betti numbers and Poincaré duality. Chapter 3 gives general criteria for two closed 4-manifolds to be homotopy equivalent, and we show that a closed 4-manifold M is aspherical if and only if $\pi_1(M)$ is a PD_4-group of type FF and $\chi(M) = \chi(\pi)$. We show that if the universal cover of a closed 4-manifold is finitely dominated then it is contractible or homotopy equivalent to S^2 or S^3 or the fundamental group is finite. We also consider at length the relationship between fundamental group and Euler characteristic for closed 4-manifolds. In Chapter 4 we show that a closed 4-manifold M fibres homotopically over S^1 with fibre a PD_3-complex if and only if $\chi(M) = 0$ and $\pi_1(M)$ is an extension of Z by a finitely presentable normal subgroup. (There remains the problem of recognizing which PD_3-complexes are homotopy equivalent to 3-manifolds). The dual problem of characterizing the total spaces of S^1-bundles over 3-dimensional bases seems more difficult. We give a criterion that applies under some restrictions on the fundamental group. In Chapter 5 we characterize the homotopy types of total spaces of surface bundles. (Our results are incomplete if the base is RP^2). In particular, a closed 4-manifold M is simple homotopy equivalent to the total space of an F-bundle over B (where B and F are closed surfaces and B is aspherical) if and only if $\chi(M) = \chi(B)\chi(F)$ and $\pi_1(M)$ is an extension of $\pi_1(B)$ by a normal subgroup isomorphic to $\pi_1(F)$. (The extension should split if $F = RP^2$). Any such extension is the fundamental group of such a bundle space; the bundle is determined by the extension of groups in the aspherical cases and by the group and Stiefel-Whitney classes if the fibre is S^2 or RP^2. This characterization is improved in Chapter 6, which considers Whitehead groups and obstructions to constructing s-cobordisms via surgery.

The next seven chapters consider geometries and geometric decompositions. Chapter 7 introduces the 4-dimensional geometries and demonstrates the limitations of geometric methods in this dimension. It also gives a brief outline of the connections between geometries, Seifert fibrations and complex surfaces. In Chapter 8 we show that a closed 4-manifold M is homeomorphic to an infrasolvmanifold if and only if $\chi(M) = 0$ and $\pi_1(M)$ has a locally nilpotent normal subgroup of Hirsch length at least 3, and two such manifolds are homeomorphic if and only if their fundamental groups are isomorphic. Moreover $\pi_1(M)$ is then a torsion free virtually poly-Z group of Hirsch length 4 and every such group is the fundamental group of an infrasolvmanifold. We also consider in detail the question of when such a manifold is the mapping torus of a self homeomorphism of a 3-manifold, and give a direct and elementary derivation of the fundamental groups of flat 4-manifolds. At the end of this chapter we show that all orientable 4-dimensional infrasolvmanifolds are determined up to diffeomorphism by their fundamental groups. (The corresponding result in other dimensions was known).

Chapters 9-12 consider the remaining 4-dimensional geometries, grouped according to whether the model is homeomorphic to R^4, $S^2 \times R^2$, $S^3 \times R$ or is compact. Aspherical geometric 4-manifolds are determined up to s-cobordism by their homotopy type. However there are only partial characterizations of the groups arising as fundamental groups of $\mathbb{H}^2 \times \mathbb{E}^2$-, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-, $\mathbb{H}^3 \times \mathbb{E}^1$- or $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds, while very little is known about $\mathbb{H}^4$- or $\mathbb{H}^2(\mathbb{C})$-manifolds. We show that the homotopy types of manifolds covered by $S^2 \times R^2$ are determined up to finite ambiguity by their fundamental groups. If the fundamental group is torsion free such a manifold is s-cobordant to the total space of an S^2-bundle over an aspherical surface. The homotopy types of manifolds covered by $S^3 \times R$ are determined by the fundamental group and first nonzero k-invariant; much is known about the possible fundamental groups, but less is known about which k-invariants are realized. Moreover, although the fundamental groups are all "good", so that in principle surgery may be used to give a classification up to homeomorphism, the problem of computing surgery obstructions seems very difficult. We conclude the geometric section of the book in Chapter 13 by considering geometric decompositions of 4-manifolds which are also mapping tori or total spaces of surface bundles, and we characterize the complex surfaces which fibre over S^1 or over a closed orientable 2-manifold.

The final five chapters are on 2-knots. Chapter 14 is an overview of knot theory; in particular it is shown how the classification of higher-dimensional knots may be largely reduced to the classification of knot manifolds. The knot exterior is determined by the knot manifold and the conjugacy class of a normal generator for the knot group, and at most two knots share a given exterior. An essential step is to characterize 2-knot groups. Kervaire gave homological conditions which characterize high dimensional knot groups and which 2-knot groups must satisfy, and showed that any high dimensional knot group with a presentation of deficiency 1 is a 2-knot group. Bridging the gap between the homological and combinatorial conditions appears to be a delicate task. In Chapter 15 we investigate 2-knot groups with infinite normal subgroups which have no noncyclic free subgroups. We show that under mild coherence hypotheses such 2-knot groups usually have nontrivial abelian normal subgroups, and we determine all 2-knot groups with finite commutator subgroup. In Chapter 16 we show that if there is an abelian normal subgroup of rank > 1 then the knot manifold is either s-cobordant to a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold or is homeomorphic to an infrasolvmanifold.

In Chapter 17 we characterize the closed 4-manifolds obtained by surgery on certain 2-knots, and show that just eight of the 4-dimensional geometries are realised by knot manifolds. We also consider when the knot manifold admits a complex structure. The final chapter considers when a fibred 2-knot with geometric fibre is determined by its exterior. We settle this question when the monodromy has finite order or when the fibre is R^3/Z^3 or is a coset space of the Lie group Nil^3.

This book arose out of two earlier books of mine, on "*2-Knots and their Groups*" and "*The Algebraic Characterization of Geometric 4-Manifolds*", published by Cambridge University Press for the Australian Mathematical Society and for the London Mathematical Society, respectively.About a quarter of the present text has been taken from these books. [1] However the arguments have been improved in many cases, notably in using Bowditch's homological criterion for virtual surface groups to streamline the results on surface bundles, using L^2-methods instead of localization, completing the characterization of mapping tori, relaxing the hypotheses on torsion or on abelian normal subgroups in the fundamental group and in deriving the results on 2-knot groups from the work on 4-manifolds. The main tools used here beyond what can be found in *Algebraic Topology* [Sp] are cohomology of groups, equivariant Poincaré duality and (to a lesser extent) L^2-(co)homology. Our references for these are the books *Homological Dimension of Discrete Groups* [Bi], *Surgery on Compact Manifolds* [Wl] and *L^2-Invariants: Theory and Applications to Geometry and K-Theory* [Lü], respectively. We also use properties of 3-manifolds (for the construction of examples) and calculations of Whitehead groups and surgery obstructions.

This work has been supported in part by ARC small grants, enabling visits by Steve Plotnick, Mike Dyer, Charles Thomas and Fang Fuquan. I would like to thank them all for their advice, and in particular Steve Plotnick for the collaboration reported in Chapter 18. I would also like to thank Robert Bieri, Robin Cobb, Peter Linnell and Steve Wilson for their collaboration, and Warren Dicks, William Dunbar, Ross Geoghegan, F.T.Farrell, Ian Hambleton, Derek Holt, K.F.Lai, Eamonn O'Brien, Peter Scott and Shmuel Weinberger for their correspondance and advice on aspects of this work.

Jonathan Hillman

[1] See the Acknowledment following this preface for a summary of the textual borrowings.

Acknowledgment

I wish to thank Cambridge University Press for their permission to use material from my earlier books [H1] and [H2]. The textual borrowings in each Chapter are outlined below.

1. §1, Lemmas 1.7 and 1.10 and Theorem 1.11, §6 (up to the discussion of $\chi(\pi)$), the first paragraph of §7 and Theorem 1.16 are from [H2:Chapter I]. (Lemma 1.1 is from [H1]). §3 is from [H2:Chapter VI].

2. §1, most of §4, part of §5 and §9 are from [H2:Chapter II and Appendix].

3. Lemma 3.1, Theorems 3.2, 3.7-3.9 and 3.12 and Corollaries 3.9.1-3.9.3 are from [H2:Chapter II]. (Theorems 3.9 and 3.12 have been improved).

4. The first half of §2, the statements of Corollaries 4.5.1-4.5.3, Theorem 4.6 and its Corollaries, and most of §8 are from [H2:Chapter III]. (Theorem 11 and the subsequent discussion have been improved).

5. Part of Lemma 5.15, Theorem 5.16 and §4-§5 are from [H2:Chapter IV]. (Theorem 5.19 and Lemmas 5.21 and 5.22 have been improved).

6. §1 (excepting Theorem 6.1), Theorem 6.12 and the proof of Theorem 6.14 are from [H2:Chapter V].

8. Part of Theorem 8.1, §6, most of §7 and §8 are from [H2:Chapter VI].

9. Theorems 9.1, 9.2 and 9.7 are from [H2:Chapter VI], with improvements.

10. Theorems 10.10-10.12 and §6 are largely from [H2:Chapter VII]. (Theorem 10.10 has been improved).

11. Theorem 11.1 is from [H2:Chapter II]. Lemma 11.3, §3 and the first three paragraphs of §5 are from [H2:Chapter VIII]. §6 is from [H2:Chapter IV].

12. The introduction, §1-§3, §5, most of §6 (from Lemma 12.5 onwards) and §7 are from [H2:Chapter IX], with improvements (particularly in §7).

14. §1-§5 are from [H1:Chapter I]. §6 and §7 are from [H1:Chapter II].

16. Most of §3 is from [H1:Chapter V].(Theorem 16.4 is new and Theorems 16.5 and 16.6 have been improved).

17. Lemma 2 and Theorem 7 are from [H1:Chapter VIII], while Corollary 17.6.1 is from [H1:Chapter VII]. The first two paragraphs of §8 and Lemma 17.12 are from [H2:Chapter X].

Part I

Manifolds and PD-complexes

Chapter 1

Group theoretic preliminaries

The key algebraic idea used in this book is to study the homology groups of covering spaces as modules over the group ring of the group of covering transformations. In this chapter we shall summarize the relevant notions from group theory, in particular, the Hirsch-Plotkin radical, amenable groups, Hirsch length, finiteness conditions, the connection between ends and the vanishing of cohomology with coefficients in a free module, Poincaré duality groups and Hilbert modules.

Our principal references for group theory are [Bi], [DD] and [Ro].

1.1 Group theoretic notation and terminology

We shall reserve the notation Z for the free (abelian) group of rank 1 (with a prefered generator) and $\mathbb{Z}$ for the ring of integers. Let $F(r)$ be the free group of rank r.

Let G be a group. Then G' and ζG denote the commutator subgroup and centre of G, respectively. The outer automorphism group of G is $Out(G) = Aut(G)/Inn(G)$, where $Inn(G) \cong G/\zeta G$ is the subgroup of $Aut(G)$ consisting of conjugations by elements of G. If H is a subgroup of G let $N_G(H)$ and $C_G(H)$ denote the normalizer and centralizer of H in G, respectively. The subgroup H is a *characteristic* subgroup of G if it is preserved under all automorphisms of G. In particular, $I(G) = \{g \in G \mid \exists n > 0,\ g^n \in G'\}$ is a characteristic subgroup of G, and the quotient $G/I(G)$ is a torsion free abelian group of rank $\beta_1(G)$. A group G is *indicable* if there is an epimorphism $p : G \to Z$, or if $G = 1$. The *normal closure* of a subset $S \subseteq G$ is $\langle\langle S\rangle\rangle_G$, the intersection of the normal subgroups of G which contain S.

If P and Q are classes of groups let PQ denote the class of ("P *by* Q") groups G which have a normal subgroup H in P such that the quotient G/H is in Q, and let ℓP denote the class of ("*locally-P*") groups such that each finitely generated subgroup is in the class P. In particular, if F is the class of finite groups ℓF is the class of *locally-finite* groups. In any group the union of all the locally-finite normal subgroups is the unique maximal locally-finite normal subgroup. Clearly there are no nontrivial homomorphisms from such a group to a torsion free group. Let *poly-*P be the class of groups with a finite composition series such that each subquotient is in P. Thus if Ab is the class of abelian groups poly-Ab is the class of solvable groups.

Let P be a class of groups which is closed under taking subgroups. A group is *virtually* P if it has a subgroup of finite index in P. Let vP be the class of groups which are virtually P. Thus a *virtually poly-*Z group is one which has a subgroup of finite index with a composition series whose factors are all infinite cyclic. The number of infinite cyclic factors is independent of the choice of finite index subgroup or composition series, and is called the *Hirsch length* of the group. We shall also say that a space virtually has some property if it has a finite regular covering space with that property.

If $p : G \to Q$ is an epimorphism with kernel N we shall say that G is an *extension of* $Q = G/N$ *by the normal subgroup* N. The action of G on N by conjugation determines a homomorphism from G to $Aut(N)$ with kernel $C_G(N)$ and hence a homomorphism from G/N to $Out(N) = Aut(N)/Inn(N)$. If $G/N \cong Z$ the extension splits: a choice of element t in G which projects to a generator of G/N determines a right inverse to p. Let θ be the automorphism of N determined by conjugation by t in G. Then G is isomorphic to the semidirect product $N \times_\theta Z$. Every automorphism of N arises in this way, and automorphisms whose images in $Out(N)$ are conjugate determine isomorphic semidirect products. In particular, $G \cong N \times Z$ if θ is an inner automorphism.

Lemma 1.1 *Let* θ *and* ϕ *automorphisms of a group* G *such that* $H_1(\theta;\mathbb{Q})-1$ *and* $H_1(\phi;\mathbb{Q})-1$ *are automorphisms of* $H_1(G;\mathbb{Q}) = (G/G')\otimes\mathbb{Q}$. *Then the semidirect products* $\pi_\theta = G\times_\theta Z$ *and* $\pi_\phi = G\times_\phi Z$ *are isomorphic if and only if* θ *is conjugate to* ϕ *or* ϕ^{-1} *in* $Out(G)$.

Proof Let t and u be fixed elements of π_θ and π_ϕ, respectively, which map to 1 in Z. Since $H_1(\pi_\theta;\mathbb{Q}) \cong H_1(\pi_\phi;\mathbb{Q}) \cong Q$ the image of G in each group is characteristic. Hence an isomorphism $h : \pi_\theta \to \pi_\phi$ induces an isomorphism $e : Z \to Z$ of the quotients, for some $e = \pm 1$, and so $h(t) = u^e g$ for some g in G. Therefore $h(\theta(h^{-1}(j))) = h(th^{-1}(j)t^{-1}) = u^e gjg^{-1}u^{-e} = \phi^e(gjg^{-1})$ for all j in G. Thus θ is conjugate to ϕ^e in $Out(G)$.

Conversely, if θ and ϕ are conjugate in $Out(G)$ there is an f in $Aut(G)$ and a g in G such that $\theta(j) = f^{-1}\phi^e f(gjg^{-1})$ for all j in G. Hence $F(j) = f(j)$ for all j in G and $F(t) = u^e f(g)$ defines an isomorphism $F : \pi_\theta \to \pi_\phi$. □

1.2 Matrix groups

In this section we shall recall some useful facts about matrices over $\mathbb{Z}$.

Lemma 1.2 *Let p be an odd prime. Then the kernel of the reduction modulo (p) homomorphism from $SL(n,\mathbb{Z})$ to $SL(n,\mathbb{F}_p)$ is torsion free.*

Proof This follows easily from the observation that if A is an integral matrix and $k=p^vq$ with q not divisible by p then $(I+p^rA)^k\equiv I+kp^rA \ mod\ (p^{2r+v})$, and $kp^r\not\equiv 0\ mod\ (p^{2r+v})$ if $r\geq 1$. $\square$

The corresponding result for $p=2$ is that the kernel of reduction *mod* (4) is torsion free.

Since $SL(n,\mathbb{F}_p)$ has order $(\Pi_{j=0}^{j=n-1}(p^n-p^j))/(p-1)$, it follows that the order of any finite subgroup of $SL(n,\mathbb{Z})$ must divide the highest common factor of these numbers, as p varies over all odd primes. In particular, finite subgroups of $SL(2,\mathbb{Z})$ have order dividing 24, and so are solvable.

Let $A=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, $B=\left(\begin{smallmatrix}0&1\\-1&1\end{smallmatrix}\right)$ and $R=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Then $A^2=B^3=-I$ and $A^4=B^6=I$. The matrices A and R generate a dihedral group of order 8, while B and R generate a dihedral group of order 12.

Theorem 1.3 *Let G be a nontrivial finite subgroup of $GL(2,\mathbb{Z})$. Then G is conjugate to one of the cyclic groups generated by A, A^2, B, B^2, R or RA, or to a dihedral subgroup generated by one of the pairs $\{A,R\}$, $\{A^2,R\}$, $\{A^2,RA\}$, $\{B,R\}$, $\{B^2,R\}$ or $\{B^2,RB\}$.*

Proof If $M\in GL(2,\mathbb{Z})$ has finite order then its characteristic polynomial has cyclotomic factors. If the characteristic polynomial is $(X\pm 1)^2$ then $M=\mp I$. (This uses the finite order of M.) If the characteristic polynomial is X^2-1 then M is conjugate to R or RA. If the characteristic polynomial is X^2+1, X^2-X+1 or X^2+X+1 then M is irreducible, and the corresponding ring of algebraic numbers is a PID. Since any $\mathbb{Z}$-torsion free module over such a ring is free it follows easily that M is conjugate to A, B or B^2.

The normalizers in $SL(2,\mathbb{Z})$ of the subgroups generated by A, B or B^2 are easily seen to be finite cyclic. Since $G\cap SL(2,\mathbb{Z})$ is solvable it must be cyclic also. As it has index at most 2 in G the theorem follows easily. $\square$

Although the 12 groups listed in the theorem represent distinct conjugacy classes in $GL(2,\mathbb{Z})$, some of these conjugacy classes coalesce in $GL(2,\mathbb{R})$. (For instance, R and RA are conjugate in $GL(2,\mathbb{Z}[\frac{1}{2}])$.)

Corollary 1.3.1 *Let G be a locally finite subgroup of $GL(2,\mathbb{R})$. Then G is finite, and is conjugate to one of the above subgroups of $GL(2,\mathbb{Z})$.*

Proof Let L be a lattice in $\mathbb{R}^2$. If G is finite then $\cup_{g\in G} gL$ is a G-invariant lattice, and so G is conjugate to a subgroup of $GL(2,\mathbb{Z})$. In general, as the finite subgroups of G have bounded order G must be finite. □

The main results of this section follow also from the fact that $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\langle \pm I\rangle$ is a free product $(Z/2Z)*(Z/3Z)$, generated by the images of A and B. (In fact $\langle A,B \mid A^2 = B^3,\ A^4 = 1\rangle$ is a presentation for $SL(2,\mathbb{Z})$.) Moreover $SL(2,\mathbb{Z})' \cong PSL(2,\mathbb{Z})'$ is freely generated by the images of $B^{-1}AB^{-2}A = \left(\begin{smallmatrix}1&1\\1&1\end{smallmatrix}\right)$ and $B^{-2}AB^{-1}A = \left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right)$, while the abelianizations are generated by the images of $B^4A = \left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$. (See §6.2 of [Ro].)

Let $\Lambda = \mathbb{Z}[t,t^{-1}]$ be the ring of integral Laurent polynomials. The next theorem is a special case of a classical result of Latimer and MacDuffee.

Theorem 1.4 *There is a 1-1 correspondance between conjugacy classes of matrices in $GL(n,\mathbb{Z})$ with irreducible characteristic polynomial $\Delta(t)$ and isomorphism classes of ideals in $\Lambda/(\Delta(t))$. The set of such ideal classes is finite.*

Proof Let $A \in GL(n,\mathbb{Z})$ have characteristic polynomial $\Delta(t)$ and let $R = \Lambda/(\Delta(t))$. As $\Delta(A) = 0$, by the Cayley-Hamilton Theorem, we may define an R-module M_A with underlying abelian group Z^n by $t.z = A(z)$ for all $z \in Z^n$. As R is a domain and has rank n as an abelian group M_A is torsion free and of rank 1 as an R-module, and so is isomorphic to an ideal of R. Conversely every R-ideal arises in this way. The isomorphism of abelian groups underlying an R-isomorphism between two such modules M_A and M_B determines a matrix $C \in GL(n,\mathbb{Z})$ such that $CA = BC$. The final assertion follows from the Jordan-Zassenhaus Theorem. □

1.3 The Hirsch-Plotkin radical

The *Hirsch-Plotkin radical* $\sqrt{G}$ of a group G is its maximal locally-nilpotent normal subgroup; in a virtually poly-Z group every subgroup is finitely generated, and so $\sqrt{G}$ is then the maximal nilpotent normal subgroup. If H is normal in G then $\sqrt{H}$ is normal in G also, since it is a characteristic subgroup of H, and in particular it is a subgroup of $\sqrt{G}$.

For each natural number $q \geq 1$ let Γ_q be the group with presentation

$$\langle x, y, z \mid xz = zx,\ yz = zy,\ xy = z^q yx \rangle.$$

Every such group Γ_q is torsion free and nilpotent of Hirsch length 3.

Theorem 1.5 *Let G be a finitely generated torsion free nilpotent group of Hirsch length $h(G) \leq 4$. Then either*

(1) *G is free abelian; or*

(2) *$h(G) = 3$ and $G \cong \Gamma_q$ for some $q \geq 1$; or*

(3) *$h(G) = 4$, $\zeta G \cong Z^2$ and $G \cong \Gamma_q \times Z$ for some $q \geq 1$; or*

(4) *$h(G) = 4$, $\zeta G \cong Z$ and $G/\zeta G \cong \Gamma_q$ for some $q \geq 1$.*

In the latter case G has characteristic subgroups which are free abelian of rank 1, 2 and 3. In all cases G is an extension of Z by a free abelian normal subgroup.

Proof The centre ζG is nontrivial and the quotient $G/\zeta G$ is again torsion free, by Proposition 5.2.19 of [Ro]. We may assume that G is not abelian, and hence that $G/\zeta G$ is not cyclic. Hence $h(G/\zeta G) \geq 2$, so $h(G) \geq 3$ and $1 \leq h(\zeta G) \leq h(G) - 2$. In all cases ζG is free abelian.

If $h(G) = 3$ then $\zeta G \cong Z$ and $G/\zeta G \cong Z^2$. On choosing elements x and y representing a basis of $G/\zeta G$ and z generating ζG we quickly find that G is isomorphic to one of the groups Γ_q, and thus is an extension of Z by Z^2.

If $h(G) = 4$ and $\zeta G \cong Z^2$ then $G/\zeta G \cong Z^2$, so $G' \subseteq \zeta G$. Since G may be generated by elements x, y, t and u where x and y represent a basis of $G/\zeta G$ and t and u are central it follows easily that G' is infinite cyclic. Therefore ζG is not contained in G' and G has an infinite cyclic direct factor. Hence $G \cong Z \times \Gamma_q$, for some $q \geq 1$, and thus is an extension of Z by Z^3.

The remaining possibility is that $h(G) = 4$ and $\zeta G \cong Z$. In this case $G/\zeta G$ is torsion free nilpotent of Hirsch length 3. If $G/\zeta G$ were abelian G' would also be infinite cyclic, and the pairing from $G/\zeta G \times G/\zeta G$ into G' defined by the commutator would be nondegenerate and skewsymmetric. But there are no such pairings on free abelian groups of odd rank. Therefore $G/\zeta G \cong \Gamma_q$, for some $q \geq 1$.

Let $\zeta_2 G$ be the preimage in G of $\zeta(G/\zeta G)$. Then $\zeta_2 G \cong Z^2$ and is a characteristic subgroup of G, so $C_G(\zeta_2 G)$ is also characteristic in G. The quotient $G/\zeta_2 G$ acts by conjugation on $\zeta_2 G$. Since $Aut(Z^2) = GL(2,\mathbb{Z})$ is virtually free and $G/\zeta_2 G \cong \Gamma_q/\zeta\Gamma_q \cong Z^2$ and since $\zeta_2 G \neq \zeta G$ it follows that $h(C_G(\zeta_2 G)) = 3$. Since $C_G(\zeta_2 G)$ is nilpotent and has centre of rank ≥ 2 it is abelian, and so $C_G(\zeta_2 G) \cong Z^3$. The preimage in G of the torsion subgroup of $G/C_G(\zeta_2 G)$ is torsion free, nilpotent of Hirsch length 3 and virtually abelian and hence is abelian. Therefore $G/C_G(\zeta_2 G) \cong Z$. □

Theorem 1.6 *Let π be a torsion free virtually poly-Z group of Hirsch length 4. Then $h(\sqrt{\pi}) \geq 3$.*

Proof Let S be a solvable normal subgroup of finite index in π. Then the lowest nontrivial term of the derived series of S is an abelian subgroup which is characteristic in S and so normal in π. Hence $\sqrt{\pi} \neq 1$. If $h(\sqrt{\pi}) \leq 2$ then $\sqrt{\pi} \cong Z$ or Z^2. Suppose π has an infinite cyclic normal subgroup A. On replacing π by a normal subgroup σ of finite index we may assume that A is central and that σ/A is poly-Z. Let B be the preimage in σ of a nontrivial abelian normal subgroup of σ/A. Then B is nilpotent (since A is central and B/A is abelian) and $h(B) > 1$ (since $B/A \neq 1$ and σ/A is torsion free). Hence $h(\sqrt{\pi}) \geq h(\sqrt{\sigma}) > 1$.

If π has a normal subgroup $N \cong Z^2$ then $Aut(N) \cong GL(2,\mathbb{Z})$ is virtually free, and so the kernel of the natural map from π to $Aut(N)$ is nontrivial. Hence $h(C_\pi(N)) \geq 3$. Since $h(\pi/N) = 2$ the quotient π/N is virtually abelian, and so $C_\pi(N)$ is virtually nilpotent.

In all cases we must have $h(\sqrt{\pi}) \geq 3$. □

1.4 Amenable groups

The class of *amenable* groups arose first in connection with the Banach-Tarski paradox. A group is amenable if it admits an invariant mean for bounded $\mathbb{C}$-valued functions [Pi]. There is a more geometric characterization of finitely presentable amenable groups that is more convenient for our purposes. Let X be a finite cell-complex with universal cover $\widetilde{X}$. Then $\widetilde{X}$ is an increasing union of finite subcomplexes $X_j \subseteq X_{j+1} \subseteq \widetilde{X} = \cup_{n\geq 1} X_n$ such that X_j is the union of $N_j < \infty$ translates of some fundamental domain D for $G = \pi_1(X)$. Let N_j' be the number of translates of D which meet the frontier of X_j in $\widetilde{X}$. The sequence $\{X_j\}$ is a *Følner exhaustion* for $\widetilde{X}$ if $\lim(N_j'/N_j) = 0$, and $\pi_1(X)$ is amenable if and only if $\widetilde{X}$ has a Følner exhaustion. This class contains all finite groups and Z, and is closed under the operations of extension, increasing union, and under the formation of sub- and quotient groups. (However nonabelian free groups are not amenable.)

The subclass EA generated from finite groups and Z by the operations of extension and increasing union is the class of *elementary amenable* groups. We may construct this class as follows. Let $U_0 = 1$ and U_1 be the class of finitely generated virtually abelian groups. If U_α has been defined for some ordinal α let $U_{\alpha+1} = (\ell U_\alpha)U_1$ and if U_α has been defined for all ordinals less than some limit ordinal β let $U_\beta = \cup_{\alpha<\beta} U_\alpha$. Let κ be the first uncountable ordinal. Then $EA = \ell U_\kappa$.

This class is well adapted to arguments by transfinite induction on the ordinal $\alpha(G) = \min\{\alpha | G \in U_\alpha\}$. It is closed under extension (in fact $U_\alpha U_\beta \subseteq U_{\alpha+\beta}$) and increasing union, and under the formation of sub- and quotient groups. As U_κ contains every countable elementary amenable group, $U_\lambda = \ell U_\kappa = EA$ if $\lambda > \kappa$. Torsion groups in EA are locally finite and elementary amenable free groups are cyclic. Every locally-finite by virtually solvable group is elementary amenable; however this inclusion is proper.

For example, let Z^∞ be the free abelian group with basis $\{x_i \mid i \in Z\}$ and let G be the subgroup of $Aut(Z^\infty)$ generated by $\{e_i \mid i \in Z\}$, where $e_i(x_i) = x_i + x_{i+1}$ and $e_i(x_j) = x_j$ if $j \neq i$. Then G is the increasing union of subgroups isomorphic to groups of upper triangular matrices, and so is locally nilpotent. However it has no nontrivial abelian normal subgroups. If we let ϕ be the automorphism of G defined by $\phi(e_i) = e_{i+1}$ for all i then $G \times_\phi Z$ is a finitely generated torsion free elementary amenable group which is not virtually solvable.

It can be shown (using the Følner condition) that finitely generated groups of subexponential growth are amenable. The class SA generated from such groups by extensions and increasing unions contains EA (since finite groups and finitely generated abelian groups have polynomial growth), and is the largest class of groups over which topological surgery techniques are known to work in dimension 4 [FT95]. Is every amenable group in SA? There is a finitely presentable group in SA which is not elementary amenable [Gr98].

A group is *restrained* if it has no noncyclic free subgroup. Amenable groups are restrained, but there are finitely presentable restrained groups which are not amenable [OS01]. There are also infinite finitely generated torsion groups. (See §14.2 of [Ro].) These are restrained, but are not elementary amenable. No known example is also finitely presentable.

1.5 Hirsch length

In this section we shall use transfinite induction to extend the notion of Hirsch length (as a measure of the size of a solvable group) to elementary amenable groups, and to establish the basic properties of this invariant.

Lemma 1.7 *Let G be a finitely generated infinite elementary amenable group. Then G has normal subgroups $K < H$ such that G/H is finite, H/K is free abelian of positive rank and the action of G/H on H/K by conjugation is effective.*

Proof We may show that G has a normal subgroup K such that G/K is an infinite virtually abelian group, by transfinite induction on $\alpha(G)$. We may assume that G/K has no nontrivial finite normal subgroup. If H is a subgroup of G which contains K and is such that H/K is a maximal abelian normal subgroup of G/K then H and K satisfy the above conditions. □

In particular, finitely generated infinite elementary amenable groups are virtually indicable.

If G is in U_1 let $h(G)$ be the rank of an abelian subgroup of finite index in G. If $h(G)$ has been defined for all G in U_α and H is in ℓU_α let

$$h(H) = \text{l.u.b.}\{h(F) | F \leq H,\ F \in U_\alpha\}.$$

Finally, if G is in $U_{\alpha+1}$, so has a normal subgroup H in ℓU_α with G/H in U_1, let $h(G) = h(H) + h(G/H)$.

Theorem 1.8 *Let G be an elementary amenable group. Then*

(1) *$h(G)$ is well defined;*

(2) *If H is a subgroup of G then $h(H) \leq h(G)$;*

(3) *$h(G) = \text{l.u.b.}\{h(F) \mid F\ is\ a\ finitely\ generated\ subgroup\ of\ G\}$;*

(4) *if H is a normal subgroup of G then $h(G) = h(H) + h(G/H)$.*

Proof We shall prove all four assertions simultaneously by induction on $\alpha(G)$. They are clearly true when $\alpha(G) = 1$. Suppose that they hold for all groups in U_α and that $\alpha(G) = \alpha + 1$. If G is in LU_α so is any subgroup, and (1) and (2) are immediate, while (3) follows since it holds for groups in U_α and since each finitely generated subgroup of G is a U_α-subgroup. To prove (4) we may assume that $h(H)$ is finite, for otherwise both $h(G)$ and $h(H) + h(G/H)$ are

∞, by (2). Therefore by (3) there is a finitely generated subgroup $J \leq H$ with $h(J) = h(H)$. Given a finitely generated subgroup Q of G/H we may choose a finitely generated subgroup F of G containing J and whose image in G/H is Q. Since F is finitely generated it is in U_α and so $h(F) = h(H) + h(Q)$. Taking least upper bounds over all such Q we have $h(G) \geq h(H) + h(G/H)$. On the other hand if F is any U_α-subgroup of G then $h(F) = h(F \cap H) + h(FH/H)$, since (4) holds for F, and so $h(G) \leq h(H) + h(G/H)$, Thus (4) holds for G also.

Now suppose that G is not in LU_α, but has a normal subgroup K in LU_α such that G/K is in U_1. If K_1 is another such subgroup then (4) holds for K and K_1 by the hypothesis of induction and so $h(K) = h(K \cap K_1) + h(KK_1/K)$. Since we also have $h(G/K) = h(G/KK_1) + h(KK_1/K)$ and $h(G/K_1) = h(G/KK_1) + h(KK_1/K_1)$ it follows that $h(K_1) + h(G/K_1) = h(K) + h(G/K)$ and so $h(G)$ is well defined. Property (2) follows easily, as any subgroup of G is an extension of a subgroup of G/K by a subgroup of K. Property (3) holds for K by the hypothesis of induction. Therefore if $h(K)$ is finite K has a finitely generated subgroup J with $h(J) = h(K)$. Since G/K is finitely generated there is a finitely generated subgroup F of G containing J and such that $FK/K = G/K$. Clearly $h(F) = h(G)$. If $h(K)$ is infinite then for every $n \geq 0$ there is a finitely generated subgroup J_n of K with $h(J_n) \geq n$. In either case, (3) also holds for G. If H is a normal subgroup of G then H and G/H are also in $U_{\alpha+1}$, while $H \cap K$ and $KH/H = K/H \cap K$ are in LU_α and $HK/K = H/H \cap K$ and G/HK are in U_1. Therefore

$$\begin{aligned} h(H) + h(G/H) &= h(H \cap K) + h(HK/K) + h(HK/H) + h(G/HK) \\ &= h(H \cap K) + h(HK/H) + h(HK/K) + h(G/HK). \end{aligned}$$

Since K is in LU_α and G/K is in U_1 this sum gives $h(G) = h(K) + h(G/K)$ and so (4) holds for G. This completes the inductive step. □

Let $\Lambda(G)$ be the maximal locally-finite normal subgroup of G.

Theorem 1.9 *There are functions d and M from $\mathbb{Z}_{\geq 0}$ to $\mathbb{Z}_{\geq 0}$ such that if G is an elementary amenable group of Hirsch length at most h and $\Lambda(G)$ is its maximal locally finite normal subgroup then $G/\Lambda(G)$ has a maximal solvable normal subgroup of derived length at most $d(h)$ and index at most $M(h)$.*

Proof We argue by induction on h. Since an elementary amenable group has Hirsch length 0 if and only if it is locally finite we may set $d(0) = 0$ and $M(0) = 1$. assume that the result is true for all such groups with Hirsch length at most h and that G is an elementary amenable group with $h(G) = h + 1$.

Suppose first that G is finitely generated. Then by Lemma 1.7 there are normal subgroups $K < H$ in G such that G/H is finite, H/K is free abelian of rank $r \geq 1$ and the action of G/H on H/K by conjugation is effective. (Note that $r = h(G/K) \leq h(G) = h+1$.) Since the kernel of the natural map from $GL(r,\mathbb{Z})$ to $GL(r,\mathbb{F}_3)$ is torsion free, by Lemma 1.2, we see that G/H embeds in $GL(r,\mathbb{F}_3)$ and so has order at most 3^{r^2}. Since $h(K) = h(G) - r \leq h$ the inductive hypothesis applies for K, so it has a normal subgroup L containing $\Lambda(K)$ and of index at most $M(h)$ such that $L/\Lambda(K)$ has derived length at most $d(h)$ and is the maximal solvable normal subgroup of $K/\Lambda(K)$. As $\Lambda(K)$ and L are characteristic in K they are normal in G. (In particular, $\Lambda(K) = K \cap \Lambda(G)$.) The centralizer of K/L in H/L is a normal solvable subgroup of G/L with index at most $[K:L]![G:H]$ and derived length at most 2. Set $M(h+1) = M(h)!3^{(h+1)^2}$ and $d(h+1) = M(h+1) + 2 + d(h)$. Then $G.\Lambda(G)$ has a maximal solvable normal subgroup of index at most the centralizer of K/L in H/L).

In general, let $\{G_i \mid i \in I\}$ be the set of finitely generated subgroups of G. By the above argument G_i has a normal subgroup H_i containing $\Lambda(G_i)$ and such that $H_i/\Lambda(G_i)$ is a maximal normal solvable subgroup of $G_i/\Lambda(G_i)$ and has derived length at most $d(h+1)$ and index at most $M(h+1)$. Let $N = \max\{[G_i : H_i] \mid i \in I\}$ and choose $\alpha \in I$ such that $[G_\alpha : H_\alpha] = N$. If $G_i \geq G_\alpha$ then $H_i \cap G_\alpha \leq H_\alpha$. Since $[G_\alpha : H_\alpha] \leq [G_\alpha : H_i \cap G_\alpha] = [H_i G_\alpha : H_i] \leq [G_i : H_i]$ we have $[G_i : H_i] = N$ and $H_i \geq H_\alpha$. It follows easily that if $G_\alpha \leq G_i \leq G_j$ then $H_i \leq H_j$.

Set $J = \{i \in I \mid H_\alpha \leq H_i\}$ and $H = \cup_{i \in J} H_i$. If $x, y \in H$ and $g \in G$ then there are indices i, k and $k \in J$ such that $x \in H_i$, $y \in H_j$ and $g \in G_k$. Choose $l \in J$ such that G_l contains $G_i \cup G_j \cup G_k$. Then xy^{-1} and gxg^{-1} are in $H_l \leq H$, and so H is a normal subgroup of G. Moreover if $x_1, \dots, x_N$ is a set of coset representatives for H_α in G_α then it remains a set of coset representatives for H in G, and so $[G;H] = N$.

Let D_i be the $d(h+1)^{th}$ derived subgroup of H_i. Then D_i is a locally-finite normal subgroup of G_i and so, bu an argument similar to that of the above paragraph $\cup_{i \in J} D_i$ is a locally-finite normal subgroup of G. Since it is easily seen that the $d(h+1)^{th}$ derived subgroup of H is contained in $\cup_{i \in J} D_i$ (as each iterated commutator involves only finitely many elements of H) it follows that $H\Lambda(G)/\Lambda(G) \cong H/H \cap \Lambda(G)$ is solvable and of derived length at most $d(h+1)$. ☐

The above result is from [HL92]. The argument can be simplified to some extent if G is countable and torsion-free. (In fact a virtually solvable group of finite Hirsch length and with no nontrivial locally-finite normal subgroup must be countable, by Lemma 7.9 of [Bi]. Moreover its Hirsch-Plotkin radical is nilpotent and the quotient is virtually abelian, by Proposition 5.5 of [BH72].)

Lemma 1.10 *Let G be an elementary amenable group. If $h(G)=\infty$ then for every $k>0$ there is a subgroup H of G with $k<h(H)<\infty$.*

Proof We shall argue by induction on $\alpha(G)$. The result is vacuously true if $\alpha(G)=1$. Suppose that it is true for all groups in U_α and G is in ℓU_α. Since $h(G)=\mathrm{l.u.b.}\{h(F)|F\leq G,\ F\in U_\alpha\}$ either there is a subgroup F of G in U_α with $h(F)=\infty$, in which case the result is true by the inductive hypothesis, or $h(G)$ is the least upper bound of a set of natural numbers and the result is true. If G is in $U_{\alpha+1}$ then it has a normal subgroup N which is in ℓU_α with quotient G/N in U_1. But then $h(N)=h(G)=\infty$ and so N has such a subgroup. □

Theorem 1.11 *Let G be a countable elementary amenable group of finite cohomological dimension. Then $h(G)\leq c.d.G$ and G is virtually solvable.*

Proof Since $c.d.G<\infty$ the group G is torsion free. Let H be a subgroup of finite Hirsch length. Then H is virtually solvable and $c.d.H\leq c.d.G$ so $h(H)\leq c.d.G$. The theorem now follows from Theorem 1.9 and Lemma 1.10. □

1.6 Modules and finiteness conditions

Let G be a group and $w:G\to Z/2Z$ a homomorphism, and let R be a commutative ring. Then $\bar{g}=(-1)^{w(g)}g^{-1}$ defines an anti-involution on $R[G]$. If L is a left $R[G]$-module $\overline{L}$ shall denote the *conjugate* right $R[G]$-module with the same underlying R-module and $R[G]$-action given by $l.g=\bar{g}.l$, for all $l\in L$ and $g\in G$. (We shall also use the overline to denote the conjugate of a right $R[G]$-module.) The conjugate of a free left (right) module is a free right (left) module of the same rank.

We shall also let Z^w denote the G-module with underlying abelian group Z and G-action given by $g.n=(-1)^{w(g)}n$ for all g in G and n in Z.

Lemma 1.12 [Wl65] *Let G and H be groups such that G is finitely presentable and there are homomorphisms $j:H\to G$ and $\rho:G\to H$ with $\rho j=id_H$. Then H is also finitely presentable.*

Proof Since G is finitely presentable there is an epimorphism $p: F \to G$ from a free group $F(X)$ with a finite basis X onto G, with kernel the normal closure of a finite set of relators R. We may choose elements w_x in $F(X)$ such that $j\rho p(x) = p(w_x)$, for all x in X. Then ρ factors through the group K with presentation $\langle X \mid R, x^{-1}w_x, \forall x \in X\rangle$, say $\rho = vu$. Now uj is clearly onto, while $vuj = \rho j = id_H$, and so v and uj are mutually inverse isomomorphisms. Therefore $H \cong K$ is finitely presentable. □

A group G is FP_n if the augmentation $\mathbb{Z}[G]$-module Z has a projective resolution which is finitely generated in degrees $\leq n$, and it is FP if it has finite cohomological dimension and is FP_n for $n = c.d.G$. It is FF if moreover Z has a finite resolution consisting of finitely generated free $\mathbb{Z}[G]$-modules. "Finitely generated" is equivalent to FP_1, while "finitely presentable" implies FP_2. Groups which are FP_2 are also said to be *almost finitely presentable.* (There are FP groups which are not finitely presentable [BB97].) An elementary amenable group G is FP_∞ if and only if it is virtually FP, and is then virtually constructible and solvable of finite Hirsch length [Kr93].

If the augmentation $\mathbb{Q}[\pi]$-module Q has a finite resolution F_* by finitely generated projective modules then $\chi(\pi) = \Sigma(-1)^i dim_{\mathbb{Q}}(\mathbb{Q}\otimes_\pi F_i)$ is independent of the resolution. (If π is the fundamental group of an aspherical finite complex K then $\chi(\pi) = \chi(K)$.) We may extend this definition to groups σ which have a subgroup π of finite index with such a resolution by setting $\chi(\sigma) = \chi(\pi)/[\sigma : \pi]$. (It is not hard to see that this is well defined.)

Let P be a finitely generated projective $\mathbb{Z}[\pi]$-module. Then P is a direct summand of $\mathbb{Z}[\pi]^r$, for some $r \geq 0$, and so is the image of some idempotent $r\times r$-matrix M with entries in $\mathbb{Z}[\pi]$. The *Kaplansky rank* $\kappa(P)$ is the coefficient of $1 \in \pi$ in the trace of M. It depends only on P and is strictly positive if $P \neq 0$. The group π satisfies the *Weak Bass Conjecture* if $\kappa(P) = dim_{\mathbb{Q}} Q\otimes_\pi P$. This conjecture has been confirmed for linear groups, solvable groups and groups of cohomological dimension ≤ 2 over $\mathbb{Q}$. (See [Dy87, Ec86, Ec96] for further details.)

The following result from [BS78] shall be useful.

Theorem 1.13 (Bieri-Strebel) *Let G be an FP_2 group such that G/G' is infinite. Then G is an HNN extension with finitely generated base and associated subgroups.*

Proof (Sketch – We shall assume that G is finitely presentable.) Let $h : F(m) \to G$ be an epimorphism, and let $g_i = h(x_i)$ for $1 \leq i \leq m$. We may assume that g_m has infinite order modulo the normal closure of $\{g_i \mid 1 \leq i < m\}$. Since G is finitely presentable the kernel of h is the normal closure of finitely many relators, of weight 0 in the letter x_m. Each such relator is a product of powers of conjugates of the generators $\{x_i \mid 1 \leq i < m\}$ by powers of x_m. Thus we may assume the relators are contained in the subgroup generated by $\{x_m^j x_i x_m^{-j} \mid 1 \leq i \leq m,\ -p \leq j \leq p\}$, for some sufficiently large p. Let U be the subgroup of G generated by $\{g_m^j g_i g_m^{-j} \mid 1 \leq i \leq m,\ -p \leq j < p\}$, and let $V = g_m U g_m^{-1}$. Let B be the subgroup of G generated by $U \cup V$ and let $\tilde{G}$ be the HNN extension with base B and associated subgroups U and V presented by $\tilde{G} = \langle B, s \mid sus^{-1} = \tau(u)\ \forall u \in U\rangle$, where $\tau : U \to V$ is the isomorphism determined by conjugation by g_m in G. There are obvious epimorphisms $\xi : F(m+1) \to \tilde{G}$ and $\psi : \tilde{G} \to G$ with composite h. It is easy to see that $\mathrm{Ker}(h) \leq \mathrm{Ker}(\xi)$ and so $\tilde{G} \cong G$. □

In particular, if G is restrained then it is an ascending HNN extension.

A ring R is *weakly finite* if every onto endomorphism of R^n is an isomorphism, for all $n \geq 0$. (In [H2] the term “SIBN ring” was used instead.) Finitely generated stably free modules over weakly finite rings have well defined ranks, and the rank is strictly positive if the module is nonzero. Skew fields are weakly finite, as are subrings of weakly finite rings. If G is a group its complex group algebra $\mathbb{C}[G]$ is weakly finite, by a result of Kaplansky. (See [Ro84] for a proof.)

A ring R is *(regular) coherent* if every finitely presentable left R-module has a (finite) resolution by finitely generated projective R-modules, and is *(regular) noetherian* if moreover every finitely generated R-module is finitely presentable. A group G is regular coherent or regular noetherian if the group ring $R[G]$ is regular coherent or regular noetherian (respectively) for any regular noetherian ring R. It is coherent as a *group* if all its finitely generated subgroups are finitely presentable.

Lemma 1.14 *If G is a group such that $\mathbb{Z}[G]$ is coherent then every finitely generated subgroup of G is FP_∞.*

Proof Let H be a subgroup of G. Since $\mathbb{Z}[H] \leq \mathbb{Z}[G]$ is a faithfully flat ring extension a left $\mathbb{Z}[H]$-module is finitely generated over $\mathbb{Z}[H]$ if and only if the induced module $\mathbb{Z}[G] \otimes_H M$ is finitely generated over $\mathbb{Z}[G]$. It follows by induction on n that M is FP_n over $\mathbb{Z}[H]$ if and only if $\mathbb{Z}[G] \otimes_H M$ is FP_n over $\mathbb{Z}[G]$.

If H is finitely generated then the augmentation $\mathbb{Z}[H]$-module Z is finitely presentable over $\mathbb{Z}[H]$. Hence $\mathbb{Z}[G] \otimes_H Z$ is finitely presentable over $\mathbb{Z}[G]$, and so is FP_∞ over $\mathbb{Z}[G]$, since that ring is coherent. Hence Z is FP_∞ over $\mathbb{Z}[H]$, i.e., H is FP_∞. □

Thus if either G is coherent (as a group) or $\mathbb{Z}[G]$ is coherent (as a ring) every finitely generated subgroup of G is FP_2. As the latter condition shall usually suffice for our purposes below, we shall say that such a group is *almost coherent.* The connection between these notions has not been much studied.

The class of groups whose integral group ring is regular coherent contains the trivial group and is closed under generalised free products and HNN extensions with amalgamation over subgroups whose group rings are regular noetherian, by Theorem 19.1 of [Wd78]. If $[G : H]$ is finite and G is torsion free then $\mathbb{Z}[G]$ is regular coherent if and only if $\mathbb{Z}[H]$ is. In particular, free groups and surface groups are coherent and their integral group rings are regular coherent, while (torsion free) virtually poly-Z groups are coherent and their integral group rings are (regular) noetherian.

1.7 Ends and cohomology with free coefficients

A finitely generated group G has 0, 1, 2 or infinitely many ends. It has 0 ends if and only if it is finite, in which case $H^0(G;\mathbb{Z}[G]) \cong Z$ and $H^q(G;\mathbb{Z}[G]) = 0$ for $q > 0$. Otherwise $H^0(G;\mathbb{Z}[G]) = 0$ and $H^1(G;\mathbb{Z}[G])$ is a free abelian group of rank $e(G) - 1$, where $e(G)$ is the number of ends of G [Sp49]. The group G has more than one end if and only if it is either a nontrivial generalised free product with amalgamation $G \cong A *_C B$ or an HNN extension $A *_C \phi$ where C is a finite group. In particular, it has two ends if and only if it is virtually Z if and only if it has a (maximal) finite normal subgroup F such that the quotient G/F is either infinite cyclic (Z) or infinite dihedral ($D = (Z/2Z) * (Z/2Z)$). (See [DD].)

Lemma 1.15 *Let N be a finitely generated restrained group. Then N is either finite or virtually Z or has one end.*

Proof Groups with infinitely many ends have noncyclic free subgroups. □

It follows that a countable restrained group is either elementary amenable of Hirsch length at most 1 or it is an increasing union of finitely generated, one-ended subgroups.

If G is a group with a normal subgroup N, and A is a left $\mathbb{Z}[G]$-module there is a *Lyndon-Hochschild-Serre spectral sequence* (LHSSS) for G as an extension of G/N by N and with coefficients A:

$$E_2 = H^p(G/N; H^q(N;A)) \Rightarrow H^{p+q}(G;A),$$

the r^{th} differential having bidegree $(r, 1-r)$. (See Section 10.1 of [Mc].)

Theorem 1.16 [Ro75] *If G has a normal subgroup N which is the union of an increasing sequence of subgroups N_n such that $H^s(N_n;\mathbb{Z}[G]) = 0$ for $s \leq r$ then $H^s(G;\mathbb{Z}[G]) = 0$ for $s \leq r$.*

Proof Let $s \leq r$. Let f be an s-cocycle for N with coefficients $\mathbb{Z}[G]$, and let f_n denote the restriction of f to a cocycle on N_n. Then there is an $(s-1)$-cochain g_n on N_n such that $\delta g_n = f_n$. Since $\delta(g_{n+1}|_{N_n} - g_n) = 0$ and $H^{s-1}(N_n;\mathbb{Z}[G]) = 0$ there is an $(s-2)$-cochain h_n on N_n with $\delta h_n = g_{n+1}|_{N_n} - g_n$. Choose an extension h_n' of h_n to N_{n+1} and let $\hat{g}_{n+1} = g_{n+1} - \delta h_n'$. Then $\hat{g}_{n+1}|_{N_n} = g_n$ and $\delta\hat{g}_{n+1} = f_{n+1}$. In this way we may extend g_0 to an $(s-1)$-cochain g on N such that $f = \delta g$ and so $H^s(N;\mathbb{Z}[G]) = 0$. The LHSSS for G as an extension of G/N by N, with coefficients $\mathbb{Z}[G]$, now gives $H^s(G;\mathbb{Z}[G]) = 0$ for $s \leq r$. □

Corollary 1.16.1 *The hypotheses are satisfied if N is the union of an increasing sequence of FP_r subgroups N_n such that $H^s(N_n;\mathbb{Z}[N_n]) = 0$ for $s \leq r$. In particular, if N is the union of an increasing sequence of finitely generated, one-ended subgroups then G has one end.*

Proof We have $H^s(N_n;\mathbb{Z}[G]) = H^s(N_n;\mathbb{Z}[N_n]) \otimes \mathbb{Z}[G/N_n] = 0$, for all $s \leq r$ and all n, since N_n is FP_r. □

In particular, G has one end if N is a countable elementary amenable group and $h(N) > 1$, by Lemma 1.15.

The following results are Theorems 8.8 of [Bi] and Theorem 0.1 of [BG85], respectively.

Theorem (Bieri) *Let G be a nonabelian group with c.d.$G = n$. Then c.d.$\zeta G \leq n-1$, and if ζG has rank $n-1$ then G' is free.* □

Theorem (Brown-Geoghegan) *Let G be an HNN extension $B*_\phi$ in which the base H and associated subgroups I and $\phi(I)$ are FP_n. If the homomorphism from $H^q(B;Z[G])$ to $H^q(I;Z[G])$ induced by restriction is injective for some $q \leq n$ then the corresponding homomorphism in the Mayer-Vietoris sequence is injective, so $H^q(G;Z[G])$ is a quotient of $H^{q-1}(I;Z[G])$.* □

The second cohomology of a group with free coefficients ($H^2(G;R[G])$, $R=\mathbb{Z}$ or a field) shall play an important role in our investigations.

Theorem (Farrell) *Let G be a finitely presentable group. If G has an element of infinite order and $R=\mathbb{Z}$ or is a field then $H^2(G;R[G])$ is either 0 or R or is not finitely generated.* □

Farrell also showed in [Fa74] that if $H^2(G;\mathbb{F}_2[G])\cong Z/2Z$ then every finitely generated subgroup of G with one end has finite index in G. Hence if G is also torsion free then subgroups of infinite index in G are locally free. Bowditch has since shown that such groups are virtually the fundamental groups of aspherical closed surfaces ([Bo99] - see §8 below).

We would also like to know when $H^2(G;\mathbb{Z}[G])$ is 0 (for G finitely presentable). In particular, we expect this to the case if G is an ascending HNN extension over a finitely generated, one-ended base, or if G has an elementary amenable, normal subgroup E such that either $h(E)=1$ and G/E has one end or $h(E)=2$ and $[G:E]=\infty$ or $h(E)\geq 3$. However our criteria here at present require finiteness hypotheses, either in order to apply an LHSSS argument or in the form of coherence.

Theorem 1.17 *Let G be a finitely presentable group with an almost coherent, locally virtually indicable, restrained normal subgroup E. Suppose that either E is abelian of rank 1 and G/E has one end or that E has a finitely generated, one-ended subgroup and G is not elementary amenable of Hirsch length 2. Then $H^s(G;\mathbb{Z}[G])=0$ for $s\leq 2$.*

Proof If E is abelian of positive rank and G/E has one end then G is 1-connected at ∞ and so $H^s(G;\mathbb{Z}[G])=0$ for $s\leq 2$, by Theorem 1 of [Mi87], and so $H^s(G;\mathbb{Z}[G])=0$ for $s\leq 2$, by [GM86].

We may assume henceforth that E is an increasing union of finitely generated one-ended subgroups $E_n\subseteq E_{n+1}\cdots\subseteq E=\cup E_n$. Since E is locally virtually indicable there are subgroups $F_n\leq E_n$ such that $[E_n:F_n]<\infty$ and which map onto Z. Since E is almost coherent these subgroups are FP_2. Hence they are HNN extensions over FP_2 bases H_n, by Theorem 1.13, and the extensions are ascending, since E is restrained. Since E_n has one end H_n has one or two ends.

If H_n has two ends then E_n is elementary amenable and $h(E_n)=2$. Therefore if H_n has two ends for all n then $[E_{n+1}:E_n]<\infty$, E is elementary amenable and $h(E) = 2$. If $[G : E] < \infty$ then G is elementary amenable and $h(G) = 2$, and so we may assume that $[G : E] = \infty$. If E is finitely generated then it is FP_2 and so $H^s(G; \mathbb{Z}[G]) = 0$ for $s \leq 2$, by an LHSSS argument. This is also the case if E is not finitely generated, for then $H^s(E; \mathbb{Z}[G]) = 0$ for $s \leq 2$, by the argument of Theorem 3.3 of [GS81], and we may again apply an LHSSS argument. (The hypothesis of [GS81] that "each G_n is FP and $c.d.G_n = h$" can be relaxed to "each G_n is FP_h".)

Otherwise we may assume that H_n has one end, for all $n \geq 1$. In this case $H^s(F_n; \mathbb{Z}[F_n]) = 0$ for $s \leq 2$, by the Theorem of Brown and Geoghegan. Therefore $H^s(G; \mathbb{Z}[G]) = 0$ for $s \leq 2$, by Theorem 1.16. □

The theorem applies if E is almost coherent and elementary amenable, and either $h(E) = 2$ and $[G : E] = \infty$ or $h(E) \geq 3$, since elementary amenable groups are restrained and locally virtually indicable. It also applies if $E = \sqrt{G}$ is large enough, since finitely generated nilpotent groups are virtually poly-Z. A similar argument shows that if $h(\sqrt{G}) \geq r$ then $H^s(G; \mathbb{Z}[G]) = 0$ for $s < r$. If moreover $[G : \sqrt{G}] = \infty$ then $H^r(G; \mathbb{Z}[G]) = 0$ also.

Are the hypotheses that E be almost coherent and locally virtually indicable necessary? Is it sufficient that E be restrained and be an increasing union of finitely generated, one-ended subgroups?

Theorem 1.18 *Let $G = B*_\phi$ be an HNN extension with FP_2 base B and associated subgroups I and $\phi(I) = J$, and which has a restrained normal subgroup $N \leq \langle\langle B \rangle\rangle$. Then $H^s(G; \mathbb{Z}[G]) = 0$ for $s \leq 2$ if either*

(1) *the HNN extension is ascending and $B = I \cong J$ has one end;*

(2) *N is locally virtually Z and G/N has one end; or*

(3) *N has a finitely generated subgroup with one end.*

Proof The first assertion follows immediately from the Brown-Geogeghan Theorem.

Let t be the stable letter, so that $tit^{-1} = \phi(i)$, for all $i \in I$. Suppose that $N \cap J \neq N \cap B$, and let $b \in N \cap B - J$. Then $b^t = t^{-1}bt$ is in N, since N is normal in G. Let a be any element of $N \cap B$. Since N has no noncyclic free subgroup there is a word $w \in F(2)$ such that $w(a, b^t) = 1$ in G. It follows from Britton's Lemma that a must be in I and so $N \cap B = N \cap I$. In particular, N is the increasing union of copies of $N \cap B$.

Hence G/N is an HNN extension with base $B/N\cap B$ and associated subgroups $I/N\cap I$ and $J/N\cap J$. Therefore if G/N has one end the latter groups are infinite, and so B, I and J each have one end. If N is virtually Z then $H^s(G;\mathbb{Z}[G])=0$ for $s\leq 2$, by an LHSSS argument. If N is locally virtually Z but is not finitely generated then it is the increasing union of a sequence of two-ended subgroups and $H^s(N;\mathbb{Z}[G])=0$ for $s\leq 1$, by Theorem 3.3 of [GS81]. Since $H^2(B;\mathbb{Z}[G])\cong H^0(B;H^2(N\cap B;\mathbb{Z}[G]))$ and $H^2(I;\mathbb{Z}[G])\cong H^0(I;H^2(N\cap I;\mathbb{Z}[G]))$, the restriction map from $H^2(B;\mathbb{Z}[G])$ to $H^2(I;\mathbb{Z}[G])$ is injective. If N has a finitely generated, one-ended subgroup N_1, we may assume that $N_1\leq N\cap B$, and so B, I and J also have one end. Moreover $H^s(N\cap B;\mathbb{Z}[G])=0$ for $s\leq 1$, by Theorem 1.16. We again see that the restriction map from $H^2(B;\mathbb{Z}[G])$ to $H^2(I;\mathbb{Z}[G])$ is injective. The result now follows in these cases from the Theorem of Brown and Geoghegan. □

1.8 Poincaré duality groups

A group G is a *PD_n-group* if it is FP, $H^p(G;\mathbb{Z}[G])=0$ for $p\neq n$ and $H^n(G;\mathbb{Z}[G])\cong Z$. The "dualizing module" $H^n(G;\mathbb{Z}[G])=Ext^n_{\mathbb{Z}[G]}(Z,\mathbb{Z}[G])$ is a right $\mathbb{Z}[G]$-module; the group is *orientable* (or is a *PD_n^+-group*) if it acts trivially on the dualizing module, i.e., if $H^n(G;\mathbb{Z}[G])$ is isomorphic to the augmentation module Z. (See [Bi].)

The only PD_1-group is Z. Eckmann, Linnell and Müller showed that every PD_2-group is the fundamental group of a closed aspherical surface. (See Chapter VI of [DD].) Bowditch has since found a much stronger result, which must be close to the optimal characterization of such groups [Bo99].

Theorem (Bowditch) *Let G be an almost finitely presentable group and F a field. Then G is virtually a PD_2-group if and only if $H^2(G;F[G])$ has a 1-dimensional G-invariant subspace.* □

In particular, this theorem applies if $H^2(G;\mathbb{Z}[G])\cong Z$. for then the image of $H^2(G;\mathbb{Z}[G])$ in $H^2(G;\mathbb{F}_2[G])$ under reduction *mod* (2) is such a subspace.

The following result from [St77] corresponds to the fact that an infinite covering space of a PL n-manifold is homotopy equivalent to a complex of dimension $<n$.

Theorem (Strebel) *Let H be a subgroup of infinite index in a PD_n-group G. Then c.d.$H<n$.* □

If R is a subring of S, A is a left R-module and C is a left S-module then the abelian groups $Hom_R(C|_R, A)$ and $Hom_S(C, Hom_R(S|_R, A))$ are naturally isomorphic, where $C|_R$ and $S|_R$ are the left R-modules underlying C and S respectively. (The maps I and J defined by $I(f)(c)(s) = f(sc)$ and $J(\theta)(c) = \theta(c)(1)$ for $f : C \to A$ and $\theta : C \to Hom_R(S, A)$ are mutually inverse isomorphisms.) When K is a subgroup of π and $R = \mathbb{Z}[K]$ and $S = \mathbb{Z}[\pi]$ these isomorphisms give rise to Shapiro's lemma. In our applications π/K shall usually be infinite cyclic and S is then a twisted Laurent extension of R.

Theorem 1.19 *Let π be a PD_n-group with an FP_r normal subgroup K such that $G = \pi/K$ is a PD_{n-r} group and $2r \geq n-1$. Then K is a PD_r-group.*

Proof It shall suffice to show that $H^s(K; F) = 0$ for any free $\mathbb{Z}[K]$-module F and all $s > r$, for then *c.d.*$K = r$ and the result follows from Theorem 9.11 of [Bi]. Let $W = Hom_{\mathbb{Z}[K]}(\mathbb{Z}[\pi], F)$ be the $\mathbb{Z}[\pi]$-module coinduced from F. Then $H^s(K; F) \cong H^s(\pi; W) \cong H_{n-s}(\pi; \overline{W})$, by Shapiro's lemma and Poincaré duality. As a $\mathbb{Z}[K]$-module $\overline{W} \cong F^G$ (the direct product of $|G|$ copies of F), and so $H_q(K; \overline{W}) = 0$ for $0 < q \leq r$ (since K is FP_r), while $H_0(K; \overline{W}) \cong A^G$, where $A = H_0(K; F)$. Moreover $A^G \cong Hom_{\mathbb{Z}}(\mathbb{Z}[G], A)$ as a $\mathbb{Z}[G]$-module, and so is coinduced from a module over the trivial group. Therefere if $n - s \leq r$ the LHSSS gives $H^s(K; F) \cong H_{n-s}(G; A^G)$. Poincaré duality for G and another application of Shapiro's lemma now give $H^s(K; F) \cong H^{s-r}(G; A^G) \cong H^{s-r}(1; A) = 0$, if $s > r$. □

If the quotient is poly-Z we can do somewhat better.

Theorem 1.20 *Let π be a PD_n-group which is an extension of Z by a normal subgroup K which is $FP_{[n/2]}$. Then K is a PD_{n-1}-group.*

Proof It is sufficient to show that $\varinjlim H^q(K; M_i) = 0$ for any direct system $\{M_i\}_{i \in I}$ with limit 0 and for all $q \leq n-1$, for then K is FP_{n-1} [Br75], and the result again follows from Theorem 9.11 of [Bi]. Since K is $FP_{[n/2]}$ we may assume $q > n/2$. We have $H^q(K; M_i) \cong H^q(\pi; W_i) \cong H_{n-q}(\pi; \overline{W_i})$, where $W_i = Hom_{\mathbb{Z}[K]}(\mathbb{Z}[\pi], M_i)$, by Shapiro's lemma and Poincaré duality. The LHSSS for π as an extension of Z by K reduces to short exact sequences

$$0 \to H_0(\pi/K; H_s(K; \overline{W_i})) \to H_s(\pi; \overline{W_i}) \to H_1(\pi/K; H_{s-1}(K; \overline{W_i})) \to 0.$$

As a $\mathbb{Z}[K]$-module $W_i \cong (M_i)^{\pi/K}$ (the direct product of countably many copies of M_i). Since K is $FP_{[n/2]}$ homology commutes with direct products in this range, and so $H_s(K;\overline{W_i}) = H_s(K;\overline{M_i})^{\pi/K}$ if $s \leq n/2$. As π/K acts on this module by shifting the entries we see that $H_s(\pi;\overline{W_i}) \cong H_{s-1}(K;\overline{M_i})$ if $s \leq n/2$, and the result now follows easily. □

A similar argument shows that if π is a PD_n-group and $\phi : \pi \to Z$ is any epimorphism then $c.d.\mathrm{Ker}(\phi) < n$. (This weak version of Strebel's Theorem suffices for some of the applications below.)

Corollary 1.20.1 *If a PD_n-group π is an extension of a virtually poly-Z group Q by an $FP_{[n/2]}$ normal subgroup K then K is a $PD_{n-h(Q)}$-group.* □

1.9 Hilbert modules

Let π be a countable group and let $\ell^2(\pi)$ be the Hilbert space completion of $\mathbb{C}[\pi]$ with respect to the inner product given by $(\Sigma a_g g, \Sigma b_h h) = \Sigma a_g \overline{b_g}$. Left and right multiplication by elements of π determine left and right actions of $\mathbb{C}[\pi]$ as bounded operators on $\ell^2(\pi)$. The (left) von Neumann algebra $\mathcal{N}(\pi)$ is the algebra of bounded operators on $\ell^2(\pi)$ which are $\mathbb{C}[\pi]$-linear with respect to the left action. By the Tomita-Takesaki theorem this is also the bicommutant in $B(\ell^2(\pi))$ of the right action of $\mathbb{C}[\pi]$, i.e., the set of operators which commute with every operator which is right $\mathbb{C}[\pi]$-linear. (See pages 45-52 of [Su].) We may clearly use the canonical involution of $\mathbb{C}[\pi]$ to interchange the roles of left and right in these definitions.

If $e \in \pi$ is the unit element we may define the von Neumann trace on $\mathcal{N}(\pi)$ by the inner product $tr(f) = (f(e), e)$. This extends to square matrices over $\mathcal{N}(\pi)$ by taking the sum of the traces of the diagonal entries. A *Hilbert $\mathcal{N}(\pi)$-module* is a Hilbert space M with a unitary left π-action which embeds isometrically and π-equivariantly into the completed tensor product $H\widehat{\otimes}\ell^2(\pi)$ for some Hilbert space H. It is finitely generated if we may take $H \cong \mathbb{C}^n$ for some integer n. (In this case we do not need to complete the ordinary tensor product over $\mathbb{C}$.) A *morphism* of Hilbert $\mathcal{N}(\pi)$-modules is a π-equivariant bounded linear operator $f : M \to N$. It is a *weak isomorphism* if it is injective and has dense image. A bounded π-linear operator on $\ell^2(\pi)^n = \mathbb{C}^n \otimes \ell^2(\pi)$ is represented by a matrix whose entries are in $\mathcal{N}(\pi)$. The von Neumann dimension of a finitely generated Hilbert $\mathcal{N}(\pi)$-module M is the real number $dim_{\mathcal{N}(\pi)}(M) = tr(P) \in [0,\infty)$, where P is any projection operator on $H \otimes \ell^2(\pi)$ with image π-isometric to M. In particular, $dim_{\mathcal{N}(\pi)}(M) = 0$ if and only if $M = 0$. The notions of finitely generated Hilbert $\mathcal{N}(\pi)$-module and finitely generated projective $\mathcal{N}(\pi)$-module are essentially equivalent, and arbitrary $\mathcal{N}(\pi)$-modules have well-defined dimensions in $[0,\infty]$ [Lü].

A sequence of bounded maps between Hilbert $\mathcal{N}(\pi)$-modules

$$M \xrightarrow{j} N \xrightarrow{p} P$$

is *weakly exact at* N if $\mathrm{Ker}(p)$ is the closure of $\mathrm{Im}(j)$. If $0 \to M \to N \to P \to 0$ is weakly exact then j is injective, $\mathrm{Ker}(p)$ is the closure of $\mathrm{Im}(j)$ and $\mathrm{Im}(p)$ is dense in P, and $dim_{\mathcal{N}(\pi)}(N) = dim_{\mathcal{N}(\pi)}(M) + dim_{\mathcal{N}(\pi)}(P)$. A finitely generated *Hilbert* $\mathcal{N}(\pi)$*-complex* C_* is a chain complex of finitely generated Hilbert $\mathcal{N}(\pi)$-modules with bounded $\mathbb{C}[\pi]$-linear operators as differentials. The *reduced* L^2-homology is defined to be $\bar{H}_p^{(2)}(C_*) = \mathrm{Ker}(d_p)/\overline{\mathrm{Im}(d_{p+1})}$. The p^{th} L^2-Betti number of C_* is then $dim_{\mathcal{N}(\pi)}\bar{H}_p^{(2)}(C_*)$. (As the images of the differentials need not be closed the *un*reduced L^2-homology modules $H_p^{(2)}(C_*) = \mathrm{Ker}(d_p)/\mathrm{Im}(d_{p+1})$ are not in general Hilbert modules.)

See [Lü] for more on modules over von Neumann algebras and L^2 invariants of complexes and manifolds.

[In this book L^2-Betti number arguments shall replace the localization arguments used in [H2]. However we shall recall the definition of *safe extension* used there. An extension of rings $\mathbb{Z}[G] < \Phi$ is a safe extension if it is faithfully flat, Φ is weakly finite and $\Phi \otimes_{\mathbb{Z}[G]} \mathbb{Z} = 0$. It was shown there that if a group has a nontrivial elementary amenable normal subgroup whose finite subgroups have bounded order and which has no nontrivial finite normal subgroup then $\mathbb{Z}[G]$ has a safe extension.]

Chapter 2

2-Complexes and PD_3-complexes

This chapter begins with a review of the notation we use for (co)homology with local coefficients and of the universal coefficient spectral sequence. We then define the L^2-Betti numbers and present some useful vanishing theorems of Lück and Gromov. These invariants are used in §3, where they are used to estimate the Euler characteristics of finite $[\pi, m]$-complexes and to give a converse to the Cheeger-Gromov-Gottlieb Theorem on aspherical finite complexes. Some of the arguments and results here may be regarded as representing in microcosm the bulk of this book; the analogies and connections between 2-complexes and 4-manifolds are well known. We then review Poincaré duality and PD_n-complexes. In §5-§9 we shall summarize briefly what is known about the homotopy types of PD_3-complexes.

2.1 Notation

Let X be a connected cell complex and let $\widetilde{X}$ be its universal covering space. If H is a normal subgroup of $G = \pi_1(X)$ we may lift the cellular decomposition of X to an equivariant cellular decomposition of the corresponding covering space X_H. The cellular chain complex C_* of X_H with coefficients in a commutative ring R is then a complex of left $R[G/H]$-modules, with respect to the action of the covering group G/H. Moreover C_* is a complex of free modules, with bases obtained by choosing a lift of each cell of X. If X is a finite complex G is finitely presentable and these modules are finitely generated. If X is finitely dominated, i.e., is a retract of a finite complex Y, then G is a retract of $\pi_1(Y)$ and so is finitely presentable, by Lemma 1.12. Moreover the chain complex C_* of the universal cover is chain homotopy equivalent over $R[G]$ to a complex of finitely generated projective modules [Wl65].

The i^{th} *equivariant homology* module of X with coefficients $R[G/H]$ is the left module $H_i(X; R[G/H]) = H_i(C_*)$, which is clearly isomorphic to $H_i(X_H; R)$ as an R-module, with the action of the covering group determining its $R[G/H]$-module structure. The i^{th} *equivariant cohomology* module of X with coefficients $R[G/H]$ is the right module $H^i(X; R[G/H]) = H^i(C^*)$, where $C^* =$

$Hom_{R[G/H]}(C_*, R[G/H])$ is the associated cochain complex of right $R[G/H]$-modules. More generally, if A and B are right and left $\mathbb{Z}[G/H]$-modules (respectively) we may define $H_j(X;A) = H_j(A \otimes_{\mathbb{Z}[G/H]} C_*)$ and $H^{n-j}(X;B) = H^{n-j}(Hom_{\mathbb{Z}[G/H]}(C_*, B))$. There is a *Universal Coefficient Spectral Sequence* (UCSS) relating equivariant homology and cohomology:

$$E_2^{pq} = Ext^q_{R[G/H]}(H_p(X;R[G/H]), R[G/H]) \Rightarrow H^{p+q}(X;R[G/H]),$$

with r^{th} differential d_r of bidegree $(1-r,r)$.

If J is a normal subgroup of G which contains H there is also a *Cartan-Leray* spectral sequence relating the homology of X_H and X_J:

$$E^2_{pq} = Tor_p^{R[G/H]}(H_q(X;R[G/H]), R[G/J]) \Rightarrow H_{p+q}(X;R[G/J]),$$

with r^{th} differential d^r of bidegree $(-r, r-1)$. (See [Mc] for more details on these spectral sequences.)

If M is a cell complex let $c_M : M \to K(\pi_1(M),1)$ denote the classifying map for the fundamental group and let $f_M : M \to P_2(M)$ denote the second stage of the Postnikov tower for M. (Thus $c_M = c_{P_2(M)} f_M$.) A map $f : X \to K(\pi_1(M),1)$ lifts to a map from X to $P_2(M)$ if and only if $f^*k_1(M) = 0$, where $k_1(M)$ is the first k-invariant of M in $H^3(\pi_1(M);\pi_2(M))$. In particular, if $k_1(M) = 0$ then $c_{P_2(M)}$ has a cross-section. The *algebraic 2-type* of M is the triple $[\pi, \pi_2(M), k_1(M)]$. Two such triples $[\pi, \Pi, \kappa]$ and $[\pi', \Pi', \kappa']$ (corresponding to M and M', respectively) are equivalent if there are isomorphisms $\alpha : \pi \to \pi'$ and $\beta : \Pi \to \Pi'$ such that $\beta(gm) = \alpha(g)\beta(m)$ for all $g \in \pi$ and $m \in \Pi$ and $\beta_*\kappa = \alpha^*\kappa'$ in $H^3(\pi, \alpha^*\Pi')$. Such an equivalence may be realized by a homotopy equivalence of $P_2(M)$ and $P_2(M')$. (The reference [Ba] gives a detailed treatment of Postnikov factorizations of nonsimple maps and spaces.)

Throughout this book *closed manifold* shall mean compact, connected TOP manifold without boundary. Every closed manifold has the homotopy type of a finite Poincaré duality complex [KS].

2.2 L^2-Betti numbers

Let X be a finite complex with fundamental group π. The *L^2-Betti numbers* of X are defined by $\beta_i^{(2)}(X) = dim_{\mathcal{N}(\pi)}(\bar{H}_2^{(2)}(\widetilde{X}))$ where the L^2-homology $\bar{H}_i^{(2)}(\widetilde{X}) = \bar{H}_i(C_*^{(2)})$ is the reduced homology of the Hilbert $\mathcal{N}(\pi)$-complex $C_*^{(2)} = \ell^2 \otimes C_*(\widetilde{X})$ of square summable chains on $\widetilde{X}$ [At76]. They are multiplicative in finite covers, and for $i = 0$ or 1 depend only on π. (In particular,

$\beta_0^{(2)}(\pi) = 0$ if π is infinite.) The alternating sum of the L^2-Betti numbers is the Euler characteristic $\chi(X)$ [At76]. The usual Betti numbers of a space or group with coefficients in a field F shall be denoted by $\beta_i(X;F) = dim_F H_i(X;F)$ (or just $\beta_i(X)$, if $F = \mathbb{Q}$).

It may be shown that $\beta_i^{(2)}(X) = dim_{\mathcal{N}(\pi)} H_i(\mathcal{N}(\pi) \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}))$, and this formulation of the definition applies to arbitrary complexes (see [CG86], [Lü]). (However we may have $\beta_i^{(2)}(X) = \infty$.) These numbers are finite if X is finitely dominated, and the Euler characteristic formula holds if also π satisfies the Strong Bass Conjecture [Ec96]. In particular, $\beta_i^{(2)}(\pi) = dim_{\mathcal{N}(\pi)} H_i(\pi;\mathcal{N}(\pi))$ is defined for any group, and $\beta_2^{(2)}(\pi_1(X)) \leq \beta_2^{(2)}(X)$. (See Theorems 1.35 and 6.54 of [Lü].)

Lemma 2.1 *Let $\pi = H*_\phi$ be a finitely presentable group which is an ascending HNN extension with finitely generated base H. Then $\beta_1^{(2)}(\pi) = 0$.*

Proof Let t be the stable letter and let H_n be the subgroup generated by H and t^n, and suppose that H is generated by g elements. Then $[\pi : H_n] = n$, so $\beta_1^{(2)}(H_n) = n\beta_1^{(2)}(\pi)$. But each H_n is also finitely presentable and generated by $g+1$ elements. Hence $\beta_1^{(2)}(H_n) \leq g+1$, and so $\beta_1^{(2)}(\pi) = 0$. □

In particular, this lemma holds if π is an extension of Z by a finitely generated normal subgroup. We shall only sketch the next theorem (from Chapter 7 of [Lü]) as we do not use it in an essential way. (See however Theorems 5.8 and 9.9.)

Theorem 2.2 (Lück) *Let π be a group with a finitely generated infinite normal subgroup Δ such that π/Δ has an element of infinite order. Then $\beta_1^{(2)}(\pi) = 0$.*

Proof (Sketch) Let $\rho \leq \pi$ be a subgroup containing Δ such that $\rho/\Delta \cong Z$. The terms in the line $p+q = 1$ of the homology LHSSS for ρ as an extension of Z by Δ with coefficients $\mathcal{N}(\rho)$ have dimension 0, by Lemma 2.1. Since $dim_{\mathcal{N}(\rho)} M = dim_{\mathcal{N}(\pi)}(\mathcal{N}(\pi) \otimes_{\mathcal{N}(\rho)} M)$ for any $\mathcal{N}(\rho)$-module M the corresponding terms for the LHSSS for π as an extension of π/Δ by Δ with coefficients $\mathcal{N}(\pi)$ also have dimension 0 and the theorem follows. □

Gaboriau has shown that the hypothesis "π/Δ has an element of infinite order" can be relaxed to "π/Δ is infinite" [Ga00]. A similar argument gives the following result.

Theorem 2.3 *Let π be a group with an infinite subnormal subgroup N such that $\beta_i^{(2)}(N)=0$ for all $i\leq s$. Then $\beta_i^{(2)}(\pi)=0$ for all $i\leq s$.*

Proof Suppose first that N is normal in π. If $[\pi:N]<\infty$ the result follows by multiplicativity of the L^2-Betti numbers, while if $[\pi:N]=\infty$ it follows from the LHSSS with coefficients $\mathcal{N}(\pi)$. We may then induct up a subnormal chain to obtain the theorem. □

In particular, we obtain the following result from page 226 of [Gr]. (Note also that if A is an amenable ascendant subgroup of π then its normal closure in π is amenable.)

Corollary 2.3.1 (Gromov) *Let π be a group with an infinite amenable normal subgroup A. Then $\beta_i^{(2)}(\pi)=0$ for all i.*

Proof If A is an infinite amenable group $\beta_i^{(2)}(A)=0$ for all i [CG86]. □

2.3 2-Complexes and finitely presentable groups

If a group π has a finite presentation P with g generators and r relators then the *deficiency* of P is $\mathrm{def}(P)=g-r$, and $\mathrm{def}(\pi)$ is the maximal deficiency of all finite presentations of π. Such a presentation determines a finite 2-complex $C(P)$ with one 0-cell, g 1-cells and r 2-cells and with $\pi_1(C(P))\cong\pi$. Clearly $\mathrm{def}(P)=1-\chi(P)=\beta_1(C(P))-\beta_2(C(P))$ and so $\mathrm{def}(\pi)\leq\beta_1(\pi)-\beta_2(\pi)$. Conversely every finite 2-complex with one 0-cell arises in this way. In general, any connected finite 2-complex X is homotopy equivalent to one with a single 0-cell, obtained by collapsing a maximal tree T in the 1-skeleton $X^{[1]}$.

We shall say that π has *geometric dimension at most* 2, written $g.d.\pi\leq 2$, if it is the fundamental group of a finite aspherical 2-complex.

Theorem 2.4 *Let X be a connected finite 2-complex with fundamental group π. Then $\chi(X)\geq\beta_2^{(2)}(\pi)-\beta_1^{(2)}(\pi)$. If $\chi(X)=-\beta_1^{(2)}(\pi)$ then X is aspherical and $\pi\neq 1$.*

Proof The lower bound follows from the Euler characteristic formula $\chi(X)=\beta_0^{(2)}(X)-\beta_1^{(2)}(X)+\beta_2^{(2)}(X)$, since $\beta_i^{(2)}(\pi)=\beta_i^{(2)}(X)$ for $i=0$ and 1 and $\beta_2^{(2)}(\pi)\leq\beta_2^{(2)}(X)$. Since X is 2-dimensional $\pi_2(X)=H_2(\widetilde{X};\mathbb{Z})$ is a subgroup of $\bar{H}_2^{(2)}(\widetilde{X})$. If $\chi(X)=-\beta_1^{(2)}(\pi)$ then $\beta_0^{(2)}(X)=0$, so π is infinite, and $\beta_2^{(2)}(X)=0$, so $\bar{H}_2^{(2)}(\widetilde{X})=0$. Therefore $\pi_2(X)=0$ and so X is aspherical. □

Corollary 2.4.1 *Let π be a finitely presentable group. Then* $\mathrm{def}(\pi) \leq 1 + \beta_1^{(2)}(\pi) - \beta_2^{(2)}(\pi)$. *If* $\mathrm{def}(\pi) = 1 + \beta_1^{(2)}(\pi)$ *then* $g.d.\pi \leq 2$.

Let $G = F(2) \times F(2)$. Then $g.d.G = 2$ and $\mathrm{def}(G) \leq \beta_1(G) - \beta_2(G) = 0$. Hence $\langle u, v, x, y \mid ux = xu,\ uy = yu,\ vx = xv,\ vy = yv \rangle$ is an optimal presentation, and $\mathrm{def}(G) = 0$. The subgroup N generated by u, vx^{-1} and y is normal in G and $G/N \cong Z$, so $\beta_1^{(2)}(G) = 0$, by Lemma 2.1. Thus asphericity need not imply equality in Theorem 2.4, in general.

Theorem 2.5 *Let π be a finitely presentable group such that $\beta_1^{(2)}(\pi) = 0$. Then $\mathrm{def}(\pi) \leq 1$, with equality if and only if $g.d.\pi \leq 2$ and $\beta_2(\pi) = \beta_1(\pi) - 1$.*

Proof The upper bound and the necessity of the conditions follow from Theorem 2.4. Conversely, if they hold and X is a finite aspherical 2-complex with $\pi_1(X) \cong \pi$ then $\chi(X) = 1 - \beta_1(\pi) + \beta_2(\pi) = 0$. After collapsing a maximal tree in X we may assume it has a single 0-cell, and then the presentation read off the 1- and 2-cells has deficiency 1. □

This theorem applies if π is a finitely presentable group which is an ascending HNN extension with finitely generated base H, or has an infinite amenable normal subgroup. In the latter case, the condition $\beta_2(\pi) = \beta_1(\pi) - 1$ is redundant. For suppose that X is a finite aspherical 2-complex with $\pi_1(X) \cong \pi$. If π has an infinite amenable normal subgroup then $\beta_i^{(2)}(\pi) = 0$ for all i, by Theorem 2.3, and so $\chi(X) = 0$.

[Similarly, if $\mathbb{Z}[\pi]$ has a safe extension Ψ and C_* is the equivariant cellular chain complex of the universal cover $\widetilde{X}$ then $\Psi \otimes_{\mathbb{Z}[\pi]} C_*$ is a complex of free left Ψ-modules with bases corresponding to the cells of X. Since Ψ is a safe extension $H_i(X; \Psi) = \Psi \otimes_{\mathbb{Z}[\pi]} H_i(X; \mathbb{Z}[\pi]) = 0$ for all i, and so again $\chi(X) = 0$.]

Corollary 2.5.1 *Let π be a finitely presentable group which is an extension of Z by an FP_2 normal subgroup N and such that $\mathrm{def}(\pi) = 1$. Then N is free.*

Proof This follows from Corollary 8.6 of [Bi]. □

The subgroup N of $F(2) \times F(2)$ defined after the Corollary to Theorem 2.4 is finitely generated, but is not free, as u and y generate a rank two abelian subgroup. (Thus N is not FP_2 and $F(2) \times F(2)$ is not almost coherent.)

The next result is a version of the "Tits alternative" for coherent groups of cohomological dimension 2. For each $m \in Z$ let $Z*_m$ be the group with presentation $\langle a, t \mid tat^{-1} = a^m \rangle$. (Thus $Z*_0 \cong Z$ and $Z*_{-1} \cong Z \times_{-1} Z$.)

Theorem 2.6 *Let π be a finitely generated group such that $c.d.\pi = 2$. Then $\pi \cong Z*_m$ for some $m \neq 0$ if and only if it is almost coherent and restrained and π/π' is infinite.*

Proof The conditions are easily seen to be necessary. Conversely, if π is almost coherent and π/π' is infinite π is an HNN extension with almost finitely presentable base H, by Theorem 1.13. The HNN extension must be ascending as π has no noncyclic free subgroup. Hence $H^2(\pi;\mathbb{Z}[\pi])$ is a quotient of $H^1(H;\mathbb{Z}[\pi]) \cong H^1(H;\mathbb{Z}[H]) \otimes \mathbb{Z}[\pi/H]$, by the Brown-Geoghegan Theorem. Now $H^2(\pi;\mathbb{Z}[\pi]) \neq 0$, since $c.d.\pi = 2$, and so $H^1(H;\mathbb{Z}[H]) \neq 0$. Since H is restrained it must have two ends, so $H \cong Z$ and $\pi \cong Z*_m$ for some $m \neq 0$. ∎

Does this remain true without any such coherence hypothesis?

Corollary 2.6.1 *Let π be an FP_2 group. Then the following are equivalent:*

(1) *$\pi \cong Z*_m$ for some $m \in Z$;*

(2) *π is torsion free, elementary amenable and $h(\pi) \leq 2$;*

(3) *π is elementary amenable and $c.d.\pi \leq 2$;*

(4) *π is elementary amenable and $\mathrm{def}(\pi) = 1$; and*

(5) *π is almost coherent and restrained and $\mathrm{def}(\pi) = 1$.*

Proof Condition (1) clearly implies the others. Suppose (2) holds. We may assume that $h(\pi) = 2$ and $h(\sqrt{\pi}) = 1$ (for otherwise $\pi \cong Z$, $Z^2 = Z*_1$ or $Z*_{-1}$). Hence $h(\pi/\sqrt{\pi}) = 1$, and so $\pi/\sqrt{\pi}$ is an extension of Z or D by a finite normal subgroup. If $\pi/\sqrt{\pi}$ maps onto D then $\pi \cong A *_C B$, where $[A:C] = [B:C] = 2$ and $h(A) = h(B) = h(C) = 1$, and so $\pi \cong Z \times_{-1} Z$. But then $h(\sqrt{\pi}) = 2$. Hence we may assume that π maps onto Z, and so π is an ascending HNN extension with finitely generated base H, by Theorem 1.13. Since H is torsion free, elementary amenable and $h(H) = 1$ it must be infinite cyclic and so (2) implies (1). If $\mathrm{def}(\pi) = 1$ then π is an ascending HNN extension with finitely generated base, so $\beta_1^{(2)}(\pi) = 0$, by Lemma 2.1. Hence (4) and (5) each imply (3) by Theorem 2.5, together with Theorem 2.6. Finally (3) implies (2), by Theorem 1.11. ∎

In fact all finitely generated solvable groups of cohomological dimension 2 are as in this corollary [Gi79]. Are these conditions also equivalent to "π is almost coherent and restrained and $c.d.\pi \leq 2$"? Note also that if $\mathrm{def}(\pi) > 1$ then π has noncyclic free subgroups [Ro77].

Let $\mathcal{X}$ be the class of groups of finite graphs of groups, all of whose edge and vertex groups are infinite cyclic. Kropholler has shown that a finitely generated, noncyclic group G is in $\mathcal{X}$ if and only if $c.d.G = 2$ and G has an infinite cyclic subgroup H which meets all its conjugates nontrivially. Moreover G is then coherent, one ended and $g.d.G = 2$ [Kr90'].

Theorem 2.7 *Let π be a finitely generated group such that $c.d.\pi = 2$. If π has a nontrivial normal subgroup E which either is almost coherent, locally virtually indicable and restrained or is elementary amenable then π is in $\mathcal{X}$ and either $E \cong Z$ or π/π' is infinite and π' is abelian.*

Proof Let F be a finitely generated subgroup of E. Then F is metabelian, by Theorem 2.6 and its Corollary, and so all words in E of the form $[[g,h],[g',h']]$ are trivial. Hence E is metabelian also. Therefore $A = \sqrt{E}$ is nontrivial, and as A is characteristic in E it is normal in π. Since A is the union of its finitely generated subgroups, which are torsion free nilpotent groups of Hirsch length ≤ 2, it is abelian. If $A \cong Z$ then $[\pi : C_\pi(A)] \leq 2$. Moreover $C_\pi(A)'$ is free, by Bieri's Theorem. If $C_\pi(A)'$ is cyclic then $\pi \cong Z^2$ or $Z \times_{-1} Z$; if $C_\pi(A)'$ is nonabelian then $E = A \cong Z$. Otherwise $c.d.A = c.d.C_\pi(A) = 2$ and so $C_\pi(A) = A$, by Bieri's Theorem. If A has rank 1 then $Aut(A)$ is abelian, so $\pi' \leq C_\pi(A)$ and π is metabelian. If $A \cong Z^2$ then π/A is isomorphic to a subgroup of $GL(2,\mathbb{Z})$, and so is virtually free. As A together with an element $t \in \pi$ of infinite order modulo A would generate a subgroup of cohomological dimension 3, which is impossible, the quotient π/A must be finite. Hence $\pi \cong Z^2$ or $Z \times_{-1} Z$. In all cases π is in $\mathcal{X}$, by Theorem C of [Kr90']. □

If $c.d.\pi = 2$, $\zeta\pi \neq 1$ and π is nonabelian then $\zeta\pi \cong Z$ and π' is free, by Bieri's Theorem. On the evidence of his work on 1-relator groups Murasugi conjectured that if G is a finitely presentable group other than Z^2 and $\mathrm{def}(G) \geq 1$ then $\zeta G \cong Z$ or 1, and is trivial if $\mathrm{def}(G) > 1$, and he verified this for classical link groups [Mu65]. Theorems 2.3, 2.5 and 2.7 together imply that if ζG is infinite then $\mathrm{def}(G) = 1$ and $\zeta G \cong Z$.

It remains an open question whether every finitely presentable group of cohomological dimension 2 has geometric dimension 2. The following partial answer to this question was first obtained by W.Beckmann under the additional assumption that the group was FF (cf. [Dy87']).

Theorem 2.8 *Let π be a finitely presentable group. Then $g.d.\pi \leq 2$ if and only if $c.d.\pi \leq 2$ and $\mathrm{def}(\pi) = \beta_1(\pi) - \beta_2(\pi)$.*

Proof The necessity of the conditions is clear. Suppose that they hold and that $C(P)$ is the 2-complex corresponding to a presentation for π of maximal deficiency. The cellular chain complex of $\widetilde{C(P)}$ gives an exact sequence

$$0 \to K = \pi_2(C(P)) \to \mathbb{Z}[\pi]^r \to \mathbb{Z}[\pi]^g \to \cdots \to \mathbb{Z}[\pi] \to 0.$$

As $c.d.\pi \leq 2$ the image of $\mathbb{Z}[\pi]^r$ in $\mathbb{Z}[\pi]^g$ is projective, by Schanuel's Lemma. Therefore the inclusion of K into $\mathbb{Z}[\pi]^r$ splits, and K is projective. Moreover $dim_{\mathbb{Q}}(\mathbb{Q}\otimes_{\mathbb{Z}[\pi]}K) = 0$, and so $K = 0$, since the Weak Bass Conjecture holds for π [Ec86]. Hence $\widetilde{C(P)}$ is contractible, and so $C(P)$ is aspherical. □

The arguments of this section may easily be extended to other highly connected finite complexes. A $[\pi, m]_f$-complex is a finite m-dimensional complex X with $\pi_1(X) \cong \pi$ and with $(m-1)$-connected universal cover $\widetilde{X}$. Such a $[\pi, m]_f$-complex X is aspherical if and only if $\pi_m(X) = 0$. In that case we shall say that π has geometric dimension at most m, written $g.d.\pi \leq m$.

Theorem 2.4′ *Let X be a $[\pi, m]_f$-complex and suppose that $\beta_i^{(2)}(\pi) = 0$ for $i < m$. Then $(-1)^m\chi(X) \geq 0$. If $\chi(X) = 0$ then X is aspherical.* □

In general the implication in the statement of this theorem cannot be reversed. For $S^1 \vee S^1$ is an aspherical $[F(2), 1]_f$-complex and $\beta_0^{(2)}(F(2)) = 0$, but $\chi(S^1 \vee S^1) = -1 \neq 0$.

One of the applications of L^2-cohomology in [CG86] was to show that if X is a finite aspherical complex such that $\pi_1(X)$ has an infinite amenable normal subgroup A then $\chi(X) = 0$. (This generalised a theorem of Gottlieb, who assumed that A was a central subgroup [Go65].) We may similarly extend Theorem 2.5 to give a converse to the Cheeger-Gromov extension of Gottlieb's Theorem.

Theorem 2.5′ *Let X be a $[\pi, m]_f$-complex and suppose that π has an infinite amenable normal subgroup. Then X is aspherical if and only if $\chi(X) = 0$.* □

2.4 Poincaré duality

The main reason for studying PD-complexes is that they represent the homotopy theory of manifolds. However they also arise in situations where the geometry does not immediately provide a corresponding manifold. For instance, under suitable finiteness assumptions an infinite cyclic covering space of a closed

4-manifold with Euler characteristic 0 will be a PD_3-complex, but need not be homotopy equivalent to a closed 3-manifold (see Chapter 11).

A *PD_n-complex* is a finitely dominated cell complex which satisfies Poincaré duality of formal dimension n with local coefficients. It is *finite* if it is homotopy equivalent to a finite cell complex. (It is most convenient for our purposes below to require that PD_n-complexes be finitely dominated. If a CW-complex X satisfies local duality then $\pi_1(X)$ is FP_2, and X is finitely dominated if and only if $\pi_1(X)$ is finitely presentable [Br72, Br75]. Ranicki uses the broader definition in his book [Rn].) All the PD_n-complexes that we consider shall be assumed to be connected.

Let P be a PD_n-complex and C_* be the cellular chain complex of $\widetilde{P}$. Then the Poincaré duality isomorphism may also be described in terms of a chain homotopy equivalence from $\overline{C^*}$ to C_{n-*}, which induces isomorphisms from $H^j(\overline{C^*})$ to $H_{n-j}(C_*)$, given by cap product with a generator $[P]$ of $H_n(P;Z^{w_1(P)}) = H_n(\bar{Z}\otimes_{\mathbb{Z}[\pi_1(P)]}C_*)$. (Here the first Stiefel-Whitney class $w_1(P)$ is considered as a homomorphism from $\pi_1(P)$ to $Z/2Z$.) From this point of view it is easy to see that Poincaré duality gives rise to ($\mathbb{Z}$-linear) isomorphisms from $H^j(P;B)$ to $H_{n-j}(P;\bar{B})$, where B is any left $\mathbb{Z}[\pi_1(P)]$-module of coefficients. (See [Wl67] or Chapter II of [Wl] for further details.) If P is a Poincaré duality complex then the L^2-Betti numbers also satisfy Poincaré duality. (This does not require that P be finite or orientable!)

A finitely presentable group is a PD_n-group (as defined in Chapter 2) if and only if $K(G,1)$ is a PD_n-complex. For every $n \geq 4$ there are PD_n-groups which are not finitely presentable [Da98].

Dwyer, Stolz and Taylor have extended Strebel's Theorem to show that if H is a subgroup of infinite index in $\pi_1(P)$ then the corresponding covering space P_H has homological dimension $< n$; hence if moreover $n \neq 3$ then P_H is homotopy equivalent to a complex of dimension $< n$ [DST96].

2.5 PD_3-complexes

In this section we shall summarize briefly what is known about PD_n-complexes of dimension at most 3. It is easy to see that a connected PD_1-complex must be homotopy equivalent to S^1. The 2-dimensional case is already quite difficult, but has been settled by Eckmann, Linnell and Müller, who showed that every PD_2-complex is homotopy equivalent to a closed surface. (See Chapter VI of [DD]. This result has been further improved by Bowditch's Theorem.)

There are PD_3-complexes with finite fundamental group which are not homotopy equivalent to any closed 3-manifold [Th77]. On the other hand, Turaev's Theorem below implies that every PD_3-complex with torsion free fundamental group is homotopy equivalent to a closed 3-manifold if every PD_3-group is a 3-manifold group. The latter is so if the Hirsch-Plotkin radical of the group is nontrivial (see §7 below), but remains open in general.

The fundamental triple of a PD_3-complex P is $(\pi_1(P), w_1(P), c_{P*}[P])$. This is a complete homotopy invariant for such complexes.

Theorem (Hendriks) *Two PD_3-complexes are homotopy equivalent if and only if their fundamental triples are isomorphic.* □

Turaev has characterized the possible triples corresponding to a given finitely presentable group and orientation character, and has used this result to deduce a basic splitting theorem [Tu90].

Theorem (Turaev) *A PD_3-complex is irreducible with respect to connected sum if and only if its fundamental group is indecomposable with respect to free product.* □

Wall has asked whether every PD_3-complex whose fundamental group has infinitely many ends is a proper connected sum [Wl67]. Since the fundamental group of a PD_3-complex is finitely presentable it is the fundamental group of a finite graph of (finitely generated) groups in which each vertex group has at most one end and each edge group is finite, by Theorem VI.6.3 of [DD]. Starting from this observation, Crisp has given a substantial partial answer to Wall's question [Cr00].

Theorem (Crisp) *Let X be an indecomposable PD_3^+-complex. If $\pi_1(X)$ is not virtually free then it has one end, and so X is aspherical.* □

With Turaev's theorem this implies that the fundamental group of any PD_3-complex is virtually torsion free, and that if X is irreducible and π has more than one end then it is virtually free. There remains the possibility that, for instance, the free product of two copies of the symmetric group on 3 letters with amalgamation over a subgroup of order 2 may be the fundamental group of an orientable PD_3-complex. (It appears difficult in practice to apply Turaev's work to the question of whether a given group can be the fundamental group of a PD_3-complex.)

2.6 The spherical cases

The possible PD_3-complexes with finite fundamental group are well understood (although it is not yet completely known which are homotopy equivalent to 3-manifolds).

Theorem 2.9 [Wl67] *Let X be a PD_3-complex with finite fundamental group F. Then*

1. (1) $\widetilde{X} \simeq S^3$, *F has cohomological period dividing 4 and X is orientable;*
2. (2) *the first nontrivial k-invariant $k(X)$ generates $H^4(F;\mathbb{Z}) \cong Z/|F|Z$.*
3. (3) *the homotopy type of X is determined by F and the orbit of $k(M)$ under $Out(F) \times \{\pm 1\}$.*

Proof Since the universal cover $\widetilde{X}$ is also a finite PD_3-complex it is homotopy equivalent to S^3. A standard Gysin sequence argument shows that F has cohomological period dividing 4. Suppose that X is nonorientable, and let C be a cyclic subgroup of F generated by an orientation reversing element. Let $\tilde{Z}$ be the nontrivial infinite cyclic $\mathbb{Z}[C]$-module. Then $H^2(X_C;\tilde{Z}) \cong H_1(X_C;\mathbb{Z}) \cong C$, by Poincaré duality. But $H^2(X_C;\tilde{Z}) \cong H^2(C;\tilde{Z}) = 0$, since the classifying map from $X_C = \widetilde{X}/C$ to $K(C,1)$ is 3-connected. Therefore X must be orientable and F must act trivially on $\pi_3(X) \cong H_3(\widetilde{X};\mathbb{Z})$.

The image μ of the orientation class of X generates $H_3(F;\mathbb{Z}) \cong Z/|F|Z$, and corresponds to the first nonzero k-invariant under the isomorphism $H_3(F;\mathbb{Z}) \cong H^4(F;\mathbb{Z})$ [Wl67]. Inner automorphisms of F act trivially on $H^4(F;\mathbb{Z})$, while changing the orientation of X corresponds to multiplication by -1. Thus the orbit of $k(M)$ under $Out(F) \times \{\pm 1\}$ is the significant invariant.

We may construct the third stage of the Postnikov tower for X by adjoining cells of dimension greater than 4 to X. The natural inclusion $j : X \to P_3(X)$ is then 4-connected. If X_1 is another such PD_3-complex and $\theta : \pi_1(X_1) \to F$ is an isomorphism which identifies the k-invariants then there is a 4-connected map $j_1 : X_1 \to P_3(X)$ inducing θ, which is homotopic to a map with image in the 4-skeleton of $P_3(X)$, and so there is a map $h : X_1 \to X$ such that j_1 is homotopic to jh. The map h induces isomorphisms on π_i for $i \leq 3$, since j and j_1 are 4-connected, and so the lift $\tilde{h} : \widetilde{X}_1 \simeq S^3 \to \widetilde{X} \simeq S^3$ is a homotopy equivalence, by the theorems of Hurewicz and Whitehead. Thus h is itself a homotopy equivalence. □

The list of finite groups with cohomological period dividing 4 is well known. Each such group F and generator $k \in H^4(F;\mathbb{Z})$ is realized by some PD_3^+-complex [Sw60, Wl67]. (See also Chapter 11 below.) In particular, there is an unique homotopy type of PD_3-complexes with fundamental group the symmetric group S_3, but there is no 3-manifold with this fundamental group.

The fundamental group of a PD_3-complex P has two ends if and only if $\widetilde{P} \simeq S^2$, and then P is homotopy equivalent to one of the four $\mathbb{S}^2 \times \mathbb{E}^1$-manifolds $S^2 \times S^1$, $S^2 \tilde{\times} S^1$, $RP^2 \times S^1$ or $RP^3 \sharp RP^3$. The following simple lemma leads to an alternative characterization.

Lemma 2.10 *Let P be a finite dimensional complex with fundamental group π and such that $H_q(\widetilde{P};\mathbb{Z}) = 0$ for all $q > 2$. If C is a cyclic subgroup of π then $H_{s+3}(C;\mathbb{Z}) \cong H_s(C;\pi_2(P))$ for all $s \geq \dim(P)$.*

Proof Since $H_2(\widetilde{P};\mathbb{Z}) \cong \pi_2(P)$ and $\dim(\widetilde{P}/C) \leq \dim(P)$ this follows either from the Cartan-Leray spectral sequence for the universal cover of $\widetilde{P}/C$ or by devissage applied to the homology of $C_*(\widetilde{P})$, considered as a chain complex over $\mathbb{Z}[C]$. □

Theorem 2.11 *Let P be a PD_3-complex whose fundamental group π has a nontrivial finite normal subgroup N. Then either P is homotopy equivalent to $RP^2 \times S^1$ or π is finite.*

Proof We may clearly assume that π is infinite. Then $H_q(\widetilde{P};\mathbb{Z}) = 0$ for $q > 2$, by Poincaré duality. Let $\Pi = \pi_2(P)$. The augmentation sequence

$$0 \to A(\pi) \to \mathbb{Z}[\pi] \to Z \to 0$$

gives rise to a short exact sequence

$$0 \to Hom_{\mathbb{Z}[\pi]}(\mathbb{Z}[\pi],\mathbb{Z}[\pi]) \to Hom_{\mathbb{Z}[\pi]}(A(\pi),\mathbb{Z}[\pi]) \to H^1(\pi;\mathbb{Z}[\pi]) \to 0.$$

Let $f : A(\pi) \to \mathbb{Z}[\pi]$ be a homomorphism and ζ be a central element of π. Then $f.\zeta(i) = f(i)\zeta = \zeta f(i) = f(\zeta i) = f(i\zeta)$ and so $(f.\zeta - f)(i) = f(i(\zeta-1)) = if(\zeta-1)$ for all $i \in A(\pi)$. Hence $f.\zeta - f$ is the restriction of a homomorphism from $\mathbb{Z}[\pi]$ to $\mathbb{Z}[\pi]$. Thus central elements of π act trivially on $H^1(\pi;\mathbb{Z}[\pi])$.

If $n \in N$ the centraliser $\gamma = C_\pi(\langle n\rangle)$ has finite index in π, and so the covering space P_γ is again a PD_3-complex with universal covering space $\widetilde{P}$. Therefore $\Pi \cong \overline{H^1(\gamma;\mathbb{Z}[\gamma])}$ as a (left) $\mathbb{Z}[\gamma]$-module. In particular, Π is a free abelian group. Since n is central in γ it acts trivially on $H^1(\gamma;\mathbb{Z}[\gamma])$ and hence via

$w(n)$ on Π. Suppose first that $w(n) = 1$. Then Lemma 2.10 gives an exact sequence

$$0 \to Z/|n|Z \to \Pi \to \Pi \to 0,$$

where the right hand homomorphism is multiplication by $|n|$, since n has finite order and acts trivially on Π. As Π is torsion free we must have $n = 1$.

Therefore if $n \in N$ is nontrivial it has order 2 and $w(n) = -1$. In this case Lemma 2.10 gives an exact sequence

$$0 \to \Pi \to \Pi \to Z/2Z \to 0,$$

where the left hand homomorphism is multiplication by 2. Since Π is a free abelian group it must be infinite cyclic, and so $\widetilde{P} \simeq S^2$. The theorem now follows from Theorem 4.4 of [Wl67]. □

If $\pi_1(P)$ has a finitely generated infinite normal subgroup of infinite index then it has one end, and so P is aspherical. We shall discuss this case next.

2.7 PD_3-groups

If Wall's question has an affirmative answer, the study of PD_3-complexes reduces largely to the study of PD_3-groups. It is not yet known whether all such groups are 3-manifold groups. The fundamental groups of 3-manifolds which are finitely covered by surface bundles or which admit one of the geometries of aspherical Seifert type may be characterized among all PD_3-groups in simple group-theoretic terms.

Theorem 2.12 *Let G be a PD_3-group with a nontrivial almost finitely presentable normal subgroup N of infinite index. Then either*

(1) *$N \cong Z$ and G/N is virtually a PD_2-group; or*

(2) *N is a PD_2-group and G/N has two ends.*

Proof Let e be the number of ends of N. If N is free then $H^3(G;\mathbb{Z}[G]) \cong H^2(G/N;H^1(N;\mathbb{Z}[G]))$. Since N is finitely generated and G/N is FP_2 this is in turn isomorphic to $H^2(G/N;\mathbb{Z}[G/N])^{(e-1)}$. Since G is a PD_3-group we must have $e-1=1$ and so $N \cong Z$. We then have $H^2(G/N;\mathbb{Z}[G/N]) \cong H^3(G;\mathbb{Z}[G]) \cong Z$, so G/N is virtually a PD_2-group, by Bowditch's Theorem.

Otherwise $c.d.N = 2$ and so $e = 1$ or ∞. The LHSSS gives an isomorphism $H^2(G;\mathbb{Z}[G]) \cong H^1(G/N;\mathbb{Z}[G/N]) \otimes H^1(N;\mathbb{Z}[N]) \cong H^1(G/N;\mathbb{Z}[G/N])^{e-1}$.

Hence either $e = 1$ or $H^1(G/N;\mathbb{Z}[G/N]) = 0$. But in the latter case we have $H^3(G;\mathbb{Z}[G]) \cong H^2(G/N;\mathbb{Z}[G/N]) \otimes H^1(N;\mathbb{Z}[N])$ and so $H^3(G;\mathbb{Z}[G])$ is either 0 or infinite dimensional. Therefore $e = 1$, and so $H^3(G;\mathbb{Z}[G]) \cong H^1(G/N;\mathbb{Z}[G/N]) \otimes H^2(N;\mathbb{Z}[N])$. Hence G/N has two ends and $H^2(N;\mathbb{Z}[N]) \cong Z$, so N is a PD_2-group. □

We shall strengthen this result in Theorem 2.16 below.

Corollary 2.12.1 *A PD_3-complex P is homotopy equivalent to the mapping torus of a self homeomorphism of a closed surface if and only if there is an epimorphism $\phi : \pi_1(P) \to Z$ with finitely generated kernel.*

Proof This follows from Theorems 1.20, 2.11 and 2.12. □

If $\pi_1(P)$ is infinite and is a nontrivial direct product then P is homotopy equivalent to the product of S^1 with a closed surface.

Theorem 2.13 *Let G be a PD_3-group. Then every almost coherent, locally virtually indicable subgroup of G is either virtually solvable or contains a noncyclic free subgroup.*

Proof Let S be a restrained, locally virtually indicable subgroup of G. Suppose first that S has finite index in G, and so is again a PD_3-group. Since S is virtually indicable we may assume without loss of generality that $\beta_1(S) > 0$. Then S is an ascending HNN extension $H*_\phi$ with finitely generated base. Since G is almost coherent H is finitely presentable, and since $H^3(S;\mathbb{Z}[S]) \cong Z$ it follows from Lemma 3.4 of [BG85] that H is normal in S and $S/H \cong Z$. Hence H is a PD_2-group, by Theorem 1.20. Since H has no noncyclic free subgroup it is virtually Z^2 and so S and G are virtually poly-Z.

If $[G : S] = \infty$ then $c.d.S \leq 2$, by Strebel's Theorem. As the finitely generated subgroups of S are virtually indicable they are metabelian, by Theorem 2.6 and its Corollary. Hence S is metabelian also. □

As the fundamental groups of virtually Haken 3-manifolds are coherent and locally virtually indicable, this implies the "Tits alternative" for such groups [EJ73]. In fact solvable subgroups of infinite index in 3-manifold groups are virtually abelian. This remains true if $K(G,1)$ is a finite PD_3-complex, by Corollary 1.4 of [KK99]. Does this hold for all PD_3-groups?

A slight modification of the argument gives the following corollary.

Corollary 2.13.1 *A PD_3-group G is virtually poly-Z if and only if it is coherent, restrained and has a subgroup of finite index with infinite abelianization.* □

If $\beta_1(G) \geq 2$ the hypothesis of coherence is redundant, for there is then an epimorphism $p : G \to Z$ with finitely generated kernel, by [BNS87], and Theorem 1.20 requires only that H be finitely generated.

The argument of Theorem 2.13 and its corollary extend to show by induction on m that a PD_m-group is virtually poly-Z if and only if it is restrained and every finitely generated subgroup is FP_{m-1} and virtually indicable.

Theorem 2.14 *Let G be a PD_3-group. Then G is the fundamental group of an aspherical Seifert fibred 3-manifold or a $\mathbb{S}ol^3$-manifold if and only if $\sqrt{G} \neq 1$. Moreover*

(1) $h(\sqrt{G}) = 1$ *if and only if G is the group of an $\mathbb{H}^2 \times \mathbb{E}^1$- or $\widetilde{\mathbb{SL}}$-manifold;*

(2) $h(\sqrt{G}) = 2$ *if and only if G is the group of a $\mathbb{S}ol^3$-manifold;*

(3) $h(\sqrt{G}) = 3$ *if and only if G is the group of an $\mathbb{E}^3$- or $\mathbb{N}il^3$-manifold.*

Proof The necessity of the conditions is clear. (See [Sc83'], or §2 and §3 of Chapter 7 below.) Certainly $h(\sqrt{G}) \leq c.d.\sqrt{G} \leq 3$. Moreover $c.d.\sqrt{G} = 3$ if and only if $[G : \sqrt{G}]$ is finite, by Strebel's Theorem. Hence G is virtually nilpotent if and only if $h(\sqrt{G}) = 3$. If $h(\sqrt{G}) = 2$ then $\sqrt{G}$ is locally abelian, and hence abelian. Moreover $\sqrt{G}$ must be finitely generated, for otherwise $c.d\sqrt{G} = 3$. Thus $\sqrt{G} \cong Z^2$ and case (2) follows from Theorem 2.12.

Suppose now that $h(\sqrt{G}) = 1$ and let $C = C_G(\sqrt{G})$. Then $\sqrt{G}$ is torsion free abelian of rank 1, so $Aut(\sqrt{G})$ is isomorphic to a subgroup of $\mathbb{Q}^\times$. Therefore G/C is abelian. If G/C is infinite then $c.d.C \leq 2$ by Strebel's Theorem and $\sqrt{G}$ is not finitely generated, so C is abelian, by Bieri's Theorem, and hence G is solvable. But then $h(\sqrt{G}) > 1$, which is contrary to our hypothesis. Therefore G/C is isomorphic to a finite subgroup of $\mathbb{Q}^\times \cong Z^\infty \oplus (Z/2Z)$ and so has order at most 2. In particular, if A is an infinite cyclic subgroup of $\sqrt{G}$ then A is normal in G, and so G/A is virtually a PD_2-group, by Theorem 2.12. If G/A is a PD_2-group then G is the fundamental group of an S^1-bundle over a closed surface. In general, a finite torsion free extension of the fundamental group of a closed Seifert fibred 3-manifold is again the fundamental group of a closed Seifert fibred 3-manifold, by [Sc83] and Section 63 of [Zi]. □

The heart of this result is the deep theorem of Bowditch. The weaker characterization of fundamental groups of $\mathbb{S}ol^3$-manifolds and aspherical Seifert fibred 3-manifolds as PD_3-groups G such that $\sqrt{G} \neq 1$ and G has a subgroup of finite index with infinite abelianization is much easier to prove [H2]. There is as yet no comparable characterization of the groups of $\mathbb{H}^3$-manifolds, although it may be conjectured that these are exactly the PD_3-groups with no noncyclic abelian subgroups. (Note also that it remains an open question whether every closed $\mathbb{H}^3$-manifold is finitely covered by a mapping torus.)

$\mathbb{N}il^3$- and $\widetilde{\mathbb{SL}}$-manifolds are orientable, and so their groups are PD_3^+-groups. This can also be seen algebraically, as every such group has a characteristic subgroup H which is a nonsplit central extension of a PD_2^+-group β by Z. An automorphism of such a group H must be orientation preserving.

Theorem 2.14 implies that if a PD_3-group G is not virtually poly-Z then its maximal elementary amenable normal subgroup is Z or 1. For this subgroup is virtually solvable, by Theorem 1.11, and if it is nontrivial then so is $\sqrt{G}$.

Lemma 2.15 *Let G be a PD_3-group with subgroups H and J such that H is almost finitely presentable, has one end and is normal in J. Then either $[J:H]$ or $[G:J]$ is finite.*

Proof Suppose that $[J:H]$ and $[G:H]$ are both infinite. Since H has one end it is not free and so $c.d.H = c.d.J = 2$, by Strebel's Theorem. Hence there is a free $\mathbb{Z}[J]$-module W such that $H^2(J;W) \neq 0$, by Proposition 5.1 of [Bi]. Since H is FP_2 and has one end $H^q(H;W) = 0$ for $q = 0$ or 1 and $H^2(H;W)$ is an induced $\mathbb{Z}[J/H]$-module. Since $[J:H]$ is infinite $H^0(J/H; H^2(H;W)) = 0$, by Lemma 8.1 of [Bi]. The LHSSS for J as an extension of J/H by H now gives $H^r(J;W) = 0$ for $r \leq 2$, which is a contradiction. $\square$

Theorem 2.16 *Let G be a PD_3-group with a nontrivial almost finitely presentable subgroup H which is subnormal and of infinite index in G. Then either H is infinite cyclic and is normal in G or G is virtually poly-Z or H is a PD_2-group, $[G:N_G(H)] < \infty$ and $N_G(H)/H$ has two ends.*

Proof Since H is subnormal in G there is a finite increasing sequence $\{J_i \mid 0 \leq i \leq n\}$ of subgroups of G with $J_0 = H$, J_i normal in J_{i+1} for each $i < n$ and $J_n = G$. Since $[G:H] = \infty$ either $c.d.H = 2$ or H is free, by Strebel's Theorem. Suppose first that $c.d.H = 2$. Let $k = \min\{i \mid [J_i:H] = \infty\}$. Then H has finite index in J_{k-1}, which therefore is also FP_2. Suppose that $c.d.J_k = 2$. If K is a finitely generated subgroup of J_k which contains J_{k-1}

then $[K : J_{k-1}]$ is finite, by Corollary 8.6 of [Bi], and so J_k is the union of a strictly increasing sequence of finite extensions of J_{k-1}. But it follows from the Kurosh subgroup theorem that the number of indecomposable factors in such intermediate groups must be strictly decreasing unless one is indecomposable (in which case all are). (See Lemma 1.4 of [Sc76].) Thus J_{k-1} is indecomposable, and so has one end (since it is torsion free but not infinite cyclic). Therefore $[G : J_k] < \infty$, by Lemma 2.15, and so J_k is a PD_3-group. Since J_{k-1} is finitely generated, normal in J_k and $[J_{k-1} : H] < \infty$ it follows easily that $[J_k : N_{J_k}(H)] < \infty$. Therefore $[G : N_G(H)] < \infty$ and so H is a PD_2-group and $N_G(H)/H$ has two ends, by Theorem 2.12.

Next suppose that $H \cong Z$. Since $\sqrt{J_i}$ is characteristic in J_i it is normal in J_{i+1}, for each $i < n$. A finite induction now shows that $H \leq \sqrt{G}$. Therefore either $\sqrt{G} \cong Z$, so $H \cong Z$ and is normal in G, or G is virtually poly-Z, by Theorem 2.14.

Suppose finally that G has a finitely generated noncyclic free subnormal subgroup. We may assume that $\{J_i \mid 0 \leq i \leq n\}$ is a chain of minimal length n among subnormal chains with $H = J_0$ a finitely generated noncyclic free group. In particular, $[J_1 : H] = \infty$, for otherwise J_1 would also be a finitely generated noncyclic free group. We may also assume that H is maximal in the partially ordered set of finitely generated free normal subgroups of J_1. (Note that ascending chains of such subgroups are always finite, for if $F(r)$ is a nontrivial normal subgroup of a free group G then G is also finitely generated, of rank s say, and and $[G : F](1 - s) = 1 - r$.)

Since J_1 has a finitely generated noncyclic free normal subgroup of infinite index it is not free, and nor is it a PD_3-group. Therefore $c.d.J_1 = 2$. The kernel of the homomorphism from J_1/H to $Out(H)$ determined by the conjugation action of J_1 on H is $HC_{J_1}(H)/H$, which is isomorphic to $C_{J_1}(H)$ since $\zeta H = 1$. As $Out(H)$ is virtually of finite cohomological dimension and $c.d.C_{J_1}(H)$ is finite $v.c.d.J_1/H < \infty$. Therefore $c.d.J_1 = c.d.H + v.c.d.J_1/H$, by Theorem 5.6 of [Bi], so $v.c.d.J_1/H = 1$ and J_1/H is virtually free.

If g normalizes J_1 then $HH^g/H = H^g/H \cap H^g$ is a finitely generated normal subgroup of J_1/H and so either has finite index or is finite. (Here $H^g = gHg^{-1}$.) In the former case J_1/H would be finitely presentable (since it is then an extension of a finitely generated virtually free group by a finitely generated free normal subgroup) and as it is subnormal in G it must be a PD_2-group, by our earlier work. But PD_2-groups do not have finitely generated noncyclic free normal subgroups. Therefore HH^g/H is finite and so $HH^g = H$, by the maximality of H. Since this holds for any $g \in J_2$ the subgroup H is normal in J_2 and so is the initial term of a subnormal chain of length $n-1$ terminating with G, contradicting the minimality of n. Therefore G has no finitely generated noncyclic free subnormal subgroups. □

The theorem as stated can be proven without appeal to Bowditch's Theorem (used here for the cases when $H \cong Z$) [BH91].

If H is a PD_2-group $N_G(H)$ is the fundamental group of a 3-manifold which is double covered by the mapping torus of a surface homeomorphism. There are however $\mathbb{N}il^3$-manifolds with no normal PD_2-subgroup (although they always have subnormal copies of Z^2).

Theorem 2.17 *Let G be a PD_3-group with an almost finitely presentable subgroup H which has one end and is of infinite index in G. Let $H_0 = H$ and $H_{i+1} = N_G(H_i)$ for $i \geq 0$. Then $\widehat{H} = \cup H_i$ is almost finitely presentable and has one end, and either $c.d.\widehat{H} = 2$ and $N_G(\widehat{H}) = \widehat{H}$ or $[G : \widehat{H}] < \infty$ and G is virtually the group of a surface bundle.*

Proof If $c.d.H_i = 2$ for all $i \geq 0$ then $[H_{i+1} : H_i] < \infty$ for all $i \geq 0$, by Lemma 2.15. Hence $h.d.\widehat{H} = 2$, by Theorem 4.7 of [Bi]. Therefore $[G : \widehat{H}] = \infty$, so $c.d.\widehat{H} = 2$ also. Hence $\widehat{H}$ is finitely generated, and so $\widehat{H} = H_i$ for i large, by Theorem 3.3 of [GS81]. In particular, $N_G(\widehat{H}) = \widehat{H}$.

Otherwise let $k = \max\{i \mid c.d.H_i = 2\}$. Then H_k is FP_2 and has one end and $[G : H_{k+1}] < \infty$, so G is virtually the group of a surface bundle, by Theorem 2.12 and the observation preceding this theorem. □

Corollary 2.17.1 *If G has a subgroup H which is a PD_2-group with $\chi(H) = 0$ (respectively, < 0) then either it has such a subgroup which is its own normalizer in G or it is virtually the group of a surface bundle.*

Proof If $c.d.\widehat{H} = 2$ then $[\widehat{H} : H] < \infty$, so $\widehat{H}$ is a PD_2-group, and $\chi(H) = [\widehat{H} : H]\chi(\widehat{H})$. □

2.8 Subgroups of PD_3-groups and 3-manifold groups

The central role played by incompressible surfaces in the geometric study of Haken 3-manifolds suggests strongly the importance of studying subgroups of infinite index in PD_3-groups. Such subgroups have cohomological dimension ≤ 2, by Strebel's Theorem.

There are substantial constraints on 3-manifold groups and their subgroups. Every finitely generated subgroup of a 3-manifold group is the fundamental group of a compact 3-manifold (possibly with boundary) [Sc73], and thus is finitely presentable and is either a 3-manifold group or has finite geometric dimension 2 or is a free group. All 3-manifold groups have *Max-c* (every strictly increasing sequence of centralizers is finite), and solvable subgroups of infinite index are virtually abelian [Kr90a]. If the Thurston Geometrization Conjecture is true every aspherical closed 3-manifold is Haken, hyperbolic or Seifert fibred. The groups of such 3-manifolds are residually finite [He87], and the centralizer of any element in the group is finitely generated [JS79]. Thus solvable subgroups are virtually poly-Z.

In contrast, any group of finite geometric dimension 2 is the fundamental group of a compact aspherical 4-manifold with boundary, obtained by attaching 1- and 2-handles to D^4. On applying the orbifold hyperbolization technique of Gromov, Davis and Januszkiewicz [DJ91] to the boundary we see that each such group embeds in a PD_4-group. Thus the question of which such groups are subgroups of PD_3-groups is critical. (In particular, which $\mathcal{X}$-groups are subgroups of PD_3-groups?)

The Baumslag-Solitar groups $\langle x, t \mid tx^pt^{-1} = x^q\rangle$ are not hopfian, and hence not residually finite, and do not have *Max-c*. As they embed in PD_4-groups there are such groups which are not residually finite and do not have *Max-c*. The product of two nonabelian PD_2^+-groups contains a copy of $F(2) \times F(2)$, and so is a PD_4^+-group which is not almost coherent.

Kropholler and Roller have shown that $F(2) \times F(2)$ is not a subgroup of any PD_3-group [KR89]. They have also proved some strong splitting theorems for PD_n-groups. Let G be a PD_3-group with a subgroup $H \cong Z^2$. If G is residually finite then it is virtually split over a subgroup commensurate with H [KR88]. If $\sqrt{G} = 1$ then G splits over an $\mathcal{X}$-group [Kr93]; if moreover G has *Max-c* then it splits over a subgroup commensurate with H [Kr90].

The geometric conclusions of Theorem 2.14 and the coherence of 3-manifold groups suggest that Theorems 2.12 and 2.16 should hold under the weaker hypothesis that N be finitely generated. (Compare Theorem 1.20.)

Is there a characterization of virtual PD_3-groups parallel to Bowditch's Theorem? (It may be relevant that homology n-manifolds are manifolds for $n \leq 2$. High dimensional analogues are known to be false. For every $k \geq 6$ there are FP_k groups G with $H^k(G; \mathbb{Z}[G]) \cong Z$ but which are not virtually torsion free [FS93].)

2.9 $\pi_2(P)$ as a $\mathbb{Z}[\pi]$-module

The cohomology group $H^2(P;\pi_2(P))$ arises in studying homotopy classes of self homotopy equivalences of P. Hendriks and Laudenbach showed that if N is a P^2-irreducible 3-manifold and $\pi_1(N)$ is virtually free then $H^2(N;\pi_2(N)) \cong Z$, and otherwise $H^2(N;\pi_2(N)) = 0$ [HL74]. Swarup showed that if N is a 3-manifold which is the connected sum of a 3-manifold whose fundamental group is free of rank r with $s \geq 1$ aspherical 3-manifolds then $\pi_2(N)$ is a finitely generated free $\mathbb{Z}[\nu]$-module of rank $2r+s-1$ [Sw73]. We shall give direct homological arguments using Schanuel's Lemma to extend these results to PD_3-complexes with torsion free fundamental group.

Theorem 2.18 *Let N be a PD_3-complex with torsion free fundamental group ν. Then*

(1) *$c.d.\nu \leq 3$;*

(2) *the $\mathbb{Z}[\nu]$-module $\pi_2(N)$ is finitely presentable and has projective dimension at most 1;*

(3) *if ν is a nontrivial free group then $H^2(N;\pi_2(N)) \cong Z$;*

(4) *if ν is not a free group then $\pi_2(N)$ is projective and $H^2(N;\pi_2(N)) = 0$;*

(5) *if ν is not a free group then any two of the conditions "ν is FF", "N is homotopy equivalent to a finite complex" and "$\pi_2(N)$ is stably free" imply the third.*

Proof We may clearly assume that $\nu \neq 1$. The PD_3-complex N is homotopy equivalent to a connected sum of aspherical PD_3-complexes and a 3-manifold with free fundamental group, by Turaev's Theorem. Therefore ν is a corresponding free product, and so it has cohomological dimension at most 3 and is FP. Since N is finitely dominated the equivariant chain complex of the universal covering space $\widetilde{N}$ is chain homotopy equivalent to a complex

$$0 \to C_3 \to C_2 \to C_1 \to C_0 \to 0$$

of finitely generated projective left $\mathbb{Z}[\nu]$-modules. Then the sequences

$$0 \to Z_2 \to C_2 \to C_1 \to C_0 \to Z \to 0$$

and

$$0 \to C_3 \to Z_2 \to \pi_2(N) \to 0$$

are exact, where Z_2 is the module of 2-cycles in C_2. Since ν is FP and $c.d.\nu \leq 3$ Schanuel's Lemma implies that Z_2 is projective and finitely generated. Hence $\pi_2(N)$ has projective dimension at most 1, and is finitely presentable.

It follows easily from the UCSS and Poincaré duality that $\pi_2(N)$ is isomorphic to $\overline{H^1(\nu;\mathbb{Z}[\nu])}$ and that there is an exact sequence

$$H^3(\nu;\mathbb{Z}[\nu]) \to H^3(N;\mathbb{Z}[\nu]) \to Ext^1_{\mathbb{Z}[\nu]}(\pi_2(N),\mathbb{Z}[\nu]) \to 0 \qquad (2.1)$$

The $w_1(N)$-twisted augmentation homomorphism from $\mathbb{Z}[\nu]$ to $\bar{Z}$ which sends $g \in \nu$ to $w_1(N)(g)$ induces an isomorphism from $H^3(N;\mathbb{Z}[\nu])$ to $H^3(N;\bar{Z}) \cong Z$. If ν is free the first term in this sequence is 0, and so $Ext^1_{\mathbb{Z}[\nu]}(\pi_2(N),\mathbb{Z}[\nu]) \cong Z$. (In particular, $\pi_2(N)$ has projective dimension 1.) There is also a short exact sequence of left modules

$$0 \to \mathbb{Z}[\nu]^r \to \mathbb{Z}[\nu] \to Z \to 0,$$

where r is the rank of ν. On dualizing we obtain the sequence of right modules

$$0 \to \mathbb{Z}[\nu] \to \mathbb{Z}[\nu]^r \to H^1(\nu;\mathbb{Z}[\nu]) \to 0.$$

The long exact sequence of homology with these coefficients includes an exact sequence

$$0 \to H_1(N;H^1(\nu;\mathbb{Z}[\nu])) \to H_0(N;\mathbb{Z}[\nu]) \to H_0(N;\mathbb{Z}[\nu]^r)$$

in which the right hand map is 0, and so $H_1(N;H^1(\nu;\mathbb{Z}[\nu])) \cong H_0(N;\mathbb{Z}[\nu]) = Z$. Hence $H^2(N;\pi_2(N) \cong H_1(N;\bar{\pi}_2(N)) = H_1(N;H^1(\nu;\mathbb{Z}[\nu])) \cong Z$, by Poincaré duality.

If ν is *not* free then the map $H^3(\nu;\mathbb{Z}[\nu]) \to H^3(N;\mathbb{Z}[\nu])$ in sequence 2.1 above is onto, as can be seen by comparison with the corresponding sequence with coefficients $\bar{Z}$. Therefore $Ext^1_{\mathbb{Z}[\nu]}(\pi_2(N),\mathbb{Z}[\nu]) = 0$. Since $\pi_2(N)$ has a short resolution by finitely generated projective modules, it follows that it is in fact projective. As $H^2(N;\mathbb{Z}[\nu]) = H_1(N;\mathbb{Z}[\nu]) = 0$ it follows that $H^2(N;P) = 0$ for any projective $\mathbb{Z}[\nu]$-module P. Hence $H^2(N;\pi_2(N)) = 0$.

The final assertion follows easily from the fact that if $\pi_2(N)$ is projective then $Z_2 \cong \pi_2(N) \oplus C_3$. □

If ν is not torsion free then the projective dimension of $\pi_2(N)$ is infinite. Does the result of [HL74] extend to all PD_3-complexes?

Chapter 3

Homotopy invariants of PD_4-complexes

The homotopy type of a 4-manifold M is largely determined (through Poincaré duality) by its algebraic 2-type and orientation character. In many cases the formally weaker invariants $\pi_1(M)$, $w_1(M)$ and $\chi(M)$ already suffice. In §1 we give criteria in such terms for a degree-1 map between PD_4-complexes to be a homotopy equivalence, and for a PD_4-complex to be aspherical. We then show in §2 that if the universal covering space of a PD_4-complex is homotopy equivalent to a finite complex then it is either compact, contractible, or homotopy equivalent to S^2 or S^3. In §3 we obtain estimates for the minimal Euler characteristic of PD_4-complexes with fundamental group of cohomological dimension at most 2 and determine the second homotopy groups of PD_4-complexes realizing the minimal value. The class of such groups includes all surface groups and classical link groups, and the groups of many other (bounded) 3-manifolds. The minima are realized by s-parallelizable PL 4-manifolds. In the final section we shall show that if $\chi(M)=0$ then $\pi_1(M)$ satisfies some stringent constraints.

3.1 Homotopy equivalence and asphericity

Many of the results of this section depend on the following lemma, in conjunction with use of the Euler characteristic to compute the rank of the surgery kernel. (This lemma and the following theorem derive from Lemmas 2.2 and 2.3 of [Wa].)

Lemma 3.1 *Let R be a ring and C_* be a finite chain complex of projective R-modules. If $H_i(C_*)=0$ for $i<q$ and $H^{q+1}(Hom_R(C_*,B))=0$ for any left R-module B then $H_q(C_*)$ is projective. If moreover $H_i(C_*)=0$ for $i>q$ then $H_q(C_*)\oplus\bigoplus_{i\equiv q+1\ (2)}C_i\cong\bigoplus_{i\equiv q\ (2)}C_i$.*

Proof We may assume without loss of generality that $q=0$ and $C_i=0$ for $i<0$. We may factor $\partial_1:C_1\to C_0$ through $B=\mathrm{Im}\partial_1$ as $\partial_1=j\beta$, where β is an epimorphism and j is the natural inclusion of the submodule

B. Since $j\beta\partial_2 = \partial_1\partial_2 = 0$ and j is injective $\beta\partial_2 = 0$. Hence β is a 1-cocycle of the complex $Hom_R(C_*, B)$. Since $H^1(Hom_R(C_*, B)) = 0$ there is a homomorphism $\sigma : C_0 \to B$ such that $\beta = \sigma\partial_1 = \sigma j\beta$. Since β is an epimorphism $\sigma j = id_B$ and so B is a direct summand of C_0. This proves the first assertion.

The second assertion follows by an induction on the length of the complex. □

Theorem 3.2 *Let N and M be finite PD_4-complexes. A map $f : M \to N$ is a homotopy equivalence if and only if $\pi_1(f)$ is an isomorphism, $f^*w_1(N) = w_1(M)$, $f_*[M] = \pm[N]$ and $\chi(M) = \chi(N)$.*

Proof The conditions are clearly necessary. Suppose that they hold. Up to homotopy type we may assume that f is a cellular inclusion of finite cell complexes, and so M is a subcomplex of N. We may also identify $\pi_1(M)$ with $\pi = \pi_1(N)$. Let $C_*(M)$, $C_*(N)$ and D_* be the cellular chain complexes of $\widetilde{M}$, $\widetilde{N}$ and $(\widetilde{N}, \widetilde{M})$, respectively. Then the sequence

$$0 \to C_*(M) \to C_*(N) \to D_* \to 0$$

is a short exact sequence of finitely generated free $\mathbb{Z}[\pi]$-chain complexes.

By the projection formula $f_*(f^*a \cap [M]) = a \cap f_*[M] = \pm a \cap [N]$ for any cohomology class $a \in H^*(N; \mathbb{Z}[\pi])$. Since M and N satisfy Poincaré duality it follows that f induces split surjections on homology and split injections on cohomology. Hence $H_q(D_*)$ is the "surgery kernel" in degree $q-1$, and the duality isomorphisms induce isomorphisms from $H^r(Hom_{\mathbb{Z}[\pi]}(D_*, B))$ to $H_{6-r}(\overline{D_*} \otimes B)$, where B is any left $\mathbb{Z}[\pi]$-module. Since f induces isomorphisms on homology and cohomology in degrees ≤ 1, with any coefficients, the hypotheses of Lemma 3.1 are satisfied for the $\mathbb{Z}[\pi]$-chain complex D_*, with $q = 3$, and so $H_3(D_*) = \mathrm{K}er(\pi_2(f))$ is projective. Moreover $H_3(D_*) \oplus \bigoplus_{i\ odd} D_i \cong \bigoplus_{i\ even} D_i$. Thus $H_3(D_*)$ is a stably free $\mathbb{Z}[\pi]$-module of rank $\chi(E, M) = \chi(M) - \chi(E) = 0$ and so it is trivial, as $\mathbb{Z}[\pi]$ is weakly finite, by a theorem of Kaplansky (see [Ro84]). Therefore f is a homotopy equivalence. □

If M and N are merely finitely dominated, rather than finite, then $H_3(D_*)$ is a finitely generated projective $\mathbb{Z}[\pi]$-module such that $H_3(D_*) \otimes_{\mathbb{Z}[\pi]} Z = 0$. If the Wall finiteness obstructions satisfy $f_*\sigma(M) = \sigma(N)$ in $\tilde{K}_0(\mathbb{Z}[\pi])$ then $H_3(D_*)$ is stably free, and the theorem remains true. This additional condition is redundant if π satisfies the Weak Bass Conjecture. (Similar comments apply elsewhere in this section.)

Corollary 3.2.1 *Let N be orientable. Then a map $f : N \to N$ which induces automorphisms of $\pi_1(N)$ and $H_4(N;\mathbb{Z})$ is a homotopy equivalence.* □

In the aspherical cases we shall see that we can relax the hypothesis that the classifying map have degree ± 1.

Lemma 3.3 *Let M be a PD_4-complex with fundamental group π. Then there is an exact sequence*

$$0 \to H^2(\pi;\mathbb{Z}[\pi]) \to \overline{\pi_2(M)} \to Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi]) \to H^3(\pi;\mathbb{Z}[\pi]) \to 0.$$

Proof Since $H_2(M;\mathbb{Z}[\pi]) \cong \pi_2(M)$ and $H^3(M;\mathbb{Z}[\pi]) \cong H_1(\widetilde{M};\mathbb{Z}) = 0$, this follows from the UCSS and Poincaré duality. □

Exactness of much of this sequence can be derived without the UCSS. The middle arrow is the composite of a Poincaré duality isomorphism and the evaluation homomorphism. Note also that $Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi])$ may be identified with $H^0(\pi;H^2(\widetilde{M};\mathbb{Z})\otimes\mathbb{Z}[\pi])$, the π-invariant subgroup of the cohomology of the universal covering space. When π is finite the sequence reduces to an isomorphism $\pi_2(M) \cong \overline{Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi])}$.

Let $ev^{(2)} : H^2_{(2)}(\widetilde{M}) \to Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\ell^2(\pi))$ be the evaluation homomorphism defined on the *un*reduced L^2-cohomology by $ev^{(2)}(f)(z) = \Sigma f(g^{-1}z)g$ for all 2-cycles z and square summable 2-cocycles f. Much of the next theorem is implicit in [Ec94].

Theorem 3.4 *Let M be a finite PD_4-complex with fundamental group π. Then*

1. *if $\beta_1^{(2)}(\pi) = 0$ then $\chi(M) \geq 0$;*
2. $\mathrm{Ker}(ev^{(2)})$ *is closed;*
3. *if $\chi(M) = \beta_1^{(2)}(\pi) = 0$ then $c_M^* : H^2(\pi;\mathbb{Z}[\pi]) \to H^2(M;\mathbb{Z}[\pi]) \cong \overline{\pi_2(M)}$ is an isomorphism.*

Proof Since M is a PD_4-complex $\chi(M) = 2\beta_0^{(2)}(\pi) - 2\beta_1^{(2)}(\pi) + \beta_2^{(2)}(M)$. Hence $\chi(M) \geq \beta_2^{(2)}(M) \geq 0$ if $\beta_1^{(2)}(\pi) = 0$.

Let $z \in C_2(\widetilde{M})$ be a 2-cycle and $f \in C_2^{(2)}(\widetilde{M})$ a square-summable 2-cocycle. As $||ev^{(2)}(f)(z)||_2 \leq ||f||_2||z||_2$, the map $f \mapsto ev^{(2)}(f)(z)$ is continuous, for fixed z. Hence if $f = lim f_n$ and $ev^{(2)}(f_n) = 0$ for all n then $ev^{(2)}(f) = 0$.

The inclusion $\mathbb{Z}[\pi] < \ell^2(\pi)$ induces a homomorphism from the exact sequence of Lemma 3.3 to the corresponding sequence with coefficients $\ell^2(\pi)$. The module $H^2(M;\ell^2(\pi))$ may be identified with the *un*reduced L^2-cohomology, and $ev^{(2)}$ may be viewed as mapping $H_2^{(2)}(\widetilde{M})$ to $H^2(\widetilde{M};\mathbb{Z})\otimes\ell^2(\pi)$ [Ec94]. As $\widetilde{M}$ is 1-connected the induced homomorphism from $H^2(\widetilde{M};\mathbb{Z})\otimes\mathbb{Z}[\pi]$ to $H^2(\widetilde{M};\mathbb{Z})\otimes\ell^2(\pi)$ is injective. As $ev^{(2)}(\delta g)(z) = ev^{(2)}(g)(\partial z) = 0$ for any square summable 1-chain g and $\mathrm{Ker}(ev^{(2)})$ is closed $ev^{(2)}$ factors through the reduced L^2-cohomology $\bar{H}^2_{(2)}(\widetilde{M})$. In particular, it is 0 if $\beta_1^{(2)}(\pi) = \chi(M) = 0$. Hence the middle arrow of the sequence in Lemma 3.3 is also 0 and c_M^* is an isomorphism. □

A related argument gives a complete and natural criterion for asphericity for closed 4-manifolds.

Theorem 3.5 *Let M be a finite PD_4-complex with fundamental group π. Then M is aspherical if and only if $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$ and $\beta_2^{(2)}(M) = \beta_2^{(2)}(\pi)$.*

Proof The conditions are clearly necessary. Suppose that they hold. Then as $\beta_i^{(2)}(M) = \beta_i^{(2)}(\pi)$ for $i \leq 2$ the classifying map $c_M : M \to K(\pi,1)$ induces weak isomorphisms on reduced L^2-cohomology $\bar{H}^i_{(2)}(\pi) \to \bar{H}^i_{(2)}(\widetilde{M})$ for $i \leq 2$.

The natural homomorphism $h : H^2(M;\ell^2(\pi)) \to H^2(\widetilde{M};\mathbb{Z})\otimes\ell^2(\pi)$ factors through $\bar{H}^2_{(2)}(\widetilde{M})$. The induced homomorphism is a homomorphism of Hilbert modules and so has closed kernel. But the image of $\bar{H}^2_{(2)}(\pi)$ is dense in $\bar{H}_2^{(2)}(\widetilde{M})$ and is in this kernel. Hence $h = 0$. Since $H^2(\pi;\mathbb{Z}[\pi]) = 0$ the homomorphism from $H^2(M;\mathbb{Z}[\pi])$ to $H^2(\widetilde{M};\mathbb{Z})\otimes\mathbb{Z}[\pi]$ obtained by forgetting $\mathbb{Z}[\pi]$-linearity is injective. Hence the composite homomorphism from $H^2(M;\mathbb{Z}[\pi])$ to $H^2(\widetilde{M};\mathbb{Z})\otimes\ell^2(\pi)$ is also injective. But this composite may also be factored as the natural map from $H^2(M;\mathbb{Z}[\pi])$ to $H^2(M;\ell^2(\pi))$ followed by h. Hence $H^2(M;\mathbb{Z}[\pi]) = 0$ and so M is aspherical, by Poincaré duality. □

Corollary 3.5.1 *M is aspherical if and only if π is an FF PD_4-group and $\chi(M) = \chi(\pi)$.* □

This also follows immediately from Theorem 3.2, if also $\beta_2(\pi) \neq 0$. For we may assume that M and π are orientable, after passing to the subgroup $Ker(w_1(M)) \cap Ker(w_1(\pi))$, if necessary. As $H_2(c_M;\mathbb{Z})$ is an epimorphism it is an isomorphism, and so c_M must have degree ±1, by Poincaré duality.

Corollary 3.5.2 *If $\chi(M)=\beta_1^{(2)}(\pi)=0$ and $H^s(\pi;\mathbb{Z}[\pi])=0$ for $s\leq 2$ then M is aspherical and π is a PD_4-group.* □

Corollary 3.5.3 *If $\pi\cong Z^r$ then $\chi(M)\geq 0$, with equality only if $r=1$, 2 or 4.*

Proof If $r>2$ then $H^s(\pi;\mathbb{Z}[\pi])=0$ for $s\leq 2$. □

Is it possible to replace the hypothesis "$\beta_2^{(2)}(M)=\beta_2^{(2)}(\pi)$" in Theorem 3.5 by "$\beta_2(M^+)=\beta_2(\mathrm{Ker}w_1(M))$", where $p_+:M^+\to M$ is the orientation cover? It is easy to find examples to show that the homological conditions on π cannot be relaxed further.

Theorem 3.5 implies that if π is a PD_4-group and $\chi(M)=\chi(\pi)$ then $c_{M*}[M]$ is nonzero. If we drop the condition $\chi(M)=\chi(\pi)$ this need not be true. Given any finitely presentable group G there is a closed orientable 4-manifold M with $\pi_1(M)\cong G$ and such that $c_{M*}[M]=0$ in $H_4(G;\mathbb{Z})$. We may take M to be the boundary of a regular neighbourhood N of some embedding in $\mathbb{R}^5$ of a finite 2-complex K with $\pi_1(K)\cong G$. As the inclusion of M into N is 2-connected and K is a deformation retract of N the classifying map c_M factors through c_K and so induces the trivial homomorphism on homology in degrees >2. However if M and π are orientable and $\beta_2(M)<2\beta_2(\pi)$ then c_M must have nonzero degree, for the image of $H^2(\pi;\mathbb{Q})$ in $H^2(M;\mathbb{Q})$ then cannot be self-orthogonal under cup-product.

Theorem 3.6 *Let π be a PD_4-group with a finite $K(\pi,1)$-complex and such that $\chi(\pi)=0$. Then $\mathrm{def}(\pi)\leq 0$.*

Proof Suppose that π has a presentation of deficiency >0, and let X be the corresponding 2-complex. Then $\beta_2^{(2)}(\pi)-\beta_1^{(2)}(\pi)\leq\beta_2^{(2)}(X)-\beta_1^{(2)}(\pi)=\chi(X)\leq 0$. We also have $\beta_2^{(2)}(\pi)-2\beta_1^{(2)}(\pi)=\chi(\pi)=0$. Hence $\beta_1^{(2)}(\pi)=\beta_2^{(2)}(\pi)=\chi(X)=0$. Therefore X is aspherical, by Theorem 2.4, and so *c.d.*$\pi\leq 2$. But this contradicts the hypothesis that π is a PD_4-group. □

Is $\mathrm{def}(\pi)\leq 0$ for any PD_4-group π? This bound is best possible for groups with $\chi=0$, since there is a poly-Z group $Z^3\times_A Z$, where $A\in SL(3,\mathbb{Z})$, with presentation $\langle s,x,\mid sxs^{-1}x=xsxs^{-1},\ s^3x=xs^3\rangle$.

The hypothesis on orientation characters in Theorem 3.2 is often redundant.

Theorem 3.7 *Let $f: M \to N$ be a 2-connected map between finite PD_4-complexes with $\chi(M) = \chi(N)$. If $H^2(N;\mathbb{F}_2) \neq 0$ then $f^*w_1(N) = w_1(M)$, and if moreover N is orientable and $H^2(N;\mathbb{Q}) \neq 0$ then f is a homotopy equivalence.*

Proof Since f is 2-connected $H^2(f;\mathbb{F}_2)$ is injective, and since $\chi(M) = \chi(N)$ it is an isomorphism. Since $H^2(N;\mathbb{F}_2) \neq 0$, the nondegeneracy of Poincaré duality implies that $H^4(f;\mathbb{F}_2) \neq 0$, and so f is a $\mathbb{F}_2$-(co)homology equivalence. Since $w_1(M)$ is characterized by the Wu formula $x \cup w_1(M) = Sq^1 x$ for all x in $H^3(M;\mathbb{F}_2)$, it follows that $f^*w_1(N) = w_1(M)$.

If $H^2(N;\mathbb{Q}) \neq 0$ then $H^2(N;\mathbb{Z})$ has positive rank and $H^2(N;\mathbb{F}_2) \neq 0$, so N orientable implies M orientable. We may then repeat the above argument with integral coefficients, to conclude that f has degree ± 1. The result then follows from Theorem 3.2. □

The argument breaks down if, for instance, $M = S^1 \tilde{\times} S^3$ is the nonorientable S^3-bundle over S^1, $N = S^1 \times S^3$ and f is the composite of the projection of M onto S^1 followed by the inclusion of a factor.

We would like to replace the hypotheses above that there be a map $f: M \to N$ realizing certain isomorphisms by weaker, more algebraic conditions. If M and N are closed 4-manifolds with isomorphic algebraic 2-types then there is a 3-connected map $f: M \to P_2(N)$. The restriction of such a map to $M_o = M \backslash D^4$ is homotopic to a map $f_o: M_o \to N$ which induces isomorphisms on π_i for $i \leq 2$. In particular, $\chi(M) = \chi(N)$. Thus if f_o extends to a map from M to N we may be able to apply Theorem 3.2. However we usually need more information on how the top cell is attached. The characteristic classes and the equivariant intersection pairing on $\pi_2(M)$ are the obvious candidates.

The following criterion arises in studying the homotopy types of circle bundles over 3-manifolds. (See Chapter 4.)

Theorem 3.8 *Let E be a finite PD_4-complex with fundamental group π and suppose that $H^4(f_E; Z^{w_1(E)})$ is a monomorphism. A finite PD_4-complex M is homotopy equivalent to E if and only if there is an isomorphism θ from $\pi_1(M)$ to π such that $w_1(M) = w_1(E)\theta$, there is a lift $\hat{c}: M \to P_2(E)$ of θc_M such that $\hat{c}_*[M] = \pm f_{E*}[E]$ and $\chi(M) = \chi(E)$.*

Proof The conditions are clearly necessary. Conversely, suppose that they hold. We shall adapt to our situation the arguments of Hendriks in analyzing the obstructions to the existence of a degree 1 map between PD_3-complexes realizing a given homomorphism of fundamental groups. For simplicity of notation we shall write $\tilde{Z}$ for $Z^{w_1(E)}$ and also for $Z^{w_1(M)}(=\theta^*\tilde{Z})$, and use θ to identify $\pi_1(M)$ with π and $K(\pi_1(M),1)$ with $K(\pi,1)$. We may suppose the sign of the fundamental class $[M]$ is so chosen that $\hat{c}_*[M]=f_{E*}[E]$.

Let $E_o=E\backslash D^4$. Then $P_2(E_o)=P_2(E)$ and may be constructed as the union of E_o with cells of dimension ≥ 4. Let

$$h:\tilde{Z}\otimes_{\mathbb{Z}[\pi]}\pi_4(P_2(E_o),E_o)\to H_4(P_2(E_o),E_o;\tilde{Z})$$

be the $w_1(E)$-twisted relative Hurewicz homomorphism, and let ∂ be the connecting homomorphism from $\pi_4(P_2(E_o),E_o)$ to $\pi_3(E_o)$ in the exact sequence of homotopy for the pair $(P_2(E_o),E_o)$. Then h and ∂ are isomorphisms since f_{E_o} is 3-connected, and so the homomorphism $\tau_E:H_4(P_2(E);\tilde{Z})\to\tilde{Z}\otimes_{\mathbb{Z}[\pi]}\pi_3(E_o)$ given by the composite of the inclusion

$$H_4(P_2(E);\tilde{Z})=H_4(P_2(E_o);\tilde{Z})\to H_4(P_2(E_o),E_o;\tilde{Z})$$

with h^{-1} and $1\otimes_{\mathbb{Z}[\pi]}\partial$ is a monomorphism. Similarly $M_o=M\backslash D^4$ may be viewed as a subspace of $P_2(M_o)$ and there is a monomorphism τ_M from $H_4(P_2(M);\tilde{Z})$ to $\tilde{Z}\otimes_{\mathbb{Z}[\pi]}\pi_3(M_o)$. These monomorphisms are natural with respect to maps defined on the 3-skeleta (i.e., E_o and M_o).

The classes $\tau_E(f_{E*}[E])$ and $\tau_M(f_{M*}[M])$ are the images of the primary obstructions to retracting E onto E_o and M onto M_o, under the Poincaré duality isomorphisms from $H^4(E,E_o;\pi_3(E_o))$ to $H_0(E\backslash E_o;\tilde{Z}\otimes_{\mathbb{Z}[\pi]}\pi_3(E_o))=\tilde{Z}\otimes_{\mathbb{Z}[\pi]}\pi_3(E_o)$ and $H^4(M,M_o;\pi_3(M_o))$ to $\tilde{Z}\otimes_{\mathbb{Z}[\pi]}\pi_3(M_o)$, respectively. Since M_o is homotopy equivalent to a cell complex of dimension ≤ 3 the restriction of $\hat{c}$ to M_o is homotopic to a map from M_o to E_o. Let $\hat{c}_\sharp$ be the homomorphism from $\pi_3(M_o)$ to $\pi_3(E_o)$ induced by $\hat{c}|M_o$. Then $(1\otimes_{\mathbb{Z}[\pi]}\hat{c}_\sharp)\tau_M(f_{M*}[M])=\tau_E(f_{E*}[E])$. It follows as in [Hn77] that the obstruction to extending $\hat{c}|M_o:M_o\to E_o$ to a map d from M to E is trivial.

Since $f_{E*}d_*[M]=\hat{c}_*[M]=f_{E*}[E]$ and f_{E*} is a monomorphism in degree 4 the map d has degree 1, and so is a homotopy equivalence, by Theorem 3.2. □

If there is such a lift $\hat{c}$ then $c_M^*\theta^*k_1(E)=0$ and $\theta_*c_{M*}[M]=c_{E*}[E]$.

3.2 Finitely dominated covering spaces

In this section we shall show that if a PD_4-complex has an infinite regular covering space which is finitely dominated then either the complex is aspherical or its universal covering space is homotopy equivalent to S^2 or S^3. In Chapters 4 and 5 we shall see that such manifolds are close to being total spaces of fibre bundles.

Theorem 3.9 *Let M be a PD_4-complex with fundamental group π. Suppose that $p : \widehat{M} \to M$ is a regular covering map, with covering group $G = Aut(p)$, and such that $\widehat{M}$ is finitely dominated. Then*

(1) *G has finitely many ends;*

(2) *if $\widehat{M}$ is acyclic then it is contractible and M is aspherical;*

(3) *if G has one end and $\pi_1(\widehat{M})$ is infinite and FP_3 then M is aspherical and $\widehat{M}$ is homotopy equivalent to an aspherical closed surface or to S^1;*

(4) *if G has one end and $\pi_1(\widehat{M})$ is finite but $\widehat{M}$ is not acyclic then $\widehat{M} \simeq S^2$ or RP^2;*

(5) *G has two ends if and only if $\widehat{M}$ is a PD_3-complex.*

Proof We may clearly assume that G is infinite and that M is orientable. As $\mathbb{Z}[G]$ has no nonzero left ideal (i.e., submodule) which is finitely generated as an abelian group $Hom_{\mathbb{Z}[G]}(H_p(\widehat{M};\mathbb{Z}),\mathbb{Z}[G]) = 0$ for all $p \geq 0$, and so the bottom row of the UCSS for the covering p is 0. From Poincaré duality and the UCSS we find that $H^1(G;\mathbb{Z}[G]) \cong \overline{H_3(\widehat{M};\mathbb{Z})}$. As this group is finitely generated, and as G is infinite, G has one or two ends.

If $\widehat{M}$ is acyclic then G is a PD_4-group and so $\widehat{M}$ is a PD_0-complex, hence contractible, by [Go79]. Hence M is aspherical.

Suppose that G has one end. Then $H_3(\widehat{M};\mathbb{Z}) = H_4(\widehat{M};\mathbb{Z}) = 0$. Since $\widehat{M}$ is finitely dominated the chain complex $C_*(\widetilde{M})$ is chain homotopy equivalent over $\mathbb{Z}[\pi_1(\widehat{M})]$ to a complex D_* of finitely generated projective $\mathbb{Z}[\pi_1(\widehat{M})]$-modules. If $\pi_1(\widehat{M})$ is FP_3 then the aumentation $\mathbb{Z}[\pi_1(\widehat{M})]$-module Z has a free resolution P_* which is finitely generated in degrees ≤ 3. On applying Schanuel's Lemma to the exact sequences

$$0 \to Z_2 \to D_2 \to D_1 \to D_0 \to Z \to 0$$

and

$$0 \to \partial P_3 \to P_2 \to P_1 \to P_0 \to Z \to 0$$

derived from these two chain complexes we find that Z_2 is finitely generated as a $\mathbb{Z}[\pi_1(\widehat{M})]$-module. Hence $\Pi = \pi_2(M) = \pi_2(\widehat{M})$ is also finitely generated as a $\mathbb{Z}[\pi_1(\widehat{M})]$-module and so $Hom_\pi(\Pi, \mathbb{Z}[\pi]) = 0$. If moreover $\pi_1(\widehat{M})$ is infinite then $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$, so $\Pi = 0$, by Lemma 3.3, and M

is aspherical. A spectral sequence corner argument then shows that either $H^2(G;\mathbb{Z}[G]) \cong Z$ and $\widehat{M}$ is homotopy equivalent to an aspherical closed surface or $H^2(G;\mathbb{Z}[G]) = 0$, $H^3(G;\mathbb{Z}[G]) \cong Z$ and $\widehat{M} \simeq S^1$. (See the following theorem.)

If $\pi_1(\widehat{M})$ is finite but $\widehat{M}$ is not acyclic then the universal covering space $\widetilde{M}$ is also finitely dominated but not contractible, and $\Pi = H_2(\widetilde{M};\mathbb{Z})$ is a nontrivial finitely generated abelian group, while $H_3(\widetilde{M};\mathbb{Z}) = H_4(\widetilde{M};\mathbb{Z}) = 0$. If C is a finite cyclic subgroup of π there are isomorphisms $H_{n+3}(C;\mathbb{Z}) \cong H_n(C;\Pi)$, for all $n \geq 4$, by Lemma 2.10. Suppose that C acts trivially on Π. Then if n is odd this isomorphism reduces to $0 = \Pi/|C|\Pi$. Since Π is finitely generated, this implies that multiplication by $|C|$ is an isomorphism. On the other hand, if n is even we have $Z/|C|Z \cong \{a \in \Pi \mid |C|a = 0\}$. Hence we must have $C = 1$. Now since Π is finitely generated any torsion subgroup of $Aut(\Pi)$ is finite. (Let T be the torsion subgroup of Π and suppose that $\Pi/T \cong Z^r$. Then the natural homomorphism from $Aut(\Pi)$ to $Aut(\Pi/T)$ has finite kernel, and its image is isomorphic to a subgroup of $GL(r,\mathbb{Z})$, which is virtually torsion free.) Hence as π is infinite it must have elements of infinite order. Since $H^2(\pi;\mathbb{Z}[\pi]) \cong \bar{\Pi}$, by Lemma 3.3, it is a finitely generated abelian group. Therefore it must be infinite cyclic, by Corollary 5.2 of [Fa74]. Hence $\widetilde{M} \simeq S^2$ and $\pi_1(\widehat{M})$ has order at most 2, so $\widehat{M} \simeq S^2$ or RP^2.

Suppose now that $\widehat{M}$ is a PD_3-complex. After passing to a finite covering of M, if necessary, we may assume that $\widehat{M}$ is orientable. Then $H^1(G;\mathbb{Z}[G]) \cong H_3(\widehat{M};\mathbb{Z})$, and so G has two ends. Conversely, if G has two ends we may assume that $G \cong Z$, after passing to a finite covering of M, if necessary. Hence $\widehat{M}$ is a PD_3-complex, by [Go79] again. (See Theorem 4.5 for an alternative argument, with weaker, algebraic hypotheses.) □

Is the hypothesis in (3) that $\pi_1(\widehat{M})$ be FP_3 redundant?

Corollary 3.9.1 *The covering space $\widehat{M}$ is homotopy equivalent to a closed surface if and only if it is finitely dominated, $H^2(G;\mathbb{Z}[G]) \cong Z$ and $\pi_1(\widehat{M})$ is FP_3.* □

In this case M has a finite covering space which is homotopy equivalent to the total space of a surface bundle over an aspherical closed surface. (See Chapter 5.)

Corollary 3.9.2 *The covering space $\widehat{M}$ is homotopy equivalent to S^1 if and only if it is finitely dominated, G has one end, $H^2(G;\mathbb{Z}[G]) = 0$ and $\pi_1(\widehat{M})$ is a nontrivial finitely generated free group.*

Proof If $\widehat{M} \simeq S^1$ then it is finitely dominated and M is aspherical, and the conditions on G follow from the LHSSS. The converse follows from part (3) of the theorem, since a nontrivial finitely generated free group is infinite and FP. □

In fact any finitely generated free normal subgroup F of a PD_n-group π must be infinite cyclic. For $\pi/C_\pi(F)$ embeds in $Out(F)$, so $v.c.d.\pi/C_\pi(F) \leq v.c.d.Out(F(r)) < \infty$. If F is nonabelian then $C_\pi(F) \cap F = 1$ and so $c.d.\pi/F < \infty$. Since F is finitely generated π/F is FP_∞. Hence we may apply Theorem 9.11 of [Bi], and an LHSSS corner argument gives a contradiction.

In the simply connected case "finitely dominated", "homotopy equivalent to a finite complex" and "having finitely generated homology" are all equivalent.

Corollary 3.9.3 *If $H_*(\widetilde{M};\mathbb{Z})$ is finitely generated then either M is aspherical or $\widetilde{M}$ is homotopy equivalent to S^2 or S^3 or $\pi_1(M)$ is finite.* □

We shall examine the spherical cases more closely in Chapters 10 and 11. (The arguments in these chapters may apply also to PD_n-complexes with universal covering space homotopy equivalent to S^{n-1} or S^{n-2}. The analogues in higher codimensions appear to be less accessible.)

The "finitely dominated" condition is used only to ensure that the chain complex of the covering is chain homotopy equivalent over $\mathbb{Z}[\pi_1(\widehat{M})]$ to a finite projective complex. Thus when M is aspherical this condition can be relaxed slightly. The following variation on the aspherical case shall be used in Theorem 4.8, but belongs most naturally here.

Theorem 3.10 *Let N be a nontrivial FP_3 normal subgroup of infinite index in a PD_4-group π, and let $G = \pi/N$. Then either*

(1) *N is a PD_3-group and G has two ends;*

(2) *N is a PD_2-group and G is virtually a PD_2-group; or*

(3) *$N \cong Z$, $H^s(G;\mathbb{Z}[G]) = 0$ for $s \leq 2$ and $H^3(G;\mathbb{Z}[G]) \cong Z$.*

Proof Since $c.d.N < 4$, by Strebel's Theorem, N and hence G are FP. The E_2 terms of the LHS spectral sequence with coefficients $\mathbb{Q}[\pi]$ can then be expressed as $E_2^{pq} = H^p(G;\mathbb{Q}[G]) \otimes H^q(N;\mathbb{Q}[N])$. If $H^j(\pi/N;\mathbb{Q}[\pi/N])$ and $H^k(N;\mathbb{Q}[N])$ are the first nonzero such cohomology groups then E_2^{jk} persists to E_∞ and hence $j+k=4$. Therefore $H^j(G;\mathbb{Q}[G]) \otimes H^{4-j}(N;\mathbb{Q}[N]) \cong Q$.

Hence $H^j(G;\mathbb{Q}[G]) \cong H^{4-j}(N;\mathbb{Q}[N]) \cong Q$. In particular, G has one or two ends and N is a PD_{4-j}-group over $\mathbb{Q}$ [Fa75]. If G has two ends then it is virtually Z, and then N is a PD_3-group (over $\mathbb{Z}$) by Theorem 9.11 of [Bi]. If $H^2(N;\mathbb{Q}[N]) \cong H^2(G;\mathbb{Q}[G]) \cong Q$ then N and G are virtually PD_2-groups, by Bowditch's Theorem. Since N is torsion free it is then in fact a PD_2-group. The only remaining possibility is (3). □

In case (1) π has a subgroup of index ≤ 2 which is a semidirect product $H \times_\theta Z$ with $N \leq H$ and $[H : N] < \infty$. Is it sufficient that N be FP_2? Must the quotient π/N be virtually a PD_3-group in case (3)?

Corollary 3.10.1 *If K is FP_2 and is subnormal in N where N is an FP_3 normal subgroup of infinite index in the PD_4-group π then K is a PD_k-group for some $k < 4$.*

Proof This follows from Theorem 3.10 together with Theorem 2.16. □

What happens if we drop the hypothesis that the covering be regular? It can be shown that a closed 3-manifold has a finitely dominated infinite covering space if and only if its fundamental group has one or two ends. We might conjecture that if a closed 4-manifold M has a finitely dominated infinite covering space $\widehat{M}$ then either M is aspherical or the universal covering space $\widetilde{M}$ is homotopy equivalent to S^2 or S^3 or M has a finite covering space which is homotopy equivalent to the mapping torus of a self homotopy equivalence of a PD_3-complex. (In particular, $\pi_1(M)$ has one or two ends.) In [Hi94'] we extend the arguments of Theorem 3.9 to show that if $\pi_1(\widehat{M})$ is FP_3 and subnormal in π the only other possibility is that $\pi_1(\widehat{M})$ has two ends, $h(\sqrt{\pi}) = 1$ and $H^2(\pi;\mathbb{Z}[\pi])$ is not finitely generated. This paper also considers in more detail FP subnormal subgroups of PD_4-groups, corresponding to the aspherical case.

3.3 Minimizing the Euler characteristic

It is well known that every finitely presentable group is the fundamental group of some closed orientable 4-manifold. Such manifolds are far from unique, for the Euler characteristic may be made arbitrarily large by taking connected sums with simply connected manifolds. Following Hausmann and Weinberger [HW85] we may define an invariant $q(\pi)$ for any finitely presentable group π by

$$q(\pi) = \min\{\chi(M) | M \textit{ is a } PD_4 \textit{ complex with } \pi_1(M) \cong \pi\}.$$

We may also define related invariants q^X where the minimum is taken over the class of PD_4-complexes whose normal fibration has an X-reduction. There are the following basic estimates for q^{SG}, which is defined in terms of PD_4^+-complexes.

Lemma 3.11 *Let π be a finitely presentable group with a subgroup H of finite index and let F be a field. Then*

(1) $1-\beta_1(H;F)+\beta_2(H;F)\leq[\pi:H](1-\mathrm{def}\pi)$;

(2) $2-2\beta_1(H;F)+\beta_2(H;F)\leq[\pi:H]q^{SG}(\pi)$;

(3) $q^{SG}(\pi)\leq 2(1-\mathrm{def}(\pi))$;

(4) *if* $H^4(\pi;F)=0$ *then* $q^{SG}(\pi)\geq 2(1-\beta_1(\pi;F)+\beta_2(\pi;F))$.

Proof Let C be the 2-complex corresponding to a presentation for π of maximal deficiency and let C_H be the covering space associated to the subgroup H. Then $\chi(C)=1-\mathrm{def}\pi$ and $\chi(C_H)=[\pi:H]\chi(\pi)$. Condition (1) follows since $\beta_1(H;F)=\beta_1(C_H;F)$ and $\beta_2(H;F)\leq\beta_2(C_H;F)$.

Condition (2) follows similarly on considering the Euler characteristics of a PD_4^+-complex M with $\pi_1(M)\cong\pi$ and of the associated covering space M_H.

The boundary of a regular neighbourhood of a PL embedding of C in R^5 is a closed orientable 4-manifold realizing the upper bound in (3).

The image of $H^2(\pi;F)$ in $H^2(M;F)$ has dimension $\beta_2(\pi;F)$, and is self-annihilating under cup-product if $H^4(\pi;F)=0$. In that case $\beta_2(M;F)\geq 2\beta_2(\pi;F)$, which implies (4). □

Condition (2) was used in [HW85] to give examples of finitely presentable superperfect groups which are not fundamental groups of homology 4-spheres. (See Chapter 14 below.)

If π is a finitely presentable, orientable PD_4-group we see immediately that $q^{SG}(\pi)\geq\chi(\pi)$. Multiplicativity then implies that $q(\pi)=\chi(\pi)$ if $K(\pi,1)$ is a finite PD_4-complex.

For groups of cohomological dimension at most two we can say more.

Theorem 3.12 *Let M be a finite PD_4-complex with fundamental group π. Suppose that $c.d._{\mathbb{Q}}\pi\leq 2$ and $\chi(M)=2\chi(\pi)=2(1-\beta_1(\pi;\mathbb{Q})+\beta_2(\pi;\mathbb{Q}))$. Then $\pi_2(M)\cong\overline{H^2(\pi;\mathbb{Z}[\pi])}$. If moreover $c.d.\pi\leq 2$ the chain complex of the universal covering space $\widetilde{M}$ is determined up to chain homotopy equivalence over $\mathbb{Z}[\pi]$ by π.*

Proof Let $A_Q(\pi)$ be the augmentation ideal of $\mathbb{Q}[\pi]$. Then there are exact sequences

$$0 \to A_Q(\pi) \to \mathbb{Q}[\pi] \to Q \to 0 \tag{3.1}$$

and

$$0 \to P \to \mathbb{Q}[\pi]^g \to A_Q(\pi) \to 0. \tag{3.2}$$

where P is a finitely generated projective module. We may assume that that $\pi \neq 1$, i.e., that π is infinite, and that M is a finite 4-dimensional cell complex. Let C_* be the cellular chain complex of $\widetilde{M}$, with coefficients $\mathbb{Q}$, and let $H_i = H_i(C_*) = H_i(\widetilde{M};\mathbb{Q})$ and $H^t = H^t(Hom_{\mathbb{Q}[\pi]}(C_*, \mathbb{Q}[\pi]))$. Since $\widetilde{M}$ is simply connected and π is infinite, $H_0 \cong Q$ and $H_1 = H_4 = 0$. Poincaré duality gives further isomorphisms $H^1 \cong \overline{H_3}$, $H^2 \cong \overline{H_2}$, $H^3 = 0$ and $H^4 \cong \overline{Q}$.

The chain complex C_* breaks up into exact sequences:

$$0 \to C_4 \to Z_3 \to H_3 \to 0, \tag{3.3}$$

$$0 \to Z_3 \to C_3 \to Z_2 \to H_2 \to 0, \tag{3.4}$$

$$0 \to Z_2 \to C_2 \to C_1 \to C_0 \to Q \to 0. \tag{3.5}$$

We shall let $e^i N = Ext^i_{\mathbb{Q}[\pi]}(N, \mathbb{Q}[\pi])$, to simplify the notation in what follows. The UCSS gives isomorphisms $H^1 \cong e^1 Q$ and $e^1 H_2 = e^2 H_3 = 0$ and another exact sequence:

$$0 \to e^2 Q \to H^2 \to e^0 H_2 \to 0. \tag{3.6}$$

Applying Schanuel's Lemma to the sequences 3.1, 3.2 and 3.5 we obtain $Z_2 \oplus C_1 \oplus \mathbb{Q}[\pi] \oplus P \cong C_2 \oplus C_0 \oplus \mathbb{Q}[\pi]^g$, so Z_2 is a finitely generated projective module. Similarly, Z_3 is projective, since $\mathbb{Q}[\pi]$ has global dimension at most 2. Since π is finitely presentable it is accessible, and hence $e^1 Q$ is finitely generated as a $\mathbb{Q}[\pi]$-module, by Theorems IV.7.5 and VI.6.3 of [DD]. Therefore Z_3 is also finitely generated, since it is an extension of $H_3 \cong \overline{e^1 Q}$ by C_4. Dualizing the sequence 3.4 and using the fact that $e^1 H_2 = 0$ we obtain an exact sequence of right modules

$$0 \to e^0 H_2 \to e^0 Z_2 \to e^0 C_3 \to e^0 Z_3 \to e^2 H_2 \to 0. \tag{3.7}$$

Since duals of finitely generated projective modules are projective it follows that $e^0 H_2$ is projective. Hence the sequence 3.6 gives $H^2 \cong e^0 H_2 \oplus e^2 Q$.

Dualizing the sequences 3.1 and 3.2, we obtain exact sequences of right modules

$$0 \to \mathbb{Q}[\pi] \to e^0 A_Q(\pi) \to e^1 Q \to 0 \tag{3.8}$$

and

$$0 \to e^0 A_Q(\pi) \to \mathbb{Q}[\pi]^g \to e^0 P \to e^2 Q \to 0. \tag{3.9}$$

Applying Schanuel's Lemma twice more, to the pairs of sequences 3.3 and the conjugate of 3.8 (using $H_3 \cong \overline{e^1Q}$) and to 3.4 and the conjugate of 3.9 (using $H_2 \cong \overline{e^0H_2} \oplus \overline{e^2Q}$) and putting all together, we obtain isomorphisms

$$Z_3 \oplus (\mathbb{Q}[\pi]^{2g} \oplus C_0 \oplus C_2 \oplus C_4) \cong Z_3 \oplus (\mathbb{Q}[\pi]^2 \oplus P \oplus \overline{e^0P} \oplus C_1 \oplus C_3 \oplus \overline{e^0H_2}).$$

On tensoring with the augmentation module we find that

$$dim_{\mathbb{Q}}(\mathbb{Q} \otimes_\pi \overline{e^0H_2}) + dim_{\mathbb{Q}}(\mathbb{Q} \otimes_\pi P) + dim_{\mathbb{Q}}(\mathbb{Q} \otimes_\pi \overline{e^0P}) = \chi(M) + 2g - 2.$$

Now

$$dim_{\mathbb{Q}}(\mathbb{Q} \otimes_\pi P) = dim_{\mathbb{Q}}(\mathbb{Q} \otimes_\pi \overline{e^0P}) = g + \beta_2(\pi;\mathbb{Q}) - \beta_1(\pi;\mathbb{Q}),$$

so $dim_{\mathbb{Q}}(\mathbb{Q}\otimes_\pi \overline{e^0H_2}) = \chi(M) - 2\chi(\pi) = 0$. Hence $e^0H_2 = 0$, since π satisfies the Weak Bass Conjecture [Ec86]. As $Hom_{\mathbb{Z}[\pi]}(H_2(\widetilde{M};\mathbb{Z}), \mathbb{Z}[\pi]) \leq e^0H_2$ it follows from Lemma 3.3 that $\pi_2(M) \cong H_2(\widetilde{M};\mathbb{Z}) \cong \overline{H^2(\pi;\mathbb{Z}[\pi])}$.

If $c.d.\pi \leq 2$ then e^1Z has a short finite projective resolution, and hence so does Z_3 (via sequence 3.2). The argument can then be modified to work over $\mathbb{Z}[\pi]$. As Z_1 is then projective, the integral chain complex of $\widetilde{M}$ is the direct sum of a projective resolution of Z with a projective resolution of $\pi_2(M)$ with degree shifted by 2. □

There are many natural examples of such manifolds for which $c.d._{\mathbb{Q}}\pi \leq 2$ and $\chi(M) = 2\chi(\pi)$ but π is not torsion free. (See Chapters 10 and 11.) However all the known examples satisfy $v.c.d.\pi \leq 2$.

Similar arguments may be used to prove the following variations.

Addendum *Suppose that $c.d._S\pi \leq 2$ for some subring $S \leq \mathbb{Q}$. Then $q(\pi) \geq 2(1 - \beta_1(\pi;S) + \beta_2(\pi;S))$. If moreover the augmentation $S[\pi]$-module S has a finitely generated free resolution then $S \otimes \pi_2(M)$ is stably isomorphic to $\overline{H^2(\pi;S[\pi])}$.* □

Corollary 3.12.1 *If $H_2(\pi;\mathbb{Q}) \neq 0$ the Hurewicz homomorphism from $\pi_2(M)$ to $H_2(M;\mathbb{Q})$ is nonzero.*

Proof By the addendum to the theorem, $H_2(M;\mathbb{Q})$ has dimension at least $2\beta_2(\pi)$, and so cannot be isomorphic to $H_2(\pi;\mathbb{Q})$ unless both are 0. □

Corollary 3.12.2 *If $\pi = \pi_1(P)$ where P is an aspherical finite 2-complex then $q(\pi) = 2\chi(P)$. The minimum is realized by an s-parallelizable PL 4-manifold.*

Proof If we choose a PL embedding $j : P \to \mathbb{R}^5$, the boundary of a regular neighbourhood N of $j(P)$ is an s-parallelizable PL 4-manifold with fundamental group π and with Euler characteristic $2\chi(P)$. $\square$

By Theorem 2.8 a finitely presentable group is the fundamental group of an aspherical finite 2-complex if and only if it has cohomological dimension ≤ 2 and is efficient, i.e. has a presentation of deficiency $\beta_1(\pi;\mathbb{Q}) - \beta_2(\pi;\mathbb{Q})$. It is not known whether every finitely presentable group of cohomological dimension 2 is efficient.

In Chapter 5 we shall see that if P is an aspherical closed surface and M is a closed 4-manifold with $\pi_1(M) \cong \pi$ then $\chi(M) = q(\pi)$ if and only if M is homotopy equivalent to the total space of an S^2-bundle over P. The homotopy types of such minimal 4-manifolds for π may be distinguished by their Stiefel-Whitney classes. Note that if π is orientable then $S^2 \times P$ is a minimal 4-manifold for π which is both s-parallelizable and also a projective algebraic complex surface. Note also that the conjugation of the module structure in the theorem involves the orientation character of M which may differ from that of the PD_2-group π.

Corollary 3.12.3 *If π is the group of an unsplittable μ-component 1-link then $q(\pi) = 0$.* $\square$

If π is the group of a μ-component n-link with $n \geq 2$ then $H_2(\pi;\mathbb{Q}) = 0$ and so $q(\pi) \geq 2(1-\mu)$, with equality if and only if π is the group of a 2-link. (See Chapter 14.)

Corollary 3.12.4 *If π is an extension of Z by a finitely generated free normal subgroup then $q(\pi) = 0$.* $\square$

In Chapter 4 we shall see that if M is a closed 4-manifold with $\pi_1(M)$ such an extension then $\chi(M) = q(\pi)$ if and only if M is homotopy equivalent to a manifold which fibres over S^1 with fibre a closed 3-manifold with free fundamental group, and then π and $w_1(M)$ determine the homotopy type.

Finite generation of the normal subgroup is essential; $F(2)$ is an extension of Z by $F(\infty)$, and $q(F(2)) = 2\chi(F(2)) = -2$.

Let π be the fundamental group of a closed orientable 3-manifold. Then $\pi \cong F * \nu$ where F is free of rank r and ν has no infinite cyclic free factors. Moreover $\nu = \pi_1(N)$ for some closed orientable 3-manifold N. If M_0 is the closed 4-manifold obtained by surgery on $\{n\} \times S^1$ in $N \times S^1$ then $M = M_0 \sharp(\sharp^r(S^1 \times S^3)$

is a smooth s-parallelisable 4-manifold with $\pi_1(M) \cong \pi$ and $\chi(M) = 2(1-r)$. Hence $q^{SG}(\pi) = 2(1-r)$, by Lemma 3.11.

The arguments of Theorem 3.12 give stronger results in this case also.

Theorem 3.13 *Let M be a finite PD_4-complex whose fundamental group π is a PD_3-group such that $w_1(\pi) = w_1(M)$. Then $\chi(M) > 0$ and $\pi_2(M)$ is stably isomorphic to the augmentation ideal $A(\pi)$ of $\mathbb{Z}[\pi]$.*

Proof The cellular chain complex for the universal covering space of M gives exact sequences

$$0 \to C_4 \to C_3 \to Z_2 \to H_2 \to 0 \tag{3.10}$$

and

$$0 \to Z_2 \to C_2 \to C_1 \to C_0 \to Z \to 0. \tag{3.11}$$

Since π is a PD_3-group the augmentation module Z has a finite projective resolution of length 3. On comparing sequence 3.11 with such a resolution and applying Schanuel's lemma we find that Z_2 is a finitely generated projective $\mathbb{Z}[\pi]$-module. Since π has one end, the UCSS reduces to an exact sequence

$$0 \to H^2 \to e^0H_2 \to e^3Z \to H^3 \to e^1H_2 \to 0 \tag{3.12}$$

and isomorphisms $H^4 \cong e^2H_2$ and $e^3H_2 = e^4H_2 = 0$. Poincaré duality implies that $H^3 = 0$ and $H^4 \cong \overline{Z}$. Hence sequence 3.12 reduces to

$$0 \to H^2 \to e^0H_2 \to e^3Z \to 0 \tag{3.13}$$

and $e^1H_2 = 0$. Hence on dualizing the sequence 3.10 we get an exact sequence of right modules

$$0 \to e^0H_2 \to e^0Z_2 \to e^0C_3 \to e^0C_4 \to e^2H_2 \to 0. \tag{3.14}$$

Schanuel's lemma again implies that e^0H_2 is a finitely generated projective module. Therefore we may splice together 3.10 and the conjugate of 3.13 to get

$$0 \to C_4 \to C_3 \to Z_2 \to \overline{e^0H_2} \to Z \to 0. \tag{3.15}$$

(Note that we have used the hypothesis on $w_1(M)$ here.) Applying Schanuel's lemma once more to the pair of sequences 3.11 and 3.15 we obtain

$$C_0 \oplus C_2 \oplus C_4 \oplus Z_2 \cong \overline{e^0H_2} \oplus C_1 \oplus C_3 \oplus Z_2.$$

Hence $\overline{e^0H_2}$ is stably free, of rank $\chi(M)$. Since sequence 3.15 is exact $\overline{e^0H_2}$ maps onto Z, and so $\chi(M) > 0$. Since π is a PD_3-group, $e^3Z \cong \overline{Z}$ and so the final assertion follows from sequence 3.13 and Schanuel's Lemma. □

Corollary 3.13.1 $1 \leq q(\pi) \leq 2$.

Proof If M is a finite PD_4-complex with $\pi_1(M) \cong \pi$ then the covering space associated to the kernel of $w_1(M) - w_1(\pi)$ satisfies the condition on w_1. Since the condition $\chi(M) > 0$ is invariant under passage to finite covers, $q(\pi) \geq 1$.

Let N be a PD_3-complex with fundamental group π. We may suppose that $N = N_o \cup D^3$, where $N_o \cap D^3 = S^2$. Let $M = N_o \times S^1 \cup S^2 \times D^2$. Then M is a finite PD_4-complex, $\chi(M) = 2$ and $\pi_1(M) \cong \pi$. Hence $q(\pi) \leq 2$. □

Can Theorem 3.13 be extended to all torsion free 3-manifold groups, or more generally to all free products of PD_3-groups?

A simple application of Schanuel's Lemma to $C_*(\tilde{M})$ shows that if M is a finite PD_4-complex with fundamental group π such that $c.d.\pi \leq 4$ and $e(\pi) = 1$ then $\pi_2(M)$ has projective dimension at most 2. If moreover π is an FF PD_4-group and c_M has degree 1 then $\pi_2(M)$ is stably free of rank $\chi(M) - \chi(\pi)$, by the argument of Lemma 3.1 and Theorem 3.2.

There has been some related work estimating the difference $\chi(M) - |\sigma(M)|$ where M is a closed orientable 4-manifold M with $\pi_1(M) \cong \pi$ and where $\sigma(M)$ is the signature of M. In particular, this difference is always ≥ 0 if $\beta_1^{(2)}(\pi) = 0$. (See [JK93] and §3 of Chapter 7 of [Lü].) The minimum value of this difference ($p(\pi) = \min\{\chi(M) - |\sigma(M)|\}$) is another numerical invariant of π, which is studied in [Ko94].

3.4 Euler Characteristic 0

In this section we shall consider the interaction of the fundamental group and Euler characteristic from another point of view. We shall assume that $\chi(M) = 0$ and show that if π is an ascending HNN extension then it satisfies some very stringent conditions. The groups $Z*_m$ shall play an important role. We shall approach our main result via several lemmas.

We begin with a simple observation relating Euler characteristic and fundamental group which shall be invoked in several of the later chapters. Recall that if G is a group then $I(G)$ is the minimal normal subgroup such that $G/I(G)$ is free abelian.

Lemma 3.14 *Let M be a PD_4-complex with $\chi(M) \leq 0$. If M is orientable then $H^1(M;\mathbb{Z}) \neq 0$ and so $\pi = \pi_1(M)$ maps onto Z. If $H^1(M;\mathbb{Z}) = 0$ then π maps onto D.*

Proof The covering space M_W corresponding to $W = \mathrm{Ker}(w_1(M))$ is orientable and $\chi(M_W) = 2 - 2\beta_1(M_W) + \beta_2(M_W) = [\pi : W]\chi(M) \leq 0$. Therefore $\beta_1(W) = \beta_1(M_W) > 0$ and so $W/I(W) \cong Z^r$ for some $r > 0$. Since $I(W)$ is characteristic in W it is normal in π. As $[\pi : W] \leq 2$ it follows easily that $\pi/I(W)$ maps onto Z or D. □

Note that if $M = RP^4\sharp RP^4$, then $\chi(M) = 0$ and $\pi_1(M) \cong D$, but $\pi_1(M)$ does not map onto Z.

Lemma 3.15 *Let M be a PD_4^+-complex such that $\chi(M) = 0$ and $\pi = \pi_1(M)$ is an extension of $Z*_m$ by a finite normal subgroup F, for some $m \neq 0$. Then the abelian subgroups of F are cyclic. If $F \neq 1$ then π has a subgroup of finite index which is a central extension of $Z*_n$ by a nontrivial finite cyclic group, where n is a power of m.*

Proof Let $\widehat{M}$ be the infinite cyclic covering space corresponding to the subgroup $I(\pi)$. Since M is compact and $\Lambda = \mathbb{Z}[Z]$ is noetherian the groups $H_i(\widehat{M};\mathbb{Z}) = H_i(M;\Lambda)$ are finitely generated as Λ-modules. Since M is orientable, $\chi(M) = 0$ and $H_1(M;\mathbb{Z})$ has rank 1 they are Λ-torsion modules, by the Wang sequence for the projection of $\widehat{M}$ onto M. Now $H_2(\widehat{M};\mathbb{Z}) \cong \overline{Ext^1_\Lambda(I(\pi)/I(\pi)',\Lambda)}$, by Poincaré duality. There is an exact sequence

$$0 \to T \to I(\pi)/I(\pi)' \to I(Z*_m) \cong \Lambda/(t-m) \to 0,$$

where T is a finite Λ-module. Therefore $Ext^1_\Lambda(I(\pi)/I(\pi)',\Lambda) \cong \Lambda/(t-m)$ and so $H_2(I(\pi);\mathbb{Z})$ is a quotient of $\Lambda/(mt-1)$, which is isomorphic to $Z[\frac{1}{m}]$ as an abelian group. Now $I(\pi)/\mathrm{Ker}(f) \cong Z[\frac{1}{m}]$ also, and $H_2(Z[\frac{1}{m}];\mathbb{Z}) \cong Z[\frac{1}{m}] \wedge Z[\frac{1}{m}] = 0$ (see page 334 of [Ro]). Hence $H_2(I(\pi);\mathbb{Z})$ is finite, by an LHSSS argument, and so is cyclic, of order relatively prime to m.

Let t in π generate $\pi/I(\pi) \cong Z$. Let A be a maximal abelian subgroup of F and let $C = C_\pi(A)$. Then $q = [\pi : C]$ is finite, since F is finite and normal in π. In particular, t^q is in C and C maps onto Z, with kernel J, say. Since J is an extension of $Z[\frac{1}{m}]$ by a finite normal subgroup its centre ζJ has finite index in J. Therefore the subgroup G generated by ζJ and t^q has finite index in π, and there is an epimorphism f from G onto $Z*_{m^q}$, with kernel A. Moreover $I(G) = f^{-1}(I(Z*_{m^q}))$ is abelian, and is an extension of $Z[\frac{1}{m}]$ by the finite abelian group A. Hence it is isomorphic to $A \oplus Z[\frac{1}{m}]$ (see page 106 of [Ro]). Now $H_2(I(G);\mathbb{Z})$ is cyclic of order prime to m. On the other hand $H_2(I(G);\mathbb{Z}) \cong (A \wedge A) \oplus (A \otimes Z[\frac{1}{m}])$ and so A must be cyclic.

If $F \neq 1$ then A is cyclic, nontrivial, central in G and $G/A \cong Z*_{m^q}$. □

Lemma 3.16 *Let M be a finite PD_4-complex with fundamental group π. Suppose that π has a nontrivial finite cyclic central subgroup F with quotient $G=\pi/F$ such that $g.d.G=2$, $e(G)=1$ and $\mathrm{def}(G)=1$. Then $\chi(M)\geq 0$. If $\chi(M)=0$ and $\mathbb{F}_p[G]$ is a weakly finite ring for some prime p dividing $|F|$ then π is virtually Z^2.*

Proof Let $\widehat{M}$ be the covering space of M with group F, and let $\Xi=\mathbb{F}_p[G]$. Let $C_*=C_*(M;\Xi)=\mathbb{F}_p\otimes C_*(M)$ be the equivariant cellular chain complex of $\widehat{M}$ with coefficients $\mathbb{F}_p$, and let c_q be the number of q-cells of M, for $q\geq 0$. Let $H_p=H_p(M;\Xi)=H_p(\widehat{M};\mathbb{F}_p)$. For any left Ξ-module H let $e^qH=Ext^q_\Xi(H,\Xi)$.

Suppose first that M is orientable. Since $\widehat{M}$ is a connected open 4-manifold $H_0=\mathbb{F}_p$ and $H_4=0$, while $H_1\cong\mathbb{F}_p$ also. Since G has one end Poincaré duality and the UCSS give $H_3=0$ and $e^2H_2\cong\mathbb{F}_p$, and an exact sequence

$$0\to e^2\mathbb{F}_p\to\overline{H_2}\to e^0H_2\to e^2H_1\to\overline{H_1}\to e^1H_2\to 0.$$

In particular, $e^1H_2\cong\mathbb{F}_p$ or is 0. Since $g.d.G=2$ and $\mathrm{def}(G)=1$ the augmentation module has a resolution

$$0\to\Xi^r\to\Xi^{r+1}\to\Xi\to\mathbb{F}_p\to 0.$$

The chain complex C_* gives four exact sequences

$$\begin{aligned}
&0\to Z_1\to C_1\to C_0\to\mathbb{F}_p\to 0,\\
&0\to Z_2\to C_2\to Z_1\to\mathbb{F}_p\to 0,\\
&0\to B_2\to Z_2\to H_2\to 0\\
\text{and}\qquad &0\to C_4\to C_3\to B_2\to 0.
\end{aligned}$$

Using Schanuel's Lemma several times we find that the cycle submodules Z_1 and Z_2 are stably free, of stable ranks c_1-c_0 and $c_2-c_1+c_0$, respectively. Dualizing the last two sequences gives two new sequences

$$\begin{aligned}
&0\to e^0B_2\to e^0C_3\to e^0C_4\to e^1B_2\to 0\\
\text{and}\qquad &0\to e^0H_2\to e^0Z_2\to e^0B_2\to e^1H_2\to 0,
\end{aligned}$$

and an isomorphism $e^1B_2\cong e^2H_2\cong\mathbb{F}_p$. Further applications of Schanuel's Lemma show that e^0B_2 is stably free of rank c_3-c_4, and hence that e^0H_2 is stably free of rank $c_2-c_1+c_0-(c_3-c_4)=\chi(M)$. (Note that we do not need to know whether $e^1H_2\cong\mathbb{F}_p$ or is 0, at this point.) Since Ξ maps onto the field $\mathbb{F}_p$ the rank must be non-negative, and so $\chi(M)\geq 0$.

If $\chi(M) = 0$ and $\Xi = \mathbb{F}_p[G]$ is a weakly finite ring then $e^0 H_2 = 0$ and so $e^2\mathbb{F}_p = e^2 H_1$ is a submodule of $\mathbb{F}_p \cong \overline{H_1}$. Moreover it cannot be 0, for otherwise the UCSS would give $H_2 = 0$ and then $H_1 = 0$, which is impossible. Therefore $e^2\mathbb{F}_p \cong \mathbb{F}_p$.

If M is nonorientable and $p > 2$ the above argument applies to the orientation cover, since p divides $|\mathrm{Ker}(w_1(M)|_F)|$, and Euler characteristic is multiplicative in finite covers. If $p = 2$ a similar argument applies directly without assuming that M is orientable.

Since G is torsion free and indicable it must be a PD_2-group, by Theorem V.12.2 of [DD]. Since $\mathrm{def}(G) = 1$ it follows that G is virtually Z^2, and hence that π is also virtually Z^2. □

We may now give the main result of this section.

Theorem 3.17 *Let M be a finite PD_4-complex whose fundamental group π is an ascending HNN extension with finitely generated base B. Then $\chi(M) \geq 0$, and hence $q(\pi) \geq 0$. If $\chi(M) = 0$ and B is FP_2 and finitely ended then either π has two ends or has a subgroup of finite index which is isomorphic to Z^2 or $\pi \cong Z*_m$ or $Z *_m \tilde{\times}(Z/2Z)$ for some $m \neq 0$ or ± 1 or M is aspherical.*

Proof The L^2 Euler characteristic formula gives $\chi(M) = \beta_2^{(2)}(M) \geq 0$, since $\beta_i^{(2)}(M) = \beta_i^{(2)}(\pi) = 0$ for $i = 0$ or 1, by Lemma 2.1.

Let $\phi : B \to B$ be the monomorphism determining $\pi \cong B*_\phi$. If B is finite then ϕ is an automorphism and so π has two ends. If B is FP_2 and has one end then $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$, by the Brown-Geoghegan Theorem. If moreover $\chi(M) = 0$ then M is aspherical, by Corollary 3.5.1.

If B has two ends then it is an extension of Z or D by a finite normal subgroup F. As ϕ must map F isomorphically to itself, F is normal in π, and is the maximal finite normal subgroup of π. Moreover $\pi/F \cong Z*_m$, for some $m \neq 0$, if $B/F \cong Z$, and is a semidirect product $Z *_m \tilde{\times}(Z/2Z)$, with a presentation $\langle a, t, u \mid tat^{-1} = a^m,\ tut^{-1} = ua^r,\ u^2 = 1,\ uau = a^{-1}\rangle$, for some $m \neq 0$ and some $r \in Z$, if $B/F \cong D$. (On replacing t by $a^{[r/2]}t$, if necessary, we may assume that $r = 0$ or 1.)

Suppose first that M is orientable, and that $F \neq 1$. Then π has a subgroup σ of finite index which is a central extension of $Z*_{m^q}$ by a finite cyclic group, for some $q \geq 1$, by Lemma 3.15. Let p be a prime dividing q. Since $Z*_{m^q}$ is a torsion free solvable group the ring $\Xi = \mathbb{F}_p[Z*_{m^q}]$ has a skew field of fractions

L, which as a right Ξ-module is the direct limit of the system $\{\Xi_\theta \mid 0 \neq \theta \in \Xi\}$, where each $\Xi_\theta = \Xi$, the index set is ordered by right divisibility ($\theta \leq \phi\theta$) and the map from Ξ_θ to $\Xi_{\phi\theta}$ sends ξ to $\phi\xi$ [KLM88]. In particular, Ξ is a weakly finite ring and so σ is torsion free, by Lemma 3.16. Therefore $F = 1$.

If M is nonorientable then $w_1(M)|_F$ must be injective, and so another application of Lemma 3.16 (with $p = 2$) shows again that $F = 1$. □

Is M still aspherical if B is assumed only finitely generated and one ended?

Corollary 3.17.1 *Let M be a finite PD_4-complex such that $\chi(M) = 0$ and $\pi = \pi_1(M)$ is almost coherent and restrained. Then either π has two ends or is virtually Z^2 or $\pi \cong Z*_m$ or $Z*_m \tilde{\times}(Z/2Z)$ for some $m \neq 0$ or ± 1 or M is aspherical.*

Proof Let $\pi^+ = \mathrm{Ker}(w_1(M))$. Then π^+ maps onto Z, by Lemma 3.14, and so is an ascending HNN extension $\pi^+ \cong B*_\phi$ with finitely generated base B. Since π is almost coherent B is FP_2, and since π has no nonabelian free subgroup B has at most two ends. Hence Lemma 3.16 and Theorem 3.17 apply, so either π has two ends or M is aspherical or $\pi^+ \cong Z*_m$ or $Z *_m \tilde{\times}(Z/2Z)$ for some $m \neq 0$ or ± 1. In the latter case $\sqrt{\pi}$ is isomorphic to a subgroup of the additive rationals Q, and $\sqrt{\pi} = C_\pi(\sqrt{\pi})$. Hence the image of π in $Aut(\sqrt{\pi}) \leq Q^\times$ is infinite. Therefore π maps onto Z and so is an ascending HNN extension $B*_\phi$, and we may again use Theorem 3.17. □

Does this corollary remain true without the hypothesis that π be almost coherent?

There are nine groups which are virtually Z^2 and are fundamental groups of PD_4-complexes with Euler characteristic 0. (See Chapter 11.) Are any of the semidirect products $Z *_m \tilde{\times}(Z/2Z)$ realized by PD_4-complexes with $\chi = 0$? If π is restrained and M is aspherical must π be virtually poly-Z? (Aspherical 4-manifolds with virtually poly-Z fundamental groups are characterized in Chapter 8.)

Let G is a group with a presentation of deficiency d and $w : G \to \{\pm 1\}$ be a homomorphism, and let $\langle x_i,\ 1 \leq i \leq m \mid r_j,\ 1 \leq j \leq n\rangle$ be a presentation for G with $m - n = d$. We may assume that $w(x_i) = +1$ for $i \leq m-1$. Let $X = \natural^m(S^1 \times D^3)$ if $w = 1$ and $X = (\natural^{m-1}(S^1 \times D^3))\natural(S^1 \tilde{\times} D^3)$ otherwise. The relators r_j may be represented by disjoint orientation preserving embeddings of S^1 in ∂X, and so we may attach 2-handles along product neighbourhoods, to get a bounded 4-manifold Y with $\pi_1(Y) = G$, $w_1(Y) = w$ and $\chi(Y) = 1 - d$. Doubling Y gives a closed 4-manifold M with $\chi(M) = 2(1-d)$ and $(\pi_1(M), w_1(M))$ isomorphic to (G, w).

Since the groups $Z{*_m}$ have deficiency 1 it follows that any homomorphism $w : Z{*_m} \to \{\pm 1\}$ may be realized as the orientation character of a closed 4-manifold with fundamental group $Z{*_m}$ and Euler characteristic 0. What other invariants are needed to determine the homotopy type of such a manifold?

Chapter 4

Mapping tori and circle bundles

Stallings showed that if M is a 3-manifold and $f : M \to S^1$ a map which induces an epimorphism $f_* : \pi_1(M) \to Z$ with infinite kernel K then f is homotopic to a bundle projection if and only if M is irreducible and K is finitely generated. Farrell gave an analogous characterization in dimensions ≥ 6, with the hypotheses that the homotopy fibre of f is finitely dominated and a torsion invariant $\tau(f) \in Wh(\pi_1(M))$ is 0. The corresponding results in dimensions 4 and 5 are constrained by the present limitations of geometric topology in these dimensions. (In fact there are counter-examples to the most natural 4-dimensional analogue of Farrell's theorem [We87].)

Quinn showed that the total space of a fibration with finitely dominated base and fibre is a Poincaré duality complex if and only if both the base and fibre are Poincaré duality complexes. (See [Go79] for a very elegant proof of this result.) The main result of this chapter is a 4-dimensional homotopy fibration theorem with hypotheses similar to those of Stallings and a conclusion similar to that of Quinn and Gottlieb.

The *mapping torus* of a self homotopy equivalence $f : X \to X$ is the space $M(f) = X \times [0,1]/\sim$, where $(x,0) \sim (f(x),1)$ for all $x \in X$. If X is finitely dominated then $\pi_1(M(f))$ is an extension of Z by a finitely presentable normal subgroup and $\chi(M(f)) = \chi(X)\chi(S^1) = 0$. We shall show that a finite PD_4-complex M is homotopy equivalent to such a mapping torus, with X a PD_3-complex, if and only if $\pi_1(M)$ is such an extension and $\chi(M) = 0$.

In the final section we consider instead bundles with fibre S^1. We give conditions for a 4-manifold to be homotopy equivalent to the total space of an S^1-bundle over a PD_3-complex, and show that these conditions are sufficient if the fundamental group of the PD_3-complex is torsion free but not free.

4.1 Some necessary conditions

Let E be a connected cell complex and let $f : E \to S^1$ be a map which induces an epimorphism $f_* : \pi_1(E) \to Z$, with kernel ν. The associated covering space with group ν is $E_\nu = E \times_{S^1} R = \{(x,y) \in E \times R \mid f(x) = e^{2\pi i y}\}$, and

$E \simeq M(\phi)$, where $\phi : E_\nu \to E_\nu$ is the generator of the covering group given by $\phi(x,y) = (x, y+1)$ for all (x,y) in E_ν. If E is a PD_4-complex and E_ν is finitely dominated then E_ν is a PD_3-complex, by Quinn's result. In particular, ν is FP_2 and $\chi(E) = 0$. The latter conditions characterize aspherical mapping tori, by the following theorem.

Theorem 4.1 *Let M be a finite PD_4-complex whose fundamental group π is an extension of Z by a finitely generated normal subgroup ν, and let M_ν be the infinite cyclic covering space corresponding to the subgroup ν. Then*

(1) *$\chi(M) \geq 0$, with equality if and only if $H_2(M_\nu;\mathbb{Q})$ is finitely generated;*

(2) *if $\chi(M) = 0$ then M is aspherical if and only if ν is infinite and $H^2(\pi;\mathbb{Z}[\pi]) = 0$;*

(3) *M_ν is an aspherical PD_3-complex if and only if $\chi(M) = 0$ and ν is almost finitely presentable and has one end.*

Proof Since M is a finite complex and $\mathbb{Q}\Lambda = \mathbb{Q}[t,t^{-1}]$ is noetherian the homology groups $H_q(M_\nu;\mathbb{Q})$ are finitely generated as $\mathbb{Q}\Lambda$-modules. Since ν is finitely generated they are finite dimensional as $\mathbb{Q}$-vector spaces if $q < 2$, and hence also if $q > 2$, by Poincaré duality. Now $H_2(M_\nu;\mathbb{Q}) \cong \mathbb{Q}^r \oplus (\mathbb{Q}\Lambda)^s$ for some $r, s \geq 0$, by the Structure Theorem for modules over a PID. It follows easily from the Wang sequence for the covering projection from M_ν to M, that $\chi(M) = s \geq 0$.

Since ν is finitely generated $\beta_1^{(2)}(\pi) = 0$, by Lemma 2.1. If M is aspherical then clearly ν is infinite and $H^2(\pi;\mathbb{Z}[\pi]) = 0$. Conversely, if these conditions hold then $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$. Hence if moreover $\chi(M) = 0$ then M is aspherical, by Corollary 3.5.2.

If ν is FP_2 and has one end then $H^2(\pi;\mathbb{Z}[\pi]) \cong H^1(\nu;\mathbb{Z}[\nu]) = 0$, by the LHSSS. As M is aspherical ν is a PD_3-group, by Theorem 1.20, and therefore is finitely presentable, by Theorem 1.1 of [KK99]. Hence $M_\nu \simeq K(\nu,1)$ is finitely dominated and so is a PD_3-complex [Br72]. □

In particular, if $\chi(M) = 0$ then $q(\pi) = 0$. This observation and the bound $\chi(M) \geq 0$ were given in Theorem 3.17. (They also follow on counting bases for the cellular chain complex of M_ν and extending coefficients to $\mathbb{Q}(t)$.)

Let F be the orientable surface of genus 2. Then $M = F \times F$ is an aspherical closed 4-manifold, and $\pi \cong G \times G$ where $G = \pi_1(F)$ has a presentation $\langle a_1, a_2, b_1, b_2 \mid [a_1,b_1] = [a_2,b_2]\rangle$. The subgroup $\nu \leq \pi$ generated by the images of (a_1, a_1) and the six elements $(x, 1)$ and $(1, x)$, for $x = a_2$, b_1 or b_2, is normal in π and $\pi/\nu \cong Z$. However ν cannot be FP_2 since $\chi(\pi) = 4 \neq 0$. Is there an aspherical 4-manifold M such that $\pi_1(M)$ is an extension of Z by a finitely generated subgroup ν which is not FP_2 and with $\chi(M) = 0$? (Note that $H_2(\nu; \mathbb{Q})$ must be finitely generated, so showing that ν is not finitely related may require some finesse.)

If $H^2(\pi; \mathbb{Z}[\pi]) = 0$ then $H^1(\nu; \mathbb{Z}[\nu]) = 0$, by an LHSSS argument, and so ν must have one end, if it is infinite. Can the hypotheses of (2) above be replaced by "$\chi(M) = 0$ and ν has one end"? It can be shown that the finitely generated subgroup N of $F(2) \times F(2)$ defined after Theorem 2.4 has one end. However $H^2(F(2) \times F(2); \mathbb{Z}[F(2) \times F(2)]) \neq 0$. (Note that $q(F(2) \times F(2)) = 2$, by Corollary 3.12.2.)

4.2 Change of rings and cup products

In the next two sections we shall adapt and extend work of Barge in setting up duality maps in the equivariant (co)homology of covering spaces.

Let π be an extension of Z by a normal subgroup ν and fix an element t of π whose image generates π/ν. Let $\alpha : \nu \to \nu$ be the automorphism determined by $\alpha(h) = tht^{-1}$ for all h in ν. This automorphism extends to a ring automorphism (also denoted by α) of the group ring $\mathbb{Z}[\nu]$, and the ring $\mathbb{Z}[\pi]$ may then be viewed as a twisted Laurent extension, $\mathbb{Z}[\pi] = \mathbb{Z}[\nu]_\alpha[t, t^{-1}]$. The quotient of $\mathbb{Z}[\pi]$ by the two-sided ideal generated by $\{h-1 | h \in \nu\}$ is isomorphic to Λ, while as a left module over itself $\mathbb{Z}[\nu]$ is isomorphic to $\mathbb{Z}[\pi]/\mathbb{Z}[\pi](t-1)$ and so may be viewed as a left $\mathbb{Z}[\pi]$-module. (Note that α is not a module automorphism unless t is central.)

If M is a left $\mathbb{Z}[\pi]$-module let $M|_\nu$ denote the underlying $\mathbb{Z}[\nu]$-module, and let $\hat{M} = Hom_{\mathbb{Z}[\nu]}(M|_\nu, \mathbb{Z}[\nu])$. Then $\hat{M}$ is a right $\mathbb{Z}[\nu]$-module via

$$(f\xi)(m) = f(m)\xi \text{ for all } \xi \in \mathbb{Z}[\nu],\ f \in \hat{M} \text{ and } m \in M.$$

If $M = \mathbb{Z}[\pi]$ then $\widehat{\mathbb{Z}[\pi]}$ is also a left $\mathbb{Z}[\pi]$-module via

$$(\phi t^r f)(\xi t^s) = \xi\alpha^{-s}(\phi) f(t^{s-r}) \text{ for all } f \in \widehat{\mathbb{Z}[\pi]},\ \phi, \xi \in \nu \text{ and } r, s \in Z.$$

As the left and right actions commute $\widehat{\mathbb{Z}[\pi]}$ is a $(\mathbb{Z}[\pi], \mathbb{Z}[\nu])$-bimodule. We may describe this bimodule more explicitly. Let $\mathbb{Z}[\nu][[t, t^{-1}]]$ be the set of doubly infinite power series $\Sigma_{n\in Z} t^n \phi_n$ with ϕ_n in $\mathbb{Z}[\nu]$ for all n in Z, with the obvious right $\mathbb{Z}[\nu]$-module structure, and with the left $\mathbb{Z}[\pi]$-module structure given by

$$\phi t^r(\Sigma t^n \phi_n) = \Sigma t^{n+r} \alpha^{-n-r}(\phi)\phi_n \text{ for all } \phi, \phi_n \in \mathbb{Z}[\nu] \text{ and } r \in Z.$$

(Note that even if $\nu = 1$ this module is not a ring in any natural way.) Then the homomorphism $j : \widehat{\mathbb{Z}[\pi]} \to \mathbb{Z}[\nu][[t, t^{-1}]]$ given by $j(f) = \Sigma t^n f(t^n)$ for all f in $\widehat{\mathbb{Z}[\pi]}$ is a $(\mathbb{Z}[\pi], \mathbb{Z}[\nu])$-bimodule isomorphism. (Indeed, it is clearly an isomorphism of right $\mathbb{Z}[\nu]$-modules, and we have defined the left $\mathbb{Z}[\pi]$-module structure on $\widehat{\mathbb{Z}[\pi]}$ by pulling back the one on $\mathbb{Z}[\nu][[t, t^{-1}]]$.)

For each f in $\hat{M}$ we may define a function $T_M f : M \to \widehat{\mathbb{Z}[\pi]}$ by the rule

$$(T_M f)(m)(t^n) = f(t^{-n} m) \text{ for all } m \in M \text{ and } n \in Z.$$

It is easily seen that $T_M f$ is $\mathbb{Z}[\pi]$-linear, and that $T_M : \hat{M} \to Hom_{\mathbb{Z}[\pi]}(M, \widehat{\mathbb{Z}[\pi]})$ is an isomorphism of abelian groups. (It is clearly a monomorphism, and if $g : M \to \widehat{\mathbb{Z}[\pi]}$ is $\mathbb{Z}[\pi]$-linear then $g = T_M f$ where $f(m) = g(m)(1)$ for all m in M. In fact if we give $Hom_{\mathbb{Z}[\pi]}(M, \widehat{\mathbb{Z}[\pi]})$ the natural right $\mathbb{Z}[\nu]$-module structure by $(\mu\phi)(m) = \mu(m)\phi$ for all $\phi \in \mathbb{Z}[\nu]$, $\mathbb{Z}[\pi]$-homomorphisms $\mu : M \to \widehat{\mathbb{Z}[\pi]}$ and $m \in M$ then T_M is an isomorphism of right $\mathbb{Z}[\nu]$-modules.) Thus we have a natural equivalence $T : Hom_{\mathbb{Z}[\nu]}(-|_\nu, \mathbb{Z}[\nu]) \Rightarrow Hom_{\mathbb{Z}[\pi]}(-, \widehat{\mathbb{Z}[\pi]})$ of functors from $\mathbf{Mod}_{\mathbb{Z}[\pi]}$ to $\mathbf{Mod}_{\mathbb{Z}[\nu]}$. If C_* is a chain complex of left $\mathbb{Z}[\pi]$-modules T induces natural isomorphisms from $H^*(C_*|_\nu; \mathbb{Z}[\nu]) = H^*(Hom_{\mathbb{Z}[\nu]}(C_*|_\nu, \mathbb{Z}[\nu])$ to $H^*(C_*; v) = H^*(Hom_{\mathbb{Z}[\pi]}(C_*, \widehat{\mathbb{Z}[\pi]}))$. In particular, since the forgetful functor $-|_\nu$ is exact and takes projectives to projectives there are isomorphisms from $Ext^*_{\mathbb{Z}[\nu]}(M|_\nu, \mathbb{Z}[\nu])$ to $Ext^*_{\mathbb{Z}[\pi]}(M, \widehat{\mathbb{Z}[\pi]})$ which are functorial in M.

If M and N are left $\mathbb{Z}[\pi]$-modules let $M \otimes N$ denote the tensor product over $\mathbb{Z}$ with the diagonal left π-action, defined by $g(m \otimes n) = gm \otimes gn$ for all $m \in M$, $n \in N$ and $g \in \pi$. The function $p_M : \Lambda \otimes M \to M$ defined by $p_M(\lambda \otimes m) = \lambda(1)m$ is then a $\mathbb{Z}[\pi]$-linear epimorphism.

We shall define products in cohomology by means of the $\mathbb{Z}[\pi]$-linear homomorphism $e : \Lambda \otimes \widehat{\mathbb{Z}[\pi]} \to \mathbb{Z}[\pi]$ given by

$$e(t^n \otimes f) = t^n f(t^n) \text{ for all } f \in \widehat{\mathbb{Z}[\pi]} \text{ and } n \in Z.$$

Let A_* be a Λ-chain complex and B_* a $\mathbb{Z}[\pi]$-chain complex and give the tensor product the total grading $A_* \otimes B_*$ and differential and the diagonal π-action. Let $e_\sharp$ be the change of coefficients homomorphism induced by e, and let $u \in H^p(A_*; \Lambda)$ and $v \in H^q(B_*; \widehat{\mathbb{Z}[\pi]})$. Then $u \otimes v \mapsto e_\sharp(u \times v)$ defines a pairing from $H^p(A_*; \Lambda) \otimes H^q(B_*; \widehat{\mathbb{Z}[\pi]})$ to $H^{p+q}(A_* \otimes B_*; \mathbb{Z}[\pi])$.

Now let A_* be the Λ-chain complex concentrated in degrees 0 and 1 with A_0 and A_1 free of rank 1, with bases $\{a_0\}$ and $\{a_1\}$, respectively, and with $\partial_1 : A_1 \to A_0$ given by $\partial_1(a_1) = (t-1)a_0$. Let $\eta_A : A_1 \to \Lambda$ be the isomorphism determined by $\eta_A(a_1) = 1$, and let $\alpha_A : A_0 \to \mathbb{Z}$ be the augmentation determined by $\alpha_A(a_0) = 1$. Then $[\eta_A]$ generates $H^1(A_*;\Lambda)$. Let B_* be a projective $\mathbb{Z}[\pi]$-chain complex and let $p_{B*} : A_* \otimes B_* \to B_*$ be the chain homotopy equivalence defined by $p_{Bj}((\lambda a_0) \otimes b_j) = \lambda(1)b_j$ and $p_{Bj}((\lambda a_1) \otimes b_{j-1}) = 0$, for all $\lambda \in \Lambda$, $b_{j-1} \in B_{j-1}$ and $b_j \in B_j$. Let $j_{B*} : B_* \to A_* \otimes B_*$ be a chain homotopy inverse to p_{B*}. Define a family of homomorphisms $h_{\mathbb{Z}[\pi]}$ from $H^q(B_*;\widehat{\mathbb{Z}[\pi]})$ to $H^{q+1}(B_*;\mathbb{Z}[\pi])$ by

$$h_{\mathbb{Z}[\pi]}([\phi]) = j_B^* e_\sharp([\eta_A] \times [\phi])$$

for $\phi : B_q \to \widehat{\mathbb{Z}[\pi]}$ such that $\phi\partial_{q+1} = 0$. Let $f : B_* \to B'_*$ be a chain homomorphism of projective $\mathbb{Z}[\pi]$-chain complexes. Then $h_{\mathbb{Z}[\pi]}([\phi f_q]) = f^* h_{\mathbb{Z}[\pi]}([\phi])$, and so these homomorphisms are functorial in B_*. In particular, if B_* is a projective resolution of the $\mathbb{Z}[\pi]$-module M we obtain homomorphisms $h_{\mathbb{Z}[\pi]} : Ext^q_{\mathbb{Z}[\pi]}(M,\widehat{\mathbb{Z}[\pi]}) \to Ext^{q+1}_{\mathbb{Z}[\pi]}(M,\mathbb{Z}[\pi])$ which are functorial in M.

Lemma 4.2 *Let M be a $\mathbb{Z}[\pi]$-module such that $M|_\nu$ is finitely generated as a $\mathbb{Z}[\nu]$-module. Then $h_{\mathbb{Z}[\pi]} : Hom_{\mathbb{Z}[\pi]}(M,\widehat{\mathbb{Z}[\pi]}) \to Ext^1_{\mathbb{Z}[\pi]}(M,\mathbb{Z}[\pi])$ is injective.*

Proof Let B_* be a projective resolution of the $\mathbb{Z}[\pi]$-module M and let $q : B_0 \to M$ be the defining epimorphism (so that $q\partial_1 = 0$). We may use composition with q to identify $Hom_{\mathbb{Z}[\pi]}(M,\widehat{\mathbb{Z}[\pi]})$ with the submodule of 0-cocycles in $Hom(B_*,\widehat{\mathbb{Z}[\pi]})$, and we set $h_{\mathbb{Z}[\pi]}(\phi) = h_{\mathbb{Z}[\pi]}([\phi q])$ for all $\phi : M \to \widehat{\mathbb{Z}[\pi]}$.

Suppose that $h_{\mathbb{Z}[\pi]}(\phi) = 0$ and let $g = \phi q : B_0 \to \widehat{\mathbb{Z}[\pi]}$. Then there is a $\mathbb{Z}[\pi]$-linear homomorphism $f : A_0 \otimes B_0 \to \mathbb{Z}[\pi]$ such that $e_\sharp([\eta_A] \times [g]) = \delta f$. We may write $g(b) = \Sigma t^n g_n(b) = \Sigma t^n g_0(t^{-n}b)$, where $g_0 : B_0 \to \mathbb{Z}[\nu]$ is $\mathbb{Z}[\nu]$-linear (and $g_0\partial_1 = 0$). We then have $g_0(b) = f((t-1)a_0 \otimes b)$ for all $b \in B_0$, while $f(1 \otimes \partial_1) = 0$. Let $k(b) = f(a_0 \otimes b)$ for $b \in B_0$. Then $k : B_0 \to \mathbb{Z}[\pi]$ is $\mathbb{Z}[\nu]$-linear, and $k\partial_1 = 0$, so k factors through M. In particular, $k(B_0)$ is finitely generated as a $\mathbb{Z}[\nu]$-submodule of $\mathbb{Z}[\pi]$. But as $\mathbb{Z}[\pi] = \bigoplus t^n\mathbb{Z}[\nu]$ and $g_0(b) = tk(t^{-1}b) - k(b)$ for all $b \in B_0$, this is only possible if $k = g_0 = 0$. Therefore $\phi = 0$ and so $h_{\mathbb{Z}[\pi]}$ is injective. □

Let B_* be a projective $\mathbb{Z}[\pi]$-chain complex such that $B_j = 0$ for $j < 0$ and $H_0(B_*) \cong \mathbb{Z}$. Then there is a $\mathbb{Z}[\pi]$-chain homomorphism $\epsilon_{B*} : B_* \to A_*$ which induces an isomorphism $H_0(B_*) \cong H_0(A_*)$, and $\alpha_B = \alpha_A\epsilon_{B_0} : B_0 \to \mathbb{Z}$ is a generator of $H^0(B_*;\mathbb{Z})$. Let $\eta_B = \eta_A\epsilon_{B_1} : B_1 \to \Lambda$. If moreover $H_1(B_*) = 0$ then $H^1(B_*;\Lambda) \cong Z$ and is generated by $[\eta_B] = \epsilon_B^*([\eta_A])$

4.3 The case $\nu = 1$

When $\nu = 1$ (so $\mathbb{Z}[\pi] = \Lambda$) we shall show that h_Λ is an equivalence, and relate it to other more explicit homomorphisms. Let S be the multiplicative system in Λ consisting of monic polynomials with constant term ± 1. Let $Lexp(f,a)$ be the Laurent expansion of the rational function f about a. Then $\ell(f) = Lexp(f,\infty) - Lexp(f,0)$ defines a homomorphism from the localization Λ_S to $\widehat{\Lambda} = \mathbb{Z}[[t,t^{-1}]]$, with kernel Λ. (Barge used a similar homomorphism to embed $\mathbb{Q}(t)/\Lambda$ in $\mathbb{Q}[[t,t^{-1}]]$ [Ba 80].) Let $\chi : \widehat{\Lambda} \to \mathbb{Z}$ be the additive homomorphism defined by $\chi(\Sigma t^n f_n) = f_0$. (This is a version of the "trace" function used by Trotter to relate Seifert forms and Blanchfield pairings on a knot module M [Tr78].)

Let M be a Λ-module which is finitely generated as an abelian group, and let N be its maximal finite submodule. Then M/N is $\mathbb{Z}$-torsion free and $Ann_\Lambda(M/N) = (\lambda_M)$, where λ_M is the minimal polynomial of t, considered as an automorphism of $(M/N)|_\mathbb{Z}$. (See Chapter 3 of [H3].) Since $M|_\mathbb{Z}$ is finitely generated $\lambda_M \in S$. The inclusion of Λ_S/Λ in $\mathbb{Q}(t)/\Lambda$ induces an isomorphism $D(M) = Hom_\Lambda(M, \Lambda_S/\Lambda) \cong Hom_\Lambda(M, \mathbb{Q}(t)/\Lambda)$. We shall show that $D(M)$ is naturally isomorphic to each of $\hat{D}(M) = Hom_\Lambda(M, \hat{\Lambda})$, $E(M) = Ext^1_\Lambda(M,\Lambda)$ and $F(M) = Hom_\mathbb{Z}(M|_\mathbb{Z}, \mathbb{Z})$.

Let $\ell_M : D(M) \to \hat{D}(M)$ and $\chi_M : \hat{D}(M) \to F(M)$ be the homomorphisms defined by composition with ℓ and χ, respectively. It is easily verified that χ_M and T_M are mutually inverse.

Let B_* be a projective resolution of M. If $\phi \in D(M)$ let $\phi_0 : B_0 \to \mathbb{Q}(t)$ be a lift of ϕ. Then $\phi_0\partial_1$ has image in Λ, and so defines a homomorphism $\phi_1 : B_1 \to \Lambda$ such that $\phi_1\partial_2 = 0$. Consideration of the short exact sequence of complexes

$$0 \to Hom_\Lambda(B_*, \Lambda) \to Hom_\Lambda(B_*, \mathbb{Q}(t)) \to Hom_\Lambda(B_*, \mathbb{Q}(t)/\Lambda) \to 0$$

shows that $\delta_M(\phi) = [\phi_1]$, where $\delta_M : D(M) \to E(M)$ is the Bockstein homomorphism associated to the coefficient sequence. (The extension corresponding to $\delta_M\phi$ is the pullback over ϕ of the sequence $0 \to \Lambda \to \mathbb{Q}(t) \to \mathbb{Q}(t)/\Lambda \to 0$.)

Lemma 4.3 *The natural transformation h_Λ is an equivalence, and $h_\Lambda\ell_M = \delta_M$.*

Proof The homomorphism j_M sending the image of g in $\Lambda/(\lambda_M)$ to the class of $g(\lambda_M)^{-1}$ in Λ_S/Λ induces an isomorphism $Hom_\Lambda(M, \Lambda/(\lambda_M)) \cong D(M)$.

Hence we may assume that $M = \Lambda/(\lambda)$ and it shall suffice to check that $h_\Lambda\ell_M(j_M) = \delta(j_M)$. Moreover we may extend coefficients to $\mathbb{C}$, and so we may reduce to the case $\lambda = (t-\alpha)^n$.

We may assume that B_1 and B_0 are freely generated by b_1 and b_0, respectively, and that $\partial(b_1) = \lambda b_0$. The chain homotopy equivalence j_{B*} may be defined by $j_0(b_0) = a_0 \otimes b_0$ and $j_1(b_1) = a_0 \otimes b_1 + \Sigma\beta_{pq}(t^p a_1) \otimes (t^q b_0)$, where $\Sigma\beta_{pq}x^p y^q = (\lambda(xy) - \lambda(y))/(x-1) = y\Sigma_{0\le r<n}(xy-\alpha)^r(y-\alpha)^{n-r-1}$. (This formula arises naturally if we identify $\Lambda \otimes_{\mathbb{Z}} \Lambda$ with $\mathbb{Z}[x,y,x^{-1},y^{-1}]$, with $t \in \Lambda$ acting via xy.) Note that $\delta(j_M)(b_1) = \beta_{0n} = 1$ and $\beta_{pq} = 0$ unless $0 \le m < q \le n$.

Now $h_\Lambda\ell_M(j_M)(b_1) = e_\sharp(\eta_A \times \ell_M(j_M))(j_*(b_1)) = \Sigma\beta_{pq}t^p\psi_{p-q}$, where ψ_{-r} is the coefficient of t^{-r} in $Lexp(\lambda^{-1},\infty)$. Clearly $\psi_r = 0$ if $-n < r < 0$ and $\psi_{-n} = 1$, since $\lambda^{-1} = t^{-n}(1-\alpha t^{-1})^{-n}$. Hence $h_\Lambda\ell_M(j_M)(b_1) = \beta_{0n} = \delta(j_M)(b_1)$, and so $h_\Lambda\ell_M = \delta_M$, by linearity and functoriality.

Since δ is a natural equivalence and h_Λ is injective, by Lemma 4.2, h_Λ is also a natural equivalence. $\square$

It can be shown that the ring Λ_S defined above is a PID.

4.4 Duality in infinite cyclic covers

Let E, f and ν be as in §1, and suppose also that E is a PD_4-complex with $\chi(E) = 0$ and that ν is finitely generated and infinite. Let $C_* = C_*(\widetilde{E})$. Then $H_0(C_*) = Z$, $H_2(C_*) \cong \pi_2(E)$ and $H_q(C_*) = 0$ if $q \neq 0$ or 2, since $\widetilde{E}$ is simply connected and π has one end. Since $H_1(\bar\Lambda \otimes_{\mathbb{Z}[\pi]} C_*) = H_1(E_\nu;\mathbb{Z}) \cong \nu/\nu'$ is finitely generated as an abelian group, $Hom_{\mathbb{Z}[\pi]}(H_1(\bar\Lambda \otimes_{\mathbb{Z}[\pi]} C_*),\Lambda) = 0$. An elementary computation then shows that $H^1(C_*;\Lambda)$ is infinite cyclic, and generated by the class $\eta = \eta_C$ defined in §2. Let $[E]$ be a fixed generator of $H_4(\bar Z \otimes_{\mathbb{Z}[\pi]} C_*) \cong Z$, and let $[E_\nu] = \eta \cap [E]$ in $H_3(E_\nu;\mathbb{Z}) = H_3(\bar\Lambda \otimes_{\mathbb{Z}[\pi]} C_*) \cong Z$.

Since $\widetilde{E}$ is also the universal covering space of E_ν, the cellular chain complex for $\widetilde{E_\nu}$ is $C_*|_\nu$. In order to verify that E_ν is a PD_3-complex (with orientation class $[E_\nu]$) it shall suffice to show that (for each $p \ge 0$) the homomorphism η_p from $\overline{H^p(C_*;\mathbb{Z}[\nu])} = \overline{H^p(C_*;\widehat{\mathbb{Z}[\pi]})}$ to $\overline{H^{p+1}(C_*;\mathbb{Z}[\pi])}$ given by cup product with η is an isomorphism, by standard properties of cap and cup products. We may identify these cup products with the degree raising homomorphisms $h_{\mathbb{Z}[\pi]}$, by the following lemma.

Lemma 4.4 *Let X be a connected space with $\pi_1(X) \cong \pi$ and let $B_* = C_*(\tilde X)$. Then $h_{\mathbb{Z}[\pi]}([\phi]) = [\eta_B] \cup [\phi]$.*

Proof The Alexander Whitney diagonal approximation d_* from B_* to $B_*\otimes B_*$ is π-equivariant, if the tensor product is given the diagonal left π-action, and we may take $j_{B*}=(\epsilon_B\otimes 1)d_*$ as a chain homotopy inverse to p_{B*}. Therefore $h_{\mathbb{Z}[\pi]}([\phi])=d^*e_\sharp([\eta_B]\times[\phi])=[\eta_B]\cup[\phi]$. □

The cohomology modules $H^p(C_*;\mathbb{Z}[\nu])$ and $H^p(C_*;\mathbb{Z}[\pi])$ may be "computed" via the UCSS. Since cross product with a 1-cycle induces a degree 1 cochain homomorphism, the functorial homomorphisms $h_{\mathbb{Z}[\pi]}$ determine homomorphisms between these spectral sequences which are compatible with cup product with η on the limit terms. In each case the E_2^{p*} columns are nonzero only for $p=0$ or 2. The E_2^{0*} terms of these spectral sequences involve only the cohomology of the groups and the homomorphisms between them may be identified with the maps arising in the LHSSS for π as an extension of Z by ν, under appropriate finiteness hypotheses on ν.

4.5 Homotopy mapping tori

In this section we shall apply the above ideas to the non-aspherical case. We use coinduced modules to transfer arguments about subgroups and covering spaces to contexts where Poincaré duality applies, and L^2-cohomology to identify $\pi_2(M)$, together with the above strategy of describing Poincaré duality for an infinite cyclic covering space in terms of cup product with a generator η of $H^1(M;\Lambda)$.

Note that most of the homology and cohomology groups defined below do not have natural module structures, and so the Poincaré duality isomorphisms are isomorphisms of abelian groups only.

Theorem 4.5 *A finite PD_4-complex M with fundamental group π is homotopy equivalent to the mapping torus of a self homotopy equivalence of a PD_3-complex if and only if $\chi(M)=0$ and π is an extension of Z by a finitely presentable normal subgroup ν.*

Proof The conditions are clearly necessary, as observed in §1 above. Suppose conversely that they hold. Let M_ν be the infinite cyclic covering space of M with fundamental group ν, and let $\tau: M_\nu\to M_\nu$ be a covering transformation corresponding to a generator of $\pi/\nu\cong Z$. Then M is homotopy equivalent to the mapping torus $M(\tau)$. Moreover $H^1(M;\Lambda)\cong H^1(\pi;\Lambda)$ is infinite cyclic, since ν is finitely generated. Let $E^r_{p,q}(M_\nu)$ and $E^r_{p,q}(M)$ be the UCSS for the cohomology of M_ν with coefficients $\mathbb{Z}[\nu]$ and for that of M with coefficients

$\mathbb{Z}[\pi]$, respectively. A choice of generator η for $H^1(M;\Lambda)$ determines homomorphisms $h_{\mathbb{Z}[\pi]} : E^r_{p,q}(M_\nu) \to E^r_{p,q+1}(M)$, giving a homomorphism of bidegree $(0,1)$ between these spectral sequences corresponding to cup product with η on the abutments, by Lemma 4.4.

Suppose first that ν is finite. The UCSS and Poincaré duality then imply that $H_i(\widetilde{M};\mathbb{Z}) \cong Z$ for $i = 0$ or 3 and is 0 otherwise. Hence $\widetilde{M} \simeq S^3$ and so $M_\nu = \widetilde{M}/\nu$ is a Swan complex for ν. (See Chapter 11 for more details.) Thus we may assume henceforth that ν is infinite. We must show that the cup product maps $\eta_p : H^p(M_\nu;\mathbb{Z}[\nu]) \to H^{p+1}(M;\mathbb{Z}[\pi])$ are isomorphisms, for $0 \leq p \leq 4$. If $p = 0$ or 4 then all the groups are 0, and so η_0 and η_4 are isomorphisms.

Applying the isomorphisms defined in §8 of Chapter 1 to the cellular chain complex C_* of $\widetilde{M}$, we see that $H^q(M_\nu;A) \cong H^q(M;Hom_{\mathbb{Z}[\nu]}(\mathbb{Z}[\pi],A))$ is isomorphic to $\overline{H_{4-q}(M;Hom_{\mathbb{Z}[\nu]}(\mathbb{Z}[\pi],A))}$ for any local coefficient system (left $\mathbb{Z}[\nu]$-module) A on M_ν. Let $t \in \pi$ represent a generator of π/ν. Since multiplication by $t-1$ is surjective on $Hom_{\mathbb{Z}[\nu]}(\mathbb{Z}[\pi],A)$, the homology Wang sequence for the covering projection of M_ν onto M gives $H_0(M;Hom_{\mathbb{Z}[\nu]}(\mathbb{Z}[\pi],A)) = 0$. Hence $H^4(M_\nu;A) = 0$ for any local coefficient system A, and so M_ν is homotopy equivalent to a 3-dimensional complex (see [Wl65]). (See also [DST96].)

Since π is an extension of Z by a finitely generated normal subgroup $\beta_1^{(2)}(\pi) = 0$, and so $\overline{\pi_2(M)} \cong H^2(M;\mathbb{Z}[\pi]) \cong H^2(\pi;\mathbb{Z}[\pi])$, by Theorem 3.4. Hence η_1 may be identified with the isomorphism $H^1(\nu;\mathbb{Z}[\nu]) \cong H^2(\pi;\mathbb{Z}[\pi])$ coming from the LHSSS for the extension. Moreover $\overline{\pi_2(M)}|_\nu \cong H^1(\nu;\mathbb{Z}[\nu])$ is finitely generated over $\mathbb{Z}[\nu]$, and so $Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi]) = 0$. Therefore $H^3(\pi;\mathbb{Z}[\pi]) = 0$, by Lemma 3.3, and so the Wang sequence map $t-1 : H^2(\nu;\mathbb{Z}[\pi]) \to H^2(\nu;\mathbb{Z}[\pi])$ is onto. Since ν is FP_2 this cohomology group is isomorphic to $H^2(\nu;\mathbb{Z}[\nu]) \otimes_Z \mathbb{Z}[\pi/\nu]$, where $\mathbb{Z}[\pi/\nu] \cong \Lambda$ acts diagonally. It is easily seen that if $H^2(\nu;\mathbb{Z}[\nu])$ has a nonzero element h then $h \otimes 1$ is not divisible by $t-1$. Hence $H^2(\nu;\mathbb{Z}[\nu]) = 0$. The differential $d^3_{2,1}(M)$ is a monomorphism, since $H^3(M;\mathbb{Z}[\pi]) = 0$, and $h_{\mathbb{Z}[\pi]} : E^2_{2,0}(M_\nu) \to E^2_{2,1}(M)$ is a monomorphism by Lemma 4.2. Therefore $d^3_{2,0}(M_\nu)$ is also a monomorphism and so $H^2(M_\nu;\mathbb{Z}[\nu]) = 0$. Hence η_2 is an isomorphism.

It remains only to check that $H^3(M_\nu;\mathbb{Z}[\nu]) \cong Z$ and that η_3 is onto. Now $H^3(M_\nu;\mathbb{Z}[\nu]) \cong \overline{H_1(M;Hom_{\mathbb{Z}[\nu]}(\mathbb{Z}[\pi],\mathbb{Z}[\nu]))} = \overline{H_1(\pi;\mathbb{Z}[\nu]^{\pi/\nu})}$. (The exponent denotes direct product indexed by π/ν rather than fixed points!) The natural homomorphism from $H_1(\pi;\mathbb{Z}[\nu]^{\pi/\nu})$ to $H_1(\pi/\nu;H_0(\nu;\mathbb{Z}[\nu]^{\pi/\nu}))$ is onto, with kernel $H_0(\pi/\nu;H_1(\nu;\mathbb{Z}[\nu]^{\pi/\nu}))$, by the LHSSS for π. Since ν is finitely generated homology commutes with direct products in this range, and it follows that

$H_1(\pi;\mathbb{Z}[\nu]^{\pi/\nu})\cong H_1(\pi/\nu;\mathbb{Z}^{\pi/\nu})$. Since $\pi/\nu\cong Z$ and acts by translation on the index set this homology group is Z. The homomorphisms from $H^3(M_\nu;\mathbb{Z}[\nu])$ to $H^3(M_\nu;\mathbb{Z})$ and from $H^4(M;\mathbb{Z}[\pi])$ to $H^4(M;\Lambda)$ induced by the augmentation homomomorphism and the epimorphism from $\mathbb{Z}[\pi]$ to $\mathbb{Z}[\pi/\nu]\cong\Lambda$ are epimorphisms, since M_ν and M are homotopy equivalent to 3- and 4-dimensional complexes, respectively. hence they are isomorphisms, since these cohomology modules are infiite cyclic as abelian groups. These isomorphisms form the vertical sides of a commutative square

$$\begin{array}{ccc} H^3(M_\nu;\mathbb{Z}[\nu]) & \xrightarrow{\eta_3} & H^4(M;\mathbb{Z}[\pi]) \\ \varepsilon\big\downarrow & & \big\downarrow \\ H^3(M_\nu;\mathbb{Z}) & \xrightarrow{-\cup\eta} & H^4(M;\Lambda). \end{array}$$

The lower horizontal edge is an isomorphism, by Lemma 4.3. Therefore η_3 is also an isomorphism.

Thus M_ν satisfies Poincaré duality of formal dimension 3 with local coefficients. Since $\pi_1(M_\nu)=\nu$ is finitely presentable M_ν is finitely dominated, and so is a PD_3-complex [Br72]. □

Note that M_ν need not be homotopy equivalent to a finite complex. If M is a *simple* PD_4-complex and a generator of $Aut(M_\nu/M)\cong\pi/\nu$ has finite order in the group of self homotopy equivalences of M_ν then M is finitely covered by a simple PD_4-complex homotopy equivalent to $M_\nu\times S^1$. In this case M_ν must be homotopy finite by [Rn86]. The hypothesis that M be finite is used in the proof of Theorem 3.4, but is probably not necessary here.

The hypothesis that ν be almost finitely presentable (FP_2) suffices to show that M_ν satisfies Poincaré duality with local coefficients. Finite presentability is used only to show that M_ν is finitely dominated. (Does the coarse Alexander duality argument of [KK99] used in part (3) of Theorem 4.1 extend to the non-aspherical case?) In view of the fact that 3-manifold groups are coherent, we might hope that the condition on ν could be weakened still further to require only that it be finitely generated.

Some argument is needed above to show that η_2 is injective. If M_ν is homotopy equivalent to a 3-manifold with more than one aspherical summand then $H^1(\nu;\mathbb{Z}[\nu])$ is a nonzero free $\mathbb{Z}[\nu]$-module and so $Hom_{\mathbb{Z}[\nu]}(\Pi|_\nu,\mathbb{Z}[\nu])\neq 0$.

A rather different proof of this theorem could be given using Ranicki's criterion for an infinite cyclic cover to be finitely dominated [Rn95] and the Quinn-Gottlieb theorem, if finitely generated stably free modules of rank 0 over the

Novikov rings $A_\pm = \mathbb{Z}[\nu]_\alpha((t^{\pm 1}))$ are trivial. (For $H_q(A_\pm \otimes_\pi C_*) = A_\pm \otimes_\pi H_*(C_*) = 0$ if $q \neq 2$, since $t-1$ is invertible in $A_\pm$. Hence $H_2(A_\pm \otimes_\pi C_*)$ is a stably free module of rank 0, by Lemma 3.1.)

An alternative strategy would be to show that $Lim_I H^q(M_\nu; A_i) = 0$ for any direct system with limit 0. We could then conclude that the cellular chain complex of $\widetilde{M} = \widetilde{M_\nu}$ is chain homotopy equivalent to a finite complex of finitely generated projective $Z[\nu]$-modules, and hence that M_ν is finitely dominated. Since ν is FP_2 this strategy applies easily when $q = 0$, 1, 3 or 4, but something else is needed when $q = 2$.

Corollary 4.5.1 *Let M be a PD_4-complex with $\chi(M) = 0$ and whose fundamental group π is an extension of Z by a normal subgroup $\nu \cong F(r)$. Then M is homotopy equivalent to a closed PL 4-manifold which fibres over the circle, with fibre $\sharp^r S^1 \times S^2$ if $w_1(M)|_\nu$ is trivial, and $\sharp^r S^1 \tilde{\times} S^2$ otherwise. The bundle is determined by the homotopy type of M.*

Proof By the theorem M_ν is a PD_3-complex with free fundamental group, and so is homotopy equivalent to $N = \sharp^r S^1 \times S^2$ if $w_1(M)|_\nu$ is trivial and to $\sharp^r S^1 \tilde{\times} S^2$ otherwise. Every self homotopy equivalence of a connected sum of S^2-bundles over S^1 is homotopic to a self-homeomorphism, and homotopy implies isotopy for such manifolds [La]. Thus M is homotopy equivalent to such a fibred 4-manifold, and the bundle is determined by the homotopy type of M. □

It is easy to see that the natural map from $Homeo(N)$ to $Out(F(r))$ is onto. If a self homeomorphism f of $N = \sharp^r S^1 \times S^2$ induces the trivial outer automorphism of $F(r)$ then f is homotopic to a product of twists about nonseparating 2-spheres [He]. How is this manifest in the topology of the mapping torus?

Since $c.d.\nu = 1$ and $c.d.\pi = 2$ the first k-invariants of M and N both lie in trivial groups, and so this Corollary also follows from Theorem 4.6 below.

Corollary 4.5.2 *Let M be a PD_4-complex with $\chi(M) = 0$ and whose fundamental group π is an extension of Z by a normal subgroup ν. If π has an infinite cyclic normal subgroup C which is not contained in ν then the covering space M_ν with fundamental group ν is a PD_3-complex.*

Proof We may assume without loss of generality that M is orientable and that C is central in π. Since $C \cap \nu = 1$ the subgroup $C\nu \cong C \times \nu$ has finite index in π. Thus by passing to a finite cover we may assume that $\pi = C \times \nu$. Hence ν is finitely presentable and so the Theorem applies. □

See [Hi89] for different proofs of Corollaries 4.5.1 and 4.5.2.

Since ν has one or two ends if it has an infinite cyclic normal subgroup, Corollary 4.5.2 remains true if $C \leq \nu$ and ν is finitely presentable. In this case ν is the fundamental group of a Seifert fibred 3-manifold, by Theorem 2.14.

Corollary 4.5.3 *Let M be a PD_4-complex with $\chi(M) = 0$ and whose fundamental group π is an extension of Z by an FP_2 normal subgroup ν. If ν is finite then it has cohomological period dividing 4. If ν has one end then M is aspherical and so π is a PD_4-group. If ν has two ends then $\nu \cong Z$, $Z \oplus (Z/2Z)$ or $D = (Z/2Z) * (Z/2Z)$. If moreover ν is finitely presentable the covering space M_ν with fundamental group ν is a PD_3-complex.*

Proof The final hypothesis is only needed if ν is one-ended, as finite groups and groups with two ends are finitely presentable. If ν is finite then $\widetilde{M} \simeq S^3$ and so the first assertion holds. (See Chapter 11 for more details.) If ν has one end then we may apply Theorem 4.1. If ν has two ends and its maximal finite normal subgroup is nontrivial then $\nu \cong Z \oplus (Z/2Z)$, by Theorem 2.11 (applied to the PD_3-complex M_ν). Otherwise $\nu \cong Z$ or D. ∎

In Chapter 6 we shall strengthen this Corollary to obtain a fibration theorem for 4-manifolds with torsion free elementary amenable fundamental group.

Our next result gives criteria (involving also the orientation character and first k-invariant) for an infinite cyclic cover of a closed 4-manifold M to be homotopy equivalent to a *particular* PD_3-complex N.

Theorem 4.6 *Let M be a PD_4-complex whose fundamental group π is an extension of Z by a torsion free normal subgroup ν which is isomorphic to the fundamental group of a PD_3-complex N. Then $\pi_2(M) \cong \pi_2(N)$ as $\mathbb{Z}[\nu]$-modules if and only if $Hom_{\mathbb{Z}[\pi]}(\pi_2(M), \mathbb{Z}[\pi]) = 0$. The infinite cyclic covering space M_ν with fundamental group ν is homotopy equivalent to N if and only if $w_1(M)|_\nu = w_1(N)$, $Hom_{\mathbb{Z}[\pi]}(\pi_2(M), \mathbb{Z}[\pi]) = 0$ and the images of $k_1(M)$ and $k_1(N)$ in $H^3(\nu; \pi_2(M)) \cong H^3(\nu; \pi_2(N))$ generate the same subgroup under the action of $Aut_{\mathbb{Z}[\nu]}(\pi_2(N))$.*

Proof If $\Pi = \pi_2(M)$ is isomorphic to $\pi_2(N)$ then it is finitely generated as a $\mathbb{Z}[\nu]$-module, by Theorem 2.18. As 0 is the only $\mathbb{Z}[\pi]$-submodule of $\mathbb{Z}[\pi]$ which is finitely generated as a $\mathbb{Z}[\nu]$-module it follows that $\Pi^* = Hom_{\mathbb{Z}[\pi]}(\pi_2(M), \mathbb{Z}[\pi])$ is trivial. It is then clear that the conditions must hold if M_ν is homotopy equivalent to N.

Suppose conversely that these conditions hold. If $\nu = 1$ then M_ν is simply connected and $\pi \cong Z$ has two ends. It follows immediately from Poincaré duality and the UCSS that $H_2(M_\nu;\mathbb{Z}) = \Pi \cong \overline{\Pi^*} = 0$ and that $H_3(M_\nu;\mathbb{Z}) \cong Z$. Therefore M_ν is homotopy equivalent to S^3. If $\nu \neq 1$ then π has one end, since it has a finitely generated infinite normal subgroup. The hypothesis that $\Pi^* = 0$ implies that $\Pi \cong \overline{H^2(\pi;\mathbb{Z}[\pi])}$, by Lemma 3.3. Hence $\Pi \cong \overline{H^1(\nu;\mathbb{Z}[\nu])}$ as a $\mathbb{Z}[\nu]$-module, by the LHSSS. (The overbar notation is unambiguous since $w_1(M)|_\nu = w_1(N)$.) But this is isomorphic to $\pi_2(N)$, by Poincaré duality for N. Since N is homotopy equivalent to a 3-dimensional complex the condition on the k-invariants implies that there is a map $f : N \to M_\nu$ which induces isomorphisms on fundamental group and second homotopy group. Since the homology of the universal covering spaces of these spaces vanishes above degree 2 the map f is a homotopy equivalence. □

We do not know whether the hypothesis on the k-invariants is implied by the other hypotheses.

Corollary 4.6.1 *Let M be a PD_4-complex whose fundamental group π is an extension of Z by a torsion free normal subgroup ν which is isomorphic to the fundamental group of a 3-manifold N whose irreducible factors are Haken, hyperbolic or Seifert fibred. Then M is homotopy equivalent to a closed PL 4-manifold which fibres over the circle with fibre N.*

Proof There is a homotopy equivalence $f : N \to M_\nu$, where N is a 3-manifold whose irreducible factors are as above, by Turaev's Theorem. (See §5 of Chapter 2.) Let $t : M_\nu \to M_\nu$ be the generator of the covering transformations. Then there is a self homotopy equivalence $u : N \to N$ such that $fu \sim tf$. As each irreducible factor of N has the property that self homotopy equivalences are homotopic to PL homeomorphisms (by [Hm], Mostow rigidity or [Sc83]), u is homotopic to a homeomorphism [HL74], and so M is homotopy equivalent to the mapping torus of this homeomorphism. □

All known PD_3-complexes with torsion free fundamental group are homotopy equivalent to connected sums of such 3-manifolds.

If the irreducible connected summands of the closed 3-manifold $N = \sharp_i N_i$ are P^2-irreducible and sufficiently large or have fundamental group Z then every self homotopy equivalence of N is realized by an unique isotopy class of homeomorphisms [HL74]. However if N is not aspherical then it admits nontrivial self-homeomorphisms ("rotations about 2-spheres") which induce the identity on ν, and so such bundles are not determined by the group alone.

Corollary 4.6.2 *Let M be a PD_4-complex whose fundamental group π is an extension of Z by a virtually torsion free normal subgroup ν. Then the infinite cyclic covering space M_ν with fundamental group ν is homotopy equivalent to a PD_3-complex if and only if ν is the fundamental group of a PD_3-complex N, $Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi])=0$ and the images of $k_1(M)$ and $k_1(N)$ in $H^3(\nu_o;\pi_2(M))\cong H^3(\nu_o;\pi_2(N))$ generate the same subgroup under the action of $Aut_{\mathbb{Z}[\nu_o]}(\pi_2(N))$, where ν_o is a torsion free subgroup of finite index in ν.*

Proof The conditions are clearly necessary. Suppose that they hold. Let $\nu_1\subseteq\nu_o\cap\nu_+\cap\pi_+$ be a torsion free subgroup of finite index in ν, where $\pi_+=\mathrm{K}erw_1(M)$ and $\nu_+=\mathrm{K}erw_1(N)$, and let $t\in\pi$ generate π modulo ν. Then each of the conjugates $t^k\nu_1t^{-k}$ in π has the same index in ν. Since ν is finitely generated the intersection $\mu=\cap t^k\nu_1t^{-k}$ of all such conjugates has finite index in ν, and is clearly torsion free and normal in the subgroup ρ generated by μ and t. If $\{r_i\}$ is a transversal for ρ in π and $f:\pi_2(M)\to\mathbb{Z}[\rho]$ is a nontrivial $\mathbb{Z}[\rho]$-linear homomorphism then $g(m)=\Sigma r_if(r_i^{-1}m)$ defines a nontrivial element of $Hom_\pi(\pi_2(M),\mathbb{Z}[\pi])$. Hence $Hom_\rho(\pi_2(M),\mathbb{Z}[\rho])=0$ and so the covering spaces M_μ and N_μ are homotopy equivalent, by the theorem. It follows easily that M_ν is also a PD_3-complex. □

All PD_3-complexes have virtually torsion free fundamental group [Cr00].

4.6 Products

If $M=N\times S^1$, where N is a closed 3-manifold, then $\chi(M)=0$, Z is a direct factor of $\pi_1(M)$, $w_1(M)$ is trivial on this factor and the Pin^--condition $w_2=w_1^2$ holds. These conditions almost characterize such products up to homotopy equivalence. We need also a constraint on the other direct factor of the fundamental group.

Theorem 4.7 *Let M be a PD_4-complex whose fundamental group π has no 2-torsion. Then M is homotopy equivalent to a product $N\times S^1$, where N is a closed 3-manifold, if and only if $\chi(M)=0$, $w_2(M)=w_1(M)^2$ and there is an isomorphism $\theta:\pi\to\nu\times Z$ such that $w_1(M)\theta^{-1}|_Z=0$, where ν is a (2-torsion free) 3-manifold group.*

Proof The conditions are clearly necessary, since the Pin^--condition holds for 3-manifolds.

If these conditions hold then the covering space M_ν with fundamental group ν is a PD_3-complex, by Theorem 4.5 above. Since ν is a 3-manifold group and has no 2-torsion it is a free product of cyclic groups and groups of aspherical closed 3-manifolds. Hence there is a homotopy equivalence $h : M_\nu \to N$, where N is a connected sum of lens spaces and aspherical closed 3-manifolds, by Turaev's Theorem. (See §5 of Chapter 2.) Let ϕ generate the covering group $Aut(M/M_\nu) \cong Z$. Then there is a self homotopy equivalence $\psi : N \to N$ such that $\psi h \sim h\phi$, and M is homotopy equivalent to the mapping torus $M(\psi)$. We may assume that ψ fixes a basepoint and induces the identity on $\pi_1(N)$, since $\pi_1(M) \cong \nu \times Z$. Moreover ψ preserves the local orientation, since $w_1(M)\theta^{-1}|_Z = 0$. Since ν has no element of order 2 N has no two-sided projective planes and so ψ is homotopic to a rotation about a 2-sphere [Hn]. Since $w_2(M) = w_1(M)^2$ the rotation is homotopic to the identity and so M is homotopy equivalent to $N \times S^1$. □

Let ρ is an essential map from S^1 to $SO(3)$, and let $M = M(\tau)$, where $\tau : S^1 \times S^2 \to S^1 \times S^2$ is the twist map, given by $\tau(x, y) = (x, \rho(x)(y))$ for all (x, y) in $S^1 \times S^2$. Then $\pi_1(M) \cong Z \times Z$, $\chi(M) = 0$, and $w_1(M) = 0$, but $w_2(M) \neq w_1(M)^2 = 0$, so M is not homotopy equivalent to a product. (Clearly however $M(\tau^2) = S^1 \times S^2 \times S^1$.)

To what extent are the constraints on ν necessary? There are orientable 4-manifolds which are homotopy equivalent to products $N \times S^1$ where $\nu = \pi_1(N)$ is finite and is *not* a 3-manifold group. (See Chapter 11.) Theorem 4.1 implies that M is homotopy equivalent to a product of an aspherical PD_3-complex with S^1 if and only if $\chi(M) = 0$ and $\pi_1(M) \cong \nu \times Z$ where ν has one end.

There are 4-manifolds which are simple homotopy equivalent to $S^1 \times RP^3$ (and thus satisfy the hypotheses of our theorem) but which are not homeomorphic to mapping tori [We87].

4.7 Subnormal subgroups

In this brief section we shall give another characterization of aspherical 4-manifolds with finite covering spaces which are homotopy equivalent to mapping tori.

Theorem 4.8 *Let M be a PD_4-complex. Then M is aspherical and has a finite cover which is homotopy equivalent to a mapping torus if and only if $\chi(M) = 0$ and $\pi = \pi_1(M)$ has an FP_3 subnormal subgroup G of infinite index and such that $H^s(G; \mathbb{Z}[G]) = 0$ for $s \leq 2$. In that case G is a PD_3-group, $[\pi : N_\pi(G)] < \infty$ and $e(N_\pi(G)/G) = 2$.*

Proof The conditions are clearly necessary. Suppose that they hold. Let $G = G_0 < G_1 < \dots G_n = \pi$ be a subnormal chain of minimal length, and let $j = \min\{i \mid [G_{i+1} : G] = \infty\}$. Then $[G_j : G] < \infty$ and $\beta_1^{(2)}(G_{j+1}) = 0$ [Ga00]. A finite induction up the subnormal chain, using LHSSS arguments (with coefficients $\mathbb{Z}[\pi]$ and $\mathcal{N}(G_j)$, respectively) shows that $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$ and that $\beta_1^{(2)}(\pi) = 0$. (See §2 of Chapter 2.) Hence M is aspherical, by Theorem 3.4.

On the other hand $H^s(G_{j+1};W) = 0$ for $s \leq 3$ and any free $\mathbb{Z}[G_{j+1}]$-module W, so $c.d.G_{j+1} = 4$. Hence $[\pi : G_{j+1}] < \infty$, by Strebel's Theorem. Therefore G_{j+1} is a PD_4-group. Hence G_j is a PD_3-group and G_{j+1}/G_j has two ends, by Theorem 3.10. The theorem now follows easily, since $[G_j : G] < \infty$ and G_j has only finitely many subgroups of index $[G_j : G]$. □

The hypotheses on G could be replaced by "*G is a PD_3-group*", for then $[\pi : G] = \infty$, by Theorem 3.12.

We shall establish an analogous result for closed 4-manifolds M such that $\chi(M) = 0$ and $\pi_1(M)$ has a subnormal subgroup of infinite index which is a PD_2-group in Chapter 5.

4.8 Circle bundles

In this section we shall consider the "dual" situation, of 4-manifolds which are homotopy equivalent to the total space of a S^1-bundle over a 3-dimensional base N. Lemma 4.9 presents a number of conditions satisfied by such manifolds. (These conditions are not all independent.) Bundles $c_N^*\xi$ induced from S^1-bundles over $K(\pi_1(N),1)$ are given equivalent characterizations in Lemma 4.10. In Theorem 4.11 we shall show that the conditions of Lemmas 4.9 and 4.10 characterize the homotopy types of such bundle spaces $E(c_N^*\xi)$, provided $\pi_1(N)$ is torsion free but not free.

Since $BS^1 \simeq K(Z,2)$ any S^1-bundle over a connected base B is induced from some bundle over $P_2(B)$. For each epimorphism $\gamma : \mu \to \nu$ with cyclic kernel and such that the action of μ by conjugation on $\mathrm{Ker}(\gamma)$ factors through multiplication by ± 1 there is an S^1-bundle $p(\gamma) : X(\gamma) \to Y(\gamma)$ whose fundamental group sequence realizes γ and which is universal for such bundles; the total space $E(p(\gamma))$ is a $K(\mu,1)$ space (cf. Proposition 11.4 of [Wl]).

Lemma 4.9 *Let $p : E \to B$ be the projection of an S^1-bundle ξ over a connected finite complex B. Then*

(1) *$\chi(E)=0$;*

(2) *the natural map $p_*:\pi=\pi_1(E)\to\nu=\pi_1(B)$ is an epimorphism with cyclic kernel, and the action of ν on $\mathrm{Ker}(p_*)$ induced by conjugation in π is given by $w=w_1(\xi):\pi_1(B)\to Z/2Z\cong\{\pm1\}\leq Aut(\mathrm{Ker}(p_*))$;*

(3) *if B is a PD-complex $w_1(E)=p^*(w_1(B)+w)$;*

(4) *if B is a PD_3-complex there are maps $\hat{c}:E\to P_2(B)$ and $y:P_2(B)\to Y(p_*)$ such that $c_{P_2(B)}=c_{Y(p_*)}y$, $y\hat{c}=p(p_*)c_E$ and $(\hat{c},c_E)_*[E]=\pm G(f_{B*}[B])$ where G is the Gysin homomorphism from $H_3(P_2(B);Z^{w_1(B)})$ to $H_4(P_2(E);Z^{w_1(E)})$;*

(5) *If B is a PD_3-complex $c_{E*}[E]=\pm G(c_{B*}[B])$, where G is the Gysin homomorphism from $H_3(\nu;Z^{w_B})$ to $H_4(\pi;Z^{w_E})$;*

(6) *$\mathrm{Ker}(p_*)$ acts trivially on $\pi_2(E)$.*

Proof Condition(1) follows from the multiplicativity of the Euler characteristic in a fibration. If α is any loop in B the total space of the induced bundle $\alpha^*\xi$ is the torus if $w(\alpha)=0$ and the Klein bottle if $w(\alpha)=1$ in $Z/2Z$; hence $gzg^{-1}=z^{\epsilon(g)}$ where $\epsilon(g)=(-1)^{w(p_*(g))}$ for g in $\pi_1(E)$ and z in $\mathrm{Ker}(p_*)$. Conditions (2) and (6) then follow from the exact homotopy sequence. If the base B is a PD-complex then so is E, and we may use naturality and the Whitney sum formula (applied to the Spivak normal bundles) to show that $w_1(E)=p^*(w_1(B)+w_1(\xi))$. (As $p^*:H^1(B;\mathbb{F}_2)\to H^1(E;\mathbb{F}_2)$ is a monomorphism this equation determines $w_1(\xi)$.)

Condition (4) implies (5), and follows from the observations in the paragraph preceding the lemma. (Note that the Gysin homomorphisms G in (4) and (5) are well defined, since $H_1(\mathrm{Ker}(\gamma);Z^{w_E})$ is isomorphic to Z^{w_B}, by (3).) □

Bundles with $\mathrm{Ker}(p_*)\cong Z$ have the following equivalent characterizations.

Lemma 4.10 *Let $p:E\to B$ be the projection of an S^1-bundle ξ over a connected finite complex B. Then the following conditions are equivalent:*

(1) *ξ is induced from an S^1-bundle over $K(\pi_1(B),1)$ via c_B;*

(2) *for each map $\beta:S^2\to B$ the induced bundle $\beta^*\xi$ is trivial;*

(3) *the induced epimorphism $p_*:\pi_1(E)\to\pi_1(B)$ has infinite cyclic kernel.*

If these conditions hold then $c(\xi)=c_B^*\Xi$, where $c(\xi)$ is the characteristic class of ξ in $H^2(B;Z^w)$ and Ξ is the class of the extension of fundamental groups in $H^2(\pi_1(B);Z^w)=H^2(K(\pi_1(B),1);Z^w)$, where $w=w_1(\xi)$.

Proof Condition (1) implies condition (2) as for any such map β the composite $c_B\beta$ is nullhomotopic. Conversely, as we may construct $K(\pi_1(B),1)$ by adjoining cells of dimension ≥ 3 to B condition (2) implies that we may extend ξ over the 3-cells, and as S^1-bundles over S^n are trivial for all $n>2$ we may then extend ξ over the whole of $K(\pi_1(B),1)$, so that (2) implies (1). The equivalence of (2) and (3) follows on observing that (3) holds if and only if $\partial\beta=0$ for all such β, where ∂ is the connecting map from $\pi_2(B)$ to $\pi_1(S^1)$ in the exact sequence of homotopy for ξ, and on comparing this with the corresponding sequence for $\beta^*\xi$.

As the natural map from the set of S^1-bundles over $K(\pi,1)$ with $w_1=w$ (which are classified by $H^2(K(\pi,1);Z^w)$) to the set of extensions of π by Z with π acting via w (which are classified by $H^2(\pi;Z^w)$) which sends a bundle to the extension of fundamental groups is an isomorphism we have $c(\xi)=c_B^*(\Xi)$. □

If N is a closed 3-manifold which has no summands of type $S^1\times S^2$ or $S^1\tilde{\times}S^2$ (i.e., if $\pi_1(N)$ has no infinite cyclic free factor) then every S^1-bundle over N with $w=0$ restricts to a trivial bundle over any map from S^2 to N. For if ξ is such a bundle, with characteristic class $c(\chi)$ in $H^2(N;\mathbb{Z})$, and $\beta:S^2\to N$ is any map then $\beta_*(c(\beta^*\xi)\cap[S^2])=\beta_*(\beta^*c(\xi)\cap[S^2])=c(\xi)\cap\beta_*[S^2]=0$, as the Hurewicz homomorphism is trivial for such N. Since β_* is an isomorphism in degree 0 it follows that $c(\beta^*\xi)=0$ and so $\beta^*\xi$ is trivial. (A similar argument applies for bundles with $w\neq 0$, provided the induced 2-fold covering space N^w has no summands of type $S^1\times S^2$ or $S^1\tilde{\times}S^2$.)

On the other hand, if η is the Hopf fibration the bundle with total space $S^1\times S^3$, base $S^1\times S^2$ and projection $id_{S^1}\times\eta$ has nontrivial pullback over any essential map from S^2 to $S^1\times S^2$, and is not induced from any bundle over $K(Z,1)$. Moreover, $S^1\times S^2$ is a 2-fold covering space of $RP^3\sharp RP^3$, and so the above hypothesis on summands of N is not stable under passage to 2-fold coverings (corresponding to a homomorphism w from $\pi_1(N)$ to $Z/2Z$).

Theorem 4.11 *Let M be a finite PD_4-complex and N a finite PD_3-complex whose fundamental group is torsion free but not free. Then M is homotopy equivalent to the total space of an S^1-bundle over N which satisfies the conditions of Lemma 4.10 if and only if*

(1) $\chi(M)=0$;

(2) *there is an epimorphism* $\gamma:\pi=\pi_1(M)\to\nu=\pi_1(N)$ *with* $\mathrm{Ker}(\gamma)\cong Z$;

(3) $w_1(M)=(w_1(N)+w)\gamma$, *where* $w:\nu\to Z/2Z\cong Aut(\mathrm{Ker}(\gamma))$ *is determined by the action of ν on* $\mathrm{Ker}(\gamma)$ *induced by conjugation in* π;

(4) $k_1(M) = \gamma^* k_1(N)$ *(and so* $P_2(M) \simeq P_2(N) \times_{K(\nu,1)} K(\pi,1)$*);*

(5) $f_{M*}[M] = \pm G(f_{N*}[N])$ *in* $H_4(P_2(M); Z^{w_1(M)})$*, where* G *is the Gysin homomorphism in degree* 3*.*

If these conditions hold then M *has minimal Euler characteristic for its fundamental group, i.e.* $q(\pi) = 0$*.*

Remark The first three conditions and Poincaré duality imply that $\pi_2(M) \cong \gamma^*\pi_2(N)$, the $\mathbb{Z}[\pi]$-module with the same underlying group as $\pi_2(N)$ and with $\mathbb{Z}[\pi]$-action determined by the homomorphism γ.

Proof Since these conditions are homotopy invariant and hold if M is the total space of such a bundle, they are necessary. Suppose conversely that they hold. As ν is torsion free N is the connected sum of a 3-manifold with free fundamental group and some aspherical PD_3-complexes [Tu90]. As ν is not free there is at least one aspherical summand. Hence *c.d.*$\nu = 3$ and $H_3(c_N; Z^{w_1(N)})$ is a monomorphism.

Let $p(\gamma) : K(\pi,1) \to K(\nu,1)$ be the S^1-bundle corresponding to γ and let $E = N \times_{K(\nu,1)} K(\pi,1)$ be the total space of the S^1-bundle over N induced by the classifying map $c_N : N \to K(\nu,1)$. The bundle map covering c_N is the classifying map c_E. Then $\pi_1(E) \cong \pi = \pi_1(M)$, $w_1(E) = (w_1(N) + w)\gamma = w_1(M)$, as maps from π to $Z/2Z$, and $\chi(E) = 0 = \chi(M)$, by conditions (1) and (3). The maps c_N and c_E induce a homomorphism between the Gysin sequences of the S^1-bundles. Since N and ν have cohomological dimension 3 the Gysin homomorphisms in degree 3 are isomorphisms. Hence $H_4(c_E; Z^{w_1(E)})$ is a monomorphism, and so *a fortiori* $H_4(f_E; Z^{w_1(E)})$ is also a monomorphism.

Since $\chi(M) = 0$ and $\beta_1^{(2)}(\pi) = 0$, by Theorem 2.3, part (3) of Theorem 3.4 implies that $\pi_2(M) \cong \overline{H^2(\pi; \mathbb{Z}[\pi])}$. It follows from conditions (2) and (3) and the LHSSS that $\pi_2(M) \cong \pi_2(E) \cong \gamma^*\pi_2(N)$ as $\mathbb{Z}[\pi]$-modules. Conditions (4) and (5) then give us a map $(\hat{c}, c_M)$ from M to $P_2(E) = P_2(N) \times_{K(\nu,1)} K(\pi,1)$ such that $(\hat{c}, c_M)_*[M] = \pm f_{E*}[E]$. Hence M is homotopy equivalent to E, by Theorem 3.8.

The final assertion now follows from part (1) of Theorem 3.4. $\square$

As $\pi_2(N)$ is a projective $\mathbb{Z}[\nu]$-module, by Theorem 2.18, it is homologically trivial and so $H_q(\pi; \gamma^*\pi_2(N) \otimes Z^{w_1(M)}) = 0$ if $q \geq 2$. Hence it follows from the spectral sequence for $c_{P_2(M)}$ that $H_4(P_2(M); Z^{w_1(M)})$ maps onto $H_4(\pi; Z^{w_1(M)})$, with kernel isomorphic to $H_0(\pi; \Gamma(\pi_2(M))) \otimes Z^{w_1(M)})$, where

$\Gamma(\pi_2(M)) = H_4(K(\pi_2(M),2);\mathbb{Z})$ is Whitehead's universal quadratic construction on $\pi_2(M)$ (see Chapter I of [Ba']). This suggests that there may be another formulation of the theorem in terms of conditions (1-3), together with some information on $k_1(M)$ and the intersection pairing on $\pi_2(M)$. If N is aspherical conditions (4) and (5) are vacuous or redundant.

Condition (4) is vacuous if ν is a free group, for then $c.d.\pi \leq 2$. In this case the Hurewicz homomorphism from $\pi_3(N)$ to $H_3(N;Z^{w_1(N)})$ is 0, and so $H_3(f_N;Z^{w_1(N)})$ is a monomorphism. The argument of the theorem would then extend if the Gysin map in degree 3 for the bundle $P_2(E) \to P_2(N)$ were a monomorphism. If $\nu = 1$ then M is orientable, $\pi \cong Z$ and $\chi(M) = 0$, so $M \simeq S^3 \times S^1$. In general, if the restriction on ν is removed it is not clear that there should be a degree 1 map from M to such a bundle space E.

It would be of interest to have a theorem with hypotheses involving only M, without reference to a model N. There is such a result in the aspherical case.

Theorem 4.12 *A finite PD_4-complex M is homotopy equivalent to the total space of an S^1-bundle over an aspherical PD_3-complex if and only if $\chi(M) = 0$ and $\pi = \pi_1(M)$ has an infinite cyclic normal subgroup A such that π/A has one end and finite cohomological dimension.*

Proof The conditions are clearly necessary. Conversely, suppose that they hold. Since π/A has one end $H^s(\pi/A;\mathbb{Z}[\pi/A]) = 0$ for $s \leq 1$ and so an LHSSS calculation gives $H^t(\pi;\mathbb{Z}[\pi]) = 0$ for $t \leq 2$. Moreover $\beta_1^{(2)}(\pi) = 0$, by Theorem 2.3. Hence M is aspherical and π is a PD_4-group, by Corollary 3.5.2. Since A is FP_∞ and $c.d.\pi/A < \infty$ the quotient π/A is a PD_3-group, by Theorem 9.11 of [Bi]. Therefore M is homotopy equivalent to the total space of an S^1-bundle over the PD_3-complex $K(\pi/A,1)$. □

Note that a finitely generated torsion free group has one end if and only if it is indecomposable as a free product and is neither infinite cyclic nor trivial.

In general, if M is homotopy equivalent to the total space of an S^1-bundle over some 3-manifold then $\chi(M) = 0$ and $\pi_1(M)$ has an infinite cyclic normal subgroup A such that $\pi_1(M)/A$ is virtually of finite cohomological dimension. Do these conditions characterize such homotopy types?

Chapter 5

Surface bundles

In this chapter we shall show that a closed 4-manifold M is homotopy equivalent to the total space of a fibre bundle with base and fibre closed surfaces if and only if the obviously necessary conditions on the Euler characteristic and fundamental group hold. When the base is S^2 we need also conditions on the characteristic classes of M, and when the base is RP^2 our results are incomplete. We shall defer consideration of bundles over RP^2 with fibre T or Kb and $\partial \neq 0$ to Chapter 11, and those with fibre S^2 or RP^2 to Chapter 12.

5.1 Some general results

If B, E and F are connected finite complexes and $p : E \to B$ is a Hurewicz fibration with fibre homotopy equivalent to F then $\chi(E) = \chi(B)\chi(F)$ and the long exact sequence of homotopy gives an exact sequence

$$\pi_2(B) \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to 1$$

in which the image of $\pi_2(B)$ under the *connecting homomorphism* ∂ is in the centre of $\pi_1(F)$. (See page 51 of [Go68].) These conditions are clearly homotopy invariant.

Hurewicz fibrations with base B and fibre X are classified by homotopy classes of maps from B to the Milgram classifying space $BE(X)$, where $E(X)$ is the monoid of all self homotopy equivalences of X, with the compact-open topology [Mi67]. If X has been given a base point the evaluation map from $E(X)$ to X is a Hurewicz fibration with fibre the subspace (and submonoid) $E_0(X)$ of base point preserving self homotopy equivalences [Go68].

Let T and Kb denote the torus and Klein bottle, respectively.

Lemma 5.1 *Let F be an aspherical closed surface and B a closed smooth manifold. There are natural bijections from the set of isomorphism classes of smooth F-bundles over B to the set of fibre homotopy equivalence classes of Hurewicz fibrations with fibre F over B and to the set $\coprod_{[\xi]} H^2(B; \zeta\pi_1(F)^\xi)$, where the union is over conjugacy classes of homomorphisms $\xi : \pi_1(B) \to Out(\pi_1(F))$ and $\zeta\pi_1(F)^\xi$ is the $\mathbb{Z}[\pi_1(F)]$-module determined by ξ.*

Proof If $\zeta\pi_1(F) = 1$ the identity components of $Diff(F)$ and $E(F)$ are contractible [EE69]. Now every automorphism of $\pi_1(F)$ is realizable by a diffeomorphism and homotopy implies isotopy for self diffeomorphisms of surfaces. (See Chapter V of [ZVC].) Therefore $\pi_0(Diff(F)) \cong \pi_0(E(F)) \cong Out(\pi_1(F))$, and the inclusion of $Diff(F)$ into $E(F)$ is a homotopy equivalence. Hence $BDiff(F) \simeq BE(F) \simeq K(Out(\pi_1(F),1)$, so smooth F-bundles over B and Hurewicz fibrations with fibre F over B are classified by the (unbased) homotopy set

$$[B, K(Out(\pi_1(F),1))] = Hom(\pi_1(B), Out(\pi_1(F)))/\backsim,$$

where $\xi \backsim \xi'$ if there is an $\alpha \in Out(\pi_1(F))$ such that $\xi'(b) = \alpha\xi(b)\alpha^{-1}$ for all $b \in \pi_1(B)$.

If $\zeta\pi_1(F) \neq 1$ then $F = T$ or Kb. Left multiplication by T on itself induces homotopy equivalences from T to the identity components of $Diff(T)$ and $E(T)$. (Similarly, the standard action of S^1 on Kb induces homotopy equivalences from S^1 to the identity components of $Diff(Kb)$ and $E(Kb)$. See Theorem III.2 of [Go65].) Let $\alpha : GL(2,\mathbb{Z}) \to Aut(T) \leq Diff(T)$ be the standard linear action. Then the natural maps from the semidirect product $T \times_\alpha GL(2,\mathbb{Z})$ to $Diff(T)$ and to $E(T)$ are homotopy equivalences. Therefore $BDiff(T)$ is a $K(Z^2,2)$-fibration over $K(GL(2,\mathbb{Z}),1)$. It follows that T-bundles over B are classified by two invariants: a conjugacy class of homomorphisms $\xi : \pi_1(B) \to GL(2,\mathbb{Z})$ together with a cohomology class in $H^2(B;(Z^2)^\xi)$. A similar argument applies if $F = Kb$. □

Theorem 5.2 *Let M be a PD_4-complex and B and F aspherical closed surfaces. Then M is homotopy equivalent to the total space of an F-bundle over B if and only if $\chi(M) = \chi(B)\chi(F)$ and $\pi_1(M)$ is an extension of $\pi_1(B)$ by $\pi_1(F)$. Moreover every extension of $\pi_1(B)$ by $\pi_1(F)$ is realized by some surface bundle, which is determined up to isomorphism by the extension.*

Proof The conditions are clearly necessary. Suppose that they hold. If $\zeta\pi_1(F) = 1$ each homomorphism $\xi : \pi_1(B) \to Out(\pi_1(F))$ corresponds to an unique equivalence class of extensions of $\pi_1(B)$ by $\pi_1(F)$, by Proposition 11.4.21 of [Ro]. Hence there is an F-bundle $p : E \to B$ with $\pi_1(E) \cong \pi_1(M)$ realizing the extension, and p is unique up to bundle isomorphism. If $F = T$ then every homomorphism $\xi : \pi_1(B) \to GL(2,\mathbb{Z})$ is realizable by an extension (for instance, the semidirect product $Z^2 \times_\xi \pi_1(B)$) and the extensions realizing ξ are classified up to equivalence by $H^2(\pi_1(B);(Z^2)^\xi)$. As B is aspherical the natural map from bundles to group extensions is a bijection. Similar arguments apply if $F = Kb$. In all cases the bundle space E is aspherical, and so $\pi_1(M)$ is an FF PD_4-group. Hence $M \simeq E$, by Corollary 3.5.1. □

Such extensions (with $\chi(F) < 0$) were shown to be realizable by bundles in [Jo79].

5.2 Bundles with base and fibre aspherical surfaces

In many cases the group $\pi_1(M)$ determines the bundle up to diffeomorphism of its base. Lemma 5.3 and Theorems 5.4 and 5.5 are based on [Jo94].

Lemma 5.3 *Let G_1 and G_2 be groups with no nontrivial abelian normal subgroup. If H is a normal subgroup of $G = G_1 \times G_2$ which contains no nontrivial direct product then either $H \leq G_1 \times \{1\}$ or $H \leq \{1\} \times G_2$.*

Proof Let P_i be the projection of H onto G_i, for $i = 1,2$. If $(h,h') \in H$, $g_1 \in G_1$ and $g_2 \in G_2$ then $([h,g_1],1) = [(h,h'),(g_1,1)]$ and $(1,[h',g_2])$ are in H. Hence $[P_1,P_1] \times [P_2,P_2] \leq H$. Therefore either P_1 or P_2 is abelian, and so is trivial, since P_i is normal in G_i, for $i = 1,2$. □

Theorem 5.4 *Let π be a group with a normal subgroup K such that K and π/K are PD_2-groups with trivial centres.*

1. *If $C_\pi(K) = 1$ and K_1 is a finitely generated normal subgroup of π then $C_\pi(K_1) = 1$ also.*
2. *The index $[\pi : KC_\pi(K)]$ is finite if and only if π is virtually a direct product of PD_2-groups.*

Proof (1) Let $z \in C_\pi(K_1)$. If $K_1 \leq K$ then $[K : K_1] < \infty$ and $\zeta K_1 = 1$. Let $M = [K : K_1]!$. Then $f(k) = k^{-1}z^M kz^{-M}$ is in K_1 for all k in K. Now $f(kk_1) = k_1^{-1}f(k)k_1$ and also $f(kk_1) = f(kk_1k^{-1}k) = f(k)$ (since K_1 is a normal subgroup centralized by z), for all k in K and k_1 in K_1. Hence $f(k)$ is central in K_1, and so $f(k) = 1$ for all k in K. Thus z^M centralizes K. Since π is torsion free we must have $z = 1$. Otherwise the image of K_1 under the projection $p : \pi \to \pi/K$ is a nontrivial finitely generated normal subgroup of π/K, and so has trivial centralizer. Hence $p(z) = 1$. Now $[K,K_1] \leq K \cap K_1$ and so $K \cap K_1 \neq 1$, for otherwise $K_1 \leq C_\pi(K)$. Since z centralizes the nontrivial normal subgroup $K \cap K_1$ in K we must again have $z = 1$.

(2) Since K has trivial centre $KC_\pi(K) \cong K \times C_\pi(K)$ and so the condition is necessary. Suppose that $f : G_1 \times G_2 \to \pi$ is an isomorphism onto a subgroup of finite index, where G_1 and G_2 are PD_2-groups. Let $L = K \cap f(G_1 \times G_2)$. Then $[K : L] < \infty$ and so L is also a PD_2-group, and is normal in $f(G_1 \times G_2)$. We may assume that $L \leq f(G_1)$, by Lemma 5.3. Then $f(G_1)/L$ is finite and is isomorphic to a subgroup of $f(G_1 \times G_2)/K \leq \pi/K$, so $L = f(G_1)$. Now $f(G_2)$ normalizes K and centralizes L, and $[K : L] < \infty$. Hence $f(G_2)$ has a subgroup of finite index which centralizes K, as in part (1). Hence $[\pi : KC_\pi(K)] < \infty$. □

It follows immediately that if π and K are as in the theorem whether

(1) $C_\pi(K) \neq 1$ and $[\pi : KC_\pi(K)] = \infty$;

(2) $[\pi : KC_\pi(K)] < \infty$; or

(3) $C_\pi(K) = 1$

depends only on π and not on the subgroup K. In [Jo94] these cases are labeled as types I, II and III, respectively. (In terms of the action: if $\mathrm{Im}(\theta)$ is infinite and $\mathrm{Ker}(\theta) \neq 1$ then π is of type I, if $\mathrm{Im}(\theta)$ is finite then π is of type II, and if θ is injective then π is of type III.)

Theorem 5.5 *Let π be a group with normal subgroups K and K_1 such that K, K_1, π/K and π/K_1 are PD_2-groups with trivial centres. If $C_\pi(K) \neq 1$ but $[\pi : KC_\pi(K)] = \infty$ then $K_1 = K$ is unique. If $[\pi : KC_\pi(K)] < \infty$ then either $K_1 = K$ or $K_1 \cap K = 1$; in the latter case K and K_1 are the only such normal subgroups which are PD_2-groups with torsion free quotients.*

Proof Let $p : \pi \to \pi/K$ be the quotient epimorphism. Then $p(C_\pi(K))$ is a nontrivial normal subgroup of π/K, since $K \cap C_\pi(K) = \zeta K = 1$. Suppose that $K_1 \cap K \neq 1$. Let $\Sigma = K_1 \cap (KC_\pi(K))$. Then Σ contains $K_1 \cap K$, and $\Sigma \not\leq C_\pi(K)$, since $K_1 \cap K \cap C_\pi(K) = K_1 \cap \zeta K = 1$. Since Σ is normal in $KC_\pi(K) \cong K \times C_\pi(K)$ we must have $\Sigma \leq K_1$, by Lemma 5.3. Hence $\Sigma \leq K_1 \cap K$. Hence $p(K_1) \cap p(C_\pi(K)) = 1$, and so $p(K_1)$ centralizes the nontrivial normal subgroup $p(C_\pi(K))$ in π/K. Therefore $K_1 \leq K$ and so $[K : K_1] < \infty$. Since π/K_1 is torsion free we must have $K_1 = K$.

If $K_1 \cap K = 1$ then $[K, K_1] = 1$ (since each subgroup is normal in π) so $K_1 \leq C_\pi(K)$ and $[\pi : KC_\pi(K)] \leq [\pi/K : p(K_1)] < \infty$. Suppose K_2 is a normal subgroup of π which is a PD_2-group with $\zeta K_2 = 1$ and such that π/K_2 is torsion free and $K_2 \cap K = 1$. Then $H = K_2 \cap (KK_1)$ is normal in $KK_1 \cong K \times K_1$ and $[K_2 : H] < \infty$, so H is a PD_2-group with $\zeta H = 1$

and $H \cap K = 1$. The projection of H to K_1 is nontrivial since $H \cap K = 1$. Therefore $H \leq K_1$, by Lemma 5.3, and so $K_1 \leq K_2$. Hence $K_1 = K_2$. □

Corollary 5.5.1 [Jo93] *Let α and β be automorphisms of π, and suppose that $\alpha(K) \cap K = 1$. Then $\beta(K) = K$ or $\alpha(K)$, and so $Aut(K \times K) \cong Aut(K)^2 \tilde{\times} (Z/2Z)$.*

We shall obtain a somewhat weaker result for groups of type III as a corollary of the next theorem.

Theorem 5.6 *Let π be a group with normal subgroups K and K_1 such that K, K_1 and π/K are PD_2-groups, π/K_1 is torsion free and $\chi(\pi/K) < 0$. Then either $K_1 = K$ or $K_1 \cap K = 1$ and $\pi \cong K \times K_1$ or $\chi(K_1) < \chi(\pi/K)$.*

Proof Let $p : \pi \to \pi/K$ be the quotient epimorphism. If $K_1 \leq K$ then $K_1 = K$, as in Theorem 5.5. Otherwise $p(K_1)$ has finite index in π/K and so $p(K_1)$ is also a PD_2-group. As the minimum number of generators of a PD_2-group G is $\beta_1(G; \mathbb{F}_2)$, we have $\chi(K_1) \leq \chi(p(K_1)) \leq \chi(\pi/K)$. We may assume that $\chi(K_1) \geq \chi(\pi/K)$. Hence $\chi(K_1) = \chi(\pi/K)$ and so $p|_{K_1}$ is an epimorphism. Therefore K_1 and π/K have the same orientation type, by the nondegeneracy of Poincaré duality with coefficients $\mathbb{F}_2$ and the Wu relation $w_1 \cup x = x^2$ for all $x \in H^1(G; \mathbb{F}_2)$ and PD_2-groups G. Hence $K_1 \cong \pi/K$. Since PD_2-groups are hopfian $p|_{K_1}$ is an isomorphism. Hence $[K, K_1] \leq K \cap K_1 = 1$ and so $\pi = K.K_1 \cong K \times \pi/K$. □

Corollary 5.6.1 [Jo98] *The group π has only finitely many such subgroups K.*

Proof We may assume given $\chi(K) < 0$ and that π is of type III. If ρ is an epimorphism from π to $Z/\chi(\pi)Z$ such that $\rho(K) = 0$ then $\chi(\mathrm{Ker}(\rho)/K) \leq \chi(K)$. Since π is not a product K is the only such subgroup of $\mathrm{Ker}(\rho)$. Since $\chi(K)$ divides $\chi(\pi)$ and $Hom(\pi, Z/\chi(\pi)Z)$ is finite the corollary follows. □

The next two corollaries follow by elementary arithmetic.

Corollary 5.6.2 *If $\chi(K) = 0$ or $\chi(K) = -1$ and π/K_1 is a PD_2-group then either $K_1 = K$ or $\pi \cong K \times K_1$.* □

Corollary 5.6.3 *If K and π/K are PD_2-groups, $\chi(\pi/K) < 0$, and $\chi(K)^2 \leq \chi(\pi)$ then either K is the unique such subgroup or $\pi \cong K \times K$.* □

Corollary 5.6.4 *Let M and M' be the total spaces of bundles ξ and ξ' with the same base B and fibre F, where B and F are aspherical closed surfaces such that $\chi(B) < \chi(F)$. Then M' is diffeomorphic to M via a fibre-preserving diffeomorphism if and only if $\pi_1(M') \cong \pi_1(M)$.* □

Compare the statement of Melvin's Theorem on total spaces of S^2-bundles (Theorem 5.13 below.)

We can often recognise total spaces of aspherical surface bundles under weaker hypotheses on the fundamental group.

Theorem 5.7 *Let M be a PD_4-complex with fundamental group π. Then the following conditions are equivalent:*

(1) *M is homotopy equivalent to the total space of a bundle with base and fibre aspherical closed surfaces:*

(2) *π has an FP_2 normal subgroup K such that π/K is a PD_2-group and $\pi_2(M) = 0$;*

(3) *π has a normal subgroup N which is a PD_2-group, π/N is torsion free and $\pi_2(M) = 0$.*

Proof Clearly (1) implies (2) and (3). Conversely they each imply that π has one end and so M is aspherical. If K is an FP_2 normal subgroup in π and π/K is a PD_2-group then K is a PD_2-group, by Theorem 1.19. If N is a normal subgroup which is a PD_2-group then an LHSSS argument gives $H^2(\pi/N; \mathbb{Z}[\pi/N]) \cong Z$. Hence π/N is virtually a PD_2-group, by Bowditch's Theorem. Since it is torsion free it is a PD_2-group and so the theorem follows from Theorem 5.2. □

If $\zeta K = 1$ we may avoid the difficult theorem of Bowditch here, for then π/K is an extension of $C_\pi(K)$ by a subgroup of $Out(K)$, so $v.c.d.\pi/K < \infty$ and thus π/K is virtually a PD_2-group, by Theorem 9.11 of [Bi].

Kapovich has given an example of an aspherical closed 4-manifold M such that $\pi_1(M)$ is an extension of a PD_2-group by a finitely generated normal subgroup which is not FP_2 [Ka98].

Theorem 5.8 *Let M be a PD_4-complex with fundamental group π and such that $\chi(M) = 0$. If π has a subnormal subgroup G of infinite index which is a PD_2-group then M is aspherical. If moreover $\zeta G = 1$ there is a subnormal chain $G < J < K \leq \pi$ such that $[\pi : K] < \infty$ and $K/J \cong J/G \cong Z$.*

Proof Let $G = G_0 < G_1 < \dots G_n = \pi$ be a subnormal chain of minimal length. Let $j = \min\{i \mid [G_{i+1} : G] = \infty\}$. Then $[G_j : G] < \infty$, so G_j is FP. It is easily seen that the theorem holds for G if it holds for G_j. Thus we may assume that $[G_1 : G] = \infty$. A finite induction up the subnormal chain using the LHSSS gives $H^s(\pi; \mathbb{Z}[\pi]) = 0$ for $s \leq 2$. Now $\beta_1^{(2)}(G_1) = 0$, since G is finitely generated and $[G_1 : G] = \infty$ [Ga00]. (This also can be deduced from Theorem 2.2 and the fact that $Out(G)$ is virtually torsion free.) Inducting up the subnormal chain gives $\beta_1^{(2)}(\pi) = 0$ and so M is aspherical, by Theorem 3.4.

If $G < \tilde{G}$ are two normal subgroups of G_1 with cohomological dimension 2 then $\tilde{G}/G$ is locally finite, by Theorem 8.2 of [Bi]. Hence $\tilde{G}/G$ is finite, since $\chi(G) = [H : G]\chi(H)$ for any finitely generated subgroup H such that $G \leq H \leq \tilde{G}$. Moreover if $\tilde{G}$ is normal in J then $[J : N_J(G)] < \infty$, since $\tilde{G}$ has only finitely many subgroups of index $[\tilde{G} : G]$.

Therefore we may assume that G is maximal among such subgroups of G_1. Let n be an element of G_2 such that $nGn^{-1} \neq G$, and let $H = G.nGn^{-1}$. Then G is normal in H and H is normal in G_1, so $[H : G] = \infty$ and $c.d.H = 3$. Moreover H is FP and $H^s(H; \mathbb{Z}[H]) = 0$ for $s \leq 2$, so either G_1/H is locally finite or $c.d.G_1 > c.d.H$, by Theorem 8.2 of [Bi]. If G_1/H is locally finite but not finite then we again have $c.d.G_1 > c.d.H$, by Theorem 3.3 of [GS81].

If $c.d.G_1 = 4$ then $[\pi : N_\pi(G)] \leq [\pi : G_1] < \infty$. An LHSSS argument gives $H^2(N_\pi(G)/G; \mathbb{Z}[N_\pi(G)/G]) \cong Z$. Hence $N_\pi(G)/G$ is virtually a PD_2-group, by [Bo99]. Therefore π has a normal subgroup $K \leq N_\pi(G)$ such that $[\pi : K] < \infty$ and K/G is a PD_2-group of orientable type. Then $\chi(G)\chi(K/G) = [\pi : K]\chi(\pi) = 0$ and so $\chi(K/G) = 0$, since $\chi(G) < 0$. Thus $K/G \cong Z^2$, and there are clearly many possibilities for J.

If $c.d.G_1 = 3$ then G_1/H is locally finite, and hence is finite, by Theorem 3.3 of [GS81]. Therefore G_1 is FP and $H^s(G_1; \mathbb{Z}[G_1]) = 0$ for $s \leq 2$. Let $k = \min\{i \mid [G_{i+1} : G_1] = \infty\}$. Then $H^s(G_k; W) = 0$ for $s \leq 3$ and any free $\mathbb{Z}[G_k]$-module W. Hence $c.d.G_k = 4$ and so $[\pi : G_k] < \infty$, by Strebel's Theorem. An LHS spectral sequence corner argument then shows that G_k/G_{k-1} has 2 ends and $H^3(G_{k-1}); \mathbb{Z}[G_{k-1}]) \cong Z$. Thus G_{k-1} is a PD_3-group, and therefore so is G_1. By a similar argument, G_1/G has two ends also. The theorem follows easily. □

Corollary 5.8.1 *If $\zeta G = 1$ and G is normal in π then M has a finite covering space which is homotopy equivalent to the total space of a surface bundle over T.*

Proof Since G is normal in π and M is aspherical M has a finite covering which is homotopy equivalent to a $K(G,1)$-bundle over an aspherical orientable surface, as in Theorem 5.7. Since $\chi(M)=0$ the base must be T. □

If π/G is virtually Z^2 then it has a subgroup of index at most 6 which maps onto Z^2 or $Z\times_{-1}Z$.

Let G be a PD_2-group such that $\zeta G=1$. Let θ be an automorphism of G whose class in $Out(G)$ has infinite order and let $\lambda: G\to Z$ be an epimorphism. Let $\pi=(G\times Z)\times_\phi Z$ where $\phi(g,n)=(\theta(g),\lambda(g)+n)$ for all $g\in G$ and $n\in Z$. Then G is subnormal in π but this group is not virtually the group of a surface bundle over a surface.

If π has a subnormal subgroup G which is a PD_2-group with $\zeta G\neq 1$ then $\sqrt{G}\cong Z^2$ is subnormal in π and hence contained in $\sqrt{\pi}$. In this case $h(\sqrt{\pi})\geq 2$ and so either Theorem 8.1 or Theorem 9.2 applies, to show that M has a finite covering space which is homotopy equivalent to the total space of a T-bundle over an aspherical closed surface.

5.3 Bundles with aspherical base and fibre S^2 or RP^2

Let $E^+(S^2)$ denote the connected component of id_{S^2} in $E(S^2)$, i.e., the submonoid of degree 1 maps. The connected component of id_{S^2} in $E_0(S^2)$ may be identified with the double loop space Ω^2S^2.

Lemma 5.9 *Let X be a finite 2-complex. Then there are natural bijections $[X;BO(3)]\cong[X;BE(S^2)]\cong H^1(X;\mathbb{F}_2)\times H^2(X;\mathbb{F}_2)$.*

Proof As a self homotopy equivalence of a sphere is homotopic to the identity if and only if it has degree $+1$ the inclusion of $O(3)$ into $E(S^2)$ is bijective on components. Evaluation of a self map of S^2 at the basepoint determines fibrations of $SO(3)$ and $E^+(S^2)$ over S^2, with fibre $SO(2)$ and Ω^2S^2, respectively, and the map of fibres induces an isomorphism on π_1. On comparing the exact sequences of homotopy for these fibrations we see that the inclusion of $SO(3)$ in $E^+(S^2)$ also induces an isomorphism on π_1. Since the Stiefel-Whitney classes are defined for any spherical fibration and w_1 and w_2 are nontrivial on suitable S^2-bundles over S^1 and S^2, respectively, the inclusion of $BO(3)$ into $BE(S^2)$ and the map $(w_1,w_2): BE(S^2)\to K(Z/2Z,1)\times K(Z/2Z,2)$ induces isomorphisms on π_i for $i\leq 2$. The lemma follows easily. □

Thus there is a natural 1-1 correspondance between S^2-bundles and spherical fibrations over such complexes, and any such bundle ξ is determined up to isomorphism over X by its total Stiefel-Whitney class $w(\xi)=1+w_1(\xi)+w_2(\xi)$. (From another point of view: if $w_1(\xi)=w_1(\xi')$ there is an isomorphism of the restrictions of ξ and ξ' over the 1-skeleton $X^{[1]}$. The difference $w_2(\xi)-w_2(\xi')$ is the obstruction to extending any such isomorphism over the 2-skeleton.)

Theorem 5.10 *Let M be a PD_4-complex and B an aspherical closed surface. Then the following conditions are equivalent:*

(1) $\pi_1(M)\cong\pi_1(B)$ *and* $\chi(M)=2\chi(B)$;

(2) $\pi_1(M)\cong\pi_1(B)$ *and* $\widetilde{M}\simeq S^2$;

(3) *M is homotopy equivalent to the total space of an S^2-bundle over B.*

Proof If (1) holds then $H_3(\widetilde{M};\mathbb{Z})=H_4(\widetilde{M};\mathbb{Z})=0$, as $\pi_1(M)$ has one end, and $\pi_2(M)\cong\overline{H^2(\pi;\mathbb{Z}[\pi])}\cong Z$, by Theorem 3.12. Hence $\widetilde{M}$ is homotopy equivalent to S^2. If (2) holds we may assume that there is a Hurewicz fibration $h:M\to B$ which induces an isomorphism of fundamental groups. As the homotopy fibre of h is $\widetilde{M}$, Lemma 5.9 implies that h is fibre homotopy equivalent to the projection of an S^2-bundle over B. Clearly (3) implies the other conditions. □

We shall summarize some of the key properties of the Stiefel-Whitney classes of such bundles in the following lemma.

Lemma 5.11 *Let ξ be an S^2-bundle over a closed surface B, with total space M and projection $p:M\to B$. Then*

(1) *ξ is trivial if and only if $w(M)=p^*w(B)$;*

(2) *$\pi_1(M)\cong\pi_1(B)$ acts on $\pi_2(M)$ by multiplication by $w_1(\xi)$;*

(3) *the intersection form on $H_2(M;\mathbb{F}_2)$ is even if and only if $w_2(\xi)=0$;*

(4) *if $q:B'\to B$ is a 2-fold covering map with connected domain B' then $w_2(q^*\xi)=0$.*

Proof (1) Applying the Whitney sum formula and naturality to the tangent bundle of the B^3-bundle associated to ξ gives $w(M)=p^*w(B)\cup p^*w(\xi)$. Since p is a 2-connected map the induced homomorphism p^* is injective in degrees ≤ 2 and so $w(M)=p^*w(B)$ if and only if $w(\xi)=1$. By Lemma 5.9 this is so if and only if ξ is trivial, since B is 2-dimensional.

(2) It is sufficient to consider the restriction of ξ over loops in B, where the result is clear.

(3) By Poincaré duality, the intersection form is even if and only if the Wu class $v_2(M) = w_2(M) + w_1(M)^2$ is 0. Now

$$\begin{aligned} v_2(M) &= p^*(w_1(B) + w_1(\xi))^2 + p^*(w_2(B) + w_1(B) \cup w_1(\xi) + w_2(\xi)) \\ &= p^*(w_2(B) + w_1(B) \cup w_1(\xi) + w_2(\xi) + w_1(B)^2 + w_1(\xi)^2) \\ &= p^*(w_2(\xi)), \end{aligned}$$

since $w_1(B) \cup \eta = \eta^2$ and $w_1(B)^2 = w_2(B)$, by the Wu relations for B. Hence $v_2(M) = 0$ if and only if $w_2(\xi) = 0$, as p^* is injective in degree 2.

(4) We have $q_*(w_2(q^*\xi) \cap [B']) = q_*((q^*w_2(\xi)) \cap [B']) = w_2(\xi) \cap q_*[B']$, by the projection formula. Since q has degree 2 this is 0, and since q_* is an isomorphism in degree 0 we find $w_2(q^*\xi) \cap [B'] = 0$. Therefore $w_2(q^*\xi) = 0$, by Poincaré duality for B'. □

Melvin has determined criteria for the total spaces of S^2-bundles over a compact surface to be diffeomorphic, in terms of their Stiefel-Whitney classes. We shall give an alternative argument for the cases with aspherical base.

Lemma 5.12 *Let B be a closed surface and w be the Poincaré dual of $w_1(B)$. If u_1 and u_2 are elements of $H_1(B; \mathbb{F}_2) - \{0, w\}$ such that $u_1.u_1 = u_2.u_2$ then there is a homeomorphism $f : B \to B$ which is a composite of Dehn twists about two-sided essential simple closed curves and such that $f_*(u_1) = u_2$.*

Proof For simplicity of notation, we shall use the same symbol for a simple closed curve u on B and its homology class in $H_1(B; \mathbb{F}_2)$. The curve u is two-sided if and only if $u.u = 0$. In that case we shall let c_u denote the automorphism of $H_1(B; \mathbb{F}_2)$ induced by a Dehn twist about u. Note also that $u.u = u.w$ and $c_v(u) = u + (u.v)v$ for all u and two-sided v in $H_1(B; \mathbb{F}_2)$.

If B is orientable it is well known that the group of isometries of the intersection form acts transitively on $H_1(B; \mathbb{F}_2)$, and is generated by the automorphisms c_u. Thus the claim is true in this case.

If $w_1(B)^2 \neq 0$ then $B \cong RP^2 \sharp T_g$, where T_g is orientable. If $u_1.u_1 = u_2.u_2 = 0$ then u_1 and u_2 are represented by simple closed curves in T_g, and so are related by a homeomorphism which is the identity on the RP^2 summand. If $u_1.u_1 = u_2.u_2 = 1$ let $v_i = u_i + w$. Then $v_i.v_i = 0$ and this case follows from the earlier one.

Suppose finally that $w_1(B) \neq 0$ but $w_1(B)^2 = 0$; equivalently, that $B \cong Kb\sharp T_g$, where T_g is orientable. Let $\{w, z\}$ be a basis for the homology of the Kb summand. In this case w is represented by a 2-sided curve. If $u_1.u_1 = u_2.u_2 = 0$ and $u_1.z = u_2.z = 0$ then u_1 and u_2 are represented by simple closed curves in T_g, and so are related by a homeomorphism which is the identity on the Kb summand. The claim then follows if $u.z = 1$ for $u = u_1$ or u_2, since we then have $c_w(u).c_w(u) = c_w(u).z = 0$. If $u.u \neq 0$ and $u.z = 0$ then $(u+z).(u+z) = 0$ and $c_{u+z}(u) = z$. If $u.u \neq 0$, $u.z \neq 0$ and $u \neq z$ then $c_{u+z+w}c_w(u) = z$. Thus if $u_1.u_1 = u_2.u_2 = 1$ both u_1 and u_2 are related to z. Thus in all cases the claim is true. □

Theorem 5.13 (Melvin) *Let ξ and ξ' be two S^2-bundles over an aspherical closed surface B. Then the following conditions are equivalent:*

(1) *there is a diffeomorphism $f : B \to B$ such that $\xi = f^*\xi'$;*
(2) *the total spaces $E(\xi)$ and $E(\xi')$ are diffeomorphic; and*
(3) *$w_1(\xi) = w_1(\xi')$ if $w_1(\xi) = 0$ or $w_1(B)$, $w_1(\xi) \cup w_1(B) = w_1(\xi') \cup w_1(B)$ and $w_2(\xi) = w_2(\xi')$.*

Proof Clearly (1) implies (2). A diffeomorphism $h : E \to E'$ induces an isomorphism on fundamental groups; hence there is a diffeomorphism $f : B \to B$ such that fp is homotopic to $p'h$. Now $h^*w(E') = w(E)$ and $f^*w(B) = w(B)$. Hence $p^*f^*w(\xi') = p^*w(\xi)$ and so $w(f^*\xi') = f^*w(\xi') = w(\xi)$. Thus $f^*\xi' = \xi$, by Theorem 5.10, and so (2) implies (1).

If (1) holds then $f^*w(\xi') = w(\xi)$. Since $w_1(B) = v_1(B)$ is the characteristic element for the cup product pairing from $H^1(B;\mathbb{F}_2)$ to $H^2(B;\mathbb{F}_2)$ and $H^2(f;\mathbb{F}_2)$ is the identity $f^*w_1(B) = w_1(B)$, $w_1(\xi) \cup w_1(B) = w_1(\xi') \cup w_1(B)$ and $w_2(\xi) = w_2(\xi')$. Hence(1) implies (3).

If $w_1(\xi) \cup w_1(B) = w_1(\xi') \cup w_1(B)$ and $w_1(\xi)$ and $w_1(\xi')$ are neither 0 nor $w_1(B)$ then there is a diffeomorphism $f : B \to B$ such that $f^*w_1(\xi') = w_1(\xi)$, by Lemma 5.12 (applied to the Poincaré dual homology classes). Hence (3) implies (1). □

Corollary 5.13.1 *There are 4 diffeomorphism classes of S^2-bundle spaces if B is orientable and $\chi(B) \leq 0$, 6 if $B = Kb$ and 8 if B is nonorientable and $\chi(B) < 0$.* □

See [Me84] for a more geometric argument, which applies also to S^2-bundles over surfaces with nonempty boundary. The theorem holds also when $B = S^2$ or RP^2; there are 2 such bundles over S^2 and 4 over RP^2. (See Chapter 12.)

Theorem 5.14 *Let M be a PD_4-complex with fundamental group π. The following are equivalent:*

(1) *M has a covering space of degree ≤ 2 which is homotopy equivalent to the total space of an S^2-bundle over an aspherical closed surface;*

(2) *the universal covering space $\widetilde{M}$ is homotopy equivalent to S^2;*

(3) *$\pi \neq 1$ and $\pi_2(M) \cong Z$.*

If these conditions hold the kernel K of the natural action of π on $\pi_2(M)$ is a PD_2-group.

Proof Clearly (1) implies (2) and (2) implies (3). Suppose that (3) holds. If π is finite and $\pi_2(M) \cong Z$ then $\widetilde{M} \simeq CP^2$, and so admits no nontrivial free group actions, by the Lefshetz fixed point theorem. Hence π must be infinite. Then $H_0(\widetilde{M};\mathbb{Z}) = Z$, $H_1(\widetilde{M};\mathbb{Z}) = 0$ and $H_2(\widetilde{M};\mathbb{Z}) = \pi_2(M)$, while $H_3(\widetilde{M};\mathbb{Z}) \cong \overline{H^1(\pi;\mathbb{Z}[\pi])}$ and $H_4(\widetilde{M};\mathbb{Z}) = 0$. Now $Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi]) = 0$, since π is infinite and $\pi_2(M) \cong Z$. Therefore $H^2(\pi;\mathbb{Z}[\pi])$ is infinite cyclic, by Lemma 3.3, and so π is virtually a PD_2-group, by Bowditch's Theorem. Hence $H_3(\widetilde{M};\mathbb{Z}) = 0$ and so $\widetilde{M} \simeq S^2$. If C is a finite cyclic subgroup of K then $H_{n+3}(C;\mathbb{Z}) \cong H_n(C;H_2(\widetilde{M};\mathbb{Z}))$ for all $n \geq 2$, by Lemma 2.10. Therefore C must be trivial, so K is torsion free. Hence K is a PD_2-group and (1) now follows from Theorem 5.10. □

A straightfoward Mayer-Vietoris argument may be used to show directly that if $H^2(\pi;\mathbb{Z}[\pi]) \cong Z$ then π has one end.

Lemma 5.15 *Let X be a finite 2-complex. Then there are natural bijections $[X;BSO(3)] \cong [X;BE(RP^2)] \cong H^2(X;\mathbb{F}_2)$.*

Proof Let $(1,0,0)$ and $[1:0:0]$ be the base points for S^2 and RP^2 respectively. A based self homotopy equivalence f of RP^2 lifts to a based self homotopy equivalence F^+ of S^2. If f is based homotopic to the identity then $deg(f^+) = 1$. Conversely, any based self homotopy equivalence is based homotopic to a map which is the identity on RP^1; if moreover $deg(f^+) = 1$ then this map is the identity on the normal bundle and it quickly follows that f is based homotopic to the identity. Thus $E_0(RP^2)$ has two components. The homeomorphism g defined by $g([x:y:z]) = [x:y:-z]$ is isotopic to the identity (rotate in the (x,y)-coordinates). However $deg(g^+) = -1$. It follows that $E(RP^2)$ is connected. As every self homotopy equivalence of RP^2 is covered by a degree 1 self map of S^2, there is a natural map from $E(RP^2)$ to $E^+(S^2)$.

We may use obstruction theory to show that $\pi_1(E_0(RP^2))$ has order 2. Hence $\pi_1(E(RP^2))$ has order at most 4. Suppose that there were a homotopy f_t through self maps of RP^2 with $f_0 = f_1 = id_{RP^2}$ and such that the loop $f_t(*)$ is essential, where $*$ is a basepoint. Let F be the map from $RP^2 \times S^1$ to RP^2 determined by $F(p,t) = f_t(p)$, and let α and β be the generators of $H^1(RP^2;\mathbb{F}_2)$ and $H^1(S^1;\mathbb{F}_2)$, respectively. Then $F^*\alpha = \alpha\otimes 1 + 1\otimes\beta$ and so $(F^*\alpha)^3 = \alpha^2\otimes\beta$ which is nonzero, contradicting $\alpha^3 = 0$. Thus there can be no such homotopy, and so the homomorphism from $\pi_1(E(RP^2))$ to $\pi_1(RP^2)$ induced by the evaluation map must be trivial. It then follows from the exact sequence of homotopy for this evaluation map that the order of $\pi_1(E(RP^2))$ is at most 2. The group $SO(3)\cong O(3)/(\pm I)$ acts isometrically on RP^2. As the composite of the maps on π_1 induced by the inclusions $SO(3)\subset E(RP^2)\subset E^+(S^2)$ is an isomorphism of groups of order 2 the first map also induces an isomorphism. It follows as in Lemma 5.9 that there are natural bijections $[X;BSO(3)]\cong[X;BE(RP^2)]\cong H^2(X;\mathbb{F}_2)$. ∎

Thus there is a natural 1-1 correspondance between RP^2-bundles and orientable spherical fibrations over such complexes. The RP^2-bundle corresponding to an orientable S^2-bundle is the quotient by the fibrewise antipodal involution. In particular, there are two RP^2-bundles over each closed aspherical surface.

Theorem 5.16 *Let M be a PD_4-complex and B an aspherical closed surface. Then M is homotopy equivalent to the total space of an RP^2-bundle over B if and only if $\pi_1(M)\cong\pi_1(B)\times(Z/2Z)$ and $\chi(M)=\chi(B)$.*

Proof If E is the total space of an RP^2-bundle over B, with projection p, then $\chi(E)=\chi(B)$ and the long exact sequence of homotopy gives a short exact sequence $1\to Z/2Z\to\pi_1(E)\to\pi_1(B)\to 1$. Since the fibre has a product neighbourhood, $j^*w_1(E)=w_1(RP^2)$, where $j: RP^2\to E$ is the inclusion of the fibre over the basepoint of B, and so $w_1(E)$ considered as a homomorphism from $\pi_1(E)$ to $Z/2Z$ splits the injection j_*. Therefore $\pi_1(E)\cong\pi_1(B)\times(Z/2Z)$ and so the conditions are necessary, as they are clearly invariant under homotopy.

Suppose that they hold, and let $w:\pi_1(M)\to Z/2Z$ be the projection onto the $Z/2Z$ factor. Then the covering space associated with the kernel of w satisfies the hypotheses of Theorem 5.10 and so $\widetilde{M}\simeq S^2$. Therefore the homotopy fibre of the map h from M to B inducing the projection of $\pi_1(M)$ onto $\pi_1(B)$ is homotopy equivalent to RP^2. The map h is fibre homotopy equivalent to the projection of an RP^2-bundle over B, by Lemma 5.15. ∎

We may use the above results to refine some of the conclusions of Theorem 3.9 on PD_4-complexes with finitely dominated covering spaces.

Theorem 5.17 *Let M be a PD_4-complex and $p:\widehat{M}\to M$ a regular covering map, with covering group $G=Aut(p)$. If the covering space $\widehat{M}$ is finitely dominated and $H^2(G;\mathbb{Z}[G])\cong Z$ then M has a finite covering space which is homotopy equivalent to a closed 4-manifold which fibres over an aspherical closed surface.*

Proof By Bowditch's Theorem G is virtually a PD_2-group. Therefore as $\widehat{M}$ is finitely dominated it is homotopy equivalent to a closed surface, by [Go79]. The result then follows as in Theorems 5.2, 5.10 and 5.16. □

Note that by Theorem 3.11 and the remarks in the paragraph preceding it the total spaces of such bundles with base an aspherical surface have minimal Euler characteristic for their fundamental groups (i.e. $\chi(M)=q(\pi)$).

Can the hypothesis that $\widehat{M}$ be finitely dominated be replaced by the more algebraic hypothesis that the chain complex of the universal cover $C_*(\tilde{M})$ be chain homotopy equivalent over $\mathbb{Z}[\pi_1(\widehat{M})]$ to a complex of free $\mathbb{Z}[\pi_1(\widehat{M})]$-modules which is finitely generated in degrees ≤ 2? One might hope to adapt the strategy of Theorem 4.5, by using cup-product with a generator of $H^2(G;\mathbb{Z}[G])\cong Z$ to relate the equivariant cohomology of $\widehat{M}$ to that of M. (See also [Ba80'].)

Theorem 5.18 *A PD_4-complex M is homotopy equivalent to the total space of a surface bundle over T or Kb if and only if $\pi=\pi_1(M)$ is an extension of Z^2 or $Z\times_{-1}Z$ (respectively) by an FP_2 normal subgroup K and $\chi(M)=0$.*

Proof The conditions are clearly necessary. If they hold then the covering space associated to the subgroup K is homotopy equivalent to a closed surface, by Corollary 4.5.3 together with Corollary 2.12.1, and so the theorem follows from Theorems 5.2, 5.10 and 5.16. □

In particular, if π is the nontrivial extension of Z^2 by $Z/2Z$ then $q(\pi)>0$.

5.4 Bundles over S^2

Since S^2 is the union of two discs along a circle, an F-bundle over S^2 is determined by the homotopy class of the clutching function, which is an element of $\pi_1(Diff(F))$.

Theorem 5.19 *Let M be a PD_4-complex with fundamental group π and F a closed surface. Then M is homotopy equivalent to the total space of an F-bundle over S^2 if and only if $\chi(M)=2\chi(F)$ and*

(1) *(when $\chi(F)<0$ and $w_1(F)=0$) $\pi\cong\pi_1(F)$ and $w_1(M)=w_2(M)=0$; or*

(2) *(when $\chi(F)<0$ and $w_1(F)\neq 0$) $\pi\cong\pi_1(F)$, $w_1(M)\neq 0$ and $w_2(M)=w_1(M)^2=(c_M^*w_1(F))^2$; or*

(3) *(when $F=T$) $\pi\cong Z^2$ and $w_1(M)=w_2(M)=0$, or $\pi\cong Z\oplus(Z/nZ)$ for some $n>0$ and, if $n=1$ or 2, $w_1(M)=0$; or*

(4) *(when $F=Kb$) $\pi\cong Z\times_{-1}Z$, $w_1(M)\neq 0$ and $w_2(M)=w_1(M)^2=0$, or π has a presentation $\langle x,y\mid yxy^{-1}=x^{-1},\ y^{2n}=1\rangle$ for some $n>0$, where $w_1(M)(x)=0$ and $w_1(M)(y)=1$, and there is a map $p:M\to S^2$ which induces an epimorphism on π_3; or*

(5) *(when $F=S^2$) $\pi=1$ and the index $\sigma(M)=0$; or*

(6) *(when $F=RP^2$) $\pi=Z/2Z$, $w_1(M)\neq 0$ and there is a class u of infinite order in $H^2(M;\mathbb{Z})$ and such that $u^2=0$.*

Proof Let $p_E:E\to S^2$ be such a bundle. Then $\chi(E)=2\chi(F)$ and $\pi_1(E)\cong\pi_1(F)/\partial\pi_2(S^2)$, where $\mathrm{Im}(\partial)\leq\zeta\pi_1(F)$ [Go68]. The characteristic classes of E restrict to the characteristic classes of the fibre, as it has a product neighbourhood. As the base is 1-connected E is orientable if and only if the fibre is orientable. Thus the conditions on χ, π and w_1 are all necessary. We shall treat the other assertions case by case.

(1) and (2) If $\chi(F)<0$ any F-bundle over S^2 is trivial, by Lemma 5.1. Thus the conditions are necessary. Conversely, if they hold then c_M is fibre homotopy equivalent to the projection of an S^2-bundle ξ with base F, by Theorem 5.10. The conditions on the Stiefel-Whitney classes then imply that $w(\xi)=1$ and hence that the bundle is trivial, by Lemma 5.11. Therefore M is homotopy equivalent to $S^2\times F$.

(3) If $\partial=0$ there is a map $q:E\to T$ which induces an isomorphism of fundamental groups, and the map $(p_E,q):E\to S^2\times T$ is clearly a homotopy equivalence, so $w(E)=1$. Conversely, if $\chi(M)=0$, $\pi\cong Z^2$ and $w(M)=1$ then M is homotopy equivalent to $S^2\times T$, by Theorem 5.10 and Lemma 5.11.

If $\chi(M)=0$ and $\pi\cong Z\oplus(Z/nZ)$ for some $n>0$ then the covering space $M_{Z/nZ}$ corresponding to the torsion subgroup Z/nZ is homotopy equivalent to a lens space L, by Corollary 4.5.3. As observed in Chapter 4 the manifold

M is homotopy equivalent to the mapping torus of a generator of the group of covering transformations $Aut(M_{Z/nZ}/M) \cong Z$. Since the generator induces the identity on $\pi_1(L) \cong Z/nZ$ it is homotopic to id_L, if $n > 2$. This is also true if $n = 1$ or 2 and M is orientable. (See Section 29 of [Co].) Therefore M is homotopy equivalent to $L \times S^1$, which fibres over S^2 via the composition of the projection to L with the Hopf fibration of L over S^2. (Hence $w(M) = 1$ in these cases also.)

(4) As in part (3), if $\pi_1(E) \cong Z \times_{-1} Z = \pi_1(Kb)$ then E is homotopy equivalent to $S^2 \times Kb$ and so $w_1(E) \neq 0$ while $w_2(E) = 0$. Conversely, if $\chi(M) = 0$, $\pi \cong \pi_1(Kb)$, M is nonorientable and $w_1(M)^2 = w_2(M) = 0$ then M is homotopy equivalent to $S^2 \times Kb$. Suppose now that $\partial \neq 0$. The homomorphism $\pi_3(p_E)$ induced by the bundle projection is an epimorphism. Conversely, if M satisfies these conditions and $q : M^+ \to M$ is the orientation double cover then M^+ satisfies the hypotheses of part (3), and so $\widetilde{M} \simeq S^3$. Therefore as $\pi_3(p)$ is onto the composition of the projection of $\widetilde{M}$ onto M with p is essentially the Hopf map, and so induces isomorphisms on all higher homotopy groups. Hence the homotopy fibre of p is aspherical. As $\pi_2(M) = 0$ the fundamental group of the homotopy fibre of p is a torsion free extension of π by Z, and so the homotopy fibre must be Kb. As in Theorem 5.2 above the map p is fibre homotopy equivalent to a bundle projection.

(5) There are just two S^2-bundles over S^2, with total spaces $S^2 \times S^2$ and $S^2 \tilde{\times} S^2 = CP^2 \sharp - CP^2$, respectively. Thus the conditions are necessary. If M satisfies these conditions then $H^2(M;\mathbb{Z}) \cong Z^2$ and there is an element u in $H^2(M;\mathbb{Z})$ which generates an infinite cyclic direct summand and has square $u \cup u = 0$. Thus $u = f^* i_2$ for some map $f : M \to S^2$, where i_2 generates $H^2(S^2;\mathbb{Z})$, by Theorem 8.4.11 of [Sp]. Since u generates a direct summand there is a homology class z in $H_2(M;\mathbb{Z})$ such that $u \cap z = 1$, and therefore (by the Hurewicz theorem) there is a map $z : S^2 \to M$ such that fz is homotopic to id_{S^2}. The homotopy fibre of f is 1-connected and has $\pi_2 \cong Z$, by the long exact sequence of homotopy. It then follows easily from the spectral sequence for f that the homotopy fibre has the homology of S^2. Therefore f is fibre homotopy equivalent to the projection of an S^2-bundle over S^2.

(6) Since $\pi_1(Diff(RP^2)) = Z/2Z$ (see page 21 of [EE69]) there are two RP^2-bundles over S^2. Again the conditions are clearly necessary. If they hold then $u = g^* i_2$ for some map $g : M \to S^2$. Let $q : M^+ \to M$ be the orientation double cover and $g^+ = gq$. Since $H_2(Z/2Z;\mathbb{Z}) = 0$ the second homology of M is spherical. As we may assume u generates an infinite cyclic direct summand of $H^2(M;\mathbb{Z})$ there is a map $z = qz^+ : S^2 \to M$ such that $gz = g^+ z^+$ is homotopic to id_{S^2}. Hence the homotopy fibre of g^+ is S^2, by case (5). Since

the homotopy fibre of g has fundamental group $Z/2Z$ and is double covered by the homotopy fibre of g^+ it is homotopy equivalent to RP^2. It follows as in Theorem 5.16 that g is fibre homotopy equivalent to the projection of an RP^2-bundle over S^2. □

Theorems 5.2, 5.10 and 5.16 may each be rephrased as giving criteria for maps from M to B to be fibre homotopy equivalent to fibre bundle projections. With the hypotheses of Theorem 5.19 (and assuming also that $\partial = 0$ if $\chi(M) = 0$) we may conclude that a map $f : M \to S^2$ is fibre homotopy equivalent to a fibre bundle projection if and only if f^*i_2 generates an infinite cyclic direct summand of $H^2(M;\mathbb{Z})$.

Is there a criterion for part (4) which does not refer to π_3? The other hypotheses are not sufficient alone. (See Chapter 11.)

It follows from Theorem 5.10 that the conditions on the Stiefel-Whitney classes are independent of the other conditions when $\pi \cong \pi_1(F)$. Note also that the nonorientable S^3- and RP^3-bundles over S^1 are not T-bundles over S^2, while if $M = CP^2\sharp CP^2$ then $\pi = 1$ and $\chi(M) = 4$ but $\sigma(M) \neq 0$. See Chapter 12 for further information on parts (5) and (6).

5.5 Bundles over RP^2

Since $RP^2 = Mb \cup D^2$ is the union of a Möbius band Mb and a disc D^2, a bundle $p : E \to RP^2$ with fibre F is determined by a bundle over Mb which restricts to a trivial bundle over ∂Mb, i.e. by a conjugacy class of elements of order dividing 2 in $\pi_0(Homeo(F))$, together with the class of a gluing map over $\partial Mb = \partial D^2$ modulo those which extend across D^2 or Mb, i.e. an element of a quotient of $\pi_1(Homeo(F))$. If F is aspherical $\pi_0(Homeo(F)) \cong Out(\pi_1(F))$, while $\pi_1(Homeo(F)) \cong \zeta\pi_1(F)$ [Go65].

We may summarize the key properties of the algebraic invariants of such bundles with F an aspherical closed surface in the following lemma. Let $\tilde{Z}$ be the non-trivial infinite cyclic $Z/2Z$-module. The groups $H^1(Z/2Z;\tilde{Z})$, $H^1(Z/2Z;\mathbb{F}_2)$ and $H^1(RP^2;\tilde{Z})$ are canonically isomorphic to $Z/2Z$.

Lemma 5.20 *Let $p : E \to RP^2$ be the projection of an F-bundle, where F is an aspherical closed surface, and let x be the generator of $H^1(RP^2;\tilde{Z})$. Then*

(1) $\chi(E) = \chi(F)$;

(2) $\partial(\pi_2(RP^2)) \leq \zeta\pi_1(F)$ *and there is an exact sequence of groups*

$$0 \to \pi_2(E) \to Z \xrightarrow{\partial} \pi_1(F) \to \pi_1(E) \to Z/2Z \to 1;$$

(3) *if* $\partial = 0$ *then* $\pi_1(E)$ *has one end and acts nontrivially on* $\pi_2(E) \cong Z$, *and the covering space* E_F *with fundamental group* $\pi_1(F)$ *is homeomorphic to* $S^2 \times F$, *so* $w_1(E)|_{\pi_1(F)} = w_1(E_F) = w_1(F)$ *(as homomorphisms from* $\pi_1(F)$ *to* $Z/2Z$*) and* $w_2(E_F) = w_1(E_F)^2$;

(4) *if* $\partial \neq 0$ *then* $\chi(F) = 0$, $\pi_1(E)$ *has two ends,* $\pi_2(E) = 0$ *and* $Z/2Z$ *acts by inversion on* $\partial(Z)$;

(5) $p^*x^3 = 0 \in H^3(E; p^*\tilde{Z})$.

Proof Condition (1) holds since the Euler characteristic is multiplicative in fibrations, while (2) is part of the long exact sequence of homotopy for p. The image of ∂ is central by [Go68], and is therefore trivial unless $\chi(F) = 0$. Conditions (3) and (4) then follow as the homomorphisms in this sequence are compatible with the actions of the fundamental groups, and E_F is the total space of an F-bundle over S^2, which is a trivial bundle if $\partial = 0$, by Theorem 5.19. Condition (5) holds since $H^3(RP^2; \tilde{Z}) = 0$. □

Let π be a group which is an extension of $Z/2Z$ by a normal subgroup G, and let $t \in \pi$ be an element which maps nontrivially to $\pi/G = Z/2Z$. Then $u = t^2$ is in G and conjugation by t determines an automorphism α of G such that $\alpha(u) = u$ and α^2 is the inner automorphism given by conjugation by u.

Conversely, let α be an automorphism of G whose square is inner, say $\alpha^2(g) = ugu^{-1}$ for all $g \in G$. Let $v = \alpha(u)$. Then $\alpha^3(g) = \alpha^2(\alpha(g)) = u\alpha(g)c^{-1} = \alpha(\alpha^2(g)) = v\alpha(g)v^{-1}$ for all $g \in G$. Therefore vu^{-1} is central. In particular, if the centre of G is trivial α fixes u, and we may define an extension

$$\xi_\alpha : 1 \to G \to \Pi_\alpha \to Z/2Z \to 1$$

in which Π_α has the presentation $\langle G, t_\alpha \mid t_\alpha g t_\alpha^{-1} = \alpha(g),\ t_\alpha^2 = u\rangle$. If β is another automorphism in the same outer automorphism class then ξ_α and ξ_β are equivalent extensions. (Note that if $\beta = \alpha.c_h$, where c_h is conjugation by h, then $\beta(\alpha(h)uh) = \alpha(h)uh$ and $\beta^2(g) = \alpha(h)uh.g.(\alpha(h)uh)^{-1}$ for all $g \in G$.)

Lemma 5.21 *If* $\chi(F) < 0$ *or* $\chi(F) = 0$ *and* $\partial = 0$ *then an* F*-bundle over* RP^2 *is determined up to isomorphism by the corresponding extension of fundamental groups.*

Proof If $\chi(F) < 0$ such bundles and extensions are each determined by an element ξ of order 2 in $Out(\pi_1(F))$. If $\chi(F) = 0$ bundles with $\partial = 0$ are the restrictions of bundles over $RP^\infty = K(Z/2Z, 1)$ (compare Lemma 4.10). Such bundles are determined by an element ξ of order 2 in $Out(\pi_1(F))$ and a cohomology class in $H^2(Z/2Z; \zeta\pi_1(F)^\xi)$, by Lemma 5.1, and so correspond bijectively to extensions also. □

Lemma 5.22 *Let M be a PD_4-complex with fundamental group π. A map $f : M \to RP^2$ is fibre homotopy equivalent to the projection of a bundle over RP^2 with fibre an aspherical closed surface if $\pi_1(f)$ is an epimorphism and either*

(1) $\chi(M) \leq 0$ *and $\pi_2(f)$ is an isomorphism; or*

(2) $\chi(M) = 0$, *π has two ends and $\pi_3(f)$ is an isomorphism.*

Proof In each case π is infinite, by Lemma 3.14. In case (1) $H^2(\pi; \mathbb{Z}[\pi]) \cong Z$ (by Lemma 3.3) and so π has one end, by Bowditch's Theorem. Hence $\widetilde{M} \simeq S^2$. Moreover the homotopy fibre of f is aspherical, and its fundamental group is a surface group. (See Chapter X for details.) In case (2) $\widetilde{M} \simeq S^3$, by Corollary 4.5.3. Hence the lift $\tilde{f} : \widetilde{M} \to S^2$ is homotopic to the Hopf map, and so induces isomorphisms on all higher homotopy groups. Therefore the homotopy fibre of f is aspherical. As $\pi_2(M) = 0$ the fundamental group of the homotopy fibre is a (torsion free) infinite cyclic extension of π and so must be either Z^2 or $Z \times_{-1} Z$. Thus the homotopy fibre of f is homotopy equivalent to T or Kb. In both cases the argument of Theorem 5.2 now shows that f is fibre homotopy equivalent to a surface bundle projection. □

5.6 Bundles over RP^2 with $\partial = 0$

If we assume that the connecting homomorphism $\partial : \pi_2(E) \to \pi_1(F)$ is trivial then conditions (2), (3) and (5) of Lemma 5.20 simplify to conditions on E and the action of $\pi_1(E)$ on $\pi_2(E)$. These conditions almost suffice to characterize the homotopy types of such bundle spaces; there is one more necesssary condition, and for nonorientable manifolds there is a further possible obstruction, of order at most 2.

Theorem 5.23 *Let M be a PD_4-complex and let $m : M_u \to M$ be the covering associated to $\kappa = \mathrm{Ker}(u)$, where $u : \pi = \pi_1(M) \to Aut(\pi_2(M))$ is the natural action. Let x be the generator of $H^1(Z/2Z; \tilde{Z})$. If M is homotopy equivalent to the total space of a fibre bundle over RP^2 with fibre an* *aspherical closed surface and with $\partial = 0$ then $\pi_2(M) \cong Z$, u is surjective, $w_2(M_u) = w_1(M_u)^2$ and u^*x^3 has image 0 in $H^3(M;\mathbb{F}_2)$. Moreover the homomorphism from $H^2(M;Z^u)$ to $H^2(S^2;Z^u)$ induced by a generator of $\pi_2(M)$ is onto. Conversely, if M is orientable these conditions imply that M is homotopy equivalent to such a bundle space. If M is nonorientable there is a further obstruction of order at most 2.*

Proof The necessity of most of these conditions follows from Lemma 5.20. The additional condition holds since the covering projection from S^2 to RP^2 induces an isomorphism $H^2(RP^2;Z^u) \cong H^2(S^2;Z^u) = H^2(S^2;\mathbb{Z})$.

Suppose that they hold. Let $g : S^2 \to P_2(RP^2)$ and $j : S^2 \to M$ represent generators for $\pi_2(P_2(RP^2))$ and $\pi_2(M)$, respectively. After replacing M by a homotopy equivalent space if necessary, we may assume that j is the inclusion of a subcomplex. We may identify u with a map from M to $K(Z/2Z,1)$, via the isomorphism $[M,K(Z/2Z,1)] \cong Hom(\pi,Z/2Z)$. The only obstruction to the construction of a map from M to $P_2(RP^2)$ which extends g and lifts u lies in $H^3(M,S^2;Z^u)$, since $u^*\pi_2(RP^2)) \cong Z^u$. This group maps injectively to $H^3(M;Z^u)$, since restriction maps $H^2(M;Z^u)$ onto $H^2(S^2;Z^u)$, and so this obstruction is 0, since its image in $H^3(M;Z^u)$ is $u^*k_1(RP^2) = u^*x^3 = 0$. Therefore there is a map $h : M \to P_2(RP^2)$ such that $\pi_1(h) = u$ and $\pi_2(h)$ is an isomorphism. The set of such maps is parametrized by $H^2(M,S^2;Z^u)$.

As $Z/2Z$ acts trivially on $\pi_3(RP^2) \cong Z$ the second k-invariant of RP^2 lies in $H^4(P_2(RP^2);Z)$. This group is infinite cyclic, and is generated by $t = k_2(RP^2)$. (See §3.12 of [Si67].) The obstruction to lifting h to a map from M to $P_3(RP^2)$ is h^*t. Let $n : \widetilde{P}_2(RP^2) \to P_2(RP^2)$ be the universal covering, and let z be a generator of $H^2(\widetilde{P}_2(RP^2);\mathbb{Z}) \cong Z$. Then h lifts to a map $h_u : M_u \to \widetilde{P}_2(RP^2)$, so that $nh_u = hm$. (Note that h_u is determined by h_u^*z, since $\widetilde{P}_2(RP^2) \simeq K(Z,2)$.)

The covering space M_u is homotopy equivalent to the total space of an S^2-bundle $q : E \to F$, where F is an aspherical closed surface, by Theorem 5.14. Since κ acts trivially on $\pi_2(M_u)$ the bundle is orientable (i.e., $w_1(q) = 0$) and so $q^*w_2(q) = w_2(E) + w_1(E)^2$, by the Whitney sum formula. Therefore $q^*w_2(q) = 0$, since $w_2(M_u) = w_1(M_u)^2$, and so $w_2(q) = 0$, since q is 2-connected. Hence the bundle is trivial, by Lemma 5.11, and so M_u is homotopy equivalent to $S^2 \times F$. Let j_F and j_S be the inclusions of the factors. Then $h_u j_S$ generates $\pi_2(P_2)$. We may choose h so that $h_u j_F$ is null homotopic. Then h_u^*z is Poincaré dual to $j_{F*}[F]$, and so $h_u^*z^2 = 0$, since $j_{F*}[F]$ has self intersection 0. As n^*t is a multiple of z^2, it follows that $m^*h^*t = 0$.

If M is orientable $m^* = H^4(m;\mathbb{Z})$ is a monomorphism and so $h^*t = 0$. Hence h lifts to a map $f : M \to P_3(RP^2)$. As $P_3(RP^2)$ may be constructed from RP^2 by adjoining cells of dimension at least 5 we may assume that f maps M into RP^2, after a homotopy if necessary. Since $\pi_1(f) = u$ is an epimorphism and $\pi_2(f)$ is an isomorphism f is fibre homotopy equivalent to the projection of an F-bundle over RP^2, by Lemma 5.22.

In general, we may assume that h maps the 3-skeleton $M^{[3]}$ to RP^2. Let w be a generator of $H^2(P_2(RP^2);\tilde{Z}) \cong H^2(RP^2;\tilde{Z}) \cong Z$ and define a function $\mu : H^2(M;Z^u) \to H^4(M;\mathbb{Z})$ by $\mu(g) = g \cup g + g \cup h^*w$ for all $g \in H^2(M;Z^u)$. If M is nonorientable $H^4(M;\mathbb{Z}) = Z/2Z$ and μ is a homomorphism. The sole obstruction to extending $h|_{M^{[3]}}$ to a map $f : M \to RP^2$ is the image of h^*t in $\mathrm{Coker}(\mu)$, which is independent of the choice of lift h. (See §3.24 of [Si67].) □

Are these hypotheses independent? A closed 4-manifold M with $\pi = \pi_1(M)$ a PD_2-group and $\pi_2(M) \cong Z$ is homotopy equivalent to the total space of an S^2-bundle $p : E \to B$, where B is an aspherical closed surface. Therefore if u is nontrivial $M_u \simeq E^+$, where $q : E^+ \to B^+$ is the bundle induced over a double cover of B. As $w_1(q) = 0$ and $q^*w_2(q) = 0$, by part (3) of Lemma 5.11, we have $w_1(E^+) = q^*w_1(B^+)$ and $w_2(E^+) = q^*w_2(B^+)$, by the Whitney sum formula. Hence $w_2(M_u) = w_1(M_u)^2$. (In particular, $w_2(M_u) = 0$ if M is orientable.) Moreover since $c.d.\pi = 2$ the condition $u^*x^3 = 0$ is automatic. (It shall follow directly from the results of Chapter 10 that any such S^2-bundle space with u nontrivial fibres over RP^2, even if it is not orientable.)

On the other hand, if $Z/2Z$ is a (semi)direct factor of π the cohomology of $Z/2Z$ is a direct summand of that of π and so the image of x^3 in $H^3(\pi;\tilde{Z})$ is nonzero.

Is the obstruction always 0 in the nonorientable cases?

Chapter 6

Simple homotopy type and surgery

The problem of determining the high-dimensional manifolds within a given homotopy type has been successfully reduced to the determination of normal invariants and surgery obstructions. This strategy applies also in dimension 4, provided that the fundamental group is in the class SA generated from groups with subexponential growth by extensions and increasing unions [FT95]. (Essentially all the groups in this class that we shall discuss in this book are in fact virtually solvable). We may often avoid this hypothesis by using 5-dimensional surgery to construct s-cobordisms.

We begin by showing that the Whitehead group of the fundamental group is trivial for surface bundles over surfaces, most circle bundles over geometric 3-manifolds and for many mapping tori. In §2 we define the modified surgery structure set, parametrizing s-cobordism classes of simply homotopy equivalences of closed 4-manifolds. This notion allows partial extensions of surgery arguments to situations where the fundamental group is not elementary amenable. Although many papers on surgery do not explicitly consider the 4-dimensional cases, their results may often be adapted to these cases. In §3 we comment briefly on approaches to the s-cobordism theorem and classification using stabilization by connected sum with copies of $S^2 \times S^2$ or by cartesian product with R.

In §4 we show that 4-manifolds M such that $\pi = \pi_1(M)$ is torsion free virtually poly-Z and $\chi(M) = 0$ are determined up to homeomorphism by their fundamental group (and Stiefel-Whitney classes, if $h(\pi) < 4$). We also characterize 4-dimensional mapping tori with torsion free, elementary amenable fundamental group and show that the structure sets for total spaces of RP^2-bundles over T or Kb are finite. In §5 we extend this finiteness to RP^2-bundle spaces over closed hyperbolic surfaces and show that total spaces of bundles with fibre S^2 or an aspherical closed surface over aspherical bases are determined up to s-cobordism by their homotopy type. (We shall consider bundles with base or fibre geometric 3-manifolds in Chapter 13).

6.1 The Whitehead group

In this section we shall rely heavily upon the work of Waldhausen in [Wd78]. The class of groups *Cl* is the smallest class of groups containing the trivial group and which is closed under generalised free products and HNN extensions with amalgamation over regular coherent subgroups and under filtering direct limit. This class is also closed under taking subgroups, by Proposition 19.3 of [Wd78]. If G is in *Cl* then $Wh(G)=0$, by Theorem 19.4 of [Wd78]. The argument for this theorem actually shows that if $G\cong A*_C B$ and C is regular coherent then there are "Mayer-Vietoris" sequences:

$$Wh(A)\oplus Wh(B)\to Wh(G)\to \tilde{K}(\mathbb{Z}[C])\to \tilde{K}(\mathbb{Z}[A])\oplus\tilde{K}(\mathbb{Z}[B])\to \tilde{K}(\mathbb{Z}[G])\to 0,$$

and similarly if $G\cong A*_C$. (See Sections 17.1.3 and 17.2.3 of [Wd78]).

The class *Cl* contains all free groups and poly-Z groups and the class $\mathcal{X}$ of Chapter 2. (In particular, all the groups $Z*_m$ are in *Cl*). Since every PD_2-group is either poly-Z or is the generalised free product of two free groups with amalgamation over infinite cyclic subgroups it is regular coherent, and is in *Cl*. Hence homotopy equivalences between S^2-bundles over aspherical surfaces are simple. The following extension implies the corresponding result for quotients of such bundle spaces by free involutions.

Theorem 6.1 *Let π be a semidirect product $\rho\tilde{\times}(Z/2Z)$ where ρ is a surface group. Then $Wh(\pi)=0$.*

Proof Assume first that $\pi\cong\rho\times(Z/2Z)$. Let $\Gamma=\mathbb{Z}[\rho]$. There is a cartesian square expressing $\Gamma[Z/2Z]=\mathbb{Z}[\rho\times(Z/2Z)]$ as the pullback of the reduction of coefficients map from Γ to $\Gamma_2=\Gamma/2\Gamma=\mathbb{Z}/2\mathbb{Z}[\rho]$ over itself. (The two maps from $\Gamma[Z/2Z]$ to Γ send the generator of $Z/2Z$ to $+1$ and -1, respectively). The Mayer-Vietoris sequence for algebraic K-theory traps $K_1(\Gamma[Z/2Z])$ between $K_2(\Gamma_2)$ and $K_1(\Gamma)^2$ (see Theorem 6.4 of [Mi]). Now since $c.d.\rho=2$ the higher K-theory of $R[\rho]$ can be computed in terms of the homology of ρ with coefficients in the K-theory of R (cf. the Corollary to Theorem 5 of the introduction of [Wd78]). In particular, the map from $K_2(\Gamma)$ to $K_2(\Gamma_2)$ is onto, while $K_1(\Gamma)=K_1(\mathbb{Z})\oplus(\rho/\rho')$ and $K_1(\Gamma_2)=\rho/\rho'$. It now follows easily that $K_1(\Gamma[Z/2Z])$ is generated by the images of $K_1(\mathbb{Z})=\{\pm1\}$ and $\rho\times(Z/2Z)$, and so $Wh(\rho\times(Z/2Z))=0$.

If $\pi=\rho\tilde{\times}(Z/2Z)$ is not such a direct product it is isomorphic to a discrete subgroup of $Isom(\mathbb{X})$ which acts properly discontinuously on X, where $\mathbb{X}=\mathbb{E}^2$ or $\mathbb{H}^2$. (See [EM82], [Zi]). The singularities of the corresponding 2-orbifold

X/π are either cone points of order 2 or reflector curves; there are no corner points and no cone points of higher order. Let $|X/\pi|$ be the surface obtained by forgetting the orbifold structure of X/π, and let m be the number of cone points. Then $\chi(|X/\pi|) - (m/2) = \chi_{orb}(X/\pi) \leq 0$, by the Riemann-Hurwitz formula [Sc83'], so either $\chi(|X/\pi|) \leq 0$ or $\chi(|X/\pi|) = 1$ and $m \geq 2$ or $|X/\pi| \cong S^2$ and $m \geq 4$.

We may separate X/π along embedded circles (avoiding the singularities) into pieces which are either (i) discs with at least two cone points; (ii) annuli with one cone point; (iii) annuli with one boundary a reflector curve; or (iv) surfaces other than D^2 with nonempty boundary. In each case the inclusions of the separating circles induce monomorphisms on orbifold fundamental groups, and so π is a generalized free product with amalgamation over copies of Z of groups of the form (i) $*^m(Z/2Z)$ (with $m \geq 2$); (ii) $Z * (Z/2Z)$; (iii) $Z \oplus (Z/2Z)$; or (iv) $*^m Z$, by the Van Kampen theorem for orbifolds [Sc83]. The Mayer-Vietoris sequences for algebraic K-theory now give $Wh(\pi) = 0$. □

The argument for the direct product case is based on one for showing that $Wh(Z \oplus (Z/2Z)) = 0$ from [Kw86].

Not all such orbifold groups arise in this way. For instance, the orbifold fundamental group of a torus with one cone point of order 2 has the presentation $\langle x, y \mid [x, y]^2 = 1\rangle$. Hence it has torsion free abelianization, and so cannot be a semidirect product as above.

The orbifold fundamental groups of flat 2-orbifolds are the 2-dimensional crystallographic groups. Their finite subgroups are cyclic or dihedral, of order properly dividing 24, and have trivial Whitehead group. In fact $Wh(\pi) = 0$ for π any such 2-dimensional crystallographic group [Pe98]. (If π is the fundamental group of an orientable hyperbolic 2-orbifold with k cone points of orders $\{n_1, \dots n_k\}$ then $Wh(\pi) \cong \oplus_{i=1}^k Wh(Z/n_iZ)$ [LS00]).

The argument for the next result is essentially due to F.T.Farrell.

Theorem 6.2 *If π is an extension of $\pi_1(B)$ by $\pi_1(F)$ where B and F are aspherical closed surfaces then $Wh(\pi) = 0$.*

Proof If $\chi(B) < 0$ then B admits a complete riemannian metric of constant negative curvature -1. Moreover the only virtually poly-Z subgroups of $\pi_1(B)$ are 1 and Z. If G is the preimage in π of such a subgroup then G is either $\pi_1(F)$ or is the group of a Haken 3-manifold. It follows easily that for any $n \geq 0$ the group $G \times Z^n$ is in Cl and so $Wh(G \times Z^n) = 0$. Therefore any such G

is K-flat and so the bundle is admissible, in the terminology of [FJ86]. Hence $Wh(\pi)=0$ by the main result of that paper.

If $\chi(B)=0$ then this argument does not work, although if moreover $\chi(F)=0$ then π is poly-Z so $Wh(\pi)=0$ by Theorem 2.13 of [FJ]. We shall sketch an argument of Farrell for the general case. Lemma 1.4.2 and Theorem 2.1 of [FJ93] together yield a spectral sequence (with coefficients in a simplicial cosheaf) whose E^2 term is $H_i(X/\pi_1(B);Wh'_j(p^{-1}(\pi_1(B)^x)))$ and which converges to $Wh'_{i+j}(\pi)$. Here $p:\pi\to\pi_1(B)$ is the epimorphism of the extension and X is a certain universal $\pi_1(B)$-complex which is contractible and such that all the nontrivial isotropy subgroups $\pi_1(B)^x$ are infinite cyclic and the fixed point set of each infinite cyclic subgroup is a contractible (nonempty) subcomplex. The Whitehead groups with negative indices are the lower K-theory of $\mathbb{Z}[G]$ (i.e., $Wh'_n(G)=K_n(\mathbb{Z}[G])$ for all $n\leq -1$), while $Wh'_0(G)=\tilde{K}_0(\mathbb{Z}[G])$ and $Wh'_1(G)=Wh(G)$. Note that $Wh'_{-n}(G)$ is a direct summand of $Wh(G\times Z^{n+1})$. If $i+j>1$ then $Wh'_{i+j}(\pi)$ agrees rationally with the higher Whitehead group $Wh_{i+j}(\pi)$. Since the isotropy subgroups $\pi_1(B)^x$ are infinite cyclic or trivial $Wh(p^{-1}(\pi_1(B)^x)\times Z^n)=0$ for all $n\geq 0$, by the argument of the above paragraph, and so $Wh'_j(p^{-1}(\pi_1(B)^x))=0$ if $j\leq 1$. Hence the spectral sequence gives $Wh(\pi)=0$. ∎

A closed 3-manifold is a *Haken manifold* if it is irreducible and contains an incompressible 2-sided surface. Every Haken 3-manifold either has solvable fundamental group or may be decomposed along a finite family of disjoint incompressible tori and Klein bottles so that the complementary components are Seifert fibred or hyperbolic. It is an open question whether every closed irreducible orientable 3-manifold with infinite fundamental group is virtually Haken (i.e., finitely covered by a Haken manifold). (Non-orientable 3-manifolds are Haken). Every virtually Haken 3-manifold is either Haken, hyperbolic or Seifert-fibred, by [CS83] and [GMT96]. A closed irreducible 3-manifold is a *graph manifold* if either it has solvable fundamental group or it may be decomposed along a finite family of disjoint incompressible tori and Klein bottles so that the complementary components are Seifert fibred. (There are several competing definitions of graph manifold in the literature).

Theorem 6.3 *Let $\pi=\nu\times_\theta Z$ where ν is torsion free and is the fundamental group of a closed 3-manifold N which is a connected sum of graph manifolds. Then ν is regular coherent and $Wh(\pi)=0$.*

Proof The group ν is a generalized free product with amalgamation along poly-Z subgroups (1, Z^2 or $Z\times_{-1}Z$) of polycyclic groups and fundamental groups of Seifert fibred 3-manifolds (possibly with boundary). The group rings of torsion free polycyclic groups are regular noetherian, and hence regular coherent. If G is the fundamental group of a Seifert fibred 3-manifold then it has a subgroup G_o of finite index which is a central extension of the fundamental group of a surface B (possibly with boundary) by Z. We may assume that G is not solvable and hence that $\chi(B) < 0$. If ∂B is nonempty then $G_o \cong Z \times F$ and so is an iterated generalized free product of copies of Z^2, with amalgamation along infinite cyclic subgroups. Otherwise we may split B along an essential curve and represent G_o as the generalised free product of two such groups, with amalgamation along a copy of Z^2. In both cases G_o is regular coherent, and therefore so is G, since $[G : G_o] < \infty$ and $c.d.G < \infty$.

Since ν is the generalised free product with amalgamation of regular coherent groups, with amalgamation along poly-Z subgroups, it is also regular coherent. Let N_i be an irreducible summand of N and let $\nu_i = \pi_1(N_i)$. If N_i is Haken then ν_i is in *Cl*. Otherwise N_i is a Seifert fibred 3-manifold which is not sufficiently large, and the argument of [Pl80] extends easily to show that $Wh(\nu_i \times Z^s) = 0$, for any $s \geq 0$. Since $\tilde{K}(\mathbb{Z}[\nu_i])$ is a direct summand of $Wh(\nu_i \times Z)$, it follows that in all cases $\tilde{K}(\mathbb{Z}[\nu_i]) = Wh(\nu_i) = 0$. The Mayer-Vietoris sequences for algebraic K-theory now give firstly that $Wh(\nu) = \tilde{K}(\mathbb{Z}[\nu]) = 0$ and then that $Wh(\pi) = 0$ also. □

All 3-manifold groups are coherent as *groups* [Hm]. If we knew that their group *rings* were regular coherent then we could use [Wd78] instead of [FJ86] to give a purely algebraic proof of Theorem 6.2, for as surface groups are free products of free groups with amalgamation over an infinite cyclic subgroup, an extension of one surface group by another is a free product of groups with $Wh = 0$, amalgamated over the group of a surface bundle over S^1. Similarly, we could deduce from [Wd78] that $Wh(\nu \times_\theta Z) = 0$ for any torsion free group $\nu = \pi_1(N)$ where N is a closed 3-manifold whose irreducible factors are Haken, hyperbolic or Seifert fibred.

Theorem 6.4 *Let μ be a group with an infinite cyclic normal subgroup A such that $\nu = \mu/A$ is torsion free and is a free product $\nu = *_{1 \leq i \leq n} \nu_i$ where each factor is the fundamental group of an irreducible 3-manifold which is Haken, hyperbolic or Seifert fibred. Then $Wh(\mu) = Wh(\nu) = 0$.*

Proof (Note that our hypotheses allow the possibility that some of the factors ν_i are infinite cyclic). Let μ_i be the preimage of ν_i in μ, for $1 \leq i \leq n$. Then μ is the generalized free product of the μ_i's, amalgamated over infinite cyclic subgroups. For all $1 \leq i \leq n$ we have $Wh(\mu_i) = 0$, by Lemma 1.1 of [St84] if $K(\nu_i, 1)$ is Haken, by the main result of [FJ86] if it is hyperbolic, by an easy extension of the argument of [Pl80] if it is Seifert fibred but not Haken and by Theorem 19.5 of [Wd78] if ν_i is infinite cyclic. The Mayer-Vietoris sequences for algebraic K-theory now give $Wh(\mu) = Wh(\nu) = 0$ also. □

Theorem 6.4 may be used to strengthen Theorem 4.11 to give criteria for a closed 4-manifold M to be *simple* homotopy equivalent to the total space of an S^1-bundle, if the irreducible summands of the base N are all virtually Haken and $\pi_1(M)$ is torsion free.

6.2 The s-cobordism structure set

Let M be a closed 4-manifold with fundamental group π and orientation character $w : \pi \to \{\pm 1\}$, and let G/TOP have the H-space multiplication determined by its loop space structure. Then the surgery obstruction maps $\sigma_{4+i} = \sigma_{4+i}^M : [M \times D^i, \partial(M \times D^i); G/TOP, \{*\}] \to L_{4+i}^s(\pi, w)$ are homomorphisms. If π is in the class SA then $L_5^s(\pi, w)$ acts on $S_{TOP}(M)$, and the surgery sequence

$$[SM; G/TOP] \xrightarrow{\sigma_5} L_5^s(\pi, w) \xrightarrow{\omega} S_{TOP}(M) \xrightarrow{\eta} [M; G/TOP] \xrightarrow{\sigma_4} L_4^s(\pi, w)$$

is an exact sequence of groups and pointed sets, i.e., the orbits of the action ω correspond to the normal invariants $\eta(f)$ of simple homotopy equivalences [FQ, FT95]. As it is not yet known whether 5-dimensional s-cobordisms over other fundamental groups are products, we shall redefine the structure set by setting

$$S_{TOP}^s(M) = \{f : N \to M \mid N \ a \ TOP \ 4\text{-}manifold, \ f \ a \ simple \ h.e.\}/\approx,$$

where $f_1 \approx f_2$ if there is a map $F : W \to M$ with domain W an s-cobordism with $\partial W = N_1 \cup N_2$ and $F|_{N_i} = f_i$ for $i = 1, 2$. If the s-cobordism theorem holds over π this is the usual TOP structure set for M. We shall usually write $L_n(\pi, w)$ for $L_n^s(\pi, w)$ if $Wh(\pi) = 0$ and $L_n(\pi)$ if moreover w is trivial. When the orientation character is nontrivial and otherwise clear from the context we shall write $L_n(\pi, -)$.

The homotopy set $[M; G/TOP]$ may be identified with the set of normal maps (f, b), where $f : N \to M$ is a degree 1 map and b is a stable framing of $T_N \oplus f^*\xi$, for some TOP R^n-bundle ξ over M. (If $f : N \to M$ is a homotopy equivalence, with homotopy inverse h, we shall let $\hat{f} = (f, b)$, where $\xi = h^*\nu_N$ and b is the framing determined by a homotopy from hf to id_N). The Postnikov 4-stage of G/TOP is homotopy equivalent to $K(Z/2Z,2)\times K(Z,4)$. Let k_2 generate $H^2(G/TOP;\mathbb{F}_2)\cong Z/2Z$ and l_4 generate $H^4(G/TOP;\mathbb{Z})\cong Z$. The function from $[M;G/TOP]$ to $H^2(M;\mathbb{F}_2)\oplus H^4(M;\mathbb{Z})$ which sends $\hat{f}$ to $(\hat{f}^*(k_2),\hat{f}^*(l_4))$ is an isomorphism.

The *Kervaire-Arf invariant* of a normal map $\hat{g}:N^{2q}\to G/TOP$ is the image of the surgery obstruction in $L_{2q}(Z/2Z,-)=Z/2Z$ under the homomorphism induced by the orientation character, $c(\hat{g})=L_{2q}(w_1(N))(\sigma_{2q}(\hat{g}))$. The argument of Theorem 13.B.5 of [Wl] may be adapted to show that there are universal classes K_{4i+2} in $H^{4i+2}(G/TOP;\mathbb{F}_2)$ (for $i\geq 0$) such that

$$c(\hat{g})=(w(M)\cup\hat{g}^*((1+Sq^2+Sq^2Sq^2)\Sigma K_{4i+2}))\cap[M].$$

Moreover $K_2=k_2$, since c induces the isomorphism $\pi_2(G/TOP)=Z/2Z$. In the 4-dimensional case this expression simplifies to

$$c(\hat{g})=(w_2(M)\cup\hat{g}^*(k_2)+\hat{g}^*(Sq^2k_2))[M]=(w_1(M)^2\cup\hat{g}^*(k_2))[M].$$

The *codimension-2 Kervaire invariant* of a 4-dimensional normal map $\hat{g}$ is $kerv(\hat{g})=\hat{g}^*(k_2)$. Its value on a 2-dimensional homology class represented by an immersion $y:Y\to M$ is the Kervaire-Arf invariant of the normal map induced over the surface Y.

The structure set may overestimate the number of homeomorphism types within the homotopy type of M, if M has self homotopy equivalences which are not homotopic to homeomorphisms. Such "exotic" self homotopy equivalences may often be constructed as follows. Given $\alpha:S^2\to M$, let $\beta:S^4\to M$ be the composition $\alpha\eta S\eta$, where η is the Hopf map, and let $s:M\to M\vee S^4$ be the pinch map obtained by shrinking the boundary of a 4-disc in M. Then the composite $f_\alpha=(id_E\vee\beta)s$ is a self homotopy equivalence of M.

Lemma 6.5 [No64] *Let M be a closed 4-manifold and let $\alpha:S^2\to M$ be a map such that $\alpha_*[S^2]\neq 0$ in $H_2(M;\mathbb{F}_2)$ and $\alpha^*w_2(M)=0$. Then $kerv(\widehat{f_\alpha})\neq 0$ and so f_α is not normally cobordant to a homeomorphism.*

Proof There is a class $u\in H_2(M;\mathbb{F}_2)$ such that $\alpha_*[S^2].u=1$, since $\alpha_*[S^2]\neq 0$. As low-dimensional homology classes may be realized by singular manifolds there is a closed surface Y and a map $y:Y\to M$ transverse to f_α and such that $f_*[Y]=u$. Then $y^*kerv(\widehat{f_\alpha})[Y]$ is the Kervaire-Arf invariant of the normal map induced over Y and is nontrivial. (See Theorem 5.1 of [CH90] for details). $\square$

The family of surgery obstruction maps may be identified with a natural transformation from $\mathbb{L}_0$-homology to L-theory. (In the nonorientable case we must use w-twisted $\mathbb{L}_0$-homology). In dimension 4 the cobordism invariance of surgery obstructions (as in §13B of [Wl]) leads to the following formula.

Theorem 6.6 [Da95] *There are homomorphisms $I_0 : H_0(\pi; Z^w) \to L_4(\pi, w)$ and $\kappa_2 : H_2(\pi; \mathbb{F}_2) \to L_4(\pi, w)$ such that for any $\hat{f} : M \to G/TOP$ the surgery obstruction is $\sigma_4(\hat{f}) = I_0 c_{M*}(\hat{f}^*(l_4) \cap [M]) + \kappa_2 c_{M*}(kerv(\hat{f}) \cap [M])$* □

If $w = 1$ the signature homomorphism from $L_4(\pi)$ to Z is a left inverse for $I_0 : Z \to L_4(\pi)$, but in general I_0 is not injective. This formula can be made somewhat more explicit as follows. Let $KS(M) \in H^4(M; \mathbb{F}_2)$ be the Kirby-Siebenmann obstruction to lifting the TOP normal fibration of M to a vector bundle. If M is orientable and $(f, b) : N \to M$ is a degree 1 normal map with classifying map $\hat{f}$ then

$$(KS(M) - (f^*)^{-1}KS(N) - kerv(\hat{f})^2)[M] \equiv (\sigma(M) - \sigma(N))/8 \ mod\ (2).$$

(See Lemma 15.5 of [Si71] - page 329 of [KS]).

Theorem [Da95, 6′] *If $\hat{f} = (f, b)$ where $f : N \to M$ is a degree 1 map then the surgery obstructions are given by*

$$\sigma_4(\hat{f}) = I_0((\sigma(N) - \sigma(M))/8) + \kappa_2 c_{M*}(kerv(\hat{f}) \cap [M]) \qquad \text{if } w = 1, \text{ and}$$

$$\sigma_4(\hat{f}) = I_0(KS(N) - KS(M) + kerv(\hat{f})^2) + \kappa_2 c_{M*}(kerv(\hat{f}) \cap [M]) \qquad \text{if } w \neq 1.$$

(In the latter case we identify $H^4(M; \mathbb{Z})$, $H^4(N; \mathbb{Z})$ and $H^4(M; \mathbb{F}_2)$ with $H_0(\pi; Z^w) = Z/2Z$). □

The homomorphism σ_4 is trivial on the image of η, but in general we do not know whether a 4-dimensional normal map with trivial surgery obstruction must be normally cobordant to a simple homotopy equivalence. In our applications we shall always have a simple homotopy equivalence in hand, and so if σ_4 is injective we can conclude that the homotopy equivalence is normally cobordant to the identity.

A more serious problem is that it is not clear how to define the action ω in general. We shall be able to circumvent this problem by *ad hoc* arguments in some cases. (There is always an action on the homological structure set, defined in terms of $\mathbb{Z}[\pi]$-homology equivalences [FQ]).

If we fix an isomorphism $i_Z : Z \to L_5(Z)$ we may define a function $I_\pi : \pi \to L_5^s(\pi)$ for any group π by $I_\pi(g) = g_*(i_Z(1))$, where $g_* : Z = L_5(Z) \to L_5^s(\pi)$ is induced by the homomorphism sending 1 in Z to g in π. Then $I_Z = i_Z$ and I_π is natural in the sense that if $f : \pi \to H$ is a homomorphism then $L_5(f)I_\pi = I_H f$. As abelianization and projection to the summands of Z^2 induce an isomorphism from $L_5(Z * Z)$ to $L_5(Z)^2$ [Ca73], it follows easily from naturality that I_π is a homomorphism (and so factors through π/π') [We83]. We shall extend this to the nonorientable case by defining $I_\pi^+ : \mathrm{Ker}(w) \to L_5^s(\pi; w)$ as the composite of $I_{\mathrm{Ker}(w)}$ with the homomorphism induced by inclusion.

Theorem 6.7 *Let M be a closed 4-manifold with fundamental group π and let $w = w_1(M)$. Given any $\gamma \in \mathrm{Ker}(w)$ there is a normal cobordism from id_M to itself with surgery obstruction $I_\pi^+(\gamma) \in L_5^s(\pi, w)$.*

Proof We may assume that γ is represented by a simple closed curve with a product neighbourhood $U \cong S^1 \times D^3$. Let P be the E_8 manifold [FQ] and delete the interior of a submanifold homeomorphic to $D^3 \times [0,1]$ to obtain P_o. There is a normal map $p : P_o \to D^3 \times [0,1]$ (*rel* boundary). The surgery obstruction for $p \times id_{S^1}$ in $L_5(Z) \cong L_4(1)$ is given by a codimension-1 signature (see §12B of [Wl]), and generates $L_5(Z)$. Let $Y = (M \backslash intU) \times [0,1] \cup P_o \times S^1$, where we identify $(\partial U) \times [0,1] = S^1 \times S^2 \times [0,1]$ with $S^2 \times [0,1] \times S^1$ in $\partial P_o \times S^1$. Matching together $id|_{(M\backslash intU)\times[0,1]}$ and $p \times id_{S^1}$ gives a normal cobordism Q from id_M to itself. The theorem now follows by the additivity of surgery obstructions and naturality of the homomorphisms I_π^+. □

Corollary 6.7.1 *Let $\lambda_* : L_5^s(\pi) \to L_5(Z)^d = Z^d$ be the homomorphism induced by a basis $\{\lambda_1, ..., \lambda_d\}$ for $Hom(\pi, Z)$. If M is orientable, $f : M_1 \to M$ is a simple homotopy equivalence and $\theta \in L_5(Z)^d$ there is a normal cobordism from f to itself whose surgery obstruction in $L_5(\pi)$ has image θ under λ_*.*

Proof If $\{\gamma_1, ..., \gamma_d\} \in \pi$ represents a "dual basis" for $H_1(\pi; \mathbb{Z})$ modulo torsion (so that $\lambda_i(\gamma_j) = \delta_{ij}$ for $1 \le i, j \le d$), then $\{\lambda_*(I_\pi(\gamma_1)), ..., \lambda_*(I_\pi(\gamma_d))\}$ is a basis for $L_5(Z)^d$. □

If π is free or is a PD_2^+-group the homomorphism λ_* is an isomorphism [Ca73]. In most of the other cases of interest to us the following corollary applies.

Corollary 6.7.2 *If M is orientable and $\mathrm{Ker}(\lambda_*)$ is finite then $S_{TOP}^s(M)$ is finite. In particular, this is so if $\mathrm{Coker}(\sigma_5)$ is finite.*

Proof The signature difference maps $[M;G/TOP] = H^4(M;\mathbb{Z}) \oplus H^2(M;\mathbb{F}_2)$ onto $L_4(1) = Z$ and so there are only finitely many normal cobordism classes of simple homotopy equivalences $f : M_1 \to M$. Moreover, $\mathrm{Ker}(\lambda_*)$ is finite if σ_5 has finite cokernel, since $[SM;G/TOP] \cong Z^d \oplus (Z/2Z)^d$. Suppose that $F : N \to M \times I$ is a normal cobordism between two simple homotopy equivalences $F_- = F|\partial_- N$ and $F_+ = F|\partial_+ N$. By Theorem 6.7 there is another normal cobordism $F' : N' \to M \times I$ from F_+ to itself with $\lambda_*(\sigma_5(F')) = \lambda_*(-\sigma_5(F))$. The union of these two normal cobordisms along $\partial_+ N = \partial_- N'$ is a normal cobordism from F_- to F_+ with surgery obstruction in $\mathrm{Ker}(\lambda_*)$. If this obstruction is 0 we may obtain an s-cobordism W by 5-dimensional surgery (rel ∂). ∎

The surgery obstruction groups for a semidirect product $\pi \cong G \times_\theta Z$, may be related to those of the (finitely presentable) normal subgroup G by means of Theorem 12.6 of [Wl]. If $Wh(\pi) = Wh(G) = 0$ this theorem asserts that there is an exact sequence

$$\dots L_m(G, w|_G) \xrightarrow{1-w(t)\theta_*} L_m(G, w|_G) \to L_m(\pi, w) \to L_{m-1}(G, w|_G) \dots,$$

where t generates π modulo G and $\theta_* = L_m(\theta, w|_G)$. The following lemma is adapted from Theorem 15.B.1 of [Wl].

Lemma 6.8 *Let M be the mapping torus of a self homeomorphism of an aspherical closed $(n-1)$-manifold N. Suppose that $Wh(\pi_1(M)) = 0$. If the homomorphisms σ_i^N are isomorphisms for all large i then so are the σ_i^M.*

Proof This is an application of the 5-lemma and periodicity, as in pages 229-230 of [Wl]. ∎

The hypotheses of this lemma are satisfied if $n = 4$ and $\pi_1(N)$ is square root closed accessible [Ca73], or N is orientable and $\beta_1(N) > 0$ [Ro00], or is hyperbolic or virtually solvable [FJ], or admits an effective S^1-action with orientable orbit space [St84, NS85]. It remains an open question whether aspherical closed manifolds with isomorphic fundamental groups must be homeomorphic. This has been verified in higher dimensions in many cases, in particular under geometric assumptions [FJ], and under assumptions on the combinatorial structure of the group [Ca73, St84, NS85]. We shall see that many aspherical 4-manifolds are determined up to s-cobordism by their groups.

There are more general "Mayer-Vietoris" sequences which lead to calculations of the surgery obstruction groups for certain generalized free products and HNN extensions in terms of those of their building blocks [Ca73, St87].

Lemma 6.9 *Let π be either the group of a finite graph of groups, all of whose vertex groups are infinite cyclic, or a square root closed accessible group of cohomological dimension 2. Then I_π^+ is an isomorphism. If M is a closed 4-manifold with fundamental group π the surgery obstruction maps $\sigma_4(M)$ and $\sigma_5(M)$ are epimorphisms.*

Proof Since π is in Cl we have $Wh(\pi) = 0$ and a comparison of Mayer-Vietoris sequences shows that the assembly map from $H_*(\pi; \mathbb{L}_0^w)$ to $L_*(\pi, w)$ is an isomorphism [Ca73, St87]. Since $c.d.\pi \leq 2$ and $H_1(\mathrm{Ker}(w); \mathbb{Z})$ maps onto $H_1(\pi; Z^w)$ the component of this map in degree 1 may be identified with I_π^+. In general, the surgery obstruction maps factor through the assembly map. Since $c.d.\pi \leq 2$ the homomorphism $c_{M*} : H_*(M; D) \to H_*(\pi; D)$ is onto for any local coefficient module D, and so the lemma follows. □

The class of groups considered in this lemma includes free groups, PD_2-groups and the groups $Z*_m$. Note however that if π is a PD_2-group w need not be the canonical orientation character.

6.3 Stabilization and h-cobordism

It has long been known that many results of high dimensional differential topology hold for smooth 4-manifolds after stabilizing by connected sum with copies of $S^2 \times S^2$ [CS71, FQ80, La79, Qu83]. In particular, if M and N are h-cobordant closed smooth 4-manifolds then $M\sharp(\sharp^k S^2 \times S^2)$ is diffeomorphic to $N\sharp(\sharp^k S^2 \times S^2)$ for some $k \geq 0$. In the spin case $w_2(M) = 0$ this is an elementary consequence of the existence of a well-indexed handle decomposition of the h-cobordism [Wa64]. In Chapter VII of [FQ] it is shown that 5-dimensional TOP cobordisms have handle decompositions relative to a component of their boundaries, and so a similar result holds for h-cobordant closed TOP 4-manifolds. Moreover, if M is a TOP 4-manifold then $KS(M) = 0$ if and only if $M\sharp(\sharp^k S^2 \times S^2)$ is smoothable for some $k \geq 0$ [LS71].

These results suggest the following definition. Two 4-manifolds M_1 and M_2 are *stably homeomorphic* if $M_1\sharp(\sharp^k S^2 \times S^2)$ and $M_2\sharp(\sharp^l S^2 \times S^2)$ are homeomorphic, for some k, $l \geq 0$. (Thus h-cobordant closed 4-manifolds are stably homeomorphic). Clearly $\pi_1(M)$, $w_1(M)$, the orbit of $c_{M*}[M]$ in $H_4(\pi_1(M), Z^{w_1(M)})$ under the action of $Out(\pi_1(M))$, and the parity of $\chi(M)$ are invariant under stabilization. If M is orientable $\sigma(M)$ is also invariant.

Kreck has shown that (in any dimension) classification up to stable homeomorphism (or diffeomorphism) can be reduced to bordism theory. There are three cases: If $w_2(\tilde{M}) \neq 0$ and $w_2(\tilde{N}) \neq 0$ then M and N are stably homeomorphic if and only if for some choices of orientations and identification of the fundamental groups the invariants listed above agree (in an obvious manner). If $w_2(M) = w_2(N) = 0$ then M and N are stably homeomorphic if and only if for some choices of orientations, Spin structures and identification of the fundamental group they represent the same element in $\Omega_4^{SpinTOP}(K(\pi,1))$. The most complicated case is when M and N are not Spin, but the universal covers are Spin. (See [Kr99], [Te] for expositions of Kreck's ideas).

We shall not pursue this notion of stabilization further (with one minor exception, in Chapter 14), for it is somewhat at odds with the tenor of this book. The manifolds studied here usually have minimal Euler characteristic, and often are aspherical. Each of these properties disappears after stabilization. We may however also stabilize by cartesian product with R, and there is then the following simple but satisfying result.

Lemma 6.10 *Closed 4-manifolds M and N are h-cobordant if and only if $M \times R$ and $N \times R$ are homeomorphic.*

Proof If W is an h-cobordism from M to N (with fundamental group $\pi = \pi_1(W)$) then $W \times S^1$ is an h-cobordism from $M \times S^1$ to $N \times S^1$. The torsion is 0 in $Wh(\pi \times Z)$, by Theorem 23.2 of [Co], and so there is a homeomorphism from $M \times S^1$ to $N \times S^1$ which carries $\pi_1(M)$ to $\pi_1(N)$. Hence $M \times R \cong N \times R$. Conversely, if $M \times R \cong N \times R$ then $M \times R$ contains a copy of N disjoint from $M \times \{0\}$, and the region W between $M \times \{0\}$ and N is an h-cobordism. □

6.4 Manifolds with π_1 elementary amenable and $\chi = 0$

In this section we shall show that closed manifolds satisfying the hypotheses of Theorem 3.17 and with torsion free fundamental group are determined up to homeomorphism by their homotopy type. As a consequence, closed 4-manifolds with torsion free elementary amenable fundamental group and Euler characteristic 0 are homeomorphic to mapping tori. We also estimate the structure sets for RP^2-bundles over T or Kb. In the remaining cases involving torsion computation of the surgery obstructions is much more difficult. We shall comment briefly on these cases in Chapters 10 and 11.

Theorem 6.11 *Let M be a closed 4-manifold with $\chi(M) = 0$ and whose fundamental group π is torsion free, coherent, locally virtually indicable and restrained. Then M is determined up to homeomorphism by its homotopy type. If moreover $h(\pi) = 4$ then every automorphism of π is realized by a self homeomorphism of M.*

Proof By Theorem 3.17 either $\pi \cong Z$ or $Z*_m$ for some $m \neq 0$, or M is aspherical, π is virtually poly-Z and $h(\pi) = 4$. Hence $Wh(\pi) = 0$, in all cases. If $\pi \cong Z$ or $Z*_m$ then the surgery obstruction homomorphisms are epimorphisms, by Lemma 6.9. We may calculate $L_4(\pi, w)$ by means of Theorem 12.6 of [Wl], or more generally §3 of [St87], and we find that if $\pi \cong Z$ or $Z*_{2n}$ then $\sigma_4(M)$ is in fact an isomorphism. If $\pi \cong Z*_{2n+1}$ then there are two normal cobordism classes of homotopy equivalences $h : X \to M$. Let ξ generate the image of $H^2(\pi; \mathbb{F}_2) \cong Z/2Z$ in $H^2(M; \mathbb{F}_2) \cong (Z/2Z)^2$, and let $j : S^2 \to M$ represent the unique nontrivial spherical class in $H_2(M; \mathbb{F}_2)$. Then $\xi^2 = 0$, since $c.d.\pi = 2$, and $\xi \cap j_*[S^2] = 0$, since $c_M j$ is nullhomotopic. It follows that $j_*[S^2]$ is Poincaré dual to ξ, and so $v_2(M) \cap j_*[S^2] = \xi^2 \cap [M] = 0$. Hence $j^*w_2(M) = j^*v_2(M) + (j^*w_1(M))^2 = 0$ and so f_j has nontrivial normal invariant, by Lemma 6.5. Therefore each of these two normal cobordism classes contains a self homotopy equivalence of M.

If M is aspherical, π is virtually poly-Z and $h(\pi) = 4$ then $S_{TOP}(M)$ has just one element, by Theorem 2.16 of [FJ]. The theorem now follows. □

Corollary 6.11.1 *Let M be a closed 4-manifold with $\chi(M) = 0$ and fundamental group $\pi \cong Z$, Z^2 or $Z \times_{-1} Z$. Then M is determined up to homeomorphism by π and $w(M)$.*

Proof If $\pi \cong Z$ then M is homotopy equivalent to the total space of an S^3-bundle over S^1, by Theorem 4.2, while if $\pi \cong Z^2$ or $Z \times_{-1} Z$ it is homotopy equivalent to the total space of an S^2-bundle over T or Kb, by Theorem 5.10. □

Is the homotopy type of M also determined by π and $w(M)$ if $\pi \cong Z*_m$ for some $|m| > 1$?

We may now give an analogue of the Farrell and Stallings fibration theorems for 4-manifolds with torsion free elementary amenable fundamental group.

Theorem 6.12 *Let M be a closed 4-manifold whose fundamental group π is torsion free and elementary amenable. A map $f : M \to S^1$ is homotopic to a fibre bundle projection if and only if $\chi(M) = 0$ and f induces an epimorphism from π to Z with almost finitely presentable kernel.*

Proof The conditions are clearly necessary. Suppose that they hold. Let $\nu = \mathrm{Ker}(\pi_1(f))$, let M_ν be the infinite cyclic covering space of M with fundamental group ν and let $t : M_\nu \to M_\nu$ be a generator of the group of covering transformations. By Corollary 4.5.3 either $\nu = 1$ (so $M_\nu \simeq S^3$) or $\nu \cong Z$ (so $M_\nu \simeq S^2 \times S^1$ or $S^2 \tilde{\times} S^1$) or M is aspherical. In the latter case π is a torsion free virtually poly-Z group, by Theorem 1.11 and Theorem 9.23 of [Bi]. Thus in all cases there is a homotopy equivalence f_ν from M_ν to a closed 3-manifold N. Moreover the self homotopy equivalence $f_\nu t f_\nu^{-1}$ of N is homotopic to a homeomorphism, g say, and so f is fibre homotopy equivalent to the canonical projection of the mapping torus $M(g)$ onto S^1. It now follows from Theorem 6.11 that any homotopy equivalence from M to $M(g)$ is homotopic to a homeomorphism. □

The structure sets of the RP^2-bundles over T or Kb are also finite.

Theorem 6.13 *Let M be the total space of an RP^2-bundle over T or Kb. Then $S_{TOP}(M)$ has order at most 32.*

Proof As M is nonorientable $H^4(M;\mathbb{Z}) = Z/2Z$ and as $\beta_1(M;\mathbb{F}_2) = 3$ and $\chi(M) = 0$ we have $H^2(M;\mathbb{F}_2) \cong (Z/2Z)^4$. Hence $[M;G/TOP]$ has order 32. Let $w = w_1(M)$. It follows from the Shaneson-Wall splitting theorem (Theorem 12.6 of [Wl]) that $L_4(\pi, w) \cong L_4(Z/2Z, -) \oplus L_2(Z/2Z, -) \cong (Z/2Z)^2$, detected by the Kervaire-Arf invariant and the codimension-2 Kervaire invariant. Similarly $L_5(\pi, w) \cong L_4(Z/2Z, -)^2$ and the projections to the factors are Kervaire-Arf invariants of normal maps induced over codimension-1 submanifolds. (In applying the splitting theorem, note that $Wh(Z \oplus (Z/2Z)) = Wh(\pi) = 0$, by Theorem 6.1 above). Hence $S_{TOP}(M)$ has order at most 128.

The Kervaire-Arf homomorphism c is onto, since $c(\hat{g}) = (w^2 \cup \hat{g}^*(k_2)) \cap [M]$, $w^2 \neq 0$ and every element of $H^2(M;\mathbb{F}_2)$ is equal to $\hat{g}^*(k_2)$ for some normal map $\hat{g} : M \to G/TOP$. Similarly there is a normal map $f_2 : X_2 \to RP^2$ with $\sigma_2(f_2) \neq 0$ in $L_2(Z/2Z, -)$. If $M = RP^2 \times B$, where $B = T$ or Kb is the base of the bundle, then $f_2 \times id_B : X_2 \times B \to RP^2 \times B$ is a normal map with surgery obstruction $(0, \sigma_2(f_2)) \in L_4(Z/2Z, -) \oplus L_2(Z/2Z, -)$. We may assume that f_2 is a homeomorphism over a disc $\Delta \subset RP^2$. As the nontrivial bundles may be obtained from the product bundles by cutting M along $RP^2 \times \partial\Delta$ and regluing via the twist map of $RP^2 \times S^1$, the normal maps for the product bundles may be compatibly modified to give normal maps with nonzero obstructions in the other cases. Hence σ_4 is onto and so $S_{TOP}(M)$ has order at most 32. □

In each case $H_2(M;\mathbb{F}_2) \cong H_2(\pi;\mathbb{F}_2)$, so the argument of Lemma 6.5 does not apply. However we can improve our estimate in the abelian case.

Theorem 6.14 *Let M be the total space of an RP^2-bundle over T. Then $L_5(\pi, w)$ acts trivially on the class of id_M in $S_{TOP}(M)$.*

Proof Let $\lambda_1, \lambda_2 : \pi \to Z$ be epimorphisms generating $Hom(\pi, Z)$ and let $t_1, t_2 \in \pi$ represent a dual basis for $\pi/(torsion)$ (i.e., $\lambda_i(t_j) = \delta_{ij}$ for $i = 1, 2$). Let u be the element of order 2 in π and let $k_i : Z \oplus (Z/2Z) \to \pi$ be the monomorphism defined by $k_i(a, b) = at_i + bu$, for $i = 1, 2$. Define splitting homomorphisms p_1, p_2 by $p_i(g) = k_i^{-1}(g - \lambda_i(g)t_i)$ for all $g \in \pi$. Then $p_i k_i = id_{Z\oplus(Z/2Z)}$ and $p_i k_{3-i}$ factors through $Z/2Z$, for $i = 1, 2$. The orientation character $w = w_1(M)$ maps the torsion subgroup of π onto $Z/2Z$, by Theorem 5.13, and t_1 and t_2 are in $\mathrm{Ker}(w)$. Therefore p_i and k_i are compatible with w, for $i = 1, 2$. As $L_5(Z/2Z, -) = 0$ it follows that $L_5(k_1)$ and $L_5(k_2)$ are inclusions of complementary summands of $L_5(\pi, w) \cong (Z/2Z)^2$, split by the projections $L_5(p_1)$ and $L_5(p_2)$.

Let γ_i be a simple closed curve in T which represents $t_i \in \pi$. Then γ_i has a product neighbourhood $N_i \cong S^1 \times [-1, 1]$ whose preimage $U_i \subset M$ is homeomorphic to $RP^2 \times S^1 \times [-1, 1]$. As in Theorem 6.13 there is a normal map $f_4 : X_4 \to RP^2 \times [-1, 1]^2$ (rel boundary) with $\sigma_4(f_4) \neq 0$ in $L_4(Z/2Z, -)$. Let $Y_i = (M \setminus intU_i) \times [-1, 1] \cup X_4 \times S^1$, where we identify $(\partial U_i) \times [-1, 1] = RP^2 \times S^1 \times S^0 \times [-1, 1]$ with $RP^2 \times [-1, 1] \times S^0 \times S^1$ in $\partial X_4 \times S^1$. If we match together $id_{(M\setminus intU_i)\times[-1,1]}$ and $f_4 \times id_{S^1}$ we obtain a normal cobordism Q_i from id_M to itself. The image of $\sigma_5(Q_i)$ in $L_4(\mathrm{Ker}(\lambda_i), w) \cong L_4(Z/2Z, -)$ under the splitting homomorphism is $\sigma_4(f_4)$. On the other hand its image in $L_4(\mathrm{Ker}(\lambda_{3-i}), w)$ is 0, and so it generates the image of $L_5(k_{3-i})$. Thus $L_5(\pi, w)$ is generated by $\sigma_5(Q_1)$ and $\sigma_5(Q_2)$, and so acts trivially on id_M. □

Does $L_5(\pi, w)$ act trivially on each class in $S_{TOP}(M)$ when M is an RP^2-bundle over T or Kb? If so, then $S_{TOP}(M)$ has order 8 in each case. Are these manifolds determined up to homeomorphism by their homotopy type?

6.5 Bundles over aspherical surfaces

The fundamental groups of total spaces of bundles over hyperbolic surfaces all contain nonabelian free subgroups. Nevertheless, such bundle spaces are determined up to s-cobordism by their homotopy type, except when the fibre is RP^2, in which case we can only show that the structure sets are finite.

Theorem 6.15 *Let M be a closed 4-manifold which is homotopy equivalent to the total space E of an F-bundle over B where B and F are aspherical closed surfaces. Then M is s-cobordant to E and $\widetilde{M}$ is homeomorphic to R^4.*

Proof Since $\pi_1(B)$ is either an HNN extension of Z or a generalised free product $F*_Z F'$, where F and F' are free groups, $\pi\times Z$ is a square root closed generalised free product with amalgamation of groups in *Cl*. Comparison of the Mayer-Vietoris sequences for $\mathbb{L}_0$-homology and L-theory (as in Proposition 2.6 of [St84]) shows that $S_{TOP}(E\times S^1)$ has just one element. (Note that even when $\chi(B)=0$ the groups arising in intermediate stages of the argument all have trivial Whitehead groups). Hence $M\times S^1\cong E\times S^1$, and so M is s-cobordant to E by Lemma 6.10 and Theorem 6.2. The final assertion follows from Corllary 7.3B of [FQ] since M is aspherical and π is 1-connected at ∞ [Ho77]. □

Davis has constructed aspherical 4-manifolds whose universal covering space is not 1-connected at ∞ [Da83].

Theorem 6.16 *Let M be a closed 4-manifold which is homotopy equivalent to the total space E of an S^2-bundle over an aspherical closed surface B. Then M is s-cobordant to E, and $\widetilde{M}$ is homeomorphic to $S^2\times R^2$.*

Proof Let $\pi=\pi_1(E)\cong\pi_1(B)$. Then $Wh(\pi)=0$, and $H_*(\pi;\mathbb{L}_0^w)\cong L_*(\pi,w)$, as in Lemma 6.9. Hence $L_4(\pi,w)\cong Z\oplus(Z/2Z)$ if $w=0$ and $(Z/2Z)^2$ otherwise. The surgery obstruction map $\sigma_4(E)$ is onto, by Lemma 6.9. Hence there are two normal cobordism classes of maps $h:X\to E$ with $\sigma_4(h)=0$. The kernel of the natural homomorphism from $H_2(E;\mathbb{F}_2)\cong(Z/2Z)^2$ to $H_2(\pi;\mathbb{F}_2)\cong Z/2Z$ is generated by $j_*[S^2]$, where $j:S^2\to E$ is the inclusion of a fibre. As $j_*[S^2]\neq -0$, while $w_2(E)(j_*[S^2])=j^*w_2(E)=0$ the normal invariant of f_j is nontrivial, by Lemma 6.5. Hence each of these two normal cobordism classes contains a self homotopy equivalence of E.

Let $f:M\to E$ be a homotopy equivalence (necessarily simple). Then there is a normal cobordism $F:V\to E\times[0,1]$ from f to some self homotopy equivalence of E. As I_π^+ is an isomorphism, by Lemma 6.9, there is an s-cobordism W from M to E, as in Corollary 6.7.2.

The universal covering space $\widetilde{W}$ is a proper s-cobordism from $\widetilde{M}$ to $\widetilde{E}\cong S^2\times R^2$. Since the end of $\widetilde{E}$ is tame and has fundamental group Z we may apply Corollary 7.3B of [FQ] to conclude that $\widetilde{W}$ is homeomorphic to a product. Hence $\widetilde{M}$ is homeomorphic to $S^2\times R^2$. □

Let ρ be a PD_2-group. As $\pi = \rho \times (Z/2Z)$ is square-root closed accessible from $Z/2Z$, the Mayer-Vietoris sequences of [Ca73] imply that $L_4(\pi, w) \cong L_4(Z/2Z, -) \oplus L_2(Z/2Z, -)$ and that $L_5(\pi, w) \cong L_4(Z/2Z, -)^\beta$, where $w = pr_2 : \pi \to Z/2Z$ and $\beta = \beta_1(\rho; \mathbb{F}_2)$. Since these L-groups are finite the structure sets of total spaces of RP^2-bundles over aspherical surfaces are also finite. (Moreover the arguments of Theorems 6.13 and 6.14 can be extended to show that σ_4 is an epimorphism and that most of $L_5(\pi, w)$ acts trivially on id_E, where E is such a bundle space).