Part II

4-dimensional Geometries

Chapter 7

Geometries and decompositions

Every closed connected surface is geometric, i.e., is a quotient of one of the three model 2-dimensional geometries $\mathbb{E}^2$, $\mathbb{H}^2$ or $\mathbb{S}^2$ by a free and properly discontinuous action of a discrete group of isometries. Much current research on 3-manifolds is guided by Thurston's Geometrization Conjecture, that every closed irreducible 3-manifold admits a finite decomposition into geometric pieces. In §1 we shall recall Thurston's definition of geometry, and shall describe briefly the 19 4-dimensional geometries. Our concern in the middle third of this book is not to show how this list arises (as this is properly a question of differential geometry; see [Fi], [Pa96] and [Wl85,86]), but rather to describe the geometries sufficiently well that we may subsequently characterize geometric manifolds up to homotopy equivalence or homeomorphism. In §2 we relate the notions of "geometry of solvable Lie type" and "infrasolvmanifold". The limitations of geometry in higher dimensions are illustrated in §3, where it is shown that a closed 4-manifold which admits a finite decomposition into geometric pieces is (essentially) either geometric or aspherical. The geometric viewpoint is nevertheless of considerable interest in connection with complex surfaces [Ue90,91, Wl85,86]. With the exception of the geometries $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{E}^2$ and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ no closed geometric manifold has a proper geometric decomposition.

A number of the geometries support natural Seifert fibrations or compatible complex structures. In §4 we characterize the groups of aspherical 4-manifolds which are orbifold bundles over flat or hyperbolic 2-orbifolds. We outline what we need about Seifert fibrations and complex surfaces in §5 and §6.

Subsequent chapters shall consider in turn geometries whose models are contractible (Chapters 8 and 9), geometries with models diffeomorphic to $S^2 \times R^2$ (Chapter 10), the geometry $\mathbb{S}^3 \times \mathbb{E}^1$ (Chapter 11) and the geometries with compact models (Chapter 12). In Chapter 13 we shall consider geometric structures and decompositions of bundle spaces. In the final chapter of the book we shall consider knot manifolds which admit geometries.

7.1 Geometries

An n-*dimensional geometry* $\mathbb{X}$ in the sense of Thurston is represented by a pair (X, G_X) where X is a complete 1-connected n-dimensional Riemannian manifold and G_X is a Lie group which acts effectively, transitively and isometrically on X and which has discrete subgroups Γ which act freely on X so that $\Gamma\backslash X$ has finite volume. (Such subgroups are called *lattices*.) Since the stabilizer of a point in X is isomorphic to a closed subgroup of $O(n)$ it is compact, and so $\Gamma\backslash X$ is compact if and only if $\Gamma\backslash G_X$ is compact. Two such pairs (X, G) and (X', G') define the same geometry if there is a diffeomorphism $f : X \to X'$ which conjugates the action of G onto that of G'. (Thus the metric is only an adjunct to the definition.) We shall assume that G is maximal among Lie groups acting thus on X, and write $Isom(\mathbb{X}) = G$, and $Isom_o(\mathbb{X})$ for the component of the identity. A closed manifold M is an $\mathbb{X}$-*manifold* if it is a quotient $\Gamma\backslash X$ for some lattice in G_X. Under an equivalent formulation, M is an $\mathbb{X}$-manifold if it is a quotient $\Gamma\backslash X$ for some discrete group Γ of isometries acting freely on a 1-connected homogeneous space $X = G/K$, where G is a connected Lie group and K is a compact subgroup of G such that the intersection of the conjugates of K is trivial, and X has a G-invariant metric. The manifold *admits a geometry of type* $\mathbb{X}$ if it is homeomorphic to such a quotient. If G is solvable we shall say that the geometry is of *solvable Lie type*. If $\mathbb{X} = (X, G_X)$ and $\mathbb{Y} = (Y, G_Y)$ are two geometries then $X \times Y$ supports a geometry in a natural way; however the maximal group of isometries $G_{X\times Y}$ may be strictly larger than $G_X \times G_Y$.

The geometries of dimension 1 or 2 are the familiar geometries of constant curvature: $\mathbb{E}^1$, $\mathbb{E}^2$, $\mathbb{H}^2$ and $\mathbb{S}^2$. Thurston showed that there are eight maximal 3-dimensional geometries ($\mathbb{E}^3$, $\mathbb{N}il^3$, $\mathbb{S}ol^3$, $\widetilde{\mathbb{SL}}$, $\mathbb{H}^2 \times \mathbb{E}^1$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{E}^1$ and $\mathbb{S}^3$.) Manifolds with one of the first five of these geometries are aspherical Seifert fibred 3-manifolds or $\mathbb{S}ol^3$-manifolds. These are determined among irreducible 3-manifolds by their fundamental groups, which are the PD_3-groups with nontrivial Hirsch-Plotkin radical. There are just four $\mathbb{S}^2 \times \mathbb{E}^1$-manifolds. It is not yet known whether every aspherical 3-manifold whose fundamental group contains no rank 2 abelian subgroup must be hyperbolic, and the determination of the closed $\mathbb{H}^3$-manifolds remains incomplete. Nor is it known whether every 3-manifold with finite fundamental group must be spherical. For a detailed and lucid account of the 3-dimensional geometries see [Sc83'].

There are 19 maximal 4-dimensional geometries; one of these ($\mathbb{S}ol^4_{m,n}$) is in fact a family of closely related geometries, and one ($\mathbb{F}^4$) is not realizable by any closed manifold [Fi]. We shall see that the geometry is determined by the fundamental group (cf. [Wl86, Ko92]). In addition to the geometries of constant curvature and products of lower dimensional geometries there are seven "new" 4-dimensional geometries. Two of these are modeled on the irreducible Riemannian symmetric spaces $CP^2 = U(3)/U(2)$ and $H^2(C) = SU(2,1)/S(U(2) \times U(1))$. The model for the geometry $\mathbb{F}^4$ is $C \times H^2$. The component of the identity in its isometry group is the semidirect product $R^2 \times_\alpha SL(2,\mathbb{R})$, where α is the natural action of $SL(2,\mathbb{R})$ on R^2. This group acts on $C \times H^2$ as follows: if $(u,v) \in R^2$ and $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$ then $(u,v)(w,z) = (u - vz + w, z)$ and $A(w,z) = (\frac{w}{cz+d}, \frac{az+b}{cz+d})$ for all $(w,z) \in C \times H^2$. The other four new geometries are of solvable Lie type, and shall be described in §2 and §3.

In most cases the model X is homeomorphic to R^4, and the corresponding geometric manifolds are aspherical. Six of these geometries ($\mathbb{E}^4$, $\mathbb{N}il^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{S}ol^4_{m,n}$, $\mathbb{S}ol^4_0$ and $\mathbb{S}ol^4_1$) are of solvable Lie type; in Chapter 8 we shall show manifolds admitting such geometries have Euler characteristic 0 and fundamental group a torsion free virtually poly-Z group of Hirsch length 4. Such manifolds are determined up to homeomorphism by their fundamental groups, and every such group arises in this way. In Chapter 9 we shall consider closed 4-manifolds admitting one of the other geometries of aspherical type ($\mathbb{H}^3 \times \mathbb{E}^1$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^4$, $\mathbb{H}^2(\mathbb{C})$ and $\mathbb{F}^4$). These may be characterised up to s-cobordism by their fundamental group and Euler characteristic. However it is unknown to what extent surgery arguments apply in these cases, and we do not yet have good characterizations of the possible fundamental groups. Although no closed 4-manifold admits the geometry $\mathbb{F}^4$, there are such manifolds with proper geometric decompositions involving this geometry; we shall give examples in Chapter 13.

Three of the remaining geometries ($\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$ and $\mathbb{S}^3 \times \mathbb{E}^1$) have models homeomorphic to $S^2 \times R^2$ or $S^3 \times R$. (Note that we shall use $\mathbb{E}^n$ or $\mathbb{H}^n$ to refer to the *geometry* and R^n to refer to the underlying *topological space*). The final three ($\mathbb{S}^4$, $\mathbb{CP}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$) have compact models, and there are only eleven such manifolds. We shall discuss these nonaspherical geometries in Chapters 10, 11 and 12.

7.2 Infranilmanifolds

The notions of "geometry of solvable Lie type" and "infrasolvmanifold" are closely related. We shall describe briefly the latter class of manifolds, from a rather utilitarian point of view. As we are only interested in closed manifolds, we shall frame our definitions accordingly. We consider the easier case of infranilmanifolds in this section, and the other infrasolvmanifolds in the next section.

A *flat n-manifold* is a quotient of R^n by a discrete torsion free subgroup of $E(n) = Isom(\mathbb{E}^n) = R^n \times_\alpha O(n)$ (where α is the natural action of $O(n)$ on R^n). A group π is a *flat n-manifold group* if it is torsion free and has a normal subgroup of finite index which is isomorphic to Z^n. (These are necessary and sufficient conditions for π to be the fundamental group of a closed flat n-manifold.) The action of π by conjugation on its *translation subgroup* $T(\pi)$ (the maximal abelian normal subgroup of π) induces a faithful action of $\pi/T(\pi)$ on $T(\pi)$. On choosing an isomorphism $T(\pi) \cong Z^n$ we may identify $\pi/T(\pi)$ with a subgroup of $GL(n,\mathbb{Z})$; this subgroup is called the *holonomy group* of π, and is well defined up to conjugacy in $GL(n,\mathbb{Z})$. We say that π is *orientable* if the holonomy group lies in $SL(n,\mathbb{Z})$. (The group is orientable if and only if the corresponding flat n-manifold is orientable.) If two discrete torsion free cocompact subgroups of $E(n)$ are isomorphic then they are conjugate in the larger group $Aff(R^n) = R^n \times_\alpha GL(n,\mathbb{R})$, and the corresponding flat n-manifolds are "affinely" diffeomorphic. There are only finitely many isomorphism classes of such flat n-manifold groups for each n.

A nilmanifold is a coset space of a 1-connected nilpotent Lie group by a discrete subgroup. More generally, an *infranilmanifold* is a quotient $\pi\backslash N$ where N is a 1-connected nilpotent Lie group and π is a discrete torsion free subgroup of the semidirect product $Aff(N) = N \times_\alpha Aut(N)$ such that $\pi \cap N$ is a lattice in N and $\pi/\pi \cap N$ is finite. Thus infranilmanifolds are finitely covered by nilmanifolds. The Lie group N is determined by $\sqrt{\pi}$, by Mal'cev's rigidity theorem, and two infranilmanifolds are diffeomorphic if and only if their fundamental groups are isomorphic. The isomorphism may then be induced by an affine diffeomorphism. The infranilmanifolds derived from the abelian Lie groups R^n are just the flat manifolds. It is not hard to see that there are just three 4-dimensional (real) nilpotent Lie algebras. (Compare the analogous argument of Theorem 1.4.) Hence there are three 1-connected 4-dimensional nilpotent Lie groups, R^4, $Nil^3 \times R$ and Nil^4.

The group Nil^3 is the subgroup of $SL(3,\mathbb{R})$ consisting of upper triangular matrices $[r,s,t] = \begin{pmatrix} 1 & r & t \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$. It has abelianization R^2 and centre $\zeta Nil^3 = Nil^{3\prime} \cong R$. The elements $[1,0,0]$, $[0,1,0]$ and $[0,0,1/q]$ generate a discrete cocompact subgroup of Nil^3 isomorphic to Γ_q, and these are essentially the only such subgroups. (Since they act orientably on R^3 they are PD_3^+-groups.)

The coset space $N_q = Nil^3/\Gamma_q$ is the total space of the S^1-bundle over $S^1 \times S^1$ with Euler number q, and the action of ζNil^3 on Nil^3 induces a free action of $S^1 = \zeta Nil/\zeta\Gamma_q$ on N_q. The group Nil^4 is the semidirect product $R^3 \times_\theta R$, where $\theta(t) = [t, t, t^2/2]$. It has abelianization R^2 and central series $\zeta Nil^4 \cong R < \zeta^2 Nil^4 = Nil^{4'} \cong R^2$.

These Lie groups have natural left invariant metrics. (See [Sc83'].) The infranilmanifolds corresponding to R^4, Nil^4 and $Nil^3 \times R$ are the $\mathbb{E}^4$-, $\mathbb{N}il^4$- and $\mathbb{N}il^3 \times \mathbb{E}^1$-manifolds. (The isometry group of $\mathbb{E}^4$ is the semidirect product $R^4 \times_\alpha O(4)$; the group Nil^4 is the identity component for its isometry group, while $\mathbb{N}il^3 \times \mathbb{E}^1$ admits an additional isometric action of S^1.)

7.3 Infrasolvmanifolds

The situation for (infra)solvmanifolds is more complicated. An *infrasolvmanifold* is a quotient $M = \Gamma\backslash S$ where S is a 1-connected solvable Lie group and Γ is a closed torsion free subgroup of the semidirect product $Aff(S) = S \times_\alpha Aut(S)$ such that Γ_o (the component of the identity of Γ) is contained in the nilradical of S (the maximal connected nilpotent normal subgroup of S), $\Gamma/\Gamma \cap S$ has compact closure in $Aut(S)$ and M is compact. The pair (S, Γ) is called a presentation for M, and is discrete if Γ is a discrete subgroup of $Aff(S)$, in which case $\pi_1(M) = \Gamma$. Every infrasolvmanifold has a presentation such that $\Gamma/\Gamma \cap S$ is finite [FJ97], but we cannot assume that Γ is discrete, and S is not determined by π.

Farrell and Jones showed that in all dimensions *except* perhaps 4 infrasolvmanifolds with isomorphic fundamental groups are diffeomorphic. However an affine diffeomorphism is not always possible [FJ97]. They showed also that 4-dimensional infrasolvmanifolds are determined up to homeomorphism by their fundamental groups (see Theorem 8.2 below). Using the Mostow orbifold bundle associated to a presentation of an infrasolvmanifold (see §5 below) and standard 3-manifold theory it is possible to show that, in most cases, 4-dimensional infrasolvmanifolds are determined up to diffeomorphism by their groups ([Cb] - see Theorem 8.9 below). However there may still be a nonorientable 4-dimensional infrasolvmanifold with virtually nilpotent fundamental group which has no discrete presentation.

An important special case includes most infrasolvmanifolds of dimension ≤ 4 (and all infranilmanifolds). Let $T_n^+(\mathbb{R})$ be the subgroup of $GL(n, \mathbb{R})$ consisting of upper triangular matrices with positive diagonal entries. A Lie group S is *triangular* if is isomorphic to a closed subgroup of $T_n^+(\mathbb{R})$ for some n. The eigenvalues of the image of each element of S under the adjoint representation are then all real, and so S is *of type R* in the terminology of [Go71]. (It can be shown that a Lie group is triangular if and only if it is 1-connected and solvable of type R.) Two infrasolvmanifolds with discrete presentations (S_i, Γ_i) where each S_i is triangular (for $i = 1,\ 2$) are affinely diffeomorphic if and only if their fundamental groups are isomorphic, by Theorem 3.1 of [Le95]. The *translation subgroup* $S \cap \Gamma$ of a discrete pair with S triangular can be characterised intrinsically as the subgroup of Γ consisting of the elements $g \in \Gamma$ such that all the eigenvalues of the automorphisms of the abelian sections of the lower central series for $\sqrt{\Gamma}$ induced by conjugation by g are positive [De97]. Does every infrasolvmanifold with a presentation (S, Γ) where S is triangular have a discrete presentation?

Since S and Γ_o are each contractible, $X = \Gamma_o \backslash S$ is contractible also. It can be shown that $\pi = \Gamma/\Gamma_o$ acts freely on X, and so is the fundamental group of $M = \pi\backslash X$. (See Chapter III.3 of [Au73] for the solvmanifold case.) Since M is aspherical π is a PD_m group, where m is the dimension of M; since π is also virtually solvable it is thus virtually poly-Z of Hirsch length m, by Theorem 9.23 of [Bi], and $\chi(M) = \chi(\pi) = 0$. Conversely, any torsion free virtually poly-Z group is the fundamental group of a closed smooth manifold which is finitely covered by the coset space of a lattice in a 1-connected solvable Lie group [AJ76].

Let S be a connected solvable Lie group of dimension m, and let N be its nilradical. If π is a lattice in S then it is torsion free and virtually poly-Z of Hirsch length m and $\pi \cap N = \sqrt{\pi}$ is a lattice in N. If S is 1-connected then S/N is isomorphic to some vector group R^n, and $\pi/\sqrt{\pi} \cong Z^n$. A complete characterization of such lattices is not known, but a torsion free virtually poly-Z group π is a lattice in a connected solvable Lie group S if and only if $\pi/\sqrt{\pi}$ is abelian. (See Sections 4.29-31 of [Rg].)

The 4-dimensional solvable Lie geometries other than the infranil geometries are $\mathbb{S}ol^4_{m,n}$, $\mathbb{S}ol^4_0$ and $\mathbb{S}ol^4_1$, and the model spaces are solvable Lie groups with left invariant metrics. The following descriptions are based on [Wl86]. The Lie group is the identity component of the isometry group for the geometries $\mathbb{S}ol^4_{m,n}$ and $\mathbb{S}ol^4_1$; the identity component of $Isom(\mathbb{S}ol^4_0)$ is isomorphic to the semidirect product $(C \oplus R) \times_\gamma C^\times$, where $\gamma(z)(u, x) = (zu, |z|^{-2}x)$ for all (u, x) in $C \oplus R$ and z in $C^\times$, and thus $\mathbb{S}ol^4_0$ admits an additional isometric action of S^1, by rotations about an axis in $C \oplus R \cong R^3$, the radical of Sol^4_0.

$Sol^4_{m,n} = R^3 \times_{\theta_{m,n}} R$, where m and n are integers such that the polynomial $f_{m,n} = X^3 - mX^2 + nX - 1$ has distinct roots e^a, e^b and e^c (with $a < b < c$ real) and $\theta_{m,n}(t)$ is the diagonal matrix $diag[e^{at}, e^{bt}, e^{ct}]$. Since $\theta_{m,n}(t) = \theta_{n,m}(-t)$ we may assume that $m \leq n$; the condition on the roots then holds if and only if $2\sqrt{n} \leq m \leq n$. The metric given by $ds^2 = e^{-2at}dx^2 + e^{-2bt}dy^2 + e^{-2ct}dz^2 + dt^2$ (in the obvious global coordinates) is left invariant, and the automorphism of $Sol^4_{m,n}$ which sends (t,x,y,z) to (t,px,qy,rz) is an isometry if and only if $p^2 = q^2 = r^2 = 1$. Let G be the subgroup of $GL(4,\mathbb{R})$ of bordered matrices $\left(\begin{smallmatrix} D & \xi \\ 0 & 1 \end{smallmatrix}\right)$, where $D = diag[\pm e^{at}, \pm e^{bt}, \pm e^{ct}]$ and $\xi \in R^3$. Then $Sol^4_{m,n}$ is the subgroup of G with positive diagonal entries, and $G = Isom(\mathbb{S}ol^4_{m,n})$ if $m \neq n$. If $m = n$ then $b = 0$ and $\mathbb{S}ol^4_{m,m} = \mathbb{S}ol^3 \times \mathbb{E}^1$, which admits the additional isometry sending (t,x,y,z) to (t^{-1},z,y,x), and G has index 2 in $Isom(\mathbb{S}ol^3 \times \mathbb{E}^1)$. The stabilizer of the identity in the full isometry group is $(Z/2Z)^3$ for $Sol^4_{m,n}$ if $m \neq n$ and $D_8 \times (Z/2Z)$ for $Sol^3 \times R$. In all cases $Isom(\mathbb{S}ol^4_{m,n}) \leq Aff(Sol^4_{m,n})$.

In general $\mathbb{S}ol^4_{m,n} = \mathbb{S}ol^4_{m',n'}$ if and only if $(a,b,c) = \lambda(a',b',c')$ for some $\lambda \neq 0$. Must λ be rational? (This is a case of the "problem of the four exponentials" of transcendental number theory.) If $m \neq n$ then $F_{m,n} = \mathbb{Q}[X]/(f_{m,n})$ is a totally real cubic number field, generated over $\mathbb{Q}$ by the image of X. The images of X under embeddings of $F_{m,n}$ in $\mathbb{R}$ are the roots e^a, e^b and e^c, and so it represents a unit of norm 1. The group of such units is free abelian of rank 2. Therefore if $\lambda = r/s \in \mathbb{Q}^\times$ this unit is an r^{th} power in $F_{m,n}$ (and its r^{th} root satisfies another such cubic). It can be shown that $|r| \leq \log_2(m)$, and so (modulo the problem of the four exponentials) there is a canonical "minimal" pair (m,n) representing each such geometry.

$Sol^4_0 = R^3 \times_\xi R$, where $\xi(t)$ is the diagonal matrix $diag[e^t, e^t, e^{-2t}]$. Note that if $\xi(t)$ preserves a lattice in R^3 then its characteristic polynomial has integral coefficients and constant term -1. Since it has e^t as a repeated root we must have $\xi(t) = I$. Therefore Sol^4_0 does not admit any lattices. The metric given by the expression $ds^2 = e^{-2t}(dx^2 + dy^2) + e^{4t}dz^2 + dt^2$ is left invariant, and $O(2) \times O(1)$ acts via rotations and reflections in the (x,y)-coordinates and reflection in the z-coordinate, to give the stabilizer of the identity. These actions are automorphisms of Sol^4_0, so $Isom(\mathbb{S}ol^4_0) = Sol^4_0 \rtimes (O(2) \times O(1)) \leq Aff(Sol^4_0)$. The identity component of $Isom(\mathbb{S}ol^4_0)$ is not triangular.

Sol^4_1 is the group of real matrices $\{\begin{pmatrix} 1 & b & c \\ 0 & \alpha & a \\ 0 & 0 & 1 \end{pmatrix} : \alpha > 0,\ a,\ b,\ c \in \mathbb{R}\}$. The metric given by $ds^2 = t^{-2}((1+x^2)(dt^2+dy^2)+t^2(dx^2+dz^2)-2tx(dtdx+dydz))$ is left invariant, and the stabilizer of the identity is D_8, generated by the isometries which send (t,x,y,z) to $(t,-x,y,-z)$ and to $t^{-1}(1,-y,-x,xy-tz)$. These are automorphisms. (The latter one is the restriction of the involution of $GL(3,\mathbb{R})$ which sends A to $J(A^{tr})^{-1}J$, where J reverses the order of the standard basis of $\mathbb{R}^3$.) Thus $Isom(\mathbb{S}ol_1^4) \leq Aff(Sol_1^4)$.

Closed $\mathbb{S}ol_{m,n}^4$- or $\mathbb{S}ol_1^4$-manifolds are clearly infrasolvmanifolds. The $\mathbb{S}ol_0^4$ case is more complicated. Let $\tilde{\gamma}(z)(u,x) = (e^z u, e^{-2\mathrm{Re}(z)}x)$ for all (u,x) in $C \oplus R$ and z in C. Then $\widetilde{I} = (C \oplus R) \times_{\tilde{\gamma}} C$ is the universal covering group of $Isom(\mathbb{S}ol_0^4)$. If M is a closed $\mathbb{S}ol_0^4$-manifold its fundamental group π is a semidirect product $Z^3 \times_\theta Z$, where $\theta(1) \in GL(3,\mathbb{Z})$ has two complex conjugate eigenvalues $\lambda \neq \bar{\lambda}$ with $|\lambda| \neq 0$ or 1 and one real eigenvalue ρ such that $|\rho| = |\lambda|^{-2}$ (see Chapter 8). If M is orientable (i.e., $\rho > 0$) then π is a lattice in $S_\pi = (C \oplus R) \times_{\tilde{\theta}} R < \widetilde{I}$, where $\tilde{\theta}(r) = \tilde{\gamma}(r\log(\lambda))$. In general, π is a lattice in $Aff(S_{\pi^+})$. The action of $\widetilde{I}$ on Sol_0^4 determines a diffeomorphism $S_{\pi^+}/\pi \cong M$, and so M is an infrasolvmanifold with a discrete presentation.

We shall see in Chapter 8 that every orientable 4-dimensional infrasolvmanifold is diffeomorphic to a geometric 4-manifold, but the argument uses the Mostow fibration and is differential-topological rather than differential-geometric.

7.4 Geometric decompositions

An n-manifold M admits a *geometric decomposition* if it has a finite collection of disjoint 2-sided hypersurfaces S such that each component of $M - \cup S$ is geometric of finite volume, i.e., is homeomorphic to $\Gamma\backslash X$, for some geometry $\mathbb{X}$ and lattice Γ. We shall call the hypersurfaces S *cusps* and the components of $M - \cup S$ *pieces* of M. The decomposition is *proper* if the set of cusps is nonempty.

Theorem 7.1 *If a closed 4-manifold M admits a geometric decomposition then either*

(1) *M is geometric; or*

(2) *M has a codimension-2 foliation with leaves S^2 or RP^2; or*

(3) *the components of $M - \cup S$ all have geometry $\mathbb{H}^2 \times \mathbb{H}^2$; or*

(4) *the components of $M - \cup S$ have geometry $\mathbb{H}^4$, $\mathbb{H}^3 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$; or*

(5) *the components of $M - \cup S$ have geometry $\mathbb{H}^2(\mathbb{C})$ or $\mathbb{F}^4$.*

In cases (3), (4) or (5) $\chi(M) \geq 0$ and in cases (4) or (5) M is aspherical.

Proof The proof consists in considering the possible ends (cusps) of complete geometric 4-manifolds of finite volume. The hypersurfaces bounding a component of $M - \cup S$ correspond to the ends of its interior. If the geometry is of solvable or compact type then there are no ends, since every lattice is then cocompact [Rg]. Thus we may concentrate on the eight geometries $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$, $\mathbb{H}^3 \times \mathbb{E}^1$, $\mathbb{H}^4$, $\mathbb{H}^2(\mathbb{C})$ and $\mathbb{F}^4$. The ends of a geometry of constant negative curvature $\mathbb{H}^n$ are flat [Eb80]; since any lattice in a Lie group must meet the radical in a lattice it follows easily that the ends are also flat in the mixed euclidean cases $\mathbb{H}^3 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^2$ and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$. Similarly, the ends of $\mathbb{S}^2 \times \mathbb{H}^2$-manifolds are $\mathbb{S}^2 \times \mathbb{E}^1$-manifolds. Since the elements of $PSL(2,\mathbb{C})$ corresponding to the cusps of finite area hyperbolic surfaces are parabolic, the ends of $\mathbb{F}^4$-manifolds are $\mathbb{N}il^3$-manifolds. The ends of $\mathbb{H}^2(\mathbb{C})$-manifolds are also $\mathbb{N}il^3$-manifolds [Ep87], while the ends of $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds are $\mathbb{S}ol^3$-manifolds in the irreducible cases [Sh63], and graph manifolds whose fundamental groups contain nonabelian free subgroups otherwise. Clearly if two pieces are contiguous their common cusps must be homeomorphic. If the piece is not a reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifold then the inclusion of a cusp into the closure of the piece induces a monomorphism on fundamental group.

If M is a closed 4-manifold with a geometric decomposition of type (2) the inclusions of the cusps into the closures of the pieces induce isomorphisms on π_2, and a Mayer-Vietoris argument in the universal covering space $\widetilde{M}$ shows that $\widetilde{M}$ is homotopy equivalent to S^2. The natural foliation of $S^2 \times H^2$ by 2-spheres induces a codimension-2 foliation on each piece, with leaves S^2 or RP^2. The cusps bounding the closure of a piece are $\mathbb{S}^2 \times \mathbb{E}^1$-manifolds, and hence also have codimension-1 foliations, with leaves S^2 or RP^2. Together these foliations give a foliation of the closure of the piece, so that each cusp is a union of leaves. The homeomorphisms identifying cusps of contiguous pieces are isotopic to isometries of the corresponding $\mathbb{S}^2 \times \mathbb{E}^1$-manifolds. As the foliations of the cusps are preserved by isometries M admits a foliation with leaves S^2 or RP^2. (In other words, it is the total space of an orbifold bundle over a hyperbolic 2-orbifold, with general fibre S^2.)

If at least one piece has an aspherical geometry other than $\mathbb{H}^2 \times \mathbb{H}^2$ then all do and M is aspherical. Since all the pieces of type $\mathbb{H}^4$, $\mathbb{H}^2(\mathbb{C})$ or $\mathbb{H}^2 \times \mathbb{H}^2$ have strictly positive Euler characteristic while those of type $\mathbb{H}^3 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ or $\mathbb{F}^4$ have Euler characteristic 0 we must have $\chi(M) \geq 0$. □

If in case (2) M admits a foliation with all leaves homeomorphic then the projection to the leaf space is a submersion and so M is the total space of an S^2-bundle or RP^2-bundle over a hyperbolic surface. In particular, the covering space M_κ corresponding to the kernel κ of the action of $\pi_1(M)$ on $\pi_2(M) \cong Z$ is the total space of an S^2-bundle over a hyperbolic surface. In Chapter 9 we shall show that S^2-bundles and RP^2-bundles over aspherical surfaces are geometric. This surely holds also for orbifold bundles (defined in the next section) over flat or hyperbolic 2-orbifolds, with general fibre S^2.

If an aspherical closed 4-manifold has a nontrivial geometric decomposition with no pieces of type $\mathbb{H}^2 \times \mathbb{H}^2$ then its fundamental group contains nilpotent subgroups of Hirsch length 3 (corresponding to the cusps).

Is there an essentially unique minimal decomposition? Since hyperbolic surfaces are connected sums of tori, and a punctured torus admits a complete hyperbolic geometry of finite area, we cannot expect that there is an unique decomposition, even in dimension 2. Any PD_n-group satisfying *Max-c* (the maximal condition on centralizers) has an essentially unique minimal finite splitting along virtually poly-Z subgroups of Hirsch length $n-1$, by Theorem A2 of [Kr90]. Do all fundamental groups of aspherical manifolds with geometric decompositions have *Max-c*? A compact non-positively curved n-manifold ($n \geq 3$) with convex boundary is either flat or has a canonical decomposition along totally geodesic closed flat hypersurfaces into pieces which are Seifert fibred or codimension-1 atoroidal [LS00]. Which 4-manifolds with geometric decompositions admit such metrics? (Closed $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds do not [KL96].)

Closed $\mathbb{H}^4$- or $\mathbb{H}^2(\mathbb{C})$-manifolds admit no proper geometric decompositions, since their fundamental groups have no noncyclic abelian subgroups [Pr43]. A similar argument shows that closed $\mathbb{H}^3 \times \mathbb{E}^1$-manifolds admit no proper decompositions, since they are finitely covered by cartesian products of $\mathbb{H}^3$-manifolds with S^1. Thus closed 4-manifolds with a proper geometric decomposition involving pieces of types other than $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ are never geometric.

Many $\mathbb{S}^2 \times \mathbb{H}^2$-, $\mathbb{H}^2 \times \mathbb{H}^2$-, $\mathbb{H}^2 \times \mathbb{E}^2$- and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds admit proper geometric decompositions. On the other hand, a manifold with a geometric decomposition into pieces of type $\mathbb{H}^2 \times \mathbb{E}^2$ need not be geometric. For instance, let $G = \langle u, v, x, y \mid [u, v] = [x, y]\rangle$ be the fundamental group of $T\sharp T$, the closed orientable surface of genus 2, and let $\theta : G \to SL(2, \mathbb{Z})$ be the epimorphism determined by $\theta(u) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\theta(x) = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$, Then the semidirect product $\pi = Z^2 \times_\theta G$ is the fundamental group of a torus bundle over $T\sharp T$ which has a geometric decomposition into two pieces of type $\mathbb{H}^2 \times \mathbb{E}^2$, but is not geometric, since π does not have a subgroup of finite index with centre Z^2.

It is easily seen that each $\mathbb{S}^2 \times \mathbb{E}^1$-manifold may be realized as the end of a complete $\mathbb{S}^2 \times \mathbb{H}^2$-manifold with finite volume and a single end. However, if the manifold is orientable the ends must be orientable, and if it is complex analytic then they must be $S^2 \times S^1$. Every flat 3-manifold is a cusp of some complete $\mathbb{H}^4$-manifold with finite volume [Ni98]. However if such a manifold has only one cusp the cusp cannot have holonomy $Z/3Z$ or $Z/6Z$ [LR00]. The fundamental group of a cusp of an $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold must have a chain of abelian normal subgroups $Z < Z^2 < Z^3$; not all orientable flat 3-manifold groups have such subgroups. The ends of complete, complex analytic $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds with finite volume and irreducible fundamental group are orientable $\mathbb{S}ol^3$-manifolds which are mapping tori, and all such may be realized in this way [Sh63].

Let M be the double of $T_o \times T_o$, where $T_o = T - intD^2$ is the once-punctured torus. Since T_o admits a complete hyperbolic geometry of finite area M admits a geometric decomposition into two pieces of type $\mathbb{H}^2 \times \mathbb{H}^2$. However as $F(2) \times F(2)$ has cohomological dimension 2 the homomorphism of fundamental groups induced by the inclusion of the cusp into $T_o \times T_o$ has nontrivial kernel, and M is not aspherical.

7.5 Orbifold bundles

An n-dimensional orbifold B has an open covering by subspaces of the form D^n/G, where G is a finite subgroup of $O(n)$. Let F be a closed manifold. An *orbifold bundle with general fibre F over B* is a map $f : M \to B$ which is locally equivalent to a projection $G\backslash(F \times D^n) \to G\backslash D^n$, where G acts freely on F and effectively and orthogonally on D^n.

If the base B has a finite regular covering $\hat{B}$ which is a manifold, then p induces a fibre bundle projection $\hat{p} : \hat{M} \to \hat{B}$ with fibre F, and the action of the covering group maps fibres to fibres. Conversely, if $p_1 : M_1 \to B_1$ is a fibre bundle projection with fibre F_1 and G is a finite group which acts freely on M_1 and maps fibres to fibres then passing to orbit spaces gives an orbifold bundle $p : M = G\backslash M_1 \to B = H\backslash B_1$ with general fibre $F = K\backslash F_1$, where H is the induced group of homeomorphisms of B_1 and K is the kernel of the epimorphism from G to H.

Theorem 7.2 [Cb] *Let M be an infrasolvmanifold. Then there is an orbifold bundle $p : M \to B$ with general fibre an infranilmanifold and base a flat orbifold.*

Proof Let (S,Γ) be a presentation for M and let R be the nilradical of S. Then $A = S/R$ is a 1-connected abelian Lie group, and so $A \cong \mathbb{R}^d$ for some

$d \geq 0$. Since R is characteristic in S there is a natural projection $q : Aff(S) \to Aff(A)$. Let $\Gamma_S = \Gamma \cap S$ and $\Gamma_R = \Gamma \cap R$. Then the action of Γ_S on S induces an action of the discrete group $q(\Gamma_S) = R\Gamma_S/R$ on A. The Mostow fibration for $M_1 = \Gamma_S \backslash S$ is the quotient map to $B_1 = q(\Gamma_S)\backslash A$, which is a bundle projection with fibre $F_1 = \Gamma_R \backslash R$. Now Γ_o is normal in R, by Corollary 3 of Theorem 2.3 of [Rg], and Γ_R/Γ_o is a lattice in the nilpotent Lie group R/Γ_o. Therefore F_1 is a nilmanifold, while B_1 is a torus.

The finite group Γ/Γ_S acts on M_1, respecting the Mostow fibration. Let $\overline{\Gamma} = q(\Gamma)$, $K = \Gamma \cap \mathrm{Ker}(q)$ and $B = \overline{\Gamma}\backslash A$. Then the induced map $p : M \to B$ is an orbifold bundle projection with general fibre the infranilmanifold $F = K\backslash R = (K/\Gamma_o)\backslash(R/\Gamma_o)$, and base a flat orbifold. □

We shall call $p : M \to B$ the *Mostow orbifold bundle* corresponding to the presentation (S, Γ). In Theorem 8.9 we shall use this construction to show that orientable 4-dimensional infrasolvmanifolds are determined up to diffeomorphism by their fundamental groups, with the possible exception of manifolds having one of two virtually abelian groups.

7.6 Realization of virtual bundle groups

Every extension of one PD_2-group by another may be realized by some surface bundle, by Theorem 5.2. The study of Seifert fibred 4-manifolds and singular fibrations of complex surfaces lead naturally to consideration of the larger class of torsion free groups which are virtually such extensions. Johnson has asked whether such " virtual bundle groups" may be realized by aspherical 4-manifolds.

Theorem 7.3 *Let π be a torsion free group with normal subgroups $K < G < \pi$ such that K and G/K are PD_2-groups and $[\pi : G] < \infty$. Then π is the fundamental group of an aspherical closed smooth 4-manifold which is the total space of an orbifold bundle with general fibre an aspherical closed surface over a 2-dimensional orbifold.*

Proof Let $p : \pi \to \pi/K$ be the quotient homomorphism. Since π is torsion free the preimage in π of any finite subgroup of π/K is a PD_2-group. As the finite subgroups of π/K have order at most $[\pi : G]$, we may assume that π/K has no nontrivial finite normal subgroup, and so is the orbifold fundamental group of some 2-dimensional orbifold B, by the solution to the Nielsen realization problem for surfaces [Ke83]. Let F be the aspherical closed surface with

$\pi_1(F) \cong K$. If π/K is torsion free then B is a closed aspherical surface, and the result follows from Theorem 5.2. In general, B is the union of a punctured surface B_o with finitely many cone discs and regular neighborhoods of reflector curves (possibly containing corner points). The latter may be further decomposed as the union of squares with a reflector curve along one side and with at most one corner point, with two such squares meeting along sides adjacent to the reflector curve. These suborbifolds U_i (i.e., cone discs and squares) are quotients of D^2 by finite subgroups of $O(2)$. Since B is finitely covered (as an orbifold) by the aspherical surface with fundamental group G/K these finite groups embed in $\pi_1^{\mathrm{orb}}(B) \cong \pi/K$, by the Van Kampen Theorem for orbifolds.

The action of π/K on K determines an action of $\pi_1(B_o)$ on K and hence an F-bundle over B_o. Let H_i be the preimage in π of $\pi_1^{orb}(U_i)$. Then H_i is torsion free and $[H_i : K] < \infty$, so H_i acts freely and cocompactly on $\mathbb{X}^2$, where $\mathbb{X}^2 = \mathbb{R}^2$ if $\chi(K) = 0$ and $\mathbb{X}^2 = \mathbb{H}^2$ otherwise, and F is a finite covering space of $H_i \backslash \mathbb{X}^2$. The obvious action of H_i on $\mathbb{X}^2 \times D^2$ determines a bundle with general fibre F over the orbifold U_i. Since self homeomorphisms of F are determined up to isotopy by the induced element of $\mathrm{Out}(K)$, bundles over adjacent suborbifolds have isomorphic restrictions along common edges. Hence these pieces may be assembled to give a bundle with general fibre F over the orbifold B, whose total space is an aspherical closed smooth 4-manifold with fundamental group π. □

We shall verify in Theorem 9.8 that torsion free groups commensurate with products of two centreless PD_2-groups are also realizable.

We can improve upon Theorem 5.7 as follows.

Corollary 7.3.1 *Let M be a closed 4-manifold M with fundamental group π. Then the following are equivalent.*

(1) *M is homotopy equivalent to the total space of an orbifold bundle with general fibre an aspherical surface over an $\mathbb{E}^2$- or $\mathbb{H}^2$-orbifold;*

(2) *π has an FP_2 normal subgroup K such that π/K is virtually a PD_2-group and $\pi_2(M) = 0$;*

(3) *π has a normal subgroup N which is a PD_2-group and $\pi_2(M) = 0$.*

Proof Condition (1) clearly implies (2) and (3). Conversely, if they hold the argument of Theorem 5.7 shows that K is a PD_2-group and N is virtually a PD_2-group. In each case (1) now follows from Theorem 7.2. □

It follows easily from the argument of part (1) of Theorem 5.4 that if π is a group with a normal subgroup K such that K and π/K are PD_2-groups with $\zeta K = \zeta(\pi/K) = 1$, ρ is a subgroup of finite index in π and $L = K \cap \rho$ then $C_\rho(L) = 1$ if and only if $C_\pi(K) = 1$. Since ρ is virtually a product of PD_2-groups with trivial centres if and only if π is, Johnson's trichotomy extends to groups commensurate with extensions of one centreless PD_2-group by another.

Theorem 7.2 settles the realization question for groups of type I. (For suppose π has a subgroup σ of finite index with a normal subgroup ν such that ν and σ/ν are PD_2-groups with $\zeta\nu = \zeta(\sigma/\nu) = 1$. Let $G = \cap h\sigma h^{-1}$ and $K = \nu \cap G$. Then $[\pi : G] < \infty$, G is normal in π, and K and G/K are PD_2-groups. If G is of type I then K is characteristic in G, by Theorem 5.5, and so is normal in π.) Groups of type II need not have such normal PD_2-subgroups - although this is almost true. It is not known whether every type III extension of centreless PD_2-groups has a characteristic PD_2-subgroup (although this is so in many cases, by the corollaries to Theorem 5.6).

If π is an extension of Z^2 by a normal PD_2-subgroup K with $\zeta K = 1$ then $C_\pi(K) = \sqrt{\pi}$, and $[\pi : KC_\pi(K)] < \infty$ if and only if π is virtually $K \times Z^2$, so Johnson's trichotomy extends to such groups. The three types may be characterized by (I) $\sqrt{\pi} \cong Z$, (II) $\sqrt{\pi} \cong Z^2$, and (III) $\sqrt{\pi} = 1$. As these properties are shared by commensurate torsion free groups the trichotomy extends further to torsion free groups which are virtually such extensions. There is at present no uniqueness result corresponding to Theorem 5.5 for such subgroups $K < \pi$, and (excepting for groups of type II) it is not known whether every such group is realized by some aspherical closed 4-manifold. (In fact, it also appears to be unknown in how many ways a 3-dimensional mapping torus may fibre over S^1.)

The Johnson trichotomy is inappropriate if $\zeta K \neq 1$, as there are then nontrivial extensions with trivial action ($\theta = 1$). Moreover $\mathrm{Out}(K)$ is virtually free and so the action θ is never injective. However all such groups π may be realized by aspherical 4-manifolds, for either $\sqrt{\pi} \cong Z^2$ and Theorem 7.2 applies, or π is virtually poly-Z and is the fundamental group of an infrasolvmanifold. (See Chapter 8.)

7.7 Seifert fibrations

A closed 4-manifold M is *Seifert fibred* if it is the total space of an orbifold bundle with general fibre a torus or Klein bottle over a 2-orbifold. (In [Zn85], [Ue90,91] it is required that the general fibre be a torus. This is always so if the manifold is orientable.) The fundamental group π of such a 4-manifold then has a rank two free abelian normal subgroup A such that π/A is virtually a surface group. If the base orbifold is good then the manifold is finitely covered by a torus bundle over a closed surface. This is in fact so in general, by the following theorem. In particular, $\chi(M) = 0$.

Theorem (Ue) *Let S be a closed orientable 4-manifold which is Seifert fibred over the 2-orbifold B. Then*

(1) *If B is spherical or bad S has geometry $\mathbb{S}^3 \times \mathbb{E}^1$ or $\mathbb{S}^2 \times \mathbb{E}^2$;*

(2) *If B is euclidean then S has geometry $\mathbb{E}^4$, $\mathbb{N}il^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$ or $\mathbb{S}ol^3 \times \mathbb{E}^1$;*

(3) *If B is orientable and hyperbolic then S is geometric if and only if it has a complex structure, in which case the geometry is either $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$.*

Conversely, excepting only two flat 4-manifolds, any orientable 4-manifold admitting one of these geometries is Seifert fibred. □

If the base is euclidean or hyperbolic then S is determined up to diffeomorphism by $\pi_1(S)$; if moreover the base is hyperbolic or S is geometric of type $\mathbb{N}il^4$ or $\mathbb{S}ol^3 \times \mathbb{E}^1$ there is a fibre-preserving diffeomorphism. If the base is bad or spherical then S may admit many inequivalent Seifert fibrations.

Less is known about the nonorientable cases. Seifert fibred 4-manifolds with general fibre a torus and base a hyperbolic orbifold with no reflector curves are determined up to fibre preserving diffeomorphism by their fundamental groups [Zi69]. Closed 4-manifolds which fibre over S^1 with fibre a small Seifert fibred 3-manifold are determined up to diffeomorphism by their fundamental groups [Oh90]. This class includes many nonorientable Seifert fibred 4-manifolds over bad, spherical or euclidean bases, but not all. It may be true in general that a Seifert fibred 4-manifold is geometric if and only if its orientable double covering space is geometric, and that aspherical Seifert fibred 4-manifolds are determined up to diffeomorphism by their fundamental groups.

The homotopy type of a $\mathbb{S}^2 \times \mathbb{E}^2$-manifold is determined up to finite ambiguity by the fundamental group (which must be virtually Z^2), Euler characteristic (which must be 0) and Stiefel-Whitney classes. There are just nine possible fundamental groups. Six of these have infinite abelianization, and the above invariants determine the homotopy type in these cases. (See Chapter 10.) The homotopy type of a $\mathbb{S}^3 \times \mathbb{E}^1$-manifold is determined by the fundamental group (which has two ends), Euler characteristic (which is 0), orientation character w_1 and first k-invariant in $H^4(\pi; \pi_3)$. (See Chapter 11.)

Every Seifert fibred 4-manifold with base an euclidean orbifold has Euler characteristic 0 and fundamental group solvable of Hirsch length 4, and so is homeomorphic to an infrasolvmanifold, by Theorem 6.11 and [AJ76]. As no group of type $\mathbb{S}ol_0^4$, $\mathbb{S}ol_1^4$ or $\mathbb{S}ol_{m,n}^4$ (with $m \neq n$) has a rank two free abelian normal subgroup, the manifold must have one of the geometries $\mathbb{E}^4$, $\mathbb{N}il^4$, $\mathbb{N}il \times \mathbb{E}^1$ or $\mathbb{S}ol \times \mathbb{E}^1$. Conversely, excepting only three flat 4-manifolds, such manifolds are Seifert fibred. The fundamental group of a closed $\mathbb{N}il^3 \times \mathbb{E}^1$- or $\mathbb{N}il^4$-manifold has a rank two free abelian normal subgroup, by Theorem 1.5. If π is the fundamental group of a $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold then the commutator subgroup of the intersection of all index 4 subgroups is such a subgroup. (In the $\mathbb{N}il^4$ and $\mathbb{S}ol^3 \times \mathbb{E}^1$ cases there is an unique maximal such subgroup, and the general fibre must be a torus.) Case-by-case inspection of the 74 flat 4-manifold groups shows that all but three have such subgroups. The only exceptions are the semidirect products $G_6 \times_\theta Z$ where $\theta = j$, cej and $abcej$. (See Chapter 8. There is a minor oversight in [Ue90]; in fact there are *two* orientable flat four-manifolds which are not Seifert fibred.)

As $\mathbb{H}^2 \times \mathbb{E}^2$- and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds are aspherical, they are determined up to homotopy equivalence by their fundamental groups. See Chapter 9 for more details.

Theorem 7.3 specializes to give the following characterization of the fundamental groups of Seifert fibred 4-manifolds.

Theorem 7.4 *A group π is the fundamental group of a closed 4-manifold which is Seifert fibred over a hyperbolic base 2-orbifold with general fibre a torus if and only if it is torsion free, $\sqrt{\pi} \cong Z^2$, $\pi/\sqrt{\pi}$ has no nontrivial finite normal subgroup and $\pi/\sqrt{\pi}$ is virtually a PD_2-group.* □

If $\sqrt{\pi}$ is central ($\zeta\pi \cong Z^2$) the corresponding Seifert fibred manifold $M(\pi)$ admits an effective torus action with finite isotropy subgroups.

7.8 Complex surfaces and related structures

In this section we shall summarize what we need from [BPV], [Ue90,91], [Wl86] and [GS], and we refer to these sources for more details.

A *complex surface* shall mean a compact connected nonsingular complex analytic manifold S of complex dimension 2. It is Kähler (and thus diffeomorphic to a projective algebraic surface) if and only if $\beta_1(S)$ is even. Since the Kähler condition is local, all finite covering spaces of such a surface must also have β_1

even. If S has a complex submanifold $L \cong CP^1$ with self-intersection -1 then L may be blown down: there is a complex surface S_1 and a holomorphic map $p: S \to S_1$ such that $p(L)$ is a point and p restricts to a biholomorphic isomorphism from $S-L$ to $S_1 - p(L)$. In particular, S is diffeomorphic to $S_1 \sharp \overline{CP^2}$. If there is no such embedded projective line L the surface is *minimal*. Excepting only the ruled surfaces, every surface has an unique minimal representative.

For many of the 4-dimensional geometries (X, G) the identity component G_o of the isometry group preserves a natural complex structure on X, and so if π is a discrete subgroup of G_o which acts freely on X the quotient $\pi\backslash X$ is a complex surface. This is clear for the geometries $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$ and $\mathbb{H}^2(\mathbb{C})$. (The corresponding model spaces may be identified with CP^2, $CP^1 \times CP^1$, $CP^1 \times C$, $CP^1 \times H^2$, $H^2 \times C$, $H^2 \times H^2$ and the unit ball in C^2, respectively, where H^2 is identified with the upper half plane.) It is also true for $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{S}ol_0^4$, $\mathbb{S}ol_1^4$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ and $\mathbb{F}^4$. In addition, the subgroups $R^4 \tilde{\times} U(2)$ of $E(4)$ and $U(2) \times R$ of $Isom(\mathbb{S}^3 \times \mathbb{E}^1)$ act biholomorphically on C^2 and $C^2 - \{0\}$, respectively, and so some $\mathbb{E}^4$- and $\mathbb{S}^3 \times \mathbb{E}^1$-manifolds have complex structures. No other geometry admits a compatible complex structure. Since none of the model spaces contain an embedded S^2 with self-intersection -1 any complex surface which admits a compatible geometry must be minimal.

Complex surfaces may be coarsely classified by their Kodaira dimension κ, which may be $-\infty$, 0, 1 or 2. Within this classification, minimal surfaces may be further classified into a number of families. We have indicated in parentheses where the geometric complex surfaces appear in this classification. (The dashes signify families which include nongeometric surfaces.)

$\kappa = -\infty$: Hopf surfaces ($\mathbb{S}^3 \times \mathbb{E}^1$, -); Inoue surfaces ($\mathbb{S}ol_0^4$, $\mathbb{S}ol_1^4$);

rational surfaces ($\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$); ruled surfaces ($\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$, -).

$\kappa = 0$: complex tori ($\mathbb{E}^4$); hyperelliptic surfaces ($\mathbb{E}^4$); Enriques surfaces (-);

K3 surfaces (-); Kodaira surfaces ($\mathbb{N}il^3 \times \mathbb{E}^1$).

$\kappa = 1$: minimal properly elliptic surfaces ($\widetilde{\mathbb{SL}} \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^2$).

$\kappa = 2$: minimal (algebraic) surfaces of general type ($\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^2(\mathbb{C})$, -).

A *Hopf surface* is a complex surface whose universal covering space is homeomorphic to $S^3 \times R \cong C^2 - \{0\}$. Some Hopf surfaces admit no compatible geometry, and there are $\mathbb{S}^3 \times \mathbb{E}^1$-manifolds that admit no complex structure. The *Inoue* surfaces are exactly the complex surfaces admitting one of the geometries $\mathbb{S}ol_0^4$ or $\mathbb{S}ol_1^4$.

A *rational surface* is a complex surface birationally equivalent to CP^2. Minimal rational surfaces are diffeomorphic to CP^2 or to $CP^1 \times CP^1$. A *ruled surface* is a complex surface which is holomorphically fibred over a smooth complex curve (closed orientable 2-manifold) of genus $g > 0$ with fibre CP^1. Rational and ruled surfaces may be characterized as the complex surfaces S with $\kappa(S) = -\infty$ and $\beta_1(S)$ even. Not all ruled surfaces admit geometries compatible with their complex structures.

A *complex torus* is a quotient of C^2 by a lattice, and a *hyperelliptic surface* is one properly covered by a complex torus. If S is a complex surface which is homeomorphic to a flat 4-manifold then S is a complex torus or is hyperelliptic, since it is finitely covered by a complex torus. Since S is orientable and $\beta_1(S)$ is even $\pi = \pi_1(S)$ must be one of the eight flat 4-manifold groups of orientable type and with $\pi \cong Z^4$ or $I(\pi) \cong Z^2$. In each case the holonomy group is cyclic, and so is conjugate (in $GL^+(4,\mathbb{R})$) to a subgroup of $GL(2,\mathbb{C})$. (See Chapter 8.) Thus all of these groups may be realized by complex surfaces. A *Kodaira surface* is finitely covered by a surface which fibres holomorphically over an elliptic curve with fibres of genus 1.

An *elliptic surface* S is a complex surface which admits a holomorphic map p to a complex curve such that the generic fibres of p are diffeomorphic to the torus T. If the elliptic surface S has no singular fibres it is Seifert fibred, and it then has a geometric structure if and only if the base is a good orbifold. An orientable Seifert fibred 4-manifold over a hyperbolic base has a geometric structure if and only if it is an elliptic surface without singular fibres [Ue90]. The elliptic surfaces S with $\kappa(S) = -\infty$ and $\beta_1(S)$ odd are the geometric Hopf surfaces. The elliptic surfaces S with $\kappa(S) = -\infty$ and $\beta_1(S)$ even are the cartesian products of elliptic curves with CP^1.

All rational, ruled and hyperelliptic surfaces are projective algebraic surfaces, as are all surfaces with $\kappa = 2$. Complex tori and surfaces with geometry $\mathbb{H}^2 \times \mathbb{E}^2$ are diffeomorphic to projective algebraic surfaces. Hopf, Inoue and Kodaira surfaces and surfaces with geometry $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ all have β_1 odd, and so are not Kähler, let alone projective algebraic.

An *almost complex structure* on a smooth $2n$-manifold M is a reduction of the structure group of its tangent bundle to $GL(n,\mathbb{C}) < GL^+(2n,\mathbb{R})$. Such a structure determines an orientation on M. If M is a closed oriented 4-manifold and $c \in H^2(M;\mathbb{Z})$ then there is an almost complex structure on M with first Chern class c and inducing the given orientation if and only if $c \equiv w_2(M)$ *mod* (2) and $c^2 \cap [M] = 3\sigma(M) + 2\chi(M)$, by a theorem of Wu. (See the Appendix to Chapter I of [GS] for a recent account.)

A *symplectic structure* on a closed smooth manifold M is a closed nondegenerate 2-form ω. Nondegenerate means that for all $x \in M$ and all $u \in T_xM$ there is a $v \in T_xM$ such that $\omega(u,v) \neq 0$. Manifolds admitting symplectic structures are even-dimensional and orientable. A condition equivalent to nondegeneracy is that the n-fold wedge $\omega^{\wedge n}$ is nowhere 0, where $2n$ is the dimension of M. The n^{th} cup power of the corresponding cohomology class $[\omega]$ is then a nonzero element of $H^{2n}(M;\mathbb{R})$. Any two of a riemannian metric, a symplectic structure and an almost complex structure together determine a third, if the given two are compatible. In dimension 4, this is essentially equivalent to the fact that $SO(4)\cap Sp(4) = SO(4)\cap GL(2,\mathbb{C}) = Sp(4)\cap GL(2,\mathbb{C}) = U(2)$, as subgroups of $GL(4,\mathbb{R})$. (See [GS] for a discussion of relations between these structures.) In particular, Kähler surfaces have natural symplectic structures, and symplectic 4-manifolds admit compatible almost complex tangential structures. However orientable $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifolds which fibre over T are symplectic [Ge92] but have no complex structure (by the classification of surfaces) and Hopf surfaces are complex manifolds with no symplectic structure (since $\beta_2 = 0$).

Chapter 8

Solvable Lie geometries

The main result of this chapter is the characterization of 4-dimensional infrasolvmanifolds up to homeomorphism, given in §1. All such manifolds are either mapping tori of self homeomorphisms of 3-dimensional infrasolvmanifolds or are unions of two twisted I-bundles over such 3-manifolds. In the rest of the chapter we consider each of the possible 4-dimensional geometries of solvable Lie type.

In §2 we determine the automorphism groups of the flat 3-manifold groups, while in §3 and §4 we determine *ab initio* the 74 flat 4-manifold groups. There have been several independent computations of these groups; the consensus reported on page 126 of [Wo] is that there are 27 orientable groups and 48 nonorientable groups. However the tables of 4-dimensional crystallographic groups in [B-Z] list only 74 torsion free groups. As these computer-generated tables give little insight into how these groups arise, and as the earlier computations were never published in detail, we shall give a direct and elementary computation, motivated by Lemma 3.14. Our conclusions as to the numbers of groups with abelianization of given rank, isomorphism type of holonomy group and orientation type agree with those of [B-Z]. (We have not attempted to make the lists correspond.)

There are infinitely many examples for each of the other geometries. In §5 we show how these geometries may be distinguished, in terms of the group theoretic properties of their lattices. In §6, §7 and §8 we consider mapping tori of self homeomorphisms of $\mathbb{E}^3$-, $\mathbb{N}il^3$- and $\mathbb{S}ol^3$-manifolds, respectively. In §9 we show directly that "most" groups allowed by Theorem 8.1 are realized geometrically and outline classifications for them, while in §10 we show that "most" 4-dimensional infrasolvmanifolds are determined up to diffeomorphism by their fundamental groups.

8.1 The characterization

In this section we show that 4-dimensional infrasolvmanifolds may be characterized up to homeomorphism in terms of the fundamental group and Euler characteristic.

Theorem 8.1 *Let M be a closed 4-manifold with fundamental group π and such that $\chi(M)=0$. The following conditions are equivalent:*

(1) *π is torsion free and virtually poly-Z and $h(\pi)=4$;*

(2) *$h(\sqrt{\pi})\geq 3$;*

(3) *π has an elementary amenable normal subgroup ρ with $h(\rho)\geq 3$, and $H^2(\pi;Z[\pi])=0$; and*

(4) *π is restrained, every finitely generated subgroup of π is FP_3 and π maps onto a virtually poly-Z group Q with $h(Q)\geq 3$.*

Moreover if these conditions hold M is aspherical, and is determined up to homeomorphism by π, and every automorphism of π may be realized by a self homeomorphism of M.

Proof If (1) holds then $h(\sqrt{\pi})\geq 3$, by Theorem 1.6, and so (2) holds. This in turn implies (3), by Theorem 1.17. If (3) holds then π has one end, by Lemma 1.15, and $\beta_1^{(2)}(\pi)=0$, by Corollary 2.3.1. Hence M is aspherical, by Corollary 3.5.2. Hence π is a PD_4-group and $3\leq h(\rho)\leq c.d.\rho\leq 4$. In particular, ρ is virtually solvable, by Theorem 1.11. If $c.d.\rho=4$ then $[\pi:\rho]$ is finite, by Strebel's Theorem, and so π is virtually solvable also. If $c.d.\rho=3$ then $c.d.\rho=h(\rho)$ and so ρ is a duality group and is FP [Kr86]. Therefore $H^q(\rho;\mathbb{Q}[\pi])\cong H^q(\rho;\mathbb{Q}[\rho])\otimes\mathbb{Q}[\pi/\rho]$ and is 0 unless $q=3$. It then follows from the LHSSS for π as an extension of π/ρ by ρ (with coefficients $\mathbb{Q}[\pi]$) that $H^4(\pi;\mathbb{Q}[\pi])\cong H^1(\pi/\rho;\mathbb{Q}[\pi/\rho])\otimes H^3(\rho;\mathbb{Q}[\rho])$. Therefore $H^1(\pi/\rho;\mathbb{Q}[\pi/\rho])\cong Q$, so π/ρ has two ends and we again find that π is virtually solvable. In all cases π is torsion free and virtually poly-Z, by Theorem 9.23 of [Bi], and $h(\pi)=4$.

If (4) holds then π is an ascending HNN extension $\pi\cong B*_\phi$ with base FP_3 and so M is aspherical, by Theorem 3.16. As in Theorem 2.13 we may deduce from [BG85] that B must be a PD_3-group and ϕ an isomorphism, and hence B and π are virtually poly-Z. Conversely (1) clearly implies (4).

The final assertions follow from Theorem 2.16 of [FJ], as in Theorem 6.11 above. □

Does the hypothesis $h(\rho)\geq 3$ in (3) imply $H^2(\pi;\mathbb{Z}[\pi])=0$? The examples $F\times S^1\times S^1$ where $F=S^2$ or is a closed hyperbolic surface show that the condition that $h(\rho)>2$ is necessary. (See also §1 of Chapter 9.)

Corollary 8.1.1 *The 4-manifold M is homeomorphic to an infrasolvmanifold if and only if the equivalent conditions of Theorem 8.1 hold.*

Proof If M is homeomorphic to an infrasolvmanifold then $\chi(M) = 0$, π is torsion free and virtually poly-Z and $h(\pi) = 4$ (see Chapter 7). Conversely, if these conditions hold then π is the fundamental group of an infrasolvmanifold, by [AJ76]. □

It is easy to see that all such groups are realizable by closed smooth 4-manifolds with Euler characteristic 0.

Theorem 8.2 *If π is torsion free and virtually poly-Z of Hirsch length 4 then it is the fundamental group of a closed smooth 4-manifold M which is either a mapping torus of a self homeomorphism of a closed 3-dimensional infrasolvmanifold or is the union of two twisted I-bundles over such a 3-manifold. Moreover, the 4-manifold M is determined up to homeomorphism by the group.*

Proof The Eilenberg-Mac Lane space $K(\pi, 1)$ is a PD_4-complex with Euler characteristic 0. By Lemma 3.14, either there is an epimorphism $\phi : \pi \to Z$, in which case π is a semidirect product $G \times_\theta Z$ where $G = \mathrm{Ker}(\phi)$, or $\pi \cong G_1 *_G G_2$ where $[G_1 : G] = [G_2 : G] = 2$. The subgroups G, G_1 and G_2 are torsion free and virtually poly-Z. Since in each case π/G has Hirsch length 1 these subgroups have Hirsch length 3 and so are fundamental groups of closed 3-dimensional infrasolvmanifolds. The existence of such a manifold now follows by standard 3-manifold topology, while its uniqueness up to homeomorphism was proven in Theorem 6.11. □

The first part of this theorem may be stated and proven in purely algebraic terms, since torsion free virtually poly-Z groups are Poincaré duality groups. (See Chapter III of [Bi].) If π is such a group then either it is virtually nilpotent or $\sqrt{\pi} \cong Z^3$ or Γ_q for some q, by Theorems 1.5 and 1.6. In the following sections we shall consider how such groups may be realized geometrically. The geometry is largely determined by $\sqrt{\pi}$. We shall consider first the virtually abelian cases.

8.2 Flat 3-manifold groups and their automorphisms

The flat n-manifold groups for $n \leq 2$ are Z, Z^2 and $K = Z \times_{-1} Z$, the Klein bottle group. There are six orientable and four nonorientable flat 3-manifold groups. The first of the orientable flat 3-manifold groups G_1 - G_6 is $G_1 = Z^3$. The next four have $I(G_i) \cong Z^2$ and are semidirect products $Z^2 \times_T Z$ where $T = -I$, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, respectively, is an element of finite order in $SL(2, \mathbb{Z})$. These groups all have cyclic holonomy groups, of orders 2, 3, 4 and 6, respectively. The group G_6 is the group of the Hantzsche-Wendt flat 3-manifold, and has a presentation

$$\langle x, y \mid xy^2x^{-1} = y^{-2},\ yx^2y^{-1} = y^{-2}\rangle.$$

Its maximal abelian normal subgroup is generated by x^2, y^2 and $(xy)^2$ and its holonomy group is the diagonal subgroup of $SL(3,\mathbb{Z})$, which is isomorphic to $(Z/2Z)^2$. (This group is the generalized free product of two copies of K, amalgamated over their maximal abelian subgroups, and so maps onto D.)

The nonorientable flat 3-manifold groups B_1 - B_4 are semidirect products $K \times_\theta Z$, corresponding to the classes in $Out(K) \cong (Z/2Z)^2$. In terms of the presentation $\langle x, y \mid xyx^{-1} = y^{-1}\rangle$ for K these classes are represented by the automorphisms θ which fix y and send x to x, xy, x^{-1} and $x^{-1}y$, respectively. The groups B_1 and B_2 are also semidirect products $Z^2 \times_T Z$, where $T = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ has determinant -1 and $T^2 = I$. They have holonomy groups of order 2, while the holonomy groups of B_3 and B_4 are isomorphic to $(Z/2Z)^2$.

All the flat 3-manifold groups either map onto Z or map onto D. The methods of this chapter may be easily adapted to find all such groups. Assuming these are all known we may use Sylow theory and a little topology to show that there are no others. We sketch here such an argument. Suppose that π is a flat 3-manifold group with finite abelianization. Then $0 = \chi(\pi) = 1 + \beta_2(\pi) - \beta_3(\pi)$, so $\beta_3(\pi) \neq 0$ and π must be orientable. Hence the holonomy group $F = \pi/T(\pi)$ is a subgroup of $SL(3,\mathbb{Z})$. Let f be a nontrivial element of F. Then f has order 2, 3, 4 or 6, and has a $+1$-eigenspace of rank 1, since it is orientation preserving. This eigenspace is invariant under the action of the normalizer $N_F(\langle f\rangle)$, and the induced action of $N_F(\langle f\rangle)$ on the quotient space is faithful. Thus $N_F(\langle f\rangle)$ is isomorphic to a subgroup of $GL(2,\mathbb{Z})$ and so is cyclic or dihedral of order dividing 24. This estimate applies to the Sylow subgroups of F, since p-groups have nontrivial centres, and so the order of F divides 24. If F has a nontrivial cyclic normal subgroup then π has a normal subgroup isomorphic to Z^2 and hence maps onto Z or D. Otherwise F has a nontrivial Sylow 3-subgroup C which is not normal in F. The number of Sylow 3-subgroups is congruent to $1\ mod\ (3)$ and divides the order of F. The action of F by conjugation on the set of such subgroups is transitive. It must also be faithful. (For otherwise $\cap_{g\in F} gN_F(C)g^{-1} \neq 1$. As $N_F(C)$ is cyclic or dihedral it would follow that F must have a nontrivial cyclic normal subgroup, contrary to hypothesis.) Hence F must be A_4 or S_4, and so contains $V \cong (Z/2Z)^2$ as a normal subgroup. But any orientable flat 3-manifold group with holonomy V must have finite abelianization. As $Z/3Z$ cannot act freely on a $\mathbb{Q}$-homology 3-sphere (by the

Lefshetz fixed point theorem) it follows that A_4 cannot be the holonomy group of a flat 3-manifold. Hence we may exclude S_4 also.

We shall now determine the (outer) automorphism groups of each of the flat 3-manifold groups. Clearly $Out(G_1) = Aut(G_1) = GL(3,\mathbb{Z})$. If $2 \leq i \leq 5$ let $t \in G_i$ represent a generator of the quotient $G_i/I(G_i) \cong Z$. The automorphisms of G_i must preserve the characteristic subgroup $I(G_i)$ and so may be identified with triples $(v, A, \epsilon) \in Z^2 \times GL(2,\mathbb{Z}) \times \{\pm 1\}$ such that $ATA^{-1} = T^\epsilon$ and which act via A on $I(G_i) = Z^2$ and send t to $t^\epsilon v$. Such an automorphism is orientation preserving if and only if $\epsilon = det(A)$. The multiplication is given by $(v, A, \epsilon)(w, B, \eta) = (\Xi v + Aw, AB, \epsilon\eta)$, where $\Xi = I$ if $\eta = 1$ and $\Xi = -T^\epsilon$ if $\eta = -1$. The inner automorphisms are generated by $(0, T, 1)$ and $((T-I)Z^2, I, 1)$.

In particular, $Aut(G_2) \cong (Z^2 \times_\alpha GL(2,\mathbb{Z})) \times \{\pm 1\}$, where α is the natural action of $GL(2,\mathbb{Z})$ on Z^2, for Ξ is always I if $T = -I$. The involution $(0, I, -1)$ is central in $Aut(G_2)$, and is orientation reversing. Hence $Out(G_2)$ is isomorphic to $((Z/2Z)^2 \times_{P\alpha} PGL(2,\mathbb{Z})) \times (Z/2Z)$, where $P\alpha$ is the induced action of $PGL(2,\mathbb{Z})$ on $(Z/2Z)^2$.

If $n = 3$, 4 or 5 the normal subgroup $I(G_i)$ may be viewed as a module over the ring $R = \mathbb{Z}[t]/(\phi(t))$, where $\phi(t) = t^2+t+1$, t^2+1 or t^2-t+1, respectively. As these rings are principal ideal domains and $I(G_i)$ is torsion free of rank 2 as an abelian group, in each case it is free of rank 1 as an R-module. Thus matrices A such that $AT = TA$ correspond to units of R. Hence automorphisms of G_i which induce the identity on $G_i/I(G_i)$ have the form $(v, \pm T^m, 1)$, for some $m \in Z$ and $v \in Z^2$. There is also an involution $(0, (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}), -1)$ which sends t to t^{-1}. In all cases $\epsilon = det(A)$. It follows that $Out(G_3) \cong S_3 \times (Z/2Z)$, $Out(G_4) \cong (Z/2Z)^2$ and $Out(G_5) = Z/2Z$. All these automorphisms are orientation preserving.

The subgroup A of G_6 generated by $\{x^2, y^2, (xy)^2\}$ is the maximal abelian normal subgroup of G_6, and $G_6/A \cong (Z/2Z)^2$. Let a, b, c, d, e, f, i and j be the automorphisms of G_6 which send x to $x^{-1}, x, x, x, y^2x, (xy)^2x, y, xy$ and y to $y, y^{-1}, (xy)^2y, x^2y, y, (xy)^2y, x, x$, respectively. The natural homomorphism from $Aut(G_6)$ to $Aut(G_6/A) \cong GL(2,\mathbb{F}_2)$ is onto, as the images of i and j generate $GL(2,\mathbb{F}_2)$, and its kernel E is generated by $\{a, b, c, d, e, f\}$. (For an automorphism which induces the identity on G_6/A must send x to $x^{2p}y^{2q}(xy)^{2r}x$, and y to $x^{2s}y^{2t}(xy)^{2u}y$. The images of x^2, y^2 and $(xy)^2$ are then x^{4p+2}, y^{4t+2} and $(xy)^{4(r-u)+2}$, which generate A if and only if $p = 0$ or -1, $t = 0$ or -1 and $r = u-1$ or u. Composing such an automorphism appropriately with a, b and c we may acheive $p = t = 0$ and $r = u$. Then by composing with powers of d, e and f we may obtain the identity automorphism.) The inner automorphisms are generated by bcd (conjugation by x) and $acef$ (conjugation by y). Then $Out(G_6)$ has a presentation

$$\langle a,b,c,e,i,j \mid a^2=b^2=c^2=e^2=i^2=j^6=1,\ a,b,c,e \text{ commute},\ iai=b,$$
$$ici=ae,\ jaj^{-1}=c,\ jbj^{-1}=abc,\ jcj^{-1}=be,\ j^3=abce,\ (ji)^2=bc\rangle.$$

The generators $a,b,c,$ and j represent orientation reversing automorphisms. (Note that $jej^{-1}=bc$ follows from the other relations. See [Zn90] for an alternative description.)

The group $B_1=Z\times K$ has a presentation

$$\langle t,x,y \mid tx=xt,\ ty=yt,\ xyx^{-1}=y^{-1}\rangle.$$

An automorphism of B_1 must preserve the centre ζB_1 (which has basis t,x^2) and $I(B_1)$ (which is generated by y). Thus the automorphisms of B_1 may be identified with triples $(A,m,\epsilon)\in\Upsilon_2\times Z\times\{\pm1\}$, where Υ_2 is the subgroup of $GL(2,\mathbb{Z})$ consisting of matrices congruent mod (2) to upper triangular matrices. Such an automorphism sends t to $t^a x^b$, x to $t^c x^d y^m$ and y to y^ϵ, and induces multiplication by A on $B_1/I(B_1)\cong Z^2$. Composition of automorphisms is given by $(A,m,\epsilon)(B,n,\eta)=(AB,m+\epsilon n,\epsilon\eta)$. The inner automorphisms are generated by $(I,1,-1)$ and $(I,2,1)$, and so $Out(B_1)\cong\Upsilon_2\times(Z/2Z)$.

The group B_2 has a presentation

$$\langle t,x,y \mid txt^{-1}=xy,\ ty=yt,\ xyx^{-1}=y^{-1}\rangle.$$

Automorphisms of B_2 may be identified with triples $(A,(m,n),\epsilon)$, where $A\in\Gamma_2$, $m,n\in Z$, $\epsilon=\pm1$ and $m=(A_{11}-\epsilon)/2$. Such an automorphism sends t to $t^a x^b y^m$, x to $t^c x^d y^n$ and y to y^ϵ, and induces multiplication by A on $B_2/I(B_2)\cong Z^2$. The automorphisms which induce the identity on $B_2/I(B_2)$ are all inner, and so $Out(B_2)\cong\Upsilon_2$.

The group B_3 has a presentation

$$\langle t,x,y \mid txt^{-1}=x^{-1},\ ty=yt,\ xyx^{-1}=y^{-1}\rangle.$$

An automorphism of B_3 must preserve $I(B_3)\cong K$ (which is generated by x,y) and $I(I(B_3))$ (which is generated by y). It follows easily that $Out(B_3)\cong(Z/2Z)^3$, and is generated by the classes of the automorphisms which fix y and send t to t^{-1},t,tx^2 and x to x,xy,x, respectively.

A similar argument using the presentation

$$\langle t,x,y \mid txt^{-1}=x^{-1}y,\ ty=yt,\ xyx^{-1}=y^{-1}\rangle$$

for B_4 shows that $Out(B_4)\cong(Z/2Z)^3$, and is generated by the classes of the automorphisms which fix y and send t to $t^{-1}y^{-1},t,tx^2$ and x to x,x^{-1},x, respectively.

8.3 Flat 4-manifold groups with infinite abelianization

We shall organize our determination of the flat 4-manifold groups π in terms of $I(\pi)$. Let π be a flat 4-manifold group, $\beta=\beta_1(\pi)$ and $h=h(I(\pi))$. Then $\pi/I(\pi)\cong Z^\beta$ and $h+\beta=4$. If $I(\pi)$ is abelian then $C_\pi(I(\pi))$ is a nilpotent normal subgroup of π and so is a subgroup of the Hirsch-Plotkin radical $\sqrt{\pi}$, which is here the maximal abelian normal subgroup $T(\pi)$. Hence $C_\pi(I(\pi))=T(\pi)$ and the holonomy group is isomorphic to $\pi/C_\pi(I(\pi))$.

$h=0$ In this case $I(\pi)=1$, so $\pi\cong Z^4$ and is orientable.

$h=1$ In this case $I(\pi)\cong Z$ and π is nonabelian, so $\pi/C_\pi(I(\pi))=Z/2Z$. Hence π has a presentation of the form

$$\langle t,x,y,z \mid txt^{-1}=xz^a,\ tyt^{-1}=yz^b,\ tzt^{-1}=z^{-1},\ x,y,z\ \mathit{commute}\rangle,$$

for some integers a, b. On replacing x by xy or interchanging x and y if necessary we may assume that a is even. On then replacing x by $xz^{a/2}$ and y by $yz^{[b/2]}$ we may assume that $a=0$ and $b=0$ or 1. Thus π is a semidirect product $Z^3\times_T Z$, where the normal subgroup Z^3 is generated by the images of x, y and z, and the action of t is determined by a matrix $T=\left(\begin{smallmatrix} I_2 & 0\\ (0,b) & -1\end{smallmatrix}\right)$ in $GL(3,\mathbb{Z})$. Hence $\pi\cong Z\times B_1=Z^2\times K$ or $Z\times B_2$. Both of these groups are nonorientable.

$h=2$ If $I(\pi)\cong Z^2$ and $\pi/C_\pi(I(\pi))$ is cyclic then we may again assume that π is a semidirect product $Z^3\times_T Z$, where $T=\left(\begin{smallmatrix} 1 & 0\\ \mu & U\end{smallmatrix}\right)$, with $\mu=\left(\begin{smallmatrix} a\\ b\end{smallmatrix}\right)$ and $U\in GL(2,\mathbb{Z})$ is of order 2, 3, 4 or 6 and does not have 1 as an eigenvalue. Thus $U=-I_2$, $\left(\begin{smallmatrix} 0 & -1\\ 1 & -1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & -1\\ 1 & 1\end{smallmatrix}\right)$. Conjugating T by $\left(\begin{smallmatrix} 1 & 0\\ \nu & I_2\end{smallmatrix}\right)$ replaces μ by $\mu+(I_2-U)\nu$. In each case the choice $a=b=0$ leads to a group of the form $\pi\cong Z\times G$, where G is an orientable flat 3-manifold group with $\beta_1(G)=1$. For each of the first three of these matrices there is one other possible group. However if $U=\left(\begin{smallmatrix} 0 & -1\\ 1 & 1\end{smallmatrix}\right)$ then I_2-U is invertible and so $Z\times G_5$ is the only possibility. All seven of these groups are orientable.

If $I(\pi)\cong Z^2$ and $\pi/C_\pi(I(\pi))$ is not cyclic then $\pi/C_\pi(I(\pi))\cong(Z/2Z)^2$. There are two conjugacy classes of embeddings of $(Z/2Z)^2$ in $GL(2,\mathbb{Z})$. One has image the subgroup of diagonal matrices. The corresponding groups π have presentations of the form

$$\langle t,u,x,y \mid tx=xt,\ tyt^{-1}=y^{-1},\ uxu^{-1}=x^{-1},\ uyu^{-1}=y^{-1},\ xy=yx,\\ tut^{-1}u^{-1}=x^my^n\rangle,$$

for some integers m, n. On replacing t by $tx^{-[m/2]}y^{[n/2]}$ if necessary we may assume that $0 \leq m,n \leq 1$. On then replacing t by tu and interchanging x and y if necessary we may assume that $m \leq n$. The only infinite cyclic subgroups of $I(\pi)$ which are normal in π are the subgroups $\langle x\rangle$ and $\langle y\rangle$. On comparing the quotients of these groups π by such subgroups we see that the three possibilities are distinct. The other embedding of $(Z/2Z)^2$ in $GL(2,\mathbb{Z})$ has image generated by $-I$ and $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. The corresponding groups π have presentations of the form

$$\langle t,u,x,y \mid txt^{-1}=y,\ tyt^{-1}=x,\ uxu^{-1}=x^{-1},\ uyu^{-1}=y^{-1},\ xy=yx,\\ tut^{-1}u^{-1}=x^my^n\rangle,$$

for some integers m, n. On replacing t by $tx^{[(m-n)/2]}$ and u by ux^{-m} if necessary we may assume that $m=0$ and $n=0$ or 1. Thus there two such groups. All five of these groups are nonorientable.

Otherwise, $I(\pi)\cong K$, $I(I(\pi))\cong Z$ and $G=\pi/I(I(\pi))$ is a flat 3-manifold group with $\beta_1(G)=2$, but with $I(G)=I(\pi)/I(I(\pi))$ not contained in G' (since it acts nontrivially on $I(I(\pi))$). Therefore $G\cong B_1=Z\times K$, and so has a presentation

$$\langle t,x,y \mid tx=xt,\ ty=yt,\ xyx^{-1}=y^{-1}\rangle.$$

If $w: G\to Aut(Z)$ is a homomorphism which restricts nontrivially to $I(G)$ then we may assume (up to isomorphism of G) that $w(x)=1$ and $w(y)=-1$. Groups π which are extensions of $Z\times K$ by Z corresponding to the action with $w(t)=w$ $(=\pm1)$ have presentations of the form

$$\langle t,x,y,z \mid txt^{-1}=xz^a,\ tyt^{-1}=yz^b,\ tzt^{-1}=z^w,\ xyx^{-1}=y^{-1}z^c,\ xz=zx,\\ yzy^{-1}=z^{-1}\rangle,$$

for some integers a,b. Any group with such a presentation is easily seen to be an extension of $Z\times K$ by a cyclic normal subgroup. However conjugating the fourth relation leads to the equation

$$txt^{-1}tyt^{-1}(txt^{-1})^{-1}=txyx^{-1}t^{-1}=ty^{-1}z^ct^{-1}=tyt^{-1}(tzt^{-1})^c$$

which simplifies to $xz^ayz^bz^{-a}x^{-1}=(yz^b)^{-1}z^{wc}$ and hence to $z^{c-2a}=z^{wc}$. Hence this cyclic normal subgroup is finite unless $2a=(1-w)c$.

Suppose first that $w=1$. Then $z^{2a}=1$ and so we must have $a=0$. On replacing t by $tz^{[b/2]}$ and x by $xz^{[c/2]}$, if necessary, we may assume that $0\leq b,c\leq 1$. If $b=0$ then $\pi\cong Z\times B_3$ or $Z\times B_4$. Otherwise, after further replacing x by txz if necessary we may assume that $c=0$. The three remaining possibilities may be distinguished by their abelianizations, and so there are three such groups. In each case the subgroup generated by $\{t, x^2, y^2, z\}$ is maximal abelian, and the holonomy group is isomorphic to $(Z/2Z)^2$.

If instead $w = -1$ then $z^{2(c-a)} = 1$ and so we must have $a = c$. On replacing y by $yz^{[b/2]}$ and x by $xz^{[c/2]}$ if necessary we may assume that $0 \leq b, c \leq 1$. If $b = 1$ then after replacing x by txy, if necessary, we may assume that $a = 0$. If $a = b = 0$ then $\pi/\pi' \cong Z^2 \oplus (Z/2Z)^2$. The remaining two possibilities both have abelianization $Z^2 \oplus (Z/2Z)$, but one has centre of rank 2 and the other has centre of rank 1. Thus there are three such groups. The subgroup generated by $\{ty, x^2, y^2, z\}$ is maximal abelian, and the holonomy group is isomorphic to $(Z/2Z)^2$. All of these groups π with $I(\pi) \cong K$ are nonorientable.

$h = 3$ In this case π is uniquely a semidirect product $\pi \cong I(\pi) \times_\theta Z$, where $I(\pi)$ is a flat 3-manifold group and θ is an automorphism of $I(\pi)$ such that the induced automorphism of $I(\pi)/I(I(\pi))$ has no eigenvalue 1, and whose image in $Out(I(\pi))$ has finite order. (The conjugacy class of the image of θ in $Out(I(\pi))$ is determined up to inversion by π.)

Since $T(I(\pi))$ is the maximal abelian normal subgroup of $I(\pi)$ it is normal in π. It follows easily that $T(\pi) \cap I(\pi) = T(I(\pi))$. Hence the holonomy group of $I(\pi)$ is isomorphic to a normal subgroup of the holonomy subgroup of π, with quotient cyclic of order dividing the order of θ in $Out(I(\pi))$. (The order of the quotient can be strictly smaller.)

If $I(\pi) \cong Z^3$ then $Out(I(\pi)) \cong GL(3, \mathbb{Z})$. If $T \in GL(3, \mathbb{Z})$ has finite order n and $\beta_1(Z^3 \times_T Z) = 1$ then either $T = -I$ or $n = 4$ or 6 and the characteristic polynomial of T is $(t+1)\phi(t)$ with $\phi(t) = t^2+1$, t^2+t+1 or t^2-t+1. In the latter cases T is conjugate to a matrix of the form $\left(\begin{smallmatrix} -1 & \mu \\ 0 & A \end{smallmatrix}\right)$, where $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, respectively. The row vector $\mu = (m_1, m_2)$ is well defined *mod* $Z^2(A+I)$. Thus there are seven such conjugacy classes. All but one pair (corresponding to $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\mu \notin Z^2(A+I)$) are self-inverse, and so there are six such groups. The holonomy group is cyclic, of order equal to the order of T. As such matrices all have determinant -1 all of these groups are nonorientable.

If $I(\pi) \cong G_i$ for $2 \leq i \leq 5$ the automorphism $\theta = (v, A, \epsilon)$ must have $\epsilon = -1$, for otherwise $\beta_1(\pi) = 2$. We have $Out(G_2) \cong ((Z/2Z)^2 \tilde{\times} PGL(2, \mathbb{Z})) \times (Z/2Z)$. The five conjugacy classes of finite order in $PGL(2, \mathbb{Z})$ are represented by the matrices I, $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$. The numbers of conjugacy classes in $Out(G_2)$ with $\epsilon = -1$ corresponding to these matrices are two, two, two, three and one, respectively. All of these conjugacy classes are self-inverse. Of these, only the two conjugacy classes corresponding to $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and the three conjugacy classes corresponding to $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ give rise to orientable groups. The holonomy groups are all isomorphic to $(Z/2Z)^2$, except when $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$, when they are isomorphic to $Z/4Z$ or $Z/6Z \oplus Z/2Z$, respectively. There are five orientable groups and five nonorientable groups.

As $Out(G_3) \cong S_3 \times (Z/2Z)$, $Out(G_4) \cong (Z/2Z)^2$ and $Out(G_5) = Z/2Z$, there are three, two and one conjugacy classes corresponding to automorphisms with $\epsilon = -1$, respectively, and all these conjugacy classes are closed under inversion. The holonomy groups are dihedral of order 6, 8 and 12, respectively. The six such groups are all orientable.

The centre of $Out(G_6)$ is generated by the image of ab, and the image of ce in the quotient $Out(G_6)/\langle ab\rangle$ generates a central $Z/2Z$ direct factor. The quotient $Out(G_6)/\langle ab, ce\rangle$ is isomorphic to the semidirect product of a normal subgroup $(Z/2Z)^2$ (generated by the images of a and c) with S_3 (generated by the images of ia and j), and has five conjugacy classes, represented by $1, a, i, j$ and ci. Hence $Out(G_6)/\langle ab\rangle$ has ten conjugacy classes, represented by $1, ce, a, ace, i, cei, j, cej, ci$ and $cice = ei$. Thus $Out(G_6)$ itself has between 10 and 20 conjugacy classes. In fact $Out(G_6)$ has 14 conjugacy classes, of which those represented by $1, ab, ace, bce, i, cej$, $abcej$ and ei are orientation preserving, and those represented by a, ce, cei, j, abj and ci are orientation reversing. All of these classes are self inverse, except for j and abj, which are mutually inverse ($j^{-1} = ai(abj)ia$). The holonomy groups corresponding to the classes $1, ab, ace$ and bce are isomorphic to $(Z/2Z)^2$, those corresponding to a and ce are isomorphic to $(Z/2Z)^3$, those corresponding to i, ei, cei and ci are dihedral of order 8, those corresponding to cej and $abcej$ are isomorphic to A_4 and the one corresponding to j has order 24. There are eight orientable groups and five nonorientable groups.

All the remaining cases give rise to nonorientable groups.

$I(\pi) \cong Z \times K$. If a matrix A in Υ_2 has finite order then as its trace is even the order must be 1, 2 or 4. If moreover A does not have 1 as an eigenvalue then either $A = -I$ or A has order 4 and is conjugate (in Υ_2) to $\left(\begin{smallmatrix} -1 & 1 \\ -2 & 1 \end{smallmatrix}\right)$. Each of the four corresponding conjugacy classes in $\Upsilon_2 \times \{\pm 1\}$ is self inverse, and so there are four such groups. The holonomy groups are isomorphic to $Z/nZ \oplus Z/2Z$, where $n = 2$ or 4 is the order of A.

$I(\pi) \cong B_2$. As $Out(B_2) \cong \Upsilon_2$ there are two relevant conjugacy classes and hence two such groups. The holonomy groups are again isomorphic to $Z/nZ \oplus Z/2Z$, where $n = 2$ or 4 is the order of A.

$I(\pi) \cong B_3$ or B_4. In each case $Out(H) \cong (Z/2Z)^3$, and there are four outer automorphism classes determining semidirect products with $\beta = 1$. (Note that here conjugacy classes are singletons and are self-inverse.) The holonomy groups are all isomorphic to $(Z/2Z)^3$.

8.4 Flat 4-manifold groups with finite abelianization

There remains the case when π/π' is finite (equivalently, $h = 4$). By Lemma 3.14 if π is such a flat 4-manifold group it is nonorientable and is isomorphic to a generalized free product $J *_\phi \tilde{J}$, where ϕ is an isomorphism from $G < J$ to $\tilde{G} < \tilde{J}$ and $[J : G] = [\tilde{J} : \tilde{G}] = 2$. The groups G, J and $\tilde{J}$ are then flat 3-manifold groups. If λ and $\tilde{\lambda}$ are automorphisms of G and $\tilde{G}$ which extend to J and $\tilde{J}$, respectively, then $J *_\phi \tilde{J}$ and $J *_{\tilde{\lambda}\phi\lambda} \tilde{J}$ are isomorphic, and so we shall say that ϕ and $\tilde{\lambda}\phi\lambda$ are equivalent isomorphisms. The major difficulty in handling these cases is that some such flat 4-manifold groups split as a generalised free product in several essentially distinct ways.

It follows from the Mayer-Vietoris sequence for $\pi \cong J *_\phi \tilde{J}$ that $H_1(G;\mathbb{Q})$ maps onto $H_1(J;\mathbb{Q}) \oplus H_1(\tilde{J};\mathbb{Q})$, and hence that $\beta_1(J) + \beta_1(\tilde{J}) \le \beta_1(G)$. Since G_3, G_4, B_3 and B_4 are only subgroups of other flat 3-manifold groups via maps inducing isomorphisms on $H_1(-;\mathbb{Q})$ and G_5 and G_6 are not index 2 subgroups of any flat 3-manifold group we may assume that $G \cong Z^3$, G_2, B_1 or B_2. If j and $\tilde{j}$ are the automorphisms of $T(J)$ and $T(\tilde{J})$ determined by conjugation in J and $\tilde{J}$, respectively, then π is a flat 4-manifold group if and only if $\Phi = jT(\phi)^{-1}\tilde{j}T(\phi)$ has finite order. In particular, the trace of Φ must have absolute value at most 3. At this point detailed computation seems unavoidable. (We note in passing that any generalised free product $J *_G \tilde{J}$ with $G \cong G_3$, G_4, B_3 or B_4, J and $\tilde{J}$ torsion free and $[J : G] = [\tilde{J} : G] = 2$ is a flat 4-manifold group, since $Out(G)$ is then finite. However all such groups have infinite abelianization.)

Suppose first that $G \cong Z^3$, with basis $\{x, y, z\}$. Then J and $\tilde{J}$ must have holonomy of order ≤ 2, and $\beta_1(J) + \beta_1(\tilde{J}) \le 3$. Hence we may assume that $J \cong G_2$ and $\tilde{J} \cong G_2$, B_1 or B_2. In each case we have $G = T(J)$ and $\tilde{G} = T(\tilde{J})$. We may assume that J and $\tilde{J}$ are generated by G and elements s and t, respectively, such that $s^2 = x$ and $t^2 \in \tilde{G}$. We may also assume that the action of s on G has matrix $j = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -I \end{smallmatrix}\right)$ with respect to the basis $\{x, y, z\}$. Fix an isomorphism $\phi : G \to \tilde{G}$ and let $T = T(\phi)^{-1}\tilde{j}T(\phi) = \left(\begin{smallmatrix} a & \delta \\ \gamma & D \end{smallmatrix}\right)$ be the matrix corresponding to the action of t on $\tilde{G}$. (Here γ is a 2×1 column vector, δ is a 1×2 row vector and D is a 2×2 matrix, possibly singular.) Then $T^2 = I$ and so the trace of T is odd. Since $j \equiv I \ mod\ (2)$ the trace of $\Phi = jT$ is also odd, and so Φ cannot have order 3 or 6. Therefore $\Phi^4 = I$. If $\Phi = I$ then

π/π' is infinite. If Φ has order 2 then $jT = Tj$ and so $\gamma = 0$, $\delta = 0$ and $D^2 = I_2$. Moreover we must have $a = -1$ for otherwise π/π' is infinite. After conjugating T by a matrix commuting with j if necessary we may assume that $D = I_2$ or $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. (Since $\tilde{J}$ must be torsion free we cannot have $D = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.) These two matrices correspond to the generalized free products $G_2 *_\phi B_1$ and $G_2 *_\phi G_2$, with presentations

$$\langle s, t, z \mid st^2s^{-1} = t^{-2},\ szs^{-1} = z^{-1},\ ts^2t^{-1} = s^{-2},\ tz = zt\rangle$$
$$\text{and}\quad \langle s, t, z \mid st^2s^{-1} = t^{-2},\ szs^{-1} = z^{-1},\ ts^2t^{-1} = s^{-2},\ tzt^{-1} = z^{-1}\rangle,$$

respectively. These groups each have holonomy group isomorphic to $(Z/2Z)^2$. If Φ has order 4 then we must have $(jT)^2 = (jT)^{-2} = (Tj)^2$ and so $(jT)^2$ commutes with j. It can then be shown that after conjugating T by a matrix commuting with j if necessary we may assume that T is the elementary matrix which interchanges the first and third rows. The corresponding group $G_2 *_\phi B_2$ has a presentation

$$\langle s, t, z \mid st^2s^{-1} = t^{-2},\ szs^{-1} = z^{-1},\ ts^2t^{-1} = z,\ tzt^{-1} = s^2\rangle.$$

Its holonomy group is isomorphic to the dihedral group of order 8.

If $G \cong B_1$ or B_2 then J and $\tilde{J}$ are nonorientable and $\beta_1(J) + \beta_1(\tilde{J}) \leq 2$. Hence J and $\tilde{J}$ are B_3 or B_4. Since neither of these groups contains B_2 as an index 2 subgroup we must have $G \cong B_1$. In each case there are two essentially different embeddings of B_1 as an index 2 subgroup of B_3 or B_4. (The image of one contains $I(B_i)$ while the other does not.) In all cases we find that j and $\tilde{j}$ are diagonal matrices with determinant -1, and that $T(\phi) = \left(\begin{smallmatrix} M & 0 \\ 0 & \pm 1 \end{smallmatrix}\right)$ for some $M \in \Gamma_2$. Calculation now shows that if Φ has finite order then M is diagonal and hence $\beta_1(J *_\phi \tilde{J}) > 0$. Thus there are no flat 4-manifold groups (with finite abelianization) which are generalized free products with amalgamation over copies of B_1 or B_2.

If $G \cong G_2$ then $\beta_1(J) + \beta_1(\tilde{J}) \leq 1$, so we may assume that $J \cong G_6$. The other factor $\tilde{J}$ must then be one of G_2, G_4, G_6, B_3 or B_4, and then every amalgamation has finite abelianization. In each case the images of any two embeddings of G_2 in one of these groups are equivalent up to composition with an automorphism of the larger group. In all cases the matrices for j and $\tilde{j}$ have the form $\left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & N \end{smallmatrix}\right)$ where $N^4 = I \in GL(2, \mathbb{Z})$, and $T(\phi) = \left(\begin{smallmatrix} \epsilon & 0 \\ 0 & M \end{smallmatrix}\right)$ for some $M \in GL(2, \mathbb{Z})$. Calculation shows that Φ has finite order if and only if M is in the dihedral subgroup D_8 of $GL(2, \mathbb{Z})$ generated by the diagonal matrices and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. (In other words, either M is diagonal or both diagonal elements of M are 0.) Now the subgroup of $Aut(G_2)$ consisting of automorphisms which extend to G_6 is $(Z^2 \times_\alpha D_8) \times \{\pm 1\}$. Hence any two such isomorphisms ϕ from

G to $\tilde{G}$ are equivalent, and so there is an unique such flat 4-manifold group $G_6 *_\phi \tilde{J}$ for each of these choices of $\tilde{J}$. The corresponding presentations are

$$\langle u,x,y \mid xux^{-1}=u^{-1},\ y^2=u^2,\ yx^2y^{-1}=x^{-2},\ u(xy)^2=(xy)^2u\rangle,$$
$$\langle u,x,y \mid yx^2y^{-1}=x^{-2},\ uy^2u^{-1}=(xy)^2,\ u(xy)^2u^{-1}=y^{-2},\ x=u^2\rangle,$$
$$\langle u,x,y \mid xy^2x^{-1}=y^{-2},\ yx^2y^{-1}=ux^2u^{-1}=x^{-2},\ y^2=u^2,\ yxy=uxu\rangle,$$
$$\langle t,x,y \mid xy^2x^{-1}=y^{-2},\ yx^2y^{-1}=x^{-2},\ x^2=t^2,\ y^2=(t^{-1}x)^2,\ t(xy)^2=(xy)^2t\rangle$$
$$\text{and}\quad \langle t,x,y \mid xy^2x^{-1}=y^{-2},\ yx^2y^{-1}=x^{-2},\ x^2=t^2(xy)^2,\ y^2=(t^{-1}x)^2,$$
$$t(xy)^2=(xy)^2t\rangle,$$

respectively. The corresponding holonomy groups are isomorphic to $(Z/2Z)^3$, D_8, $(Z/2Z)^2$, $(Z/2Z)^3$ and $(Z/2Z)^3$, respectively.

Thus we have found eight generalized free products $J *_G \tilde{J}$ which are flat 4-manifold groups with $\beta=0$. The groups $G_2 *_\phi B_1$, $G_2 *_\phi G_2$ and $G_6 *_\phi G_6$ are all easily seen to be semidirect products of G_6 with an infinite cyclic normal subgroup, on which G_6 acts nontrivially. It follows easily that these three groups are in fact isomorphic, and so there is just one flat 4-manifold group with finite abelianization and holonomy isomorphic to $(Z/2Z)^2$.

The above presentations of $G_2 *_\phi B_2$ and $G_6 *_\phi G_4$ are in fact equivalent; the function sending s to y, t to yu^{-1} and z to uy^2u^{-1} determines an isomorphism between these groups. Thus there is just one flat 4-manifold group with finite abelianization and holonomy isomorphic to D_8.

The above presentations of $G_6 *_\phi G_2$ and $G_6 *_\phi B_4$ are also equivalent; the function sending x to xt^{-1}, y to yt and u to $xy^{-1}t$ determines an isomorphism between these groups (with inverse sending x to $uy^{-1}x^{-2}$, y to ux^{-1} and t to xuy^{-1}). (This isomorphism and the one in the paragraph above were found by Derek Holt, using the program described in [HR92].) The translation subgroups of $G_6 *_\phi B_3$ and $G_6 *_\phi B_4$ are generated by the images of $U=(ty)^2$, $X=x^2$, $Y=y^2$ and $Z=(xy)^2$ (with respect to the above presentations). In each case the images of t, x and y act diagonally, via the matrices $diag[-1,1,-1,1]$, $diag[1,1,-1,-1]$ and $diag[-1,-1,1,-1]$, respectively. However the maximal orientable subgroups have abelianization $Z\oplus(Z/2)^3$ and $Z\oplus(Z/4Z)\oplus(Z/2Z)$, respectively, and so $G_6 *_\phi B_3$ is not isomorphic to $G_6 *_\phi B_4$. Thus there are two flat 4-manifold groups with finite abelianization and holonomy isomorphic to $(Z/2Z)^3$.

In summary, there are 27 orientable flat 4-manifold groups (all with $\beta>0$), 43 nonorientable flat 4-manifold groups with $\beta>0$ and 4 (nonorientable) flat 4-manifold groups with $\beta=0$. (We suspect that the discrepancy with the results reported in [Wo] may be explained by an unnoticed isomorphism between two examples with finite abelianization.)

8.5 Distinguishing between the geometries

Let M be a closed 4-manifold with fundamental group π and with a geometry of solvable Lie type. We shall show that the geometry is largely determined by the structure of $\sqrt{\pi}$. (See also Proposition 10.4 of [Wl86].) As a geometric structure on a manifold lifts to each covering space of the manifold it shall suffice to show that the geometries on suitable finite covering spaces (corresponding to subgroups of finite index in π) can be recognized.

If M is an infranilmanifold then $[\pi:\sqrt{\pi}]<\infty$. If it is flat then $\sqrt{\pi}\cong Z^4$, while if it has the geometry $\mathbb{N}il^3\times\mathbb{E}^1$ or $\mathbb{N}il^4$ then $\sqrt{\pi}$ is nilpotent of class 2 or 3 respectively. (These cases may also be distinguished by the rank of $\zeta\sqrt{\pi}$.) All such groups have been classified, and may be realized geometrically. (See [De] for explicit representations of the $\mathbb{N}il^3\times\mathbb{E}^1$- and $\mathbb{N}il^4$-groups as lattices in $Aff(Nil^3\times R)$ and $Aff(Nil^4)$, respectively.)

If M is a $\mathbb{S}ol_0^4$- or $\mathbb{S}ol_{m,n}^4$-manifold then $\sqrt{\pi}\cong Z^3$. Hence $h(\pi/\sqrt{\pi})=1$ and so π has a normal subgroup σ of finite index which is a semidirect product $\sqrt{\pi}\times_\theta Z$, where the action of a generator t of Z by conjugation on $\sqrt{\pi}$ is given by a matrix θ in $GL(3,\mathbb{Z})$. We may further assume that θ is in $SL(3,\mathbb{Z})$ and has no negative eigenvalues, and that σ is maximal among such normal subgroups. The characteristic polynomial of θ is X^3-mX^2+nX-1, where $m=trace(\theta)$ and $n=trace(\theta^{-1})$. The matrix θ has infinite order, for otherwise the subgroup generated by $\sqrt{\pi}$ and a suitable power of t would be abelian of rank 4. Moreover the eigenvalues must be distinct. For otherwise they would be all 1, so $(\theta-I)^3=0$ and π would be virtually nilpotent.

If M is a $\mathbb{S}ol_0^4$-manifold two of the eigenvalues are complex conjugates. They cannot be roots of unity, since θ has infinite order, and so the real eigenvalue is not 1. If M is a $\mathbb{S}ol_{m,n}^4$-manifold the eigenvalues of θ are distinct and real. The geometry is $\mathbb{S}ol^3\times\mathbb{E}^1$ ($=\mathbb{S}ol_{m,m}^4$ for any $m\geq 4$) if and only if θ has 1 as a simple eigenvalue.

The groups of $\mathbb{E}^4$-, $\mathbb{N}il^3\times\mathbb{E}^1$- and $\mathbb{N}il^4$-manifolds also have finite index subgroups $\sigma\cong Z^3\times_\theta Z$. We may assume that all the eigenvalues of θ are 1, so $N=\theta-I$ is nilpotent. If the geometry is $\mathbb{E}^4$ then $N=0$; if it is $\mathbb{N}il^3\times\mathbb{E}^1$ then $N\neq 0$ but $N^2=0$, while if it is $\mathbb{N}il^4$ then $N^2\neq 0$ but $N^3=0$. (Conversely, it is easy to see that such semidirect products may be realized by lattices in the corresponding Lie groups.)

Finally, if M is a $\mathbb{S}ol_1^4$-manifold then $\sqrt{\pi} \cong \Gamma_q$ for some $q \geq 1$ (and so is nonabelian, of Hirsch length 3).

If $h(\sqrt{\pi}) = 3$ then π is an extension of Z or D by a normal subgroup ν which contains $\sqrt{\pi}$ as a subgroup of finite index. Hence either M is the mapping torus of a self homeomorphism of a flat 3-manifold or a $\mathbb{N}il^3$-manifold, or it is the union of two twisted I-bundles over such 3-manifolds and is doubly covered by such a mapping torus. (Compare Theorem 8.2.)

We shall consider the converse question of realizing geometrically such torsion free virtually poly-Z groups π (with $h(\pi) = 4$ and $h(\sqrt{\pi}) = 3$) in §9.

8.6 Mapping tori of self homeomorphisms of $\mathbb{E}^3$-manifolds

It follows from the above that a 4-dimensional infrasolvmanifold M admits one of the product geometries of type $\mathbb{E}^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$ or $\mathbb{S}ol^3 \times \mathbb{E}^1$ if and only if $\pi_1(M)$ has a subgroup of finite index of the form $\nu \times Z$, where ν is abelian, nilpotent of class 2 or solvable but not virtually nilpotent, respectively. In the next two sections we shall examine when M is the mapping torus of a self homeomorphism of a 3-dimensional infrasolvmanifold. (Note that if M is orientable then it must be a mapping torus, by Lemma 3.14 and Theorem 6.11.)

Theorem 8.3 *Let ν be the fundamental group of a flat 3-manifold, and let θ be an automorphism of ν. Then*

(1) *$\sqrt{\nu}$ is the maximal abelian subgroup of ν and $\nu/\sqrt{\nu}$ embeds in $Aut(\sqrt{\nu})$;*
(2) *$Out(\nu)$ is finite if and only if $[\nu : \sqrt{\nu}] > 2$;*
(3) *the kernel of the restriction homomorphism from $Out(\nu)$ to $Aut(\sqrt{\nu})$ is finite;*
(4) *if $[\nu : \sqrt{\nu}] = 2$ then $(\theta|_{\sqrt{\nu}})^2$ has 1 as an eigenvalue;*
(5) *if $[\nu : \sqrt{\nu}] = 2$ and $\theta|_{\sqrt{\nu}}$ has infinite order but all of its eigenvalues are roots of unity then $((\theta|_{\sqrt{\nu}})^2 - I)^2 = 0$.*

Proof It follows immediately from Theorem 1.5 that $\sqrt{\nu} \cong Z^3$ and is thus the maximal abelian subgroup of ν. The kernel of the homomorphism from ν to $Aut(\sqrt{\nu})$ determined by conjugation is the centralizer $C = C_\nu(\sqrt{\nu})$. As $\sqrt{\nu}$ is central in C and $[C : \sqrt{\nu}]$ is finite, C has finite commutator subgroup, by Schur's Theorem (Proposition 10.1.4 of [Ro]). Since C is torsion free it must be abelian and so $C = \sqrt{\nu}$. Hence $H = \nu/\sqrt{\nu}$ embeds in $Aut(\sqrt{\nu}) \cong GL(3, \mathbb{Z})$. (This is just the holonomy representation.)

If H has order 2 then θ induces the identity on H; if H has order greater than 2 then some power of θ induces the identity on H, since $\sqrt{\nu}$ is a characteristic subgroup of finite index. The matrix $\theta|_{\sqrt{\nu}}$ then commutes with each element of the image of H in $GL(3,\mathbb{Z})$, and the remaining assertions follow from simple calculations, on considering the possibilities for π and H listed in §3 above. □

Corollary 8.3.1 *The mapping torus $M(\phi) = N \times_\phi S^1$ of a self homeomorphism ϕ of a flat 3-manifold N is flat if and only if the outer automorphism $[\phi_*]$ induced by ϕ has finite order.* □

If N is flat and $[\phi_*]$ has infinite order then $M(\phi)$ may admit one of the other product geometries $\mathbb{S}ol^3 \times \mathbb{E}^1$ or $\mathbb{N}il^3 \times \mathbb{E}^1$; otherwise it must be a $\mathbb{S}ol^4_{m,n}$-, $\mathbb{S}ol^4_0$- or $\mathbb{N}il^4$-manifold. (The latter can only happen if $N = R^3/Z^3$, by part (v) of the theorem.)

Theorem 8.4 *Let M be an infrasolvmanifold with fundamental group π such that $\sqrt{\pi} \cong Z^3$ and $\pi/\sqrt{\pi}$ is an extension of D by a finite normal subgroup. Then M is a $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold.*

Proof Let $p : \pi \to D$ be an epimorphism with kernel K containing $\sqrt{\pi}$ as a subgroup of finite index, and let t and u be elements of π whose images under p generate D and such that $p(t)$ generates an infinite cyclic subgroup of index 2 in D. Then there is an $N > 0$ such that the image of $s = t^N$ in $\pi/\sqrt{\pi}$ generates a normal subgroup. In particular, the subgroup generated by s and $\sqrt{\pi}$ is normal in π and usu^{-1} and s^{-1} have the same image in $\pi/\sqrt{\pi}$. Let θ be the matrix of the action of s on $\sqrt{\pi}$, with respect to some basis $\sqrt{\pi} \cong Z^3$. Then θ is conjugate to its inverse, since usu^{-1} and s^{-1} agree modulo $\sqrt{\pi}$. Hence one of the eigenvalues of θ is ± 1. Since π is not virtually nilpotent the eigenvalues of θ must be distinct, and so the geometry must be of type $\mathbb{S}ol^3 \times \mathbb{E}^1$. □

Corollary 8.4.1 *If M admits one of the geometries $\mathbb{S}ol^4_0$ or $\mathbb{S}ol^4_{m,n}$ with $m \neq n$ then it is the mapping torus of a self homeomorphism of R^3/Z^3, and so $\pi \cong Z^3 \times_\theta Z$ for some θ in $GL(3,\mathbb{Z})$ and is a metabelian poly-Z group.*

Proof This follows immediately from Theorems 8.3 and 8.4. □

We may use the idea of Theorem 8.2 to give examples of $\mathbb{E}^4$-, $\mathbb{N}il^4$-, $\mathbb{N}il^3 \times \mathbb{E}^1$- and $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifolds which are not mapping tori. For instance, the groups with presentations

$$\langle u,v,x,y,z \mid xy=yx,\ xz=zx,\ yz=zy,\ uxu^{-1}=x^{-1},\ u^2=y,\ uzu^{-1}=z^{-1},$$
$$v^2=z,\ vxv^{-1}=x^{-1},\ vyv^{-1}=y^{-1}\rangle,$$
$$\langle u,v,x,y,z \mid xy=yx,\ xz=zx,\ yz=zy,\ u^2=x,\ uyu^{-1}=y^{-1},\ uzu^{-1}=z^{-1},$$
$$v^2=x,\ vyv^{-1}=v^{-4}y^{-1},\ vzv^{-1}=z^{-1}\rangle$$

and $$\langle u,v,x,y,z \mid xy=yx,\ xz=zx,\ yz=zy,\ u^2=x,\ v^2=y,$$
$$uyu^{-1}=x^4y^{-1},\ vxv^{-1}=x^{-1}y^2,\ uzu^{-1}=vzv^{-1}=z^{-1}\rangle$$

are each generalised free products of two copies of $Z^2\times_{-I}Z$ amalgamated over their maximal abelian subgroups. The Hirsch-Plotkin radicals of these groups are isomorphic to Z^4 (generated by $\{(uv)^2,x,y,z\}$), $\Gamma_2\times Z$ (generated by $\{uv,x,y,z\}$) and Z^3 (generated by $\{x,y,z\}$), respectively. The group with presentation

$$\langle u,v,x,y,z \mid xy=yx,\ xz=zx,\ yz=zy,\ u^2=x,\ uz=zu,\ uyu^{-1}=x^2y^{-1},$$
$$v^2=y,\ vxv^{-1}=x^{-1},\ vzv^{-1}=v^4z^{-1}\rangle$$

is a generalised free product of copies of $(Z\times_{-1}Z)\times Z$ (generated by $\{u,y,z\}$) and $Z^2\times_{-I}Z$ (generated by $\{v,x,z,\}$) amalgamated over their maximal abelian subgroups. Its Hirsch-Plotkin radical is the subgroup of index 4 generated by $\{(uv)^2,x,y,z\}$, and is nilpotent of class 3. The manifolds corresponding to these groups admit the geometries $\mathbb{E}^4$, $\mathbb{N}il^3\times\mathbb{E}^1$, $\mathbb{S}ol^3\times\mathbb{E}^1$ and $\mathbb{N}il^4$, respectively. However they cannot be mapping tori, as these groups each have finite abelianization.

8.7 Mapping tori of self homeomorphisms of $\mathbb{N}il^3$-manifolds

Let ϕ be an automorphism of Γ_q, sending x to $x^ay^bz^m$ and y to $x^cy^dz^n$ for some $a\dots n$ in Z. Then $A=\left(\begin{smallmatrix}a&c\\b&d\end{smallmatrix}\right)$ is in $GL(2,\mathbb{Z})$ and $\phi(z)=z^{det(A)}$. (In particular, the PD_3-group Γ_q is orientable, as already observed in §2 of Chapter 7, and ϕ is orientation preserving, by the criterion of page 177 of [Bi], or by the argument of §3 of Chapter 18 below.) Every pair (A,μ) in the set $GL(2,\mathbb{Z})\times Z^2$ determines an automorphism (with $\mu=(m,n)$). However $Aut(\Gamma_q)$ is not the direct product of $GL(2,\mathbb{Z})$ and Z^2, as

$$(A,\mu)(B,\nu)=(AB,\mu B+det(A)\nu+q\omega(A,B)),$$

where $\omega(A,B)$ is biquadratic in the entries of A and B. The natural map $p: Aut(\Gamma_q)\to Aut(\Gamma_q/\zeta\Gamma_q)=GL(2,\mathbb{Z})$ sends (A,μ) to A and is an epimorphism, with $\mathrm{Ker}(p)\cong Z^2$. The inner automorphisms are represented by $q\mathrm{Ker}(p)$, and $Out(\Gamma_q)$ is the semidirect product of $GL(2,\mathbb{Z})$ with the normal subgroup $(Z/qZ)^2$. (Let $[A,\mu]$ be the image of (A,μ) in $Out(\Gamma_q)$. Then $[A,\mu][B,\nu] = [AB,\mu B+det(A)\nu]$.) In particular, $Out(\Gamma_1)=GL(2,\mathbb{Z})$.

Theorem 8.5 *Let ν be the fundamental group of a $\mathbb{N}il^3$-manifold N. Then*

(1) *$\nu/\sqrt{\nu}$ embeds in $Aut(\sqrt{\nu}/\zeta\sqrt{\nu})\cong GL(2,\mathbb{Z})$;*

(2) *$\bar{\nu}=\nu/\zeta\sqrt{\nu}$ is a 2-dimensional crystallographic group;*

(3) *the images of elements of $\bar{\nu}$ of finite order under the holonomy representation in $Aut(\sqrt{\bar{\nu}})\cong GL(2,\mathbb{Z})$ have determinant 1;*

(4) *$Out(\bar{\nu})$ is infinite if and only if $\bar{\nu}\cong Z^2$ or $Z^2\times_{-I}(Z/2Z)$;*

(5) *the kernel of the natural homomorphism from $Out(\nu)$ to $Out(\bar{\nu})$ is finite.*

(6) *ν is orientable and every automorphism of ν is orientation preserving.*

Proof Let $h:\nu\to Aut(\sqrt{\nu}/\zeta\sqrt{\nu})$ be the homomorphism determined by conjugation, and let $C=\mathrm{Ker}(h)$. Then $\sqrt{\nu}/\zeta\sqrt{\nu}$ is central in $C/\zeta\sqrt{\nu}$ and $[C/\zeta\sqrt{\nu}:\sqrt{\nu}/\zeta\sqrt{\nu}]$ is finite, so $C/\zeta\sqrt{\nu}$ has finite commutator subgroup, by Schur's Theorem (Proposition 10.1.4 of [Ro].) Since C is torsion free it follows easily that C is nilpotent and hence that $C=\sqrt{\nu}$. This proves (1) and (2). In particular, h factors through the holonomy representation for $\bar{\nu}$, and $gzg^{-1}=z^{d(g)}$ for all $g\in\nu$ and $z\in\zeta\sqrt{\nu}$, where $d(g)=det(h(g))$. If $g\in\nu$ is such that $g\neq 1$ and $g^k\in\zeta\sqrt{\nu}$ for some $k>0$ then $g^k\neq 1$ and so g must commute with elements of $\zeta\sqrt{\nu}$, i.e., the determinant of the image of g is 1. Condition (4) follows as in Theorem 8.3, on considering the possible finite subgroups of $GL(2,\mathbb{Z})$. (See Theorem 1.3.)

If $\zeta\nu\neq 1$ then $\zeta\nu=\zeta\sqrt{\nu}\cong Z$ and so the kernel of the natural homomorphism from $Aut(\nu)$ to $Aut(\bar{\nu})$ is isomorphic to $Hom(\nu/\nu',Z)$. If ν/ν' is finite this kernel is trivial. If $\bar{\nu}\cong Z^2$ then $\nu=\sqrt{\nu}\cong\Gamma_q$, for some $q\geq 1$, and the kernel is isomorphic to $(Z/qZ)^2$. Otherwise $\bar{\nu}\cong Z\times_{-1}Z$, $Z\times D$ or $D\times_\tau Z$ (where τ is the automorphism of $D=(Z/2Z)*(Z/2Z)$ which interchanges the factors). But then $H^2(\bar{\nu};\mathbb{Z})$ is finite and so any central extension of such a group by Z is virtually abelian, and thus not a $\mathbb{N}il^3$-manifold group.

If $\zeta\nu=1$ then $\nu/\sqrt{\nu}<GL(2,\mathbb{Z})$ has an element of order 2 with determinant -1. No such element can be conjugate to $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, for otherwise ν would not be torsion free. Hence the image of $\nu/\sqrt{\nu}$ in $GL(2,\mathbb{Z})$ is conjugate to a subgroup of the group of diagonal matrices $\left(\begin{smallmatrix}\epsilon&0\\0&\epsilon'\end{smallmatrix}\right)$, with $|\epsilon|=|\epsilon'|=1$. If $\nu/\sqrt{\nu}$ is generated by $\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ then $\nu/\zeta\sqrt{\nu}\cong Z\times_{-1}Z$ and $\nu\cong Z^2\times_\theta Z$, where $\theta=\left(\begin{smallmatrix}-1&r\\0&-1\end{smallmatrix}\right)$ for

some nonzero integer r, and N is a circle bundle over the Klein bottle. If $\nu/\sqrt{\nu} \cong (Z/2Z)^2$ then ν has a presentation

$$\langle t, u, z \mid u^2 = z, tzt^{-1} = z^{-1}, ut^2u^{-1} = t^{-2}z^s \rangle,$$

and N is a Seifert bundle over the orbifold $P(22)$. It may be verified in each case that the kernel of the natural homomorphism from $Out(\nu)$ to $Out(\bar{\nu})$ is finite. Therefore (5) holds.

Since $\sqrt{\nu} \cong \Gamma_q$ is a PD_3^+-group, $[\nu : \sqrt{\nu}] < \infty$ and every automorphism of Γ_q is orientation preserving ν must also be orientable. Since $\sqrt{\nu}$ is characteristic in ν and the image of $H_3(\sqrt{\nu}; \mathbb{Z})$ in $H_3(\nu; \mathbb{Z})$ has index $[\nu : \sqrt{\nu}]$ it follows easily that any automorphism of ν must be orientation preserving. □

In fact every $\mathbb{N}il^3$-manifold is a Seifert bundle over a 2-dimensional euclidean orbifold [Sc83']. The base orbifold must be one of the seven such with no reflector curves, by (3).

Theorem 8.6 *The mapping torus $M(\phi) = N \times_\phi S^1$ of a self homeomorphism ϕ of a $\mathbb{N}il^3$-manifold N is orientable, and is a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold if and only if the outer automorphism $[\phi_*]$ induced by ϕ has finite order.*

Proof Since N is orientable and ϕ is orientation preserving (by part (6) of Theorem 8.5) $M(\phi)$ must be orientable.

The subgroup $\zeta\sqrt{\nu}$ is characteristic in ν and hence normal in π, and $\nu/\zeta\sqrt{\nu}$ is virtually Z^2. If $M(\phi)$ is a $\mathbb{N}il^3 \times \mathbb{E}^1$-manifold then $\pi/\zeta\sqrt{\nu}$ is also virtually abelian. It follows easily that that the image of ϕ_* in $Aut(\nu/\zeta\sqrt{\nu})$ has finite order. Hence $[\phi_*]$ has finite order also, by Theorem 8.5. Conversely, if $[\phi_*]$ has finite order in $Out(\nu)$ then π has a subgroup of finite index which is isomorphic to $\nu \times Z$, and so $M(\phi)$ has the product geometry, by the discussion above. □

Theorem 4.2 of [KLR83] (which extends Bieberbach's theorem to the virtually nilpotent case) may be used to show directly that every outer automorphism class of finite order of the fundamental group of an $\mathbb{E}^3$- or $\mathbb{N}il^3$-manifold is realizable by an isometry of an affinely equivalent manifold.

The image of an automorphism θ of Γ_q in $Out(\Gamma_q)$ has finite order if and only if the induced automorphism $\bar{\theta}$ of $\bar{\Gamma}_q = \Gamma_q/\zeta\Gamma_q \cong Z^2$ has finite order in $Aut(\bar{\Gamma}_q) \cong GL(2, \mathbb{Z})$. If $\bar{\theta}$ has infinite order but has trace ± 2 (i.e., if $\bar{\theta}^2 - I$ is a nonzero nilpotent matrix) then $\pi = \Gamma_q \times_\theta Z$ is virtually nilpotent of class 3. If the trace of $\bar{\theta}$ has absolute value greater than 2 then $h(\sqrt{\pi}) = 3$.

Theorem 8.7 *Let M be a closed 4-manifold which admits one of the geometries $\mathbb{N}il^4$ or $\mathbb{S}ol_1^4$. Then M is the mapping torus of a self homeomorphism of a $\mathbb{N}il^3$-manifold if and only if it is orientable.*

Proof If M is such a mapping torus then it is orientable, by Theorem 8.6. Conversely, if M is orientable then $\pi = \pi_1(M)$ has infinite abelianization, by Lemma 3.14. Let $p : \pi \to Z$ be an epimorphism with kernel K, and let t be an element of π such that $p(t)$ generates Z. If K is virtually nilpotent of class 2 we are done, by Theorem 6.12. (Note that this must be the case if M is a $\mathbb{S}ol_1^4$-manifold.) If K is virtually abelian then $K \cong Z^3$, by part (5) of Theorem 8.3. The matrix corresponding to the action of t on K by conjugation must be orientation preserving, since M is orientable. It follows easily that π is nilpotent. Hence there is another epimorphism with kernel nilpotent of class 2, and so the theorem is proven. □

Corollary 8.7.1 *Let M be a closed $\mathbb{S}ol_1^4$-manifold with fundamental group π. Then $\beta_1(M) \leq 1$ and M is orientable if and only if $\beta_1(M) = 1$.*

Proof The first assertion is clear if π is a semidirect product $\Gamma_q \times_\theta Z$, and then follows in general. Hence if there is an epimorphism $p\colon \pi \to Z$ with kernel K then K must be virtually nilpotent of class 2 and the result follows from the theorem. □

If M is a $\mathbb{N}il^3 \times \mathbb{E}^1$- or $\mathbb{N}il^4$-manifold then $\beta_1(\pi) \leq 3$ or 2, respectively, with equality if and only if π is nilpotent. In the latter case M is orientable, and is a mapping torus, both of a self homeomorphism of R^3/Z^3 and also of a self homeomorphism of a $\mathbb{N}il^3$-manifold. We have already seen that $\mathbb{N}il^3 \times \mathbb{E}^1$- and $\mathbb{N}il^4$-manifolds need not be mapping tori at all. We shall round out this discussion with examples illustrating the remaining combinations of mapping torus structure and orientation compatible with Lemma 3.14 and Theorem 8.7. As the groups have abelianization of rank 1 the corresponding manifolds are mapping tori in an essentially unique way. The groups with presentations

$$\langle t,x,y,z \mid xz = zx,\ yz = zy,\ txt^{-1} = x^{-1},\ tyt^{-1} = y^{-1},\ tzt^{-1} = yz^{-1}\rangle,$$
$$\langle t,x,y,z \mid xyx^{-1}y^{-1} = z,\ xz = zx,\ yz = zy,\ txt^{-1} = x^{-1},\ tyt^{-1} = y^{-1}\rangle$$
$$\text{and}\quad \langle t,x,y,z \mid xy = yx,\ zxz^{-1} = x^{-1},\ zyz^{-1} = y^{-1},\ txt^{-1} = x^{-1},\ ty = yt,\ tzt^{-1} = z^{-1}\rangle$$

are each virtually nilpotent of class 2. The corresponding $\mathbb{N}il^3 \times \mathbb{E}^1$-manifolds are mapping tori of self homeomorphisms of R^3/Z^3, a $\mathbb{N}il^3$-manifold and a flat manifold, respectively. The latter two of these manifolds are orientable. The groups with presentations

$$\langle t,x,y,z \mid xz=zx,\ yz=zy,\ txt^{-1}=x^{-1},\ tyt^{-1}=xy^{-1},\ tzt^{-1}=yz^{-1}\rangle$$

and $\langle t,x,y,z \mid xyx^{-1}y^{-1}=z,\ xz=zx,\ yz=zy,\ txt^{-1}=x^{-1},\ tyt^{-1}=xy^{-1}\rangle$

are each virtually nilpotent of class 3. The corresponding $\mathbb{N}il^4$-manifolds are mapping tori of self homeomorphisms of R^3/Z^3 and of a $\mathbb{N}il^3$-manifold, respectively.

The group with presentation

$$\langle t,u,x,y,z \mid xyx^{-1}y^{-1}=z^2,\ xz=zx,\ yz=zy,\ txt^{-1}=x^2y,\ tyt^{-1}=xy,$$
$$tz=zt,\ u^4=z,\ uxu^{-1}=y^{-1},\ uyu^{-1}=x,\ utu^{-1}=t^{-1}\rangle$$

has Hirsch-Plotkin radical isomorphic to Γ_2 (generated by $\{x,y,z\}$), and has finite abelianization. The corresponding $\mathbb{S}ol_1^4$-manifold is nonorientable and is not a mapping torus.

8.8 Mapping tori of self homeomorphisms of $\mathbb{S}ol^3$-manifolds

The arguments in this section are again analogous to those of §6.

Theorem 8.8 *Let σ be the fundamental group of a $\mathbb{S}ol^3$-manifold. Then*

(1) $\sqrt{\sigma}\cong Z^2$ *and* $\sigma/\sqrt{\sigma}\cong Z$ *or* D*;*

(2) $Out(\sigma)$ *is finite.*

Proof The argument of Theorem 1.6 implies that $h(\sqrt{\sigma})>1$. Since σ is not virtually nilpotent $h(\sqrt{\sigma})<3$. Hence $\sqrt{\sigma}\cong Z^2$, by Theorem 1.5. Let $\tilde{F}$ be the preimage in σ of the maximal finite normal subgroup of $\sigma/\sqrt{\nu}$, let t be an element of σ whose image generates the maximal abelian subgroup of $\sigma/\tilde{F}$ and let τ be the automorphism of $\tilde{F}$ determined by conjugation by t. Let σ_1 be the subgroup of σ generated by $\tilde{F}$ and t. Then $\sigma_1\cong\tilde{F}\times_\tau Z$, $[\sigma:\sigma_1]\leq 2$, $\tilde{F}$ is torsion free and $h(\tilde{F})=2$. If $\tilde{F}\neq\sqrt{\sigma}$ then $\tilde{F}\cong Z\times_{-1}Z$. But extensions of Z by $Z\times_{-1}Z$ are virtually abelian, since $Out(Z\times_{-1}Z)$ is finite. Hence $\tilde{F}=\sqrt{\sigma}$ and so $\sigma/\sqrt{\sigma}\cong Z$ or D.

Every automorphism of σ induces automorphisms of $\sqrt{\sigma}$ and of $\sigma/\sqrt{\sigma}$. Let $Out^+(\sigma)$ be the subgroup of $Out(\sigma)$ represented by automorphisms which induce the identity on $\sigma/\sqrt{\sigma}$. The restriction of any such automorphism to $\sqrt{\sigma}$ commutes with τ. We may view $\sqrt{\sigma}$ as a module over the ring $R=$

$\mathbb{Z}[X]/(\lambda(X))$, where $\lambda(X) = X^2 - tr(\tau)X + det(\tau)$ is the characteristic polynomial of τ. The polynomial λ is irreducible and has real roots which are not roots of unity, for otherwise $\sqrt{\sigma} \times_\tau Z$ would be virtually nilpotent. Therefore R is a domain and its field of fractions $\mathbb{Q}[X]/(\lambda(X))$ is a real quadratic number field. The R-module $\sqrt{\sigma}$ is clearly finitely generated, R-torsion free and of rank 1. Hence the endomorphism ring $End_R(\sqrt{\sigma})$ is a subring of $\tilde{R}$, the integral closure of R. Since $\tilde{R}$ is the ring of integers in $\mathbb{Q}[X]/(\lambda(X))$ the group of units $\tilde{R}^\times$ is isomorphic to $\{\pm 1\} \times Z$. Since τ determines a unit of infinite order in $R^\times$ the index $[\tilde{R}^\times : \tau^Z]$ is finite.

Suppose now that $\sigma/\sqrt{\sigma} \cong Z$. If f is an automorphism which induces the identity on $\sqrt{\sigma}$ and on $\sigma/\sqrt{\sigma}$ then $f(t) = tw$ for some w in $\sqrt{\sigma}$. If w is in the image of $\tau - 1$ then f is an inner automorphism. Now $\sqrt{\sigma}/(\tau-1)\sqrt{\sigma}$ is finite, of order $det(\tau - 1)$. Since τ is the image of an inner automorphism of σ it follows that $Out^+(\sigma)$ is an extension of a subgroup of $\tilde{R}^\times/\tau^Z$ by $\sqrt{\sigma}/(\tau-1)\sqrt{\sigma}$. Hence $Out(\sigma)$ has order dividing $2[\tilde{R}^\times : \tau^Z]det(\tau - 1)$.

If $\sigma/\sqrt{\sigma} \cong D$ then σ has a characteristic subgroup σ_1 such that $[\sigma : \sigma_1] = 2$, $\sqrt{\sigma} < \sigma_1$ and $\sigma_1/\sqrt{\sigma} \cong Z = \sqrt{D}$. Every automorphism of σ restricts to an automorphism of σ_1. It is easily verified that the restriction from $Aut(\sigma)$ to $Aut(\sigma_1)$ is a monomorphism. Since $Out(\sigma_1)$ is finite it follows that $Out(\sigma)$ is also finite. □

Corollary 8.8.1 *The mapping torus of a self homeomorphism of a $\mathbb{S}ol^3$-manifold is a $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold.* □

The group with presentation

$$\langle x, y, t \mid xy = yx,\ txt^{-1} = x^3y^2,\ tyt^{-1} = x^2y\rangle$$

is the fundamental group of a nonorientable $\mathbb{S}ol^3$-manifold Σ. The nonorientable $\mathbb{S}ol^3 \times \mathbb{E}^1$-manifold $\Sigma \times S^1$ is the mapping torus of id_Σ and is also the mapping torus of a self homeomorphism of R^3/Z^3.

The groups with presentations

$$\langle t, x, y, z \mid xy = yx,\ zxz^{-1} = x^{-1},\ zyz^{-1} = y^{-1},\ txt^{-1} = xy,\ tyt^{-1} = x,\ tzt^{-1} = z^{-1}\rangle,$$

$$\langle t, x, y, z \mid xy = yx,\ zxz^{-1} = x^2y,\ zyz^{-1} = xy,\ tx = xt,\ tyt^{-1} = x^{-1}y^{-1},\ tzt^{-1} = z^{-1}\rangle,$$

$$\langle t, x, y, z \mid xy = yx,\ xz = zx,\ yz = zy,\ txt^{-1} = x^2y,\ tyt^{-1} = xy,\ tzt^{-1} = z^{-1}\rangle$$

and $$\langle t, u, x, y \mid xy = yx,\ txt^{-1} = x^2y,\ tyt^{-1} = xy,\ uxu^{-1} = y^{-1},\ uyu^{-1} = x,\ utu^{-1} = t^{-1}\rangle$$

have Hirsch-Plotkin radical Z^3 and abelianization of rank 1. The corresponding $\mathbb{S}ol^3\times\mathbb{E}^1$-manifolds are mapping tori in an essentially unique way. The first two are orientable, and are mapping tori of self homeomorphisms of the orientable flat 3-manifold with holonomy of order 2 and of an orientable $\mathbb{S}ol^3$-manifold, respectively. The latter two are nonorientable, and are mapping tori of orientation reversing self homeomorphisms of R^3/Z^3 and of the same orientable $\mathbb{S}ol^3$-manifold, respectively.

8.9 Realization and classification

Let π be a torsion free virtually poly-Z group of Hirsch length 4. If π is virtually abelian then it is the fundamental group of a flat 4-manifold, by the work of Bieberbach, and such groups are listed in §2-§4 above.

If π is virtually nilpotent but not virtually abelian then $\sqrt{\pi}$ is nilpotent of class 2 or 3. In the first case it has a characteristic chain $\sqrt{\pi}'\cong Z<C=\zeta\sqrt{\pi}\cong Z^2$. Let $\theta:\pi\to Aut(C)\cong GL(2,\mathbb{Z})$ be the homomorphism induced by conjugation in π. Then $\mathrm{Im}(\theta)$ is finite and triangular, and so is 1, $Z/2Z$ or $(Z/2Z)^2$. Let $K=C_\pi(C)=\mathrm{Ker}(\theta)$. Then K is torsion free and $\zeta K=C$, so K/C is a flat 2-orbifold group. Moreover as $K/\sqrt{K}$ acts trivially on $\sqrt{\pi}'$ it must act orientably on $\sqrt{K}/C$, and so $K/\sqrt{K}$ is cyclic of order 1, 2, 3, 4 or 6. As $\sqrt{\pi}$ is the preimage of $\sqrt{K}$ in π we see that $[\pi:\sqrt{\pi}]\leq 24$. (In fact $\pi/\sqrt{\pi}\cong F$ or $F\oplus(Z/2Z)$, where F is a finite subgroup of $GL(2,\mathbb{Z})$, excepting only direct sums of the dihedral groups of order 6, 8 or 12 with $(Z/2Z)$ [De].) Otherwise (if $\sqrt{\pi}'\not\leq\zeta\sqrt{\pi}$) it has a subgroup of index ≤ 2 which is a semidirect product $Z^3\times_\theta Z$, by part (5) of Theorem 8.3. Since (θ^2-I) is nilpotent it follows that $\pi/\sqrt{\pi}=1$, $Z/2Z$ or $(Z/2Z)^2$. All these possibilities occur.

Such virtually nilpotent groups are fundamental groups of $\mathbb{N}il^3\times\mathbb{E}^1$- and $\mathbb{N}il^4$-manifolds (respectively), and are classified in [De]. Dekimpe observes that π has a characteristic subgroup Z such that $Q=\pi/Z$ is a $\mathbb{N}il^3$- or $\mathbb{E}^3$-orbifold group and classifies the torsion free extensions of such Q by Z. There are 61 families of $\mathbb{N}il^3\times\mathbb{E}^1$-groups and 7 families of $\mathbb{N}il^4$-groups. He also gives a faithful affine representation for each such group.

We shall sketch an alternative approach for the geometry $\mathbb{N}il^4$, which applies also to $\mathbb{S}ol^4_{m,n}$, $\mathbb{S}ol^4_0$ and $\mathbb{S}ol^4_1$. Each such group π has a characteristic subgroup ν of Hirsch length 3, and such that $\pi/\nu\cong Z$ or D. The preimage in π of $\sqrt{\pi/\nu}$ is characteristic, and is a semidirect product $\nu\times_\theta Z$. Hence it is determined up to isomorphism by the union of the conjugacy classes of θ and θ^{-1} in $Out(\nu)$, by Lemma 1.1. All such semidirect products may be realized as lattices and have faithful affine representations.

If the geometry is $\mathbb{N}il^4$ then $\nu = C_{\sqrt{\pi}}(\zeta_2\sqrt{\pi}) \cong Z^3$, by Theorem 1.5 and part (5) of Theorem 8.3. Moreover ν has a basis x, y, z such that $\langle z\rangle = \zeta\sqrt{\pi}$ and $\langle y, z\rangle = \zeta_2\sqrt{\pi}$. As these subgroups are characteristic the matrix of θ with respect to such a basis is $\pm(I+N)$, where N is strictly lower triangular and $n_{21}n_{32} \neq 0$. (See §5 above.) The conjugacy class of θ is determined by $(det(\theta), |n_{21}|, |n_{32}|, [n_{31}\ mod\ (n_{32})])$. (Thus θ is conjugate to θ^{-1} if and only if n_{32} divides $2n_{31}$.) The classification is more complicated if $\pi/\nu \cong D$.

If the geometry is $\mathbb{S}ol^4_{m,n}$ for some $m \neq n$ then $\pi \cong Z^3 \times_\theta Z$, where the eigenvalues of θ are distinct and real, and not ± 1, by the Corollary to Theorem 8.4. The translation subgroup $\pi \cap Sol^4_{m,n}$ is $Z^3 \times_A Z$, where $A = \theta$ or θ^2 is the least nontrivial power of θ with all eigenvalues positive, and has index ≤ 2 in π. Conversely, it is clear from the description of the isometries of $Sol^4_{m,n}$ in §3 of Chapter 7 that every such group is a lattice in $Isom(\mathbb{S}ol^4_{m,n})$. The conjugacy class of θ is determined by its characteristic polynomial $\Delta_\theta(t)$ and the ideal class of $\nu \cong Z^3$, considered as a rank 1 module over the order $\Lambda/(\Delta_\theta(t))$, by Theorem 1.4. (No such θ is conjugate to its inverse, as neither 1 nor -1 is an eigenvalue.)

A similar argument applies for $\mathbb{S}ol^4_0$. Although Sol^4_0 has no lattice subgroups, any semidirect product $Z^3 \times_\theta Z$ where θ has a pair of complex conjugate roots which are not roots of unity is a lattice in $Isom(\mathbb{S}ol^4_0)$. Such groups are again classified by the characteristic polynomial and an ideal class.

If the geometry is $\mathbb{S}ol^4_1$ then $\sqrt{\pi} \cong \Gamma_q$ for some $q \geq 1$, and either $\nu = \sqrt{\pi}$ or $\nu/\sqrt{\pi} = Z/2Z$ and $\nu/\zeta\sqrt{\pi} \cong Z^2 \times_{-I} (Z/2Z)$. (In the latter case ν is uniquely determined by q.) Moreover π is orientable if and only if $\beta_1(\pi) = 1$. In particular, $\mathrm{Ker}(w_1(\pi)) \cong \nu \times_\theta Z$ for some $\theta \in Aut(\nu)$. Let $A = \theta|_{\sqrt{\pi}}$ and let $\overline{A}$ be its image in $Aut(\sqrt{\pi}/\zeta\sqrt{\pi}) \cong GL(2,\mathbb{Z})$. If $\nu = \sqrt{\pi}$ the translation subgroup $\pi \cap Sol^4_1$ is $T = \Gamma_q \times_B Z$, where $B = A$ or A^2 is the least nontrivial power of A such that both eigenvalues of $\overline{A}$ are positive. If $\nu \neq \sqrt{\pi}$ the conjugacy class of $\overline{A}$ is only well-defined up to sign. If moreover $\pi/\nu \cong D$ then $\overline{A}$ is conjugate to its inverse, and so $det(\overline{A}) = 1$, since $\overline{A}$ has infinite order. We can then choose θ and hence A so that $T = \sqrt{\pi} \times_A Z$. In all cases we find that $[\pi : T]$ divides 4. (Note that $Isom(\mathbb{S}ol^4_1)$ has 8 components.)

Conversely, it is fairly easy to verify that a torsion free semidirect product $\nu \times_\theta Z$ (with $[\nu : \Gamma_q] \leq 2$ and ν as above) which is not virtually nilpotent is a lattice in the group of upper triangular matrices generated by Sol^4_1 and the diagonal matrix $diag[\pm 1, 1, \pm 1]$, which is contained in $Isom(\mathbb{S}ol^4_1)$. The conjugacy class of θ is determined up to a finite ambiguity by the characteristic polynomial of $\overline{A}$. Realization and classification of the nonorientable groups seems more difficult.

In the remaining case $\mathbb{S}ol^3 \times \mathbb{E}^1$ the subgroup ν is one of the four flat 3-manifold groups Z^3, $Z^2 \times_{-I} Z$, B_1 or B_2, and $\theta|\sqrt{\nu}$ has distinct real eigenvalues, one being ± 1. The index of the translation subgroup $\pi \cap (Sol^3 \times R)$ in π divides 8. (Note that $Isom(\mathbb{S}ol^3 \times \mathbb{E}^1)$ has 16 components.) Conversely any such semidirect product $\nu \times_\theta Z$ can be realized as a lattice in the index 2 subgroup $G < Isom(Sol^3 \times \mathbb{E}^1)$ defined in §3 of Chapter 7. Realization and classification of the groups with $\pi/\nu \cong D$ seems more difficult. (The number of subcases to be considered makes any classification an uninviting task. See however [Cb].)

8.10 Diffeomorphism

In all dimensions $n \neq 4$ it is known that infrasolvmanifolds with isomorphic fundamental group are diffeomorphic [FJ97]. In general one cannot expect to find affine diffeomorphisms, and the argument of Farrell and Jones uses differential topology rather than Lie theory for the cases $n \geq 5$. The cases with $n \leq 3$ follow from standard results of low dimensional topology. We shall show that related arguments also cover most 4-dimensional infrasolvmanifolds. The following theorem extends the main result of [Cb] (in which it was assumed that π is not virtually nilpotent).

Theorem 8.9 *Let M and M' be 4-manifolds which are total spaces of orbifold bundles $p : M \to B$ and $p' : M' \to B'$ with flat orbifold bases and infranilmanifold fibres, and suppose that $\pi_1(M) \cong \pi_1(M') \cong \pi$. Suppose that either π is orientable or $\beta_1(\pi) = 3$ or $\beta_1(\pi) = 2$ and $(\sqrt{\pi})' \cong Z$. Then M and M' are diffeomorphic.*

Proof We may assume that $d = dim(B) \leq d' = dim(B')$. Clearly $d' \leq 4 - \beta_1(\sqrt{\pi})$. Suppose first that π is not virtually abelian or virtually nilpotent of class 2 (i.e., suppose that $(\sqrt{\pi})' \not\leq \zeta\sqrt{\pi}$). Then all subgroups of finite index in π have $\beta_1 \leq 2$, and so $1 \leq d \leq d' \leq 2$. Moreover π has a characteristic nilpotent subgroup $\tilde{\nu}$ such that $h(\pi/\tilde{\nu}) = 1$, by Theorems 1.5 and 1.6. Let ν be the preimage in π of the maximal finite normal subgroup of $\pi/\tilde{\nu}$. Then ν is a characteristic virtually nilpotent subgroup (with $\sqrt{\nu} = \tilde{\nu}$) and $\pi/\nu \cong Z$ or D. If $d = 1$ then $\pi_1(F) = \nu$ and $p : M \to B$ induces this isomorphism. If $d = 2$ the image of ν in $\pi_1^{orb}(B)$ is normal. Hence there is an orbifold map q from B to the circle S^1 or the reflector interval $\mathbb{I}$ such that qp is an orbifold bundle projection. A similar analysis applies to M'. In either case, M and M' are canonically mapping tori or the unions of two twisted I-bundles, and the theorem follows via standard 3-manifold theory.

If π is virtually nilpotent it is realized by an infranilmanifold M_0 [DeK]. Hence we may assume that $M = M_0$, $d = 0$ or 4 and $(\sqrt{\pi})' \leq \zeta\sqrt{\pi}$. If $d' = 0$ or 4 then M' is also an infranilmanifold and the result is clear. If $d' = 1$ or if $\beta_1(\pi) + d' > 4$ then M' is a mapping torus or the union of twisted I-bundles, and π is a semidirect product $\kappa \rtimes Z$ or a generalized free product with amalgamation $G *_J H$ where $[G : J] = [H : J] = 2$. Hence the model M_0 is also a mapping torus or the union of twisted I-bundles, and we may argue as before.

Therefore we may assume that either $d' = 2$ and $\beta_1(\pi) \leq 2$ or $d' = 3$ and $\beta_1(\pi) \leq 1$. If $d' = 2$ then M and M' are Seifert fibred. If moreover π is orientable then M is diffeomorphic to M', by [Ue90]. If $\beta_1(\pi) = 2$ then either $\pi_1^{orb}(B')$ maps onto Z or π is virtually abelian.

If π is orientable then $\beta_1(\pi) > 0$, by Lemma 3.14. Therefore the remaining possibility is that $d' = 3$ and $\beta_1(\pi) = 1$. If $\pi_1^{orb}(B')$ maps onto Z then we may argue as before. Otherwise $\pi_1(F) \cap \pi' = 1$, so π is virtually abelian and the kernel of the induced homomorphism from π to $\pi_1^{orb}(B)$ is infinite cyclic and central. Hence the orbifold projection is the orbit map of an S^1-action on M. If M is orientable it is determined up to diffeomorphism by the orbifold data and an Euler class corresponding to the central extension of $\pi_1^{orb}(B)$ by Z [Fi78]. Thus M and M' are diffeomorphic. □

It is highly probable that the arguments of Ue and of Fintushel can be extended to all 4-manifolds which are Seifert fibred or admit smooth S^1-actions, and the theorem is surely true without any restrictions on π. (If $d' = 3$ and $\beta_1(\pi) = 0$ then π maps onto D, by Lemma 3.14, and $\pi_1(F) \cong Z$. It is not difficult to determine the maximal infinite cyclic normal subgroups of the flat 4-manifold groups π with $\beta_1(\pi) = 0$, and to verify that in each case the quotient maps onto D. Otherwise $\pi_1(F) = (\sqrt{\pi})'$, since $d' = 3$, and any epimorphism from π to D must factor through $\pi_1^{orb}(B') \cong \pi/(\sqrt{\pi})'$.)

We may now compare the following notions for M a closed smooth 4-manifold:

(1) *M is geometric of solvable Lie type;*

(2) *M is an infrasolvmanifold;*

(3) *M is the total space of an orbifold bundle with infranilmanifold fibre and flat base.*

Geometric 4-manifolds of solvable Lie type are infrasolvmanifolds, by the observations in §3 of Chapter 7, and the Mostow orbifold bundle of an infrasolvmanifold is as in (3), by Theorem 7.2. If π is orientable then it is realized geometrically and determines the total space of such an orbifold bundle up to diffeomorphism. Hence orientable smooth 4-manifolds admitting such orbifold fibrations are diffeomorphic to geometric 4-manifolds of solvable Lie type.

Are these three notions equivalent in general?

Chapter 9

The other aspherical geometries

The aspherical geometries of nonsolvable type which are realizable by closed 4-manifolds are the "mixed" geometries $\mathbb{H}^2 \times \mathbb{E}^2$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$, $\mathbb{H}^3 \times \mathbb{E}^1$ and the "semisimple" geometries $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^4$ and $\mathbb{H}^2(\mathbb{C})$. (We shall consider the geometry $\mathbb{F}^4$ briefly in Chapter 13.) Closed $\mathbb{H}^2 \times \mathbb{E}^2$- or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds are Seifert fibred, have Euler characteristic 0 and their fundamental groups have Hirsch-Plotkin radical Z^2. In §1 and §2 we examine to what extent these properties characterize such manifolds and their fundamental groups. Closed $\mathbb{H}^3 \times \mathbb{E}^1$-manifolds also have Euler characteristic 0, but we have only a conjectural characterization of their fundamental groups (§3). In §4 we determine the mapping tori of self homeomorphisms of geometric 3-manifolds which admit one of these mixed geometries. (We return to this topic in Chapter 13.) In §5 we consider the three semisimple geometries. All closed 4-manifolds with product geometries other than $\mathbb{H}^2 \times \mathbb{H}^2$ are finitely covered by cartesian products. We characterize the fundamental groups of $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds with this property; there are also "irreducible" $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds which are not virtually products. Little is known about manifolds admitting one of the two hyperbolic geometries.

Although it is not yet known whether the disk embedding theorem holds over lattices for such geometries, we can show that the fundamental group and Euler characteristic determine the manifold up to s-cobordism (§6). Moreover an aspherical orientable closed 4-manifold which is finitely covered by a geometric manifold is homotopy equivalent to a geometric manifold (excepting perhaps if the geometry is $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$).

9.1 Aspherical Seifert fibred 4-manifolds

In Chapter 8 we saw that if M is a closed 4-manifold with fundamental group π such that $\chi(M) = 0$ and $h(\sqrt{\pi}) \geq 3$ then M is homeomorphic to an infrasolvmanifold. Here we shall show that if $\chi(M) = 0$, $h(\sqrt{\pi}) = 2$ and $[\pi : \sqrt{\pi}] = \infty$ then M is homotopy equivalent to a 4-manifold which is Seifert fibred over a hyperbolic 2-orbifold. (We shall consider the case when $\chi(M) = 0$, $h(\sqrt{\pi}) = 2$ and $[\pi : \sqrt{\pi}] < \infty$ in Chapter 10.)

Theorem 9.1 *Let M be a closed 4-manifold with fundamental group π. If $\chi(M)=0$ and π has an elementary amenable normal subgroup ρ with $h(\rho)=2$ and such that either $H^2(\pi;\mathbb{Z}[\pi])=0$ or ρ is torsion free and $[\pi:\rho]=\infty$ then M is aspherical and ρ is virtually abelian.*

Proof Since π has one end, by Corollary 1.16.1, and $\beta_1^{(2)}(\pi)=0$, by Theorem 2.3, M is aspherical if also $H^2(\pi;\mathbb{Z}[\pi])=0$, by Corollary 3.5.2. In this case ρ is torsion free and of infinite index in π, and so we may assume this henceforth. Since ρ is torsion free elementary amenable and $h(\rho)=2$ it is virtually solvable, by Theorem 1.11. Therefore $A=\sqrt{\rho}$ is nontrivial, and as it is characteristic in ρ it is normal in π. Since A is torsion free and $h(A)\leq 2$ it is abelian, by Theorem 1.5.

Suppose first that $h(A)=1$. Then A is isomorphic to a subgroup of Q and the homomorphism from $B=\rho/A$ to $Aut(A)$ induced by conjugation in ρ is injective. Since $Aut(A)$ is isomorphic to a subgroup of $\mathbb{Q}^\times$ and $h(B)=1$ either $B\cong Z$ or $B\cong Z\oplus(Z/2Z)$. We must in fact have $B\cong Z$, since ρ is torsion free. Moreover A is not finitely generated and the centre of ρ is trivial. The quotient group π/A has one end as the image of ρ is an infinite cyclic normal subgroup of infinite index. Therefore π is 1-connected at ∞, by Theorem 1 of [Mi87], and so $H^s(\pi;\mathbb{Z}[\pi])=0$ for $s\leq 2$ [GM86]. Hence M is aspherical and π is a PD_4-group.

As A is a characteristic subgroup every automorphism of ρ restricts to an automorphism of A. This restriction from $Aut(\rho)$ to $Aut(A)$ is an epimorphism, with kernel isomorphic to A, and so $Aut(\rho)$ is solvable. Let $C=C_\pi(\rho)$ be the centralizer of ρ in π. Then C is nontrivial, for otherwise π would be isomorphic to a subgroup of $Aut(\rho)$ and hence would be virtually poly-Z. But then A would be finitely generated, ρ would be virtually abelian and $h(A)=2$. Moreover $C\cap\rho=\zeta\rho=1$, so $C\rho\cong C\times\rho$ and $c.d.C+c.d.\rho=c.d.C\rho\leq c.d.\pi=4$. The quotient group $\pi/C\rho$ is isomorphic to a subgroup of $Out(\rho)$.

If $c.d.C\rho\leq 3$ then as C is nontrivial and $h(\rho)=2$ we must have $c.d.C=1$ and $c.d.\rho=h(\rho)=2$. Therefore C is free and ρ is of type FP [Kr86]. By Theorem 1.13 ρ is an ascending HNN group with base a finitely generated subgroup of A and so has a presentation $\langle a,t\mid tat^{-1}=a^n\rangle$ for some nonzero integer n. We may assume $|n|>1$, as ρ is not virtually abelian. The subgroup of $Aut(\rho)$ represented by $(n-1)A$ consists of inner automorphisms. Since $n>1$ the quotient $A/(n-1)A\cong Z/(n-1)Z$ is finite, and as $Aut(A)\cong\mathbb{Z}[1/n]^\times$ it follows that $Out(\rho)$ is virtually abelian. Therefore π has a subgroup σ of finite index which contains $C\rho$ and such that $\sigma/C\rho$ is a finitely generated free abelian group, and in particular $c.d.\sigma/C\rho$ is finite. As σ is a PD_4-group it follows from Theorem 9.11 of [Bi] that $C\rho$ is a PD_3-group and hence that ρ is a PD_2-group. We reach the same conclusion if $c.d.C\rho = 4$, for then $[\pi : C\rho]$ is finite, by Strebel's Theorem, and so $C\rho$ is a PD_4-group. As a solvable PD_2-group is virtually Z^2 our original assumption must have been wrong.

Therefore $h(A) = 2$. As π/A is finitely generated and infinite π is not elementary amenable of Hirsch length 2. Hence $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$, by Theorem 1.17, and so M is aspherical. Moreover as every finitely generated subgroup of ρ is either isomorphic to $Z \times_{-1} Z$ or is abelian $[\rho : A] \leq 2$. □

The group $Z*_n$ (with presentation $\langle a,t \mid tat^{-1} = a^n\rangle$) is torsion free and solvable of Hirsch length 2, and is the fundamental group of a closed orientable 4-manifold M with $\chi(M) = 0$. (See Chapter 3.) Thus the hypothesis that the subgroup ρ have infinite index in π is necessary for the above theorem. Do the other hypotheses imply that ρ must be torsion free?

Theorem 9.2 *Let M be a closed 4-manifold with fundamental group π. If $h(\sqrt{\pi}) = 2$, $[\pi : \sqrt{\pi}] = \infty$ and $\chi(M) = 0$ then M is aspherical and $\sqrt{\pi} \cong Z^2$.*

Proof As $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$, by Theorem 1.17, M is aspherical, by Theorem 9.1. We may assume henceforth that $\sqrt{\pi}$ is a torsion free abelian group of rank 2 which is not finitely generated.

Suppose first that $[\pi : C] = \infty$, where $C = C_\pi(\sqrt{\pi})$. Then $c.d.C \leq 3$, by Strebel's Theorem. Since $\sqrt{\pi}$ is not finitely generated $c.d.\sqrt{\pi} = h(\sqrt{\pi}) + 1 = 3$, by Theorem 7.14 of [Bi]. Hence $C = \sqrt{\pi}$, by Theorem 8.8 of [Bi], so the homomorphism from $\pi/\sqrt{\pi}$ to $Aut(\sqrt{\pi})$ determined by conjugation in π is a monomorphism. Since $\sqrt{\pi}$ is torsion free abelian of rank 2 $Aut(\sqrt{\pi})$ is isomorphic to a subgroup of $GL(2,\mathbb{Q})$ and therefore any torsion subgroup of $Aut(\sqrt{\pi})$ is finite, by Corollary 1.3.1. Thus if $\pi'\sqrt{\pi}/\sqrt{\pi}$ is a torsion group $\pi'\sqrt{\pi}$ is elementary amenable and so π is itself elementary amenable, contradicting our assumption. Hence we may suppose that there is an element g in π' which has infinite order modulo $\sqrt{\pi}$. The subgroup $\langle\sqrt{\pi}, g\rangle$ generated by $\sqrt{\pi}$ and g is an extension of Z by $\sqrt{\pi}$ and has infinite index in π, for otherwise π would be virtually solvable. Hence $c.d.\langle\sqrt{\pi}, g\rangle = 3 = h(\langle\sqrt{\pi}, g\rangle)$, by Strebel's Theorem. By Theorem 7.15 of [Bi], $L = H_2(\sqrt{\pi};\mathbb{Z})$ is the underlying abelian group of a subring $\mathbb{Z}[m^{-1}]$ of Q, and the action of g on L is multiplication by a rational number a/b, where a and b are relatively prime and ab and m have the same prime divisors. But g acts on $\sqrt{\pi}$ as an element of $GL(2,\mathbb{Q})' \leq SL(2,\mathbb{Q})$. Since $L = \sqrt{\pi} \wedge \sqrt{\pi}$, by Proposition 11.4.16 of [Ro], g acts on L via $det(g) = 1$.

Therefore $m = 1$ and so L must be finitely generated. But then $\sqrt{\pi}$ must also be finitely generated, again contradicting our assumption.

Thus we may assume that C has finite index in π. Let $A < \sqrt{\pi}$ be a subgroup of $\sqrt{\pi}$ which is free abelian of rank 2. Then A_1 is central in C and C/A is finitely presentable. Since $[\pi : C]$ is finite A has only finitely many distinct conjugates in π, and they are all subgroups of ζC. Let N be their product. Then N is a finitely generated torsion free abelian normal subgroup of π and $2 \leq h(N) \leq h(\sqrt{C}) \leq h(\sqrt{\pi}) = 2$. An LHSSS argument gives $H^2(\pi/N; \mathbb{Z}[\pi/N]) \cong Z$, and so π/N is virtually a PD_2-group, by Bowditch's Theorem. Since $\sqrt{\pi}/N$ is a torsion group it must be finite, and so $\sqrt{\pi} \cong Z^2$. □

Corollary 9.2.1 *The manifold M is homotopy equivalent to one which is Seifert fibred with general fibre T or Kb over a hyperbolic 2-orbifold if and only if $h(\sqrt{\pi}) = 2$, $[\pi : \sqrt{\pi}] = \infty$ and $\chi(M) = 0$.*

Proof This follows from the theorem together with Theorem 7.3. □

9.2 The Seifert geometries: $\mathbb{H}^2 \times \mathbb{E}^2$ and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$

A manifold with geometry $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ is Seifert fibred with base a hyperbolic orbifold. However not all such Seifert fibred 4-manifolds are geometric. An orientable Seifert fibred 4-manifold over an orientable hyperbolic base is geometric if and only if it is an elliptic surface; the relevant geometries are then $\mathbb{H}^2 \times \mathbb{E}^2$ and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ [Ue90,91].

In this section we shall show that such manifolds may be characterized up to homotopy equivalence in terms of their fundamental groups.

Theorem 9.3 *Let M be a closed $\mathbb{H}^3 \times \mathbb{E}^1$-, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$- or $\mathbb{H}^2 \times \mathbb{E}^2$-manifold. Then M has a finite covering space which is diffeomorphic to a product $N \times S^1$.*

Proof If M is an $\mathbb{H}^3 \times \mathbb{E}^1$-manifold then $\pi = \pi_1(M)$ is a discrete cocompact subgroup of $G = Isom(\mathbb{H}^3 \times \mathbb{E}^1)$. The radical of this group is $Rad(G) \cong R$, and $G_o/Rad(G) \cong PSL(2, \mathbb{C})$, where G_o is the component of the identity in G. Therefore $A = \pi \cap Rad(G)$ is a lattice subgroup, by Proposition 8.27 of [Rg]. Since R/A is compact the image of π/A in $Isom(\mathbb{H}^3)$ is again a discrete cocompact subgroup. Hence $\sqrt{\pi} = A \cong Z$. Moreover π preserves the foliation of the model space by euclidean lines, so M is an orbifold bundle with general fibre S^1 over an $\mathbb{H}^3$-orbifold with orbifold fundamental group $\pi/\sqrt{\pi}$.

On passing to a 2-fold covering space, if necessary, we may assume that $\pi \leq Isom(\mathbb{H}^3) \times R$ and (hence) $\zeta\pi = \sqrt{\pi}$. Projection to the second factor maps $\sqrt{\pi}$ monomorphically to R. Hence on passing to a further finite covering space, if necessary, we may assume that $\pi \cong \nu \times Z$, where $\nu = \pi/\sqrt{\pi} \cong \pi_1(N)$ for some closed orientable $\mathbb{H}^3$-manifold N. (Note that we do *not* claim that $\pi = \nu \times Z$ as a subgroup of $PSL(2,\mathbb{R}) \times R$.) The foliation of $H^3 \times R$ by lines induces an S^1-bundle structure on M, with base N. As such bundles (with aspherical base) are determined by their fundamental groups, M is diffeomorphic to $N \times S^1$.

Similar arguments apply in the other two cases. If $G = Isom(\mathbb{X})$ where $\mathbb{X} = \mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$, then $Rad(G) \cong R^2$, and $G_o/R^2 \cong PSL(2,\mathbb{R})$. The intersection $A = \pi \cap Rad(G)$ is again a lattice subgroup, and the image of π/A in $PSL(2,\mathbb{R})$ is a discrete cocompact subgroup. Hence $\sqrt{\pi} = A \cong Z^2$ and $\pi/\sqrt{\pi}$ is virtually a PD_2-group. If $\mathbb{X} = \widetilde{\mathbb{SL}} \times \mathbb{E}^1$ then (after passing to a 2-fold covering space, if necessary) we may assume that $\pi \leq Isom(\widetilde{\mathbb{SL}}) \times R$. If $\mathbb{X} = \mathbb{H}^2 \times \mathbb{E}^2$ then $PSL(2,\mathbb{R}) \times R^2$ is a cocompact subgroup of $Isom(\mathbb{X})$. Hence $\pi \cap PSL(2,\mathbb{R}) \times R^2$ has finite index in π. In each case projection to the second factor maps $\sqrt{\pi}$ monomorphically. Moreover π preserves the foliation of the model space by copies of the euclidean factor. As before, M is virtually a product. □

In general, there may not be such a covering which is geometrically a cartesian product. Let ν be a discrete cocompact subgroup of $Isom(\mathbb{X})$ where $\mathbb{X} = \mathbb{H}^3$ or $\widetilde{\mathbb{SL}}$ which admits an epimorphism $\alpha : \nu \to Z$. Define a homomorphism $\theta : \nu \times Z \to Isom(\mathbb{X} \times \mathbb{E}^1)$ by $\theta(g,n)(x,r) = (g(x), r + n + \alpha(g)\sqrt{2})$ for all $g \in \nu$, $n \in Z$, $x \in X$ and $r \in R$. Then θ is a monomorphism onto a discrete subgroup which acts freely and cocompactly on $X \times R$, but the image of $\theta(\nu \times Z)$ in $E(1)$ has rank 2.

Orientable $\mathbb{H}^2 \times \mathbb{E}^2$- and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds are determined up to diffeomorphism (among such geometric manifolds) by their fundamental groups [Ue91]. However we do not yet have a complete characterization of the possible groups.

Corollary 9.3.1 *Let M be a closed 4-manifold with fundamental group π. Then M has a covering space of degree dividing 4 which is homotopy equivalent to a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$- or $\mathbb{H}^2 \times \mathbb{E}^2$-manifold if and only if $\sqrt{\pi} \cong Z^2$, $[\pi : \sqrt{\pi}] = \infty$, $[\pi : C_\pi(\sqrt{\pi})] < \infty$ and $\chi(M) = 0$.*

Proof The necessity of most of these conditions is clear from the proof of the theorem. If $\mathbb{X} = \mathbb{H}^2 \times \mathbb{E}^2$ then π has a subgroup of finite index which is isomorphic to $\tau \times Z^2$, where $\zeta\tau = 1$. If $\mathbb{X} = \widetilde{\mathbb{SL}} \times \mathbb{E}^1$ then π has a normal subgroup of finite index which is isomorphic to a product $Z \times \sigma$, and $\sqrt{\pi}$ has a characteristic infinite cyclic subgroup. Hence $\pi/C_\pi(\sqrt{\pi})$ is isomorphic to a finite upper triangular subgroup of $GL(2,\mathbb{Z})$. Since M is aspherical and $\sqrt{\pi}$ is infinite $\chi(M) = 0$.

If these conditions hold $\beta_1^{(2)}(\pi) = 0$ and $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$, and so M is aspherical, by Corollary 3.5.2. Hence M is homotopy equivalent to a manifold $M(\pi)$ which is Seifert fibred over a hyperbolic base orbifold, by Theorem 7.3. On passing to a covering space of degree dividing 4, if necessary, we may assume that M and the base orbifold are each orientable. Since π must then act on $\sqrt{\pi}$ through a finite subgroup of $SL(2,\mathbb{Z})$ (which is upper triangular if $\sqrt{\pi}$ is not a direct factor of a subgroup of finite index in π) the result follows from Theorem B of §5 of [Ue91]. $\square$

Corollary 9.3.2 *A group π is the fundamental group of a closed orientable $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ - or $\mathbb{H}^2 \times \mathbb{E}^2$-manifold with orientable base orbifold if and only if it is a PD_4^+-group, $\sqrt{\pi} \cong Z^2$, $[\pi : \sqrt{\pi}] = \infty$ and π acts on $\sqrt{\pi}$ through a finite cyclic subgroup of $SL(2,\mathbb{Z})$.* $\square$

The geometry is $\mathbb{H}^2 \times \mathbb{E}^2$ if and only if $\sqrt{\pi}$ is virtually a direct factor in π. This case may also be distinguished as follows.

Theorem 9.4 *Let M be a closed 4-manifold with fundamental group π. Then M has a covering space of degree dividing 4 which is homotopy equivalent to a $\mathbb{H}^2 \times \mathbb{E}^2$-manifold if and only if π has a finitely generated infinite subgroup ρ such that $[\pi : N_\pi(\rho)] < \infty$, $\sqrt{\rho} = 1$, $\zeta C_\pi(\rho) \cong Z^2$ and $\chi(M) = 0$.*

Proof The necessity of the conditions follows from Theorem 9.3. Suppose that they hold. Then M is aspherical and so π is a PD_4-group. Let $C = C_\pi(\rho)$. Then C is also normal in $\nu = N_\pi(\rho)$, and $C \cap \rho = 1$, since $\sqrt{\rho} = 1$. Hence $\rho \times C \cong \rho.C \leq \pi$. Now ρ is nontrivial. If ρ were free then an argument using the LHSSS for $H^*(\pi;\mathbb{Q}[\pi])$ would imply that ρ has two ends, and hence that $\sqrt{\rho} = \rho \cong Z$. Hence $c.d.\rho \geq 2$. Since moreover $Z^2 \leq C$ we must have $c.d.\rho = c.d.C = 2$ and $[\pi : \rho.C] < \infty$. It follows easily that $\sqrt{\pi} \cong Z^2$ and that $[\pi : C_\pi(\sqrt{\pi})] < \infty$. Hence we may apply Corollary 9.3.1. Since π is virtually a product it must be of type $\mathbb{H}^2 \times \mathbb{E}^2$. $\square$

Is it possible to give a more self-contained argument for this case? It is not hard to see that $\pi/\sqrt{\pi}$ acts discretely, cocompactly and isometrically on H^2. However it is more difficult to find a suitable homomorphism from π to $E(2)$.

Theorems 9.1 and 9.2 suggest that there should be a characterization of closed $\mathbb{H}^2 \times \mathbb{E}^2$- and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds parallel to Theorem 8.1, i.e., in terms of the conditions "$\chi(M) = 0$" and "π has an elementary amenable normal subgroup of Hirsch length 2 and infinite index".

9.3 $\mathbb{H}^3 \times \mathbb{E}^1$-manifolds

We have only conjectural characterizations of manifolds homotopy equivalent to $\mathbb{H}^3 \times \mathbb{E}^1$-manifolds and of their fundamental groups. An argument similar to that of Corollary 9.3.1 shows that a 4-manifold M with fundamental group π is virtually simple homotopy equivalent to an $\mathbb{H}^3 \times \mathbb{E}^1$-manifold if and only if $\chi(M) = 0$, $\sqrt{\pi} = Z$ and π has a normal subgroup of finite index which is isomorphic to $\rho \times Z$ where ρ is a discrete cocompact subgroup of $PSL(2, \mathbb{C})$. If every PD_3-group is the fundamental group of an aspherical closed 3-manifold and if every atoroidal aspherical closed 3-manifold is hyperbolic we could replace the last assertion by the more intrinsic conditions that ρ have one end (which would suffice with the other conditions to imply that M is aspherical and hence that ρ is a PD_3-group), no noncyclic abelian subgroups and $\sqrt{\rho} = 1$ (which would imply that any irreducible 3-manifold with fundamental group ρ is atoroidal). Similarly, a group G should be the fundamental group of an $\mathbb{H}^3 \times \mathbb{E}^1$-manifold if and only if it is torsion free and has a normal subgroup of finite index isomorphic to $\rho \times Z$ where ρ is a PD_3-group with $\sqrt{\rho} = 1$ and no noncyclic abelian subgroups.

Lemma 9.5 *Let π be a finitely generated group with $\sqrt{\pi} \cong Z$, and which has a subgroup G of finite index such that $\sqrt{\pi} \cap G' = 1$. Then there is a homomorphism $\lambda : \pi \to D$ which is injective on $\sqrt{\pi}$.*

Proof We may assume that G is normal in π and that $G < C_\pi(\sqrt{\pi})$. Let $H = \pi/I(G)$ and let A be the image of $\sqrt{\pi}$ in H. Then H is an extension of the finite group π/G by the finitely generated free abelian group $G/I(G)$, and $A \cong Z$. Conjugation in H determines a homomorphism w from π/G to $Aut(A) = \{\pm 1\}$. Since the rational group ring $\mathbb{Q}[\pi/G]$ is semisimple $\mathbb{Q} \otimes A$ is a direct summand of $\mathbb{Q} \otimes (G/I(G))$, and so there is a $\mathbb{Z}[\pi/G]$-linear homomorphism $p : G/I(G) \to Z^w$ which is injective on A. The kernel is a normal subgroup of H, and $H/\mathrm{Ker}(p)$ has two ends. The lemma now follows easily. □

The foliation of $H^3 \times R$ by copies of H^3 induces a codimension 1 foliation of any closed $\mathbb{H}^3 \times \mathbb{E}^1$-manifold. If all the leaves are compact, then it is either a mapping torus or the union of two twisted I-bundles.

Theorem 9.6 *Let M be a closed $\mathbb{H}^3 \times \mathbb{E}^1$-manifold. If $\zeta\pi \cong Z$ then M is homotopy equivalent to a mapping torus of a self homeomorphism of an $\mathbb{H}^3$-manifold; otherwise M is homotopy equivalent to the union of two twisted I-bundles over $\mathbb{H}^3$-manifold bases.*

Proof Let $\lambda : \pi \to D$ be a homomorphism as in Lemma 9.5 and let $K = \mathrm{Ker}(\lambda)$. Then $K \cap \sqrt{\pi} = 1$, so K is isomorphic to a subgroup of finite index in $\pi/\sqrt{\pi}$. Therefore $K \cong \pi_1(N)$ for some closed $\mathbb{H}^3$-manifold, since it is torsion free. If $\zeta\pi = Z$ then $\mathrm{Im}(\lambda) \cong Z$ (since $\zeta D = 1$); if $\zeta\pi = 1$ then $w \neq 1$ and so $\mathrm{Im}(\lambda) \cong D$. The theorem now follows easily. □

Is M itself such a mapping torus or union of I-bundles?

9.4 Mapping tori

In this section we shall use 3-manifold theory to characterize mapping tori with one of the geometries $\mathbb{H}^3 \times \mathbb{E}^1$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ or $\mathbb{H}^2 \times \mathbb{E}^2$.

Theorem 9.7 *Let ϕ be a self homeomorphism of a closed 3-manifold N which admits the geometry $\mathbb{H}^2 \times \mathbb{E}^1$ or $\widetilde{\mathbb{SL}}$. Then the mapping torus $M(\phi) = N \times_\phi S^1$ admits the corresponding product geometry if and only if the outer automorphism $[\phi_*]$ induced by ϕ has finite order. The mapping torus of a self homeomorphism ϕ of an $\mathbb{H}^3$-manifold N admits the geometry $\mathbb{H}^3 \times \mathbb{E}^1$.*

Proof Let $\nu = \pi_1(N)$ and let t be an element of $\pi = \pi_1(M(\phi))$ which projects to a generator of $\pi_1(S^1)$. If $M(\phi)$ has geometry $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ then after passing to the 2-fold covering space $M(\phi^2)$, if necessary, we may assume that π is a discrete cocompact subgroup of $Isom(\widetilde{\mathbb{SL}}) \times R$. As in Theorem 9.3 the intersection of π with the centre of this group is a lattice subgroup $L \cong Z^2$. Since the centre of ν is Z the image of L in π/ν is nontrivial, and so π has a subgroup σ of finite index which is isomorphic to $\nu \times Z$. In particular, conjugation by $t^{[\pi:\sigma]}$ induces an inner automorphism of ν.

If $M(\phi)$ has geometry $\mathbb{H}^2 \times \mathbb{E}^2$ a similar argument implies that π has a subgroup σ of finite index which is isomorphic to $\rho \times Z^2$, where ρ is a discrete cocompact subgroup of $PSL(2, \mathbb{R})$, and is a subgroup of ν. It again follows that $t^{[\pi:\sigma]}$ induces an inner automorphism of ν.

Conversely, suppose that N has a geometry of type $\mathbb{H}^2 \times \mathbb{E}^1$ or $\widetilde{\mathbb{SL}}$ and that $[\phi_*]$ has finite order in $Out(\nu)$. Then ϕ is homotopic to a self homeomorphism of (perhaps larger) finite order [Zn80] and is therefore isotopic to such a self homeomorphism [Sc85,BO91], which may be assumed to preserve the geometric structure [MS86]. Thus we may assume that ϕ is an isometry. The self homeomorphism of $N \times R$ sending (n,r) to $(\phi(n), r+1)$ is then an isometry for the product geometry and the mapping torus has the product geometry.

If N is hyperbolic then ϕ is homotopic to an isometry of finite order, by Mostow rigidity [Ms68], and is therefore isotopic to such an isometry [GMT96], so the mapping torus again has the product geometry. ∎

A closed 4-manifold M which admits an effective T-action with hyperbolic base orbifold is homotopy equivalent to such a mapping torus. For then $\zeta\pi = \sqrt{\pi}$ and the LHSSS for homology gives an exact sequence

$$H_2(\pi/\zeta\pi;\mathbb{Q}) \to H_1(\zeta\pi;\mathbb{Q}) \to H_1(\pi;\mathbb{Q}).$$

As $\pi/\zeta\pi$ is virtually a PD_2-group $H_2(\pi/\zeta\pi;\mathbb{Q}) \cong Q$ or 0, so $\zeta\pi/\zeta\pi\cap\pi'$ has rank at least 1. Hence $\pi \cong \nu \times_\theta Z$ where $\zeta\nu \cong Z$, $\nu/\zeta\nu$ is virtually a PD_2-group and $[\theta]$ has finite order in $Out(\nu)$. If moreover M is orientable then it is geometric ([Ue90,91] - see also §7 of Chapter 7). Note also that if M is a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold then $\zeta\pi = \sqrt{\pi}$ if and only if $\pi \leq Isom_o(\widetilde{\mathbb{SL}} \times \mathbb{E}^1)$.

Let F be a closed hyperbolic surface and $\alpha : F \to F$ a pseudo-Anasov homeomorphism. Let $\Theta(f,z) = (\alpha(f), \bar{z})$ for all (f,z) in $N = F \times S^1$. Then N is an $\mathbb{H}^2 \times \mathbb{E}^1$-manifold. The mapping torus of Θ is homeomorphic to an $\mathbb{H}^3 \times \mathbb{E}^1$-manifold which is not a mapping torus of any self-homeomorphism of an $\mathbb{H}^3$-manifold. In this case $[\Theta_*]$ has infinite order. However if N is a $\widetilde{\mathbb{SL}}$-manifold and $[\phi_*]$ has infinite order then $M(\phi)$ admits no geometric structure, for then $\sqrt{\pi} \cong Z$ but is not a direct factor of any subgroup of finite index.

If $\zeta\nu \cong Z$ and $\zeta(\nu/\zeta\nu) = 1$ then $Hom(\nu/\nu', \zeta\nu)$ embeds in $Out(\nu)$, and thus ν has outer automorphisms of infinite order, in most cases [CR77].

Let N be an aspherical closed $\mathbb{X}^3$-manifold where $\mathbb{X}^3 = \mathbb{H}^3$, $\widetilde{\mathbb{SL}}$ or $\mathbb{H}^2 \times \mathbb{E}^1$, and suppose that $\beta_1(N) > 0$ but N is not a mapping torus. Choose an epimorphism $\lambda : \pi_1(N) \to Z$ and let $\widehat{N}$ be the 2-fold covering space associated to the subgroup $\lambda^{-1}(2Z)$. If $\nu : \widehat{N} \to \widehat{N}$ is the covering involution then $\mu(n,z) = (\nu(n), \bar{z})$ defines a free involution on $N \times S^1$, and the orbit space M is an $\mathbb{X}^3 \times \mathbb{E}^1$-manifold with $\beta_1(M) > 0$ which is not a mapping torus.

9.5 The semisimple geometries: $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^4$ and $\mathbb{H}^2(\mathbb{C})$

In this section we shall consider the remaining three geometries realizable by closed 4-manifolds. (Not much is known about $\mathbb{H}^4$ or $\mathbb{H}^2(\mathbb{C})$.)

Let $P = PSL(2,\mathbb{R})$ be the group of orientation preserving isometries of $\mathbb{H}^2$. Then $Isom(\mathbb{H}^2 \times \mathbb{H}^2)$ contains $P \times P$ as a normal subgroup of index 8. If M is a closed $\mathbb{H}^2 \times \mathbb{H}^2$-manifold then $\sigma(M) = 0$ and $\chi(M) > 0$. It is *reducible* if it has a finite cover isometric to a product of closed surfaces. The model space for $\mathbb{H}^2 \times \mathbb{H}^2$ may be taken as the unit polydisc

$$\{(w, z) \in C^2 : |w| < 1, |z| < 1\}.$$

Thus M is a complex surface if (and only if) $\pi_1(M)$ is a subgroup of $P \times P$.

We have the following characterizations of the fundamental groups of reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds.

Theorem 9.8 *A group π is the fundamental group of a reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifold if and only if it is torsion free, $\sqrt{\pi} = 1$ and π has a subgroup of finite index which is isomorphic to a product of PD_2-groups.*

Proof The conditions are clearly necessary. Suppose that they hold. Then π is a PD_4-group and has a subgroup of finite index which is a direct product $\alpha.\beta \cong \alpha \times \beta$, where α and β are PD_2-groups. Let N be the intersection of the conjugates of $\alpha.\beta$ in π. Then N is normal in π, so $\sqrt{N} = 1$ also, and $[\pi : N] < \infty$. Let $K = \alpha \cap N$ and $L = \beta \cap N$. Then K and L are PD_2-groups with trivial centre, and $K.L \cong K \times L$ is normal in N and has finite index in π. Moreover N/K and N/L are isomorphic to subgroups of finite index in β and α, respectively, and so are also PD_2-groups. Since any automorphism of N must either fix these subgroups or interchange them, by Theorem 5.6, $K.L$ is normal in π and $[\pi : N_\pi(K)] \leq 2$.

Let $\nu = N_\pi(K)$. Then $L \leq C_\pi(K) \leq \nu$ and $\nu = N_\pi(L)$ also. After enlarging K and L, if necessary, we may assume that $L = C_\pi(K)$ and $K = C_\pi(L)$. Hence ν/K and ν/L have no nontrivial finite normal subgroup. (For if K_1 is normal in ν and contains K as a subgroup of finite index then $K_1 \cap L$ is finite, hence trivial, and so $K_1 \leq C_\pi(L)$.) The action of ν/L by conjugation on K has finite image in $Out(K)$, and so ν/L embeds as a discrete cocompact subgroup of $Isom(\mathbb{H}^2)$, by the Nielsen conjecture [Ke83]. Together with a similar embedding for ν/K we obtain a homomorphism from ν to a discrete cocompact subgroup of $Isom(\mathbb{H}^2 \times \mathbb{H}^2)$.

If $[\pi : \nu] = 2$ let t be an element of $\pi - \nu$, and let $j : \nu/K \to Isom(\mathbb{H}^2)$ be an embedding onto a discrete cocompact subgroup S. Then $tKt^{-1} = L$ and conjugation by t induces an isomorphism $f : \nu/K \to \nu/L$. The homomorphisms j and $j \circ f^{-1}$ determine an embedding $J : \nu \to Isom(\mathbb{H}^2 \times \mathbb{H}^2)$ onto a discrete cocompact subgroup of finite index in $S \times S$. Now $t^2 \in \nu$ and $J(t^2) = (s,s)$, where $s = j(t^2K)$. We may extend J to an embedding of π in $Isom(\mathbb{H}^2 \times \mathbb{H}^2)$ by defining $J(t)$ to be the isometry sending (x,y) to $(y, s.x)$. Thus (in either case) π acts isometrically and properly discontinuously on $H^2 \times H^2$. Since π is torsion free the action is free, and so $\pi = \pi_1(M)$, where $M = \pi\backslash(H^2 \times H^2)$. □

Corollary 9.8.1 *Let M be a $\mathbb{H}^2 \times \mathbb{H}^2$-manifold. Then M is reducible if and only if it has a 2-fold covering space which is homotopy equivalent to the total space of an orbifold bundle over a hyperbolic 2-orbifold.*

Proof That reducible manifolds have such coverings was proven in the theorem. Conversely, an irreducible lattice in $P \times P$ cannot have any nontrivial normal subgroups of infinite index, by Theorem IX.6.14 of [Ma]. Hence an $\mathbb{H}^2 \times \mathbb{H}^2$-manifold which is finitely covered by the total space of a surface bundle is virtually a cartesian product. □

Is the 2-fold covering space itself such a bundle space over a 2-orbifold?

In general, we cannot assume that M is itself fibred over a 2-orbifold. Let G be a PD_2-group with $\zeta G = 1$ and let x be a nontrivial element of G. A cocompact free action of G on H^2 determines a cocompact free action of

$$\pi = \langle G \times G, t \mid t(g_1,g_2)t^{-1} = (xg_2x^{-1}, g_1) \text{ for all } (g_1,g_2) \in G \times G, t^2 = (x,x)\rangle$$

on $H^2 \times H^2$, by $(g_1,g_2).(h_1,h_2) = (g_1.h_1, g_2.h_2)$ and $t.(h_1,h_2) = (x.h_2, h_1)$, for all $(g_1,g_2) \in G \times G$ and $(h_1,h_2) \in H^2 \times H^2$. The group π has no normal subgroup which is a PD_2-group. (Note also that if G is orientable $\pi\backslash(H^2 \times H^2)$ is a compact complex surface.)

We may use Theorem 9.8 to give several characterizations of the homotopy types of such manifolds.

Theorem 9.9 *Let M be a closed 4-manifold with fundamental group π. Then the following are equivalent:*

(1) *M is homotopy equivalent to a reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifold;*

(2) *π has a subnormal subgroup G which is FP_2, has one end and such that $C_\pi(G)$ is not a free group, $\pi_2(M) = 0$ and $\chi(M) \neq 0$;*

(3) *π has a subgroup ρ of finite index which is isomorphic to a product of two PD_2-groups and $\chi(M)[\pi:\rho] = \chi(\rho) \neq 0$.*

(4) *π is virtually a PD_4-group, $\sqrt{\pi}=1$ and π has a torsion free subgroup of finite index which is isomorphic to a nontrivial product $\sigma\times\tau$ where $\chi(M)[\pi:\sigma\times\tau]=(2-\beta_1(\sigma))(2-\beta_1(\tau))$.*

Proof If (1) holds then M is aspherical and so (2) holds, by Theorem 9.8 and its Corollary.

Suppose now that (2) holds. Then π has one end, by an iterated LHSSS argument, since G does. Hence M is aspherical and π is a PD_4-group, since $\pi_2(M)=0$. Since $\chi(M)\neq 0$ we must have $\sqrt{\pi}=1$. (For otherwise $\beta_i^{(2)}(\pi)=0$ for all i, by Theorem 2.3, and so $\chi(M)=0$.) In particular, every subnormal subgroup of π has trivial centre. Therefore $G\cap C_\pi(G)=\zeta G=1$ and so $G\times C_\pi(G)\cong\rho=G.C_\pi(G)\leq\pi$. Hence $c.d.C_\pi(G)\leq 2$. Since $C_\pi(G)$ is not free $c.d.G\times C_\pi(G)=4$ and so ρ has finite index in π. (In particular, $[C_\pi(C_\pi(G)):G]$ is finite.) Hence ρ is a PD_4-group and G and $C_\pi(G)$ are PD_2-groups, so π is virtually a product. Thus (2) implies (1), by Theorem 9.8.

It is clear that (1) implies (3). If (3) holds then on applying Theorems 2.2 and 3.5 to the finite covering space associated to ρ we see that M is aspherical, so π is a PD_4-group and (4) holds. Similarly, M is asperical if (4) holds. In particular, π is a PD_4-group and so is torsion free. Since $\sqrt{\pi}=1$ neither σ nor τ can be infinite cyclic, and so they are each PD_2-groups. Therefore π is the fundamental group of a reducible $\mathbb{H}^2\times\mathbb{H}^2$-manifold, by Theorem 9.8, and $M\simeq\pi\backslash H^2\times H^2$, by asphericity. □

The asphericity of M could be ensured by assuming that π be PD_4 and $\chi(M)=\chi(\pi)$, instead of assuming that $\pi_2(M)=0$.

For $\mathbb{H}^2\times\mathbb{H}^2$-manifolds we can give more precise criteria for reducibility.

Theorem 9.10 *Let M be a closed $\mathbb{H}^2\times\mathbb{H}^2$-manifold with fundamental group π. Then the following are equivalent:*

(1) *π has a subgroup of finite index which is a nontrivial direct product;*

(2) *$Z^2<\pi$;*

(3) *π has a nontrivial element with nonabelian centralizer;*

(4) *$\pi\cap(\{1\}\times P)\neq 1$;*

(5) *$\pi\cap(P\times\{1\})\neq 1$;*

(6) *M is reducible.*

Proof Since π is torsion free each of the above conditions is invariant under passage to subgroups of finite index, and so we may assume without loss of generality that $\pi \leq P \times P$. Suppose that σ is a subgroup of finite index in π which is a nontrivial direct product. Since $\chi(\sigma) \neq 0$ neither factor can be infinite cyclic, and so the factors must be PD_2-groups. In particular, $Z^2 < \sigma$ and the centraliser of any element of either direct factor is nonabelian. Thus (1) implies (2) and (3).

Suppose that (a,b) and (a',b') generate a subgroup of π isomorphic to Z^2. Since centralizers of elements of infinite order in P are cyclic the subgroup of P generated by $\{a,a'\}$ is infinite cyclic or is finite. Hence we may assume without loss of generality that $a' = 1$, and so (2) implies (4). Similarly, (2) implies (5).

Let $g = (g_1, g_2) \in P \times P$ be nontrivial. Since centralizers of elements of infinite order in P are infinite cyclic and $C_{P\times P}(\langle g\rangle) = C_P(\langle g_1\rangle) \times C_P(\langle g_2\rangle)$ it follows that if $C_\pi(\langle g\rangle)$ is nonabelian then either g_1 or g_2 has finite order. Thus (3) implies (4) and (5).

Let $K_1 = \pi \cap (\{1\} \times P)$ and $K_2 = \pi \cap (P \times \{1\})$. Then K_i is normal in π, and there are exact sequences

$$1 \to K_i \to \pi \to L_i \to 1,$$

where $L_i = pr_i(\pi)$ is the image of π under projection to the i^{th} factor of $P \times P$, for $i = 1$ and 2. Moreover K_i is normalised by L_{3-i}, for $i = 1$ and 2. Suppose that $K_1 \neq 1$. Then K_1 is non abelian, since it is normal in π and $\chi(\pi) \neq 0$. If L_2 were not discrete then elements of L_2 sufficiently close to the identity would centralize K_1. As centralizers of nonidentity elements of P are abelian, this would imply that K_1 is abelian. Hence L_2 is discrete. Now $L_2\backslash H^2$ is a quotient of $\pi\backslash H \times H$ and so is compact. Therefore L_2 is virtually a PD_2-group. Now $c.d.K_2 + v.c.d.L_2 \geq c.d.\pi = 4$, so $c.d.K_2 \geq 2$. In particular, $K_2 \neq 1$ and so a similar argument now shows that $c.d.K_1 \geq 2$. Hence $c.d.K_1 \times K_2 \geq 4$. Since $K_1 \times K_2 \cong K_1.K_2 \leq \pi$ it follows that π is virtually a product, and M is finitely covered by $(K_1\backslash H^2) \times (K_2\backslash H^2)$. Thus (4) and (5) are equivalent, and imply (6). Clearly (6) implies (1). □

The idea used in showing that (4) implies (5) and (6) derives from one used in the proof of Theorem 6.3 of [Wl85].

If Γ is a discrete cocompact subgroup of $P \times P$ such that $M = \Gamma\backslash H^2 \times H^2$ is irreducible then $\Gamma \cap P \times \{1\} = \Gamma \cap \{1\} \times P = 1$, by the theorem. Hence the natural foliations of $H^2 \times H^2$ descend to give a pair of transverse foliations of M by copies of H^2. (Conversely, if M is a closed Riemannian 4-manifold with a codimension 2 metric foliation by totally geodesic surfaces then M has a finite cover which either admits the geometry $\mathbb{H}^2 \times \mathbb{E}^2$ or $\mathbb{H}^2 \times \mathbb{H}^2$ or is the total space of an S^2 or T-bundle over a closed surface or is the mapping torus of a self homeomorphism of R^3/Z^3, $S^2 \times S^1$ or a lens space [Ca90]).

An irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-lattice is an arithmetic subgroup of $Isom(\mathbb{H}^2 \times \mathbb{H}^2)$, and has no nontrivial normal subgroups of infinite index, by Theorems IX.6.5 and 14 of [Ma]. Such irreducible lattices are rigid, and so the argument of Theorem 8.1 of [Wa72] implies that there are only finitely many irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds with given Euler characteristic. What values of χ are realized by such manifolds?

Examples of irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds may be constructed as follows. Let F be a totally real number field, with ring of integers O_F. Let H be a skew field which is a quaternion algebra over F such that $H \otimes_\sigma \mathbb{R} \cong M_2(\mathbb{R})$ for exactly two embeddings σ of F in $\mathbb{R}$. If A is an order in H (a subring which is also a finitely generated O_F-submodule and such that $F.A = H$) then the quotient of the group of units $A^\times$ by ± 1 embeds as a discrete cocompact subgroup of $P \times P$, and the corresponding $\mathbb{H}^2 \times \mathbb{H}^2$-manifold is irreducible. (See Chapter IV of [Vi].) It can be shown that every irreducible, cocompact $\mathbb{H}^2 \times \mathbb{H}^2$-lattice is commensurable with such a subgroup.

Much less is known about $\mathbb{H}^4$- or $\mathbb{H}^2(\mathbb{C})$-manifolds. If M is a closed orientable $\mathbb{H}^4$-manifold then $\sigma(M) = 0$ and $\chi(M) > 0$ [Ko92]. If M is a closed $\mathbb{H}^2(\mathbb{C})$-manifold it is orientable and $\chi(M) = 3\sigma(M) > 0$ [Wl86]. The isometry group of $\mathbb{H}^2(\mathbb{C})$ has two components; the identity component is $SU(2,1)$ and acts via holomorphic isomorphisms on the unit ball

$$\{(w,z) \in C^2 : |w|^2 + |z|^2 < 1\}.$$

(No closed $\mathbb{H}^4$-manifold admits a complex structure.) There are only finitely many closed $\mathbb{H}^4$- or $\mathbb{H}^2(\mathbb{C})$-manifolds with a given Euler characteristic (see Theorem 8.1 of [Wa72]). The 120-*cell space* of Davis is a closed orientable $\mathbb{H}^4$-manifold with $\chi = 26$ and $\beta_1 = 24 > 0$ [Da85, TS01], so all positive multiples of 26 are realized. Examples of $\mathbb{H}^2(\mathbb{C})$-manifolds due to Mumford and Hirzebruch have the homology of CP^2 (so $\chi = 3$), and $\chi = 15$ and $\beta_1 > 0$, respectively [HP96]. It is not known whether all positive multiples of 3 are realized. Since H^4 and $H^2(\mathbb{C})$ are rank 1 symmetric spaces the fundamental groups can contain no noncyclic abelian subgroups [Pr43]. In each case there are cocompact lattices which are not arithmetic. At present there are not even conjectural intrinsic characterizations of such groups. (See also [Rt] for the geometries $\mathbb{H}^n$ and [Go] for the geometries $\mathbb{H}^n(\mathbb{C})$.)

Each of the geometries $\mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^4$ and $\mathbb{H}^2(\mathbb{C})$ admits cocompact lattices which are not almost coherent (see §1 of Chapter 4 above, [BM94] and [Ka98], respectively). Is this true of every such lattice for one of these geometries? (Lattices for the other geometries are coherent.)

9.6 Miscellany

A homotopy equivalence between two closed $\mathbb{H}^n$- or $\mathbb{H}^n(\mathbb{C})$-manifolds of dimension ≥ 3 is homotopic to an isometry, by *Mostow rigidity* [Ms68]. Farrell and Jones have established "topological" analogues of Mostow rigidity, which apply when the model manifold has a geometry of nonpositive curvature and dimension ≥ 5. By taking cartesian products with S^1, we can use their work in dimension 4 also.

Theorem 9.11 *Let M be a closed 4-manifold M with fundamental group π. Then M is s-cobordant to an $\mathbb{X}^4$-manifold where $\mathbb{X}^4 = \mathbb{H}^2 \times \mathbb{H}^2$, $\mathbb{H}^4$, $\mathbb{H}^2(\mathbb{C})$, $\mathbb{H}^3 \times \mathbb{E}^1$ or $\mathbb{H}^2 \times \mathbb{E}^2$ if and only if π is isomorphic to a cocompact lattice in $Isom(\mathbb{X}^4)$ and $\chi(M) = \chi(\pi)$.*

Proof The conditions are clearly necessary. If they hold M is aspherical and so $c_M : M \to \pi\backslash X$ is a homotopy equivalence, by Theorem 3.5. In all cases the geometry has nonpositive sectional curvatures, so $Wh(\pi) = Wh(\pi \times Z) = 0$ and $M \times S^1$ is homeomorphic to $(\pi\backslash X) \times S^1$ [FJ93']. Hence M and $\pi\backslash X$ are s-cobordant, by Lemma 6.10. □

A similar result holds for $\widetilde{\mathbb{SL}}\times\mathbb{E}^1$-manifolds, provided that $\pi \leq Isom_o(\widetilde{\mathbb{SL}}\times\mathbb{E}^1)$. This is equivalent to the condition "$\zeta\pi = \sqrt{\pi}$". Although closed $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds do not admit metrics of nonpositive curvature [KL96], they do admit effective T-actions if $\zeta\pi = \sqrt{\pi}$, and we then may appeal to [NS85] instead of [FJ93']. (See also Theorem 13.2 below.) The hypothesis that the Seifert structure derive from a toral group action may well be unnecessary.

Does a similar result hold for aspherical closed 4-manifolds with a geometric decomposition? Let M be such a manifold and let $\pi = \pi_1(M)$. Then π is built from the fundamental groups of the pieces by amalgamation along torsion free virtually poly-Z subgroups. As the Whitehead groups of the geometric pieces are trivial (by the argument of [FJ86]) and the amalgamated subgroups are regular noetherian it follows from the K-theoretic Mayer-Vietoris sequence of Waldhausen that $Wh(\pi) = 0$. Is there a corresponding argument in L-theory?

For the semisimple geometries we may avoid the appeal to L^2-methods to establish asphericity as follows. Since $\chi(M) > 0$ and π is infinite and residually finite there is a subgroup σ of finite index such that the associated covering spaces M_σ and $\sigma\backslash X$ are orientable and $\chi(M_\sigma) = \chi(\sigma) > 2$. In particular, $H^2(M_\sigma;\mathbb{Z})$ has elements of infinite order. Since the classifying map $c_{M_\sigma} : M_\sigma \to \sigma\backslash X$ is 2-connected it induces an isomorphism on H^2 and hence is a degree-1 map, by Poincaré duality. Therefore it is a homotopy equivalence, by Theorem 3.2.

Theorem 9.12 *If M is an aspherical closed 4-manifold which is finitely covered by a manifold with a geometry other than $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ then M is homotopy equivalent to a geometric 4-manifold.*

Proof The result is clear for infrasolvmanifolds, and follows from Theorem 9.8 if M is finitely covered by a reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifold. It holds for the other closed $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds and for the geometries $\mathbb{H}^4$ and $\mathbb{H}^2(\mathbb{C})$ by Mostow rigidity.

If the geometry is $\mathbb{H}^3 \times \mathbb{E}^1$ then $\sqrt{\pi} \cong Z$ and $\pi/\sqrt{\pi}$ is virtually the group of a $\mathbb{H}^3$-manifold. Hence $\pi/\sqrt{\pi}$ acts isometrically and properly discontinuously on $\mathbb{H}^3$, by Mostow rigidity. Moreover as the hypotheses of Lemma 9.5 are satisfied, by Theorem 9.3, there is a homomorphism $\lambda : \pi \to D < Isom(\mathbb{E}^1)$ which maps $\sqrt{\pi}$ injectively. Together these actions determine a discrete and cocompact action of π by isometries on $H^3 \times R$. Since π is torsion free this action is free, and so M is homotopy equivalent to an $\mathbb{H}^3 \times \mathbb{E}^1$-manifold. $\square$

The result is not yet clear for $\mathbb{H}^2 \times \mathbb{E}^2$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$, $\mathbb{S}^2 \times \mathbb{E}^2$ or $\mathbb{S}^2 \times \mathbb{H}^2$. The theorem holds also for $\mathbb{S}^4$ and $\mathbb{CP}^2$, but fails for $\mathbb{S}^3 \times \mathbb{E}^1$ or $\mathbb{S}^2 \times \mathbb{S}^2$. In particular, there is a closed nonorientable 4-manifold which is doubly covered by $S^2 \times S^2$ but is not homotopy equivalent to an $\mathbb{S}^2 \times \mathbb{S}^2$-manifold. (See Chapters 11 and 12.)

If π is the fundamental group of an aspherical closed geometric 4-manifold then $\beta_s^{(2)}(\pi) = 0$ for $s = 0$ or 1, and so $\beta_2^{(2)}(\pi) = \chi(\pi)$, by Theorem 1.35 of [Lü]. Therefore $\mathrm{def}(\pi) \leq \min\{0, 1 - \chi(\pi)\}$, by Theorems 2.4 and 2.5. If π is orientable this gives $\mathrm{def}(\pi) \leq 2\beta_1(\pi) - \beta_2(\pi) - 1$. When $\beta_1(\pi) = 0$ this is an improvement on the estimate $\mathrm{def}(\pi) \leq \beta_1(\pi) - \beta_2(\pi)$ derived from the ordinary homology of a 2-complex with fundamental group π.

Chapter 10

Manifolds covered by $S^2 \times R^2$

If the universal covering space of a closed 4-manifold with infinite fundamental group is homotopy equivalent to a finite complex then it is either contractible or homotopy equivalent to S^2 or S^3, by Theorem 3.9. The cases when M is aspherical have been considered in Chapters 8 and 9. In this chapter and the next we shall consider the spherical cases. We show first that if $\widetilde{M} \simeq S^2$ then M has a finite covering space which is s-cobordant to a product $S^2 \times B$, where B is an aspherical surface, and π is the group of a $\mathbb{S}^2 \times \mathbb{E}^2$- or $\mathbb{S}^2 \times \mathbb{H}^2$-manifold. In §2 we show that there are only finitely many homotopy types of such manifolds for each such group π. In §3 we show that all S^2- and RP^2-bundles over aspherical closed surfaces are geometric. We shall then determine the nine possible elementary amenable groups (corresponding to the geometry $\mathbb{S}^2 \times \mathbb{E}^2$). Six of these groups have infinite abelianization, and in §5 we show that for these groups the homotopy types may be distinguished by their Stiefel-Whitney classes. We conclude with some remarks on the homeomorphism classification.

For brevity, we shall let $\mathbb{X}^2$ denote both $\mathbb{E}^2$ and $\mathbb{H}^2$.

10.1 Fundamental groups

The determination of the closed 4-manifolds with universal covering space homotopy equivalent to S^2 rests on Bowditch's Theorem, via Theorem 5.14.

Theorem 10.1 *Let M be a closed 4-manifold with fundamental group π. Then the following conditions are equivalent:*

(1) *π is virtually a PD_2-group and $\chi(M) = 2\chi(\pi)$;*

(2) *$\pi \neq 1$ and $\pi_2(M) \cong Z$;*

(3) *M has a covering space of degree dividing 4 which is s-cobordant to $S^2 \times B$, where B is an aspherical closed orientable surface;*

(4) *M is virtually s-cobordant to an $\mathbb{S}^2 \times \mathbb{X}^2$-manifold.*

If these conditions hold then $\widetilde{M}$ is homeomorphic to $S^2 \times R^2$.

Proof If (1) holds then $\pi_2(M) \cong Z$, by Theorem 5.10, and so (2) holds. If (2) holds then the covering space associated to the kernel of the natural action of π on $\pi_2(M)$ is homotopy equivalent to the total space of an S^2-bundle ξ over an aspherical closed surface with $w_1(\xi) = 0$, by Lemma 5.11 and Theorem 5.14. On passing to a 2-fold covering space, if necessary, we may assume that $w_2(\xi) = w_1(M) = 0$ also. Hence ξ is trivial and so the corresponding covering space of M is s-cobordant to a product $S^2 \times B$ with B orientable. Moreover $\widetilde{M} \cong S^2 \times R^2$, by Theorem 6.16. It is clear that (3) implies (4) and (4) implies (1). ∎

This follows also from [Fa74] instead of [Bo99] if we know also that $\chi(M) \leq 0$. If π is infinite and $\pi_2(M) \cong Z$ then π may be realized geometrically.

Theorem 10.2 *Let M be a closed 4-manifold with fundamental group π and such that $\pi_2(M) \cong Z$. Then π is the fundamental group of a closed manifold admitting the geometry $\mathbb{S}^2 \times \mathbb{E}^2$, if π is virtually Z^2, or $\mathbb{S}^2 \times \mathbb{H}^2$ otherwise.*

Proof If π is torsion free then it is itself a surface group. If π has a nontrivial finite normal subgroup then it is a direct product $\mathrm{Ker}(u) \times (Z/2Z)$, where $u : \pi \to \{\pm 1\} = Aut(\pi_2(M))$ is the natural homomorphism. (See Theorem 5.14). In either case π is the fundamental group of a corresponding product of surfaces. Otherwise π is a semidirect product $\mathrm{Ker}(u) \tilde{\times} (Z/2Z)$ and is a plane motion group, by a theorem of Nielsen ([Zi]; see also Theorem A of [EM82]). This means that there is a monomorphism $f : \pi \to Isom(\mathbb{X}^2)$ with image a discrete subgroup which acts cocompactly on X, where X is the Euclidean or hyperbolic plane, according as π is virtually abelian or not. The homomorphism $(u, f) : \pi \to \{\pm I\} \times Isom(\mathbb{X}^2) \leq Isom(\mathbb{S}^2 \times \mathbb{X}^2)$ is then a monomorphism onto a discrete subgroup which acts freely and cocompactly on $S^2 \times R^2$. In all cases such a group may be realised geometrically. ∎

The orbit space of the geometric action of π described above is a cartesian product with S^2 if u is trivial and fibres over RP^2 otherwise.

10.2 Homotopy type

In this section we shall extend an argument of Hambleton and Kreck to show that there are only finitely many homotopy types of manifolds with universal cover $S^2 \times R^2$ and given fundamental group.

We shall first show that the orientation character and the action of π on π_2 determine each other.

Lemma 10.3 *Let M be a closed 4-manifold with fundamental group $\pi \neq 1$ and such that $\pi_2(M) \cong Z$. Then $H^2(\pi;\mathbb{Z}[\pi]) \cong Z$ and $u = w_1(M) + v$, where $u : \pi \to Aut(\pi_2(M)) = Z/2Z$ and $v : \pi \to Aut(H^2(\pi;\mathbb{Z}[\pi])) = Z/2Z$ are the natural actions.*

Proof Since π is infinite $Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi]) = 0$ and so $\overline{H^2(\pi;\mathbb{Z}[\pi])} \cong \pi_2(M)$, by Lemma 3.3. Now $\overline{H^2(\pi;\mathbb{Z}[\pi])} \cong H^2(\pi;\mathbb{Z}[\pi]) \otimes Z^{w_1(M)}$, (where the tensor product is over Z and has the diagonal π-action). Therefore $Z^u = Z^v \otimes Z^{w_1(M)}$ and so $u = w_1(M) + v$. □

Note that u and $w_1(M)$ are constrained by the further conditions that $K = \mathrm{Ker}(u)$ is torsion free and $\mathrm{Ker}(w_1(M))$ has infinite abelianization if $\chi(M) \leq 0$. If $\pi < Isom(\mathbb{X}^2)$ is a plane motion group then $v(g)$ detects whether $g \in \pi$ preserves the orientation of X^2. If π is torsion free then M is homotopy equivalent to the total space of an S^2-bundle ξ over an aspherical closed surface B, and the equation $u = w_1(M) + v$ follows from Lemma 5.11.

Let β^u be the Bockstein operator associated with the exact sequence of coefficients

$$0 \to Z^u \to Z^u \to \mathbb{F}_2 \to 0,$$

and let $\overline{\beta^u}$ be the composition with reduction modulo (2). In general β^u is NOT the Bockstein operator for the untwisted sequence $0 \to Z \to Z \to \mathbb{F}_2 \to 0$, and $\overline{\beta^u}$ is not Sq^1, as can be seen already for cohomology of the group $Z/2Z$ acting nontrivially on Z.

Lemma 10.4 *Let M be a closed 4-manifold with fundamental group π and such that $\pi_2(M) \cong Z$. If π has nontrivial torsion $H^s(M;\mathbb{F}_2) \cong H^s(\pi;\mathbb{F}_2)$ for $s \leq 2$. The Bockstein operator $\beta^u : H^2(\pi;\mathbb{F}_2) \to H^3(\pi;Z^u)$ is onto, and reduction mod 2 from $H^3(\pi;Z^u)$ to $H^3(\pi;\mathbb{F}_2)$ is a monomorphism. The restriction of $k_1(M)$ to each subgroup of order 2 is nontrivial. Its image in $H^3(M;Z^u)$ is 0.*

Proof Most of these assertions hold vacuously if π is torsion free, so we may assume that π has an element of order 2. Then M has a covering space $\hat{M}$ homotopy equivalent to RP^2, and so the mod-2 Hurewicz homomorphism from $\pi_2(M)$ to $H_2(M;\mathbb{F}_2)$ is trivial, since it factors through $H_2(\hat{M};\mathbb{F}_2)$. Since we may construct $K(\pi,1)$ from M by adjoining cells to kill the higher homotopy of M the first assertion follows easily.

The group $H^3(\pi;Z^u)$ has exponent dividing 2, since the composition of restriction to $H^3(K;\mathbb{Z}) = 0$ with the corestriction back to $H^3(\pi;Z^u)$ is multiplication by the index $[\pi : K]$. Consideration of the long exact sequence associated to the coefficient sequence shows that β^u is onto. If $f : Z/2Z \to \pi$ is a monomorphism then $f^*k_1(M)$ is the first k-invariant of $\widetilde{M}/f(Z/2Z) \simeq RP^2$, which generates $H^3(Z/2Z;\pi_2(M)) = Z/2Z$. The final assertion is clear. □

Theorem 10.5 *Let M be a closed 4-manifold such that $\pi_2(M) \cong Z$. Then there are only finitely many homotopy types of such manifolds with fundamental group π and orientation character $w_1(M)$. If $w_1(M) \neq 0$ there are at most two such homotopy types with given first k-invariant.*

Proof By the lemma, the action of π on $\pi_2(M)$ is determined by $w_1(M)$. As $c.d.\rho = 2$, an LHSSS calculation shows that $H^3(\pi;\pi_2(M))$ is finite, so there are only finitely many possible k-invariants. The action and the first k-invariant $k_1(M)$ determine $P = P_2(M)$, the second stage of the Postnikov tower for M. Let $\widetilde{P} \simeq K(Z,2)$ denote the universal covering space of P.

As $f_M : M \to P$ is 3-connected we may define a class w in $H^1(P;\mathbb{Z}/2\mathbb{Z})$ by $f_M^*w = w_1(M)$. Let $S_4^{PD}(P)$ be the set of "polarized" PD_4-complexes (X,f) where $f : X \to P$ is 3-connected and $w_1(X) = f^*w$, modulo homotopy equivalence over P. (Note that as π is one-ended the universal cover of X is homotopy equivalent to S^2). Let $[X]$ be the fundamental class of X in $H_4(X;Z^w)$. It follows as in Lemma 1.3 of [HK88] that given two such polarized complexes (X,f) and (Y,g) there is a map $h : X \to Y$ with $gh = f$ if and only if $f_*[X] = g_*[Y]$ in $H_4(P;Z^w)$. Since $\widetilde{X} \simeq \widetilde{Y} \simeq S^2$ and f and g are 3-connected such a map h must be a homotopy equivalence.

From the Cartan-Leray homology spectral sequence for the classifying map $c_P : P \to K = K(\pi,1)$ we see that there is an exact sequence

$$0 \to H_2(\pi;H_2(\widetilde{P}) \otimes Z^w)/im(d^2_{5,0}) \to H_4(P;Z^w)/J \to H_4(\pi;Z^w),$$

where $J = H_0(\pi;H_4(\widetilde{P};\mathbb{Z}) \otimes Z^w)/im(d^2_{3,2} + d^4_{5,0})$ is the image of $H_4(\widetilde{P};\mathbb{Z}) \otimes Z^w$ in $H_4(P;Z^w)$. On comparing this spectral sequence with that for c_X we see that f induces an isomorphism from $H_4(X;Z^w)$ to $H_4(P;Z^w)/J$. We also see that $H_3(f;Z^w)$ is an isomorphism. Hence the cokernel of $H_4(f;Z^w)$ is $H_4(P,X;Z^w) \cong H_0(\pi;H_4(\widetilde{P},\widetilde{X};\mathbb{Z}) \otimes Z^w)$, by the exact sequence of homology with coefficients Z^w for the pair (P,X). Since $H_4(\widetilde{P},\widetilde{X};\mathbb{Z}) \cong Z$ as a π-module this cokernel is Z if $w = 0$ and $Z/2Z$ otherwise. Hence $J \cong \mathrm{Coker}(H_4(f;Z^w))$. (Note that $H_2(\pi;H_2(\widetilde{P}) \otimes Z^w)/(torsion) \cong Z$ and the groups $H_p(\pi;Z^w)$ are finite if $p > 2$). Thus if $w \neq 0$ there are at most two possible values for $f_*[X]$, up to sign. If $w = 0$ we shall show that there are only finitely many orbits of fundamental classes of such polarized complexes under the action of the group

G of (based) self homotopy equivalences of P which induce the identity on π and $\pi_2(P)$.

The cohomology spectral sequence for c_P gives rise to an exact sequence

$$0 \to H^2(\pi; Z^u) \to H^2(P; Z^u) \to H^0(\pi; H^2(\widetilde{P}; \mathbb{Z}) \otimes Z^u) \cong Z \to H^3(\pi; Z^u).$$

Note that $H^2(\pi; Z^u) \cong Z$ modulo 2-torsion (since $w = 0$), $H^2(\widetilde{P}; \mathbb{Z}) \cong Z^u$ and $Z^u \otimes Z^u \cong Z$ as π-modules. Moreover the right hand map is the transgression, with image generated by $k_1(M)$. There is a parallel exact sequence with rational coefficients

$$0 \to H^2(\pi; Q^u) \cong Q \to H^2(P; Q^u) \to H^0(\pi; H^2(\widetilde{P}; \mathbb{Z}) \otimes Q^u) \cong Q \to 0.$$

Thus $H^2(P; Q^u)$ has a $\mathbb{Q}$-basis t, z in the image of $H^2(P; Z^u)$ such that t is the image of a generator of $H^2(\pi; Z^u)/(torsion)$ and z has nonzero restriction to $H^2(\widetilde{P}; \mathbb{Z})$. The spectral sequence also gives an exact sequence

$$0 \to H^2(\pi; H^2(\widetilde{P}; \mathbb{Q})) \to H^4(P; \mathbb{Q}) \to H^0(\pi; H^4(\widetilde{P}; \mathbb{Q})) \cong Q \to 0.$$

(Note that $H^2(\widetilde{P}; \mathbb{Q}) \cong Q^u$ as a $\mathbb{Q}[\pi]$-module). Since $cd_{\mathbb{Q}}\pi = 2$ we have $t^2 = 0$ in $H^4(P; Q^u \otimes Q^u) = H^4(P; \mathbb{Q})$; since $\widetilde{P} \simeq K(Z, 2)$ we have $z^2 \neq 0$. Thus tz, z^2 is a $\mathbb{Q}$-basis for $H^4(P; \mathbb{Q})$. A self homotopy h in G induces the identity on π, and its lift to a self map of $\widetilde{P}$ is homotopic to the identity. Hence $h^*t = t$ and $h^*z \equiv z$ modulo $\mathbb{Q}t$. Nevertheless we shall see that the action of G on $H^2(P; Q^u)$ is nontrivial.

Suppose first that $u = 0$, so π is an orientable surface group and $k_1(M) = 0$. Then $P \simeq K(\pi, 1) \times K(Z, 2)$ and $G \cong [K(\pi, 1), K(Z, 2)]$. Let $f : K(\pi, 1) \to K(Z, 2)$ be a map which induces an isomorphism on H^2 and fix a generator ζ for $H^2(K(Z, 2); \mathbb{Z})$. Then $t = pr_1^* f^* \zeta$ and $z = pr_2^* \zeta$ freely generate $H^2(P; \mathbb{Z})$, and so tz, z^2 freely generate $H^4(P; \mathbb{Z})$. Each $g \in [K(\pi, 1), K(Z, 2)]$ determines a self homotopy equivalence $\tilde{g} : P \to P$ by $\tilde{g}(k, n) = (k, g(k).n)$, where $K(Z, 2) = \Omega K(Z, 3)$ has the natural loop multiplication. Clearly $\tilde{g}$ is in G, and all elements of G are of this form [Ts80]. Let $d : G \to Z$ be the isomorphism determined by the equation $g^*\zeta = d(g) f^*\zeta$. Then $\tilde{g}^*t = (f pr_1 \tilde{g})^*\zeta = t$ and $\tilde{g}^*z = (pr_2 \tilde{g})^*\zeta = (g pr_1)^*\zeta + pr_2^*\zeta = pr_1^*(g^*\zeta) + z = z + d(g)t$. On taking cup products we have $h^*(tz) = tz$ and $h^*(z^2) = z^2 + 2d(g)tz$. On passing to homology we see that there are two G-orbits of elements in $H_4(P; \mathbb{Z})$ whose images generate $H_4(P; \mathbb{Z})/J$.

In general let P_K denote the covering space corresponding to the subgroup K, and let G_K be the image of G in the group of self homotopy equivalences of P_K. Then lifting self homotopy equivalences defines a homomorphism from G to G_K, which by [Ts80] may be identified with the restriction from $H^2(\pi; Z^u)$

to $H^2(K;\mathbb{Z}) \cong Z$, which has image of index ≤ 2. Moreover the projection induces an isomorphism from $H^4(P;\mathbb{Q})$ to $H^4(P_K;\mathbb{Q})$. Hence the action of G on $H_4(P;\mathbb{Z})/(torsion) \cong Z^2$ is nontrivial, and so there are only finitely many G-orbits of elements whose images generate $H_4(P;\mathbb{Z})/J$. This proves the theorem. $\square$

As a consequence of Lemma 10.4 we may assume that the cohomology class z in the above theorem restricts to 2 times a generator of $H^2(\widetilde{P};\mathbb{Z})$, if $k_1(M) \neq 0$. A closer study of the action of G on $H^2(P;Z^u)$ suggests that in general there are at most 4 homotopy types with given π, w_1 and k-invariant. However we have not succeeded in proving this.

Significant features of the duality pairing such as $w_2(M)$ are not reflected in the Postnikov 2-stage. If π is torsion free $k_1(M) = 0$ and w_2 is the only other invariant needed. For then π is a surface group and each such manifold is homotopy equivalent to the total space of an S^2-bundle. There are two such bundle spaces for each group and orientation character, distinguished by the value of $w_2(M)$.

For RP^2-bundles $u = w_1$ and $\pi \cong K \times (Z/2Z)$. The element of order 2 in π is unique, and the splitting is unique up to composition with an automorphism of π. There are two such bundle spaces for each surface group K, again distinguished by $w_2(M)$. Can it be seen *a priori* that the k-invariant must be standard?

If $w_1(M) = 0$, $w_2(M)$ restricts to 0 in $H^2(K;\mathbb{F}_2)$, $u \neq 0$ and $H^3(u;Z^u)$ is 0 then M is homotopy equivalent to the total space of a surface bundle over RP^2, by Theorem 5.23.

In general, we may view the classifying map $c_M : M \to K(\pi,1)$ as a fibration with fibre S^2. Fix a homotopy equivalence $\widetilde{M} \simeq S^2$. Then the action of π on $\widetilde{M}$ determines a homomorphism $j : \pi \to Homeo(\widetilde{M}) \to E(S^2)$, and the fibration c_M is induced from the universal S^2-fibration over $BE(S^2)$ by the map $Bj : K(\pi,1) \to BE(S^2)$. The orientation character of this fibration is $w_1(c_M) = u$, and is induced by the composite $c_{BE(S^2)}Bj : K(\pi,1) \to K(\pi_0(E(S^2)),1)$. The (twisted) Euler class is the first obstruction to a cross-section of c_M, and so equals $k_1(M)$. Hence the reduction modulo (2) of $k_1(M)$ is $w_3(c_M) \in H^3(\pi;\mathbb{F}_2)$. Calculation show that $\beta^u : H^2(BE(S^2);\mathbb{F}_2) \to H^3(BE(S^2);Z^u)$ is an isomorphism, and so $w_3(c_M)$ also determines $k_1(M)$. In particular, if j factors through $\{\pm I\} < O(3)$ then $k_1(M) = \beta^u(U^2)$, where $U \in H^1(\pi;\mathbb{F}_2)$ is the cohomology class determined by u. (This is so when M is a $\mathbb{S}^2 \times \mathbb{X}^2$-manifold and π is generated by elements of order 2, by Lemma 10.6 below).

As M is finitely covered by a cartesian product $S^2 \times B$, where B is a closed orientable surface, $w_2(M)$ restricts to 0 in $H^2(\widetilde{M};\mathbb{F}_2)$ and so is induced from π. The Wu formulae for M then imply that the total Stiefel-Whitney class $w(M)$ is induced from π. It can be shown that $c_M^*(w(c_M))$ is determined by $w(M)$ and π; unfortunately as $c_M^*(w_3(c_M)) = 0$ (by exactness of the Gysin sequence for c_M) we do not know whether $k_1(M)$ is also determined by these invariants.

Is the homotopy type of M determined by $\pi_1(M)$, $w(M)$ and $k_1(M)$? What is the role of the exotic class in $H^3(BE(S^2);\mathbb{F}_2)$? Are there any PD_4-complexes M with $\widetilde{M} \simeq S^2$ and such that the image of this class under $(Bj)^*$ is nonzero?

10.3 Bundle spaces are geometric

All $\mathbb{S}^2\times\mathbb{X}^2$-manifolds are total spaces of orbifold bundles over $\mathbb{X}^2$-orbifolds. We shall determine the S^2- and RP^2-bundle spaces among them in terms of their fundamental groups, and then show that all such bundle spaces are geometric.

Lemma 10.6 *Let $J = (A,\theta) \in O(3) \times Isom(\mathbb{X}^2)$ be an isometry of order 2 which is fixed point free. Then $A = -I$. If moreover J is orientation reversing then $\theta = id_X$ or has a single fixed point.*

Proof Since any involution of R^2 (such as θ) must fix a point, a line or be the identity, $A \in O(3)$ must be a fixed point free involution, and so $A = -I$. If J is orientation reversing then θ is orientation preserving, and so must fix a point or be the identity. ∎

Theorem 10.7 *Let M be a closed $\mathbb{S}^2 \times \mathbb{X}^2$-manifold with fundamental group π. Then*

1. *M is the total space of an orbifold bundle with base an $\mathbb{X}^2$-orbifold and general fibre S^2 or RP^2;*
2. *M is the total space of an S^2-bundle over a closed aspherical surface if and only if π is torsion free;*
3. *M is the total space of an RP^2-bundle over a closed aspherical surface if and only if $\pi \cong (Z/2Z) \times K$, where K is torsion free.*

Proof (1) The group π is a discrete subgroup of the isometry group $Isom(\mathbb{S}^2 \times \mathbb{X}^2) = O(3) \times Isom(\mathbb{X}^2)$ which acts freely and cocompactly on $S^2 \times R^2$. In particular, $N = \pi \cap (O(3) \times \{1\})$ is finite and acts freely on S^2, so has order ≤ 2. Let p_1 and p_2 be the projections of $Isom(\mathbb{S}^2 \times \mathbb{X}^2)$ onto $O(3)$ and $Isom(\mathbb{X}^2)$, respectively. Then $p_2(\pi)$ is a discrete subgroup of $Isom(\mathbb{X}^2)$ which acts cocompactly on R^2, and so has no nontrivial finite normal subgroup. Hence N is the maximal finite normal subgroup of π. Projection of $S^2 \times R^2$ onto R^2 induces an orbifold bundle projection of M onto $p_2(\pi)\backslash R^2$ and general fibre $N\backslash S^2$. If $N \neq 1$ then $N \cong Z/2Z$ and $\pi \cong (Z/2Z) \times K$, where K is a PD_2-group, by Theorem 5.14.

(2) The condition is clearly necessary. (See Theorem 5.10). The kernel of the projection of π onto its image in $Isom(\mathbb{X}^2)$ is the subgroup N. Therefore if π is torsion free it is isomorphic to its image in $Isom(\mathbb{X}^2)$, which acts freely on R^2. The projection $\rho : S^2 \times R^2 \to R^2$ induces a map $r : M \to \pi\backslash R^2$, and we have a commutative diagram:

$$\begin{tikzcd} S^2 \times R^2 \arrow[r, "\rho"] \arrow[d, "f"] & R^2 \arrow[d, "\bar{f}"] \\ M = \pi\backslash(S^2 \times R^2) \arrow[r, "r"] & \pi\backslash R^2 \end{tikzcd}$$

where f and $\bar{f}$ are covering projections. It is easily seen that r is an S^2-bundle projection.

(3) The condition is necessary, by Theorem 5.16. Suppose that it holds. Then K acts freely and properly discontinuously on R^2, with compact quotient. Let g generate the torsion subgroup of π. Then $p_1(g) = -I$, by Lemma 10.6. Since $p_2(g)^2 = \mathrm{id}_{R^2}$ the fixed point set $F = \{x \in R^2 \mid p_2(g)(x) = x\}$ is nonempty, and is either a point, a line, or the whole of R^2. Since $p_2(g)$ commutes with the action of K on R^2 we have $KF = F$, and so K acts freely and properly discontinuously on F. But K is neither trivial nor infinite cyclic, and so we must have $F = R^2$. Hence $p_2(g) = \mathrm{id}_{R^2}$. The result now follows, as $K\backslash(S^2 \times R^2)$ is the total space of an S^2-bundle over $K\backslash R^2$, by part (1), and g acts as the antipodal involution on the fibres. □

If the $\mathbb{S}^2 \times \mathbb{X}^2$-manifold M is the total space of an S^2-bundle ξ then $w_1(\xi)$ is detected by the determinant: $\det(p_1(g)) = (-1)^{w_1(\xi)(g)}$ for all $g \in \pi$.

The total space of an RP^2-bundle over B is the quotient of its orientation double cover (which is an S^2-bundle over B) by the fibrewise antipodal involution and so there is a bijective correspondance between orientable S^2-bundles over B and RP^2-bundles over B.

Let $(A, \beta, C) \in O(3) \times E(2) = O(3) \times (R^2 \tilde{\times} O(2))$ be the $\mathbb{S}^2 \times \mathbb{E}^2$-isometry which sends $(v, x) \in S^2 \times R^2$ to $(Av, Cx + \beta)$.

Theorem 10.8 *Let M be the total space of an S^2- or RP^2-bundle over T or Kb. Then M admits the geometry $\mathbb{S}^2 \times \mathbb{E}^2$.*

Proof Let $R_i \in O(3)$ be the reflection of R^3 which changes the sign of the i^{th} coordinate, for $i = 1, 2, 3$. If A and B are products of such reflections then the subgroups of $Isom(\mathbb{S}^2 \times \mathbb{E}^2)$ generated by $\alpha = (A, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), I)$ and $\beta = (B, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), I)$ are discrete, isomorphic to Z^2 and act freely and cocompactly on $S^2 \times R^2$. Taking

(1) $A = B = I$;

(2) $A = R_1R_2, B = R_1R_3$;

(3) $A = R_1, B = I$; and

(4) $A = R_1, B = R_1R_2$

gives four S^2-bundles η_i over the torus. If instead we use the isometries $\alpha = (A, \left(\begin{smallmatrix}\frac{1}{2}\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}1 & 0\\0 & -1\end{smallmatrix}\right))$ and $\beta = (B, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), I)$ we obtain discrete subgroups isomorphic to $Z \times_{-1} Z$ which act freely and cocompactly. Taking

(1) $A = R_1, B = I$;

(2) $A = R_1, B = R_2R_3$;

(3) $A = I, B = R_1$;

(4) $A = R_1R_2, B = R_1$;

(5) $A = B = I$; and

(6) $A = I, B = R_1R_2$

gives six S^2-bundles ξ_i over the Klein bottle.

To see that these are genuinely distinct, we check first the fundamental groups, then the orientation character of the total space; consecutive pairs of generators determine bundles with the same orientation character, and we distinguish these by means of the second Stiefel-Whitney classes, by computing the self-intersections of cross-sections. (See Lemma 5.11.(2). We shall use the stereographic projection of $S^2 \subset R^3 = C \times R$ onto $\widehat{C} = C \cup \{\infty\}$, to identify the reflections $R_i : S^2 \to S^2$ with the antiholomorphic involutions:

$$z \xmapsto{R_1} \overline{z}, \quad z \xmapsto{R_2} -\overline{z}, \quad z \xmapsto{R_3} \overline{z}^{-1}.$$

Let $\mathfrak{T} = \{(s,t) \in R^2 | 0 \le s, t \le 1\}$ be the fundamental domain for the standard action of Z^2 on R^2. A section $\sigma : \mathfrak{T} \to S^2 \times R^2$ of the projection to R^2 over $\mathfrak{T}$ such that $\sigma(1,t) = \alpha\sigma(0,t)$ and $\sigma(s,1) = \beta\sigma(s,0)$ induces a section of the bundle ξ_i.

As the orientable cases (η_1, η_2, ξ_1 and ξ_2) have been treated in [Ue90] we may concentrate on the nonorientable cases. In the case of η_3 each fixed point P of A determines a section σ_P with $\sigma_P(s,t) = (P,s,t)$. Since A fixes a circle on S^2 it follows that sections determined by distinct fixed points are isotopic and disjoint. Therefore $\sigma \cdot \sigma = 0$, so $v_2(M) = 0$ and hence $w_2(\eta_3) = 0$.

We may define a 1-parameter family of sections for η_4 by

$$\sigma_\lambda(s,t) = ((1-\lambda)(2t-1) + \lambda(4t^2-2))e^{\pi i \lambda (s-\frac{1}{2})}.$$

Now σ_0 and σ_1 intersect transversely in a single point, corresponding to $s = 1/2$ and $t = (1+\sqrt{5})/4$. Hence $\sigma \cdot \sigma = 1$, so $v_2(M) \neq 0$ and $w_2(\eta_4) \neq 0$.

The remaining cases correspond to S^2-bundles over Kb with nonorientable total space. We now take $\mathfrak{K} = \{(s,t) \in R^2 | 0 \leq s \leq 1, |t| \leq \frac{1}{2}\}$ as the fundamental domain for the action of $Z \times_{-1} Z$ on R^2. In this case it suffices to find $\sigma : \mathfrak{K} \to S^2 \times R^2$ such that $\sigma(1,t) = \alpha\sigma(0,-t)$ and $\sigma(s,\frac{1}{2}) = \beta\sigma(s,-\frac{1}{2})$.

The cases of ξ_3 and ξ_5 are similar to that of η_3: there are obvious one-parameter families of disjoint sections, and so $w_2(\xi_3) = w_2(\xi_5) = 0$. However $w_1(\xi_3) \neq w_1(\xi_5)$. (In fact ξ_5 is the product bundle).

The functions $\sigma_\lambda(s,t) = \lambda(2s-1+it)$ define a 1-parameter family of sections for ξ_4 such that σ_0 and σ_1 intersect transversely in one point, so that $\sigma \cdot \sigma = 1$. Hence $v_2(M) \neq 0$ and so $w_2(\xi_4) \neq 0$.

For ξ_6 the functions $\sigma_\lambda(s,t) = \lambda(2s-1)t + i(1-\lambda)(4t^2-1)$ define a 1-parameter family of sections such that σ_0 and $\sigma_1(s,t)$ intersect transversely in one point, so that $\sigma \cdot \sigma = 1$. Hence $v_2(M) \neq 0$ and so $w_2(\xi_6) \neq 0$.

Thus these bundles are all distinct, and so all S^2-bundles over T or Kb are geometric of type $\mathbb{S}^2 \times \mathbb{E}^2$.

Adjoining the fixed point free involution $(-I, 0, I)$ to any one of the above ten sets of generators for the S^2-bundle groups amounts to dividing out the S^2 fibres by the antipodal map and so we obtain the corresponding RP^2-bundles. (Note that there are just four such RP^2-bundles - but each has several distinct double covers which are S^2-bundles). □

Theorem 10.9 *Let M be the total space of an S^2- or RP^2-bundle over a closed hyperbolic surface. Then M admits the geometry $\mathbb{S}^2 \times \mathbb{H}^2$.*

Proof Let T_g be the closed orientable surface of genus g, and let $\mathfrak{T}^g \subset \mathbb{H}^2$ be a $2g$-gon representing the fundamental domain of T_g. The map $\Omega : \mathfrak{T}^g \to \mathfrak{T}$

that collapses $2g-4$ sides of $\mathfrak{T}^g$ to a single vertex in the rectangle $\mathfrak{T}$ induces a degree 1 map $\widehat{\Omega}$ from T_g to T that collapses $g-1$ handles on T^g to a single point on T. (We may assume the induced epimorphism from

$$\pi_1(T_g) = \langle a_1, b_1, \dots, a_g, b_g \mid \Pi_{i=1}^g [a_i, b_i] = 1 \rangle$$

to Z^2 kills the generators a_j, b_j for $j > 1$). Hence given an S^2-bundle ξ over T with total space $M_\xi = \Gamma_\xi \backslash (S^2 \times E^2)$, where

$$\Gamma_\xi = \{(\xi(h), h) \mid h \in \pi_1(T)\} \leq Isom(S^2 \times E^2)$$

and $\xi : \mathbb{Z}^2 \to O(3)$ is as in Theorem 10.8, the pullback $\widehat{\Omega}^*(\xi)$ is an S^2-bundle over T_g, with total space $M_{\xi\Omega} = \Gamma_{\xi\Omega} \backslash (S^2 \times \mathbb{H}^2)$, where $\Gamma_{\xi\Omega} = \{(\xi\Omega(h), h) \mid h \in \Pi_1(T^g)\} \leq Isom(S^2 \times \mathbb{H}^2)$. As $\widehat{\Omega}$ is of degree 1 it induces monomorphisms in cohomology, so $w(\xi)$ is nontrivial if and only if $w(\widehat{\Omega}^*(\xi)) = \widehat{\Omega}^* w(\xi)$ is nontrivial. Hence all S^2-bundles over T^g for $g \geq 2$ are geometric of type $S^2 \times \mathbb{H}^2$.

Suppose now that B is the closed surface $\#^3 RP^2 = T \# RP^2 = Kb \# RP^2$. Then there is a map $\widehat{\Omega} : T\#RP^2 \to RP^2$ that collapses the torus summand to a single point. This map $\widehat{\Omega}$ again has degree 1 and so induces monomorphisms in cohomology. In particular $\widehat{\Omega}^*$ preserves the orientation character, that is $w_1(\widehat{\Omega}^*(\xi)) = \widehat{\Omega}^* w_1(RP^2) = w_1(B)$, and is an isomorphism on H^2. We may pull back the four S^2-bundles ξ over RP^2 along $\widehat{\Omega}$ to obtain the four bundles over B with first Stiefel-Whitney class $w_1(\widehat{\Omega}^*\xi)$ either 0 or $w_1(B)$.

Similarly there is a map $\widehat{\Upsilon} : Kb\#RP^2 \to RP^2$ that collapses the Klein bottle summand to a single point. This map $\widehat{\Upsilon}$ has degree 1 mod 2 so that $\widehat{\Upsilon}^* w_1(RP^2)$ has nonzero square since $w_1(RP^2)^2 \neq 0$. Note that in this case $\widehat{\Upsilon}^* w_1(RP^2) \neq w_1(B)$. Hence we may pull back the two S^2-bundles ξ over RP^2 with $w_1(\xi) = w_1(RP^2)$ to obtain a further two bundles over B with $w_1(\widehat{\Upsilon}^*(\xi))^2 = \widehat{\Upsilon}^* w_1(\xi)^2 \neq 0$, as $\widehat{\Upsilon}$ is a ring monomorphism.

There is again a map $\widehat{\Theta} : Kb\#RP^2 \to Kb$ that collapses the Klein bottle summand to a single point. Once again $\widehat{\Theta}$ is of degree 1 mod 2 so that we may pull back the two S^2-bundles ξ over Kb with $w_1(\xi) = w_1(Kb)$ along $\widehat{\Theta}$ to obtain the remaining two S^2-bundles over B. These two bundles $\widehat{\Theta}^*(\xi)$ have $w_1(\widehat{\Theta}^*(\xi)) \neq 0$ but $w_1(\widehat{\Theta}^*(\xi))^2 = 0$; as $w_1(Kb) \neq 0$ but $w_1(Kb)^2 = 0$ and $\widehat{\Theta}^*$ is a monomorphism.

Similar arguments apply to bundles over $\#^n RP^2$ where $n > 3$.

Thus all S^2-bundles over all closed aspherical surfaces are geometric. Furthermore since the antipodal involution of a geometric S^2-bundle is induced by an isometry $(-I, id_{\mathbb{H}^2}) \in O(3) \times Isom(\mathbb{H}^2)$ we have that all RP^2-bundles over closed aspherical surfaces are geometric. □

An alternative route to Theorems 10.8 and 10.9 would be to first show that orientable 4-manifolds which are total spaces of S^2-bundles are geometric, then deduce that RP^2-bundles are geometric (as above); and finally observe that every S^2-bundle space double covers an RP^2-bundle space.

The other $\mathbb{S}^2 \times \mathbb{X}^2$-manifolds are orbifold bundles over flat or hyperbolic orbifolds, with general fibre S^2. In other words, they have codimension-2 foliation whose leaves are homeomorphic to S^2 or RP^2. Is every such closed 4-manifold geometric?

If $\chi(F) < 0$ or $\chi(F) = 0$ and $\partial = 0$ then every F-bundle over RP^2 is geometric, by Lemma 5.21 and the remark following Theorem 10.2.

However it is not generally true that the projection of $S^2 \times X$ onto S^2 induces an orbifold bundle projection from M to an $\mathbb{S}^2$-orbifold. For instance, if ρ and ρ' are rotations of S^2 about a common axis which generate a rank 2 abelian subgroup of $SO(3)$ then $(\rho,(1,0))$ and $(\rho',(0,1))$ generate a discrete subgroup of $SO(3) \times R^2$ which acts freely, cocompactly and isometrically on $S^2 \times R^2$. The orbit space is homeomorphic to $S^2 \times T$. (It is an orientable S^2-bundle over the torus, with disjoint sections, detemined by the ends of the axis of the rotations). Thus it is Seifert fibred over S^2, but the fibration is not canonically associated to the metric structure, for $\langle \rho, \rho' \rangle$ does not act properly discontinuously on S^2.

10.4 Fundamental groups of $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds

We shall show first that if M is a closed 4-manifold any two of the conditions "$\chi(M) = 0$", "$\pi_1(M)$ is virtually Z^2" and "$\pi_2(M) \cong Z$" imply the third, and then determine the possible fundamental groups.

Theorem 10.10 *Let M be a closed 4-manifold with fundamental group π. Then the following conditions are equivalent:*

(1) *π is virtually Z^2 and $\chi(M) = 0$;*

(2) *π has an infinite restrained normal subgroup and $\pi_2(M) \cong Z$;*

(3) *$\chi(M) = 0$ and $\pi_2(M) \cong Z$; and*

(4) *M has a covering space of degree dividing 4 which is homeomorphic to $S^2 \times T$.*

(5) *M is virtually homeomorphic to an $\mathbb{S}^2 \times \mathbb{E}^2$-manifold.*

Proof If π is virtually a PD_2-group and either $\chi(\pi) = 0$ or π has an infinite restrained normal subgroup then π is virtually Z^2. Hence the equivalence of these conditions follows from Theorem 10.1, with the exception of the assertions regarding homeomorphisms, which then follow from Theorem 6.11. □

We shall assume henceforth that the conditions of Theorem 10.10 hold, and shall show next that there are nine possible groups. Seven of them are 2-dimensional crystallographic groups, and we shall give also the name of the corresponding $\mathbb{E}^2$-orbifold, following Appendix A of [Mo]. (The restriction on finite subgroups eliminates the remaining ten $\mathbb{E}^2$-orbifold groups from consideration).

Theorem 10.11 *Let M be a closed 4-manifold such that $\pi = \pi_1(M)$ is virtually Z^2 and $\chi(M) = 0$. Let A and F be the maximal abelian and maximal finite normal subgroups (respectively) of π. If π is torsion free then either*

(1) *$\pi = A \cong Z^2$ (the torus); or*

(2) *$\pi \cong Z \times_{-1} Z$ (the Klein bottle).*
If $F = 1$ but π has nontrivial torsion and $[\pi : A] = 2$ then either

(3) *$\pi \cong D \times Z \cong (Z \oplus (Z/2Z)) *_Z (Z \oplus (Z/2Z))$, with the presentation $\langle s, x, y \mid x^2 = y^2 = 1,\ sx = xs,\ sy = ys\rangle$ (the silvered annulus); or*

(4) *$\pi \cong D\tilde{\times} Z \cong Z *_Z (Z \oplus (Z/2Z))$, with the presentation $\langle t, x \mid x^2 = 1,\ t^2x = xt^2\rangle$ (the silvered Möbius band); or*

(5) *$\pi \cong (Z^2) \times_{-I} (Z/2Z) \cong D *_Z D$, with the presentations $\langle s, t, x \mid x^2 = 1,\ xsx = s^{-1},\ xtx = t^{-1},\ st = ts\rangle$ and (setting $y = xt$) $\langle s, x, y \mid x^2 = y^2 = 1,\ xsx = ysy = s^{-1}\rangle$ (the pillowcase $S(2222)$).*
If $F = 1$ and $[\pi : A] = 4$ then either

(6) *$\pi \cong D *_Z (Z \oplus (Z/2Z))$, with the presentations $\langle s, t, x \mid x^2 = 1,\ xsx = s^{-1},\ xtx = t^{-1},\ tst^{-1} = s^{-1}\rangle$ and (setting $y = xt$) $\langle s, x, y \mid x^2 = y^2 = 1,\ xsx = s^{-1},\ ys = sy\rangle$ ($D(22)$); or*

(7) *$\pi \cong Z *_Z D$, with the presentations $\langle r, s, x \mid x^2 = 1,\ xrx = r^{-1},\ xsx = rs^{-1},\ srs^{-1} = r^{-1}\rangle$ and (setting $t = xs$) $\langle t, x \mid x^2 = 1,\ xt^2x = t^{-2}\rangle$ ($P(22)$).*
If F is nontrivial then either

(8) *$\pi \cong Z^2 \oplus (Z/2Z)$; or*

(9) *$\pi \cong (Z \times_{-1} Z) \times (Z/2Z)$.*

Proof Let $u : \pi \to \{\pm 1\} = Aut(\pi_2(M))$ be the natural homomorphism. Since $\mathrm{Ker}(u)$ is torsion free it is either Z^2 or $Z \times_{-1} Z$; since it has index at most 2 it follows that $[\pi : A]$ divides 4 and that F has order at most 2. If $F = 1$ then $A \cong Z^2$ and π/A acts effectively on A, so π is a 2-dimensional crystallographic group. If $F \neq 1$ then it is central in π and u maps F isomorphically to $Z/2Z$, so $\pi \cong (Z/2Z) \times \mathrm{Ker}(u)$. $\square$

Each of these groups may be realised geometrically, by Theorem 10.2. It is easy to see that any $\mathbb{S}^2 \times E^2$-manifold whose fundamental group has infinite abelianization is a mapping torus, and hence is determined up to diffeomorphism by its homotopy type. (See Theorems 10.8 and 10.12). We shall show next that there are four affine diffeomorphism classes of $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds whose fundamental groups have finite abelianization.

Let Ω be a discrete subgroup of $Isom(\mathbb{S}^2 \times \mathbb{E}^2) = O(3) \times E(2)$ which acts freely and cocompactly on $S^2 \times R^2$. If $\Omega \cong D *_Z D$ or $D *_Z (Z \oplus (Z/2Z))$ it is generated by elements of order 2, and so $p_1(\Omega) = \{\pm I\}$, by Lemma 10.6. Since $p_2(\Omega) < E(2)$ is a 2-dimensional crystallographic group it is determined up to conjugacy in $Aff(2) = R^2 \tilde{\times} GL(2, \mathbb{R})$ by its isomorphism type, Ω is determined up to conjugacy in $O(3) \times Aff(2)$ and the corresponding geometric 4-manifold is determined up to affine diffeomorphism.

Although $Z *_Z D$ is not generated by involutions, a similar argument applies. The isometries $T = (\tau, \left(\begin{smallmatrix} 0 \\ \frac{1}{2} \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right))$ and $X = (-I, \left(\begin{smallmatrix} \frac{1}{2} \\ \frac{1}{2} \end{smallmatrix}\right), -I)$ generate a discrete subgroup of $Isom(\mathbb{S}^2 \times \mathbb{E}^2)$ isomorphic to $Z *_Z D$ and which acts freely and cocompactly on $S^2 \times R^2$, provided $\tau^2 = I$. Since $x^2 = (xt^2)^2 = 1$ this condition is necessary, by Lemma 10.6. We shall see below that we may assume that T is orientation preserving, i.e., that $\det(\tau) = -1$. (The isometries T^2 and XT generate $\mathrm{Ker}(u)$). Thus there are two affine diffeomorphism classes of such manifolds, corresponding to the choices $\tau = -I$ or R_3.

None of these manifolds fibre over S^1, since in each case π/π' is finite. However if Ω is a $\mathbb{S}^2 \times \mathbb{E}^2$-lattice such that $p_1(\Omega) \leq \{\pm I\}$ then $\Omega \backslash (S^2 \times R^2)$ fibres over RP^2, since the map sending $(v, x) \in S^2 \times R^2$ to $[\pm v] \in RP^2$ is compatible with the action of Ω. If $p_1(\Omega) = \{\pm I\}$ the fibre is $\omega \backslash R^2$, where $\omega = \Omega \cap (\{1\} \times E(2))$; otherwise it has two components. Thus three of these four manifolds fibre over RP^2 (excepting perhaps only the case $\Omega \cong Z *_Z D$ and $R_3 \in p_1(\Omega)$).

10.5 Homotopy types of $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds

Our next result shows that if M satisfies the conditions of Theorem 10.10 and its fundamental group has infinite abelianization then it is determined up to homotopy by $\pi_1(M)$ and its Stiefel-Whitney classes.

Theorem 10.12 *Let M be a closed 4-manifold with $\chi(M) = 0$ and such that $\pi = \pi_1(M)$ is virtually Z^2. If π/π' is infinite then M is homotopy equivalent to an $\mathbb{S}^2 \times \mathbb{E}^2$-manifold which fibres over S^1.*

Proof The infinite cyclic covering space of M determined by an epimorphism $\lambda : \pi \to Z$ is a PD_3-complex, by Theorem 4.5, and therefore is homotopy equivalent to

(1) $S^2 \times S^1$ (if $\mathrm{Ker}(\lambda) \cong Z$ is torsion free and $w_1(M)|_{\mathrm{Ker}(\lambda)} = 0$),
(2) $S^2 \tilde{\times} S^1$ (if $\mathrm{Ker}(\lambda) \cong Z$ and $w_1(M)|_{\mathrm{Ker}(\lambda)} \neq 0$),
(3) $RP^2 \times S^1$ (if $\mathrm{Ker}(\lambda) \cong Z \oplus (Z/2Z)$) or
(4) $RP^3 \sharp RP^3$ (if $\mathrm{Ker}(\lambda) \cong D$).

Therefore M is homotopy equivalent to the mapping torus $M(\phi)$ of a self homotopy equivalence of one of these spaces.

The group of free homotopy classes of self homotopy equivalences $E(S^2 \times S^1)$ is generated by the reflections in each factor and the twist map, and has order 8. The group $E(S^2 \tilde{\times} S^1)$ has order 4 [KR90]. Two of the corresponding mapping tori also arise from self homeomorphisms of $S^2 \times S^1$. The other two have nonintegral w_1. The group $E(RP^2 \times S^1)$ is generated by the reflection in the second factor and by a twist map, and has order 4. As all these mapping tori are also S^2- or RP^2-bundles over the torus or Klein bottle, they are geometric by Theorem 10.8.

The group $E(RP^3 \sharp RP^3)$ is generated by the reflection interchanging the summands and the fixed point free involution (cf. page 251 of [Ba']), and has order 4. Let $\alpha = (-I, 0, \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right))$, $\beta = (I, \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right), I)$ $\gamma = (I, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), I)$ and $\delta = (-I, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), I)$ Then the subgroups generated by $\{\alpha, \beta, \gamma\}$, $\{\alpha, \beta, \delta\}$, $\{\alpha, \beta\gamma\}$ and $\{\alpha, \beta\delta\}$, respectively, give the four $RP^3 \sharp RP^3$-bundles. (Note that these may be distinguished by their groups and orientation characters). □

A T-bundle over RP^2 which does not also fibre over S^1 has fundamental group $D *_Z D$, while the group of a Kb-bundle over RP^2 which does not also fibre over S^1 is $D *_Z (Z \oplus (Z/2Z))$ or $Z *_Z D$ (assuming throughout that π is virtually Z^2).

When π is torsion free every homomorphism from π to $Z/2Z$ arises as the orientation character for some M with fundamental group π. However if $\pi \cong D \times Z$ or $D \tilde{\times} Z$ the orientation character must be trivial on all elements of order 2, while if $F \neq 1$ the orientation character is determined up to composition with an automorphism of π.

Theorem 10.13 *Let M be a closed 4-manifold with $\chi(M)=0$ and such that $\pi=\pi_1(M)$ is an extension of Z by an almost finitely presentable infinite normal subgroup N with a nontrivial finite normal subgroup F. Then M is homotopy equivalent to the mapping torus of a self homeomorphism of $RP^2 \times S^1$.*

Proof Let $\widetilde{M}$ be the universal covering space of M. Since N is infinite and finitely generated π has one end, and so $H_i(\widetilde{M};\mathbb{Z})=0$ for $i \neq 0$ or 2. Let $\Pi=\pi_2(M)=H_2(\widetilde{M};\mathbb{Z})$. We wish to show that $\Pi \cong Z$, and that $w=w_1(M)$ maps F isomorphically onto $\{\pm 1\}$. Since $\beta_1^{(2)}(\pi)=0$ by Lemma 2.1, there is an isomorphism of left $\mathbb{Z}[\pi]$-modules $\Pi \cong \overline{H^2(\pi;\mathbb{Z}[\pi])}$, by Theorem 3.4. An LHSSS argument then gives $\Pi \cong \overline{H^1(N;\mathbb{Z}[N])}$, which is a free abelian group.

The normal closure of F in π is the product of the conjugates of F, which are finite normal subgroups of N, and so is locally finite. If it is infinite then N has one end and so $H^s(\pi;\mathbb{Z}[\pi])=0$ for $s \leq 2$, by an LHSSS argument. Since locally finite groups are amenable $\beta_1^{(2)}(\pi)=0$, by Theorem 2.3, and so M must be aspherical, by Corollary 3.5.2, contradicting the hypothesis that π has nontrivial torsion. Hence we may assume that F is normal in π.

Let f be a nontrivial element of F. Since F is normal in π the centralizer $C_\pi(f)$ of f has finite index in π, and we may assume without loss of generality that F is generated by f and is central in π. It follows from the spectral sequence for the projection of $\widetilde{M}$ onto $F\backslash\widetilde{M}$ that there are isomorphisms $H_{s+3}(F;\mathbb{Z}) \cong H_s(F;\Pi)$ for all $s \geq 4$, since $F\backslash\widetilde{M}$ is a 4-dimensional complex. Here F acts trivially on Z, but we must determine its action on Π.

Now central elements n of N act trivially on $H^1(N;\mathbb{Z}[N])$ and hence via $w(n)$ on Π. (See Theorem 2.11). Thus if $w(f)=1$ the sequence

$$0 \to Z/|f|Z \to \Pi \to \Pi \to 0$$

is exact, where the right hand homomorphism is multiplication by $|f|$. As Π is torsion free this contradicts $f \neq 1$. Therefore if f is nontrivial it has order 2 and $w(f)=-1$. Hence $w: F \to \{\pm 1\}$ is an isomorphism and there is an exact sequence

$$0 \to \Pi \to \Pi \to Z/2Z \to 0,$$

where the left hand homomorphism is multiplication by 2. Since Π is a free abelian group it must be infinite cyclic, and so $\widetilde{M} \simeq S^2$. The theorem now follows from Theorems 10.10 and 10.12. □

The possible orientation characters for the groups with finite abelianization are restricted by Lemma 3.14, which implies that $\mathrm{Ker}(w_1)$ must have infinite abelianization. For $D *_Z D$ we must have $w_1(x) = w_1(y) = 1$ and $w_1(s) = 0$. For $D *_Z (Z \oplus (Z/2Z))$ we must have $w_1(s) = 0$ and $w_1(x) = 1$; since the subgroup generated by the commutator subgroup and y is isomorphic to $D \times Z$ we must also have $w_1(y) = 0$. Thus the orientation characters are uniquely determined for these groups. For $Z *_Z D$ we must have $w_1(x) = 1$, but $w_1(t)$ may be either 0 or 1. As there is an automorphism ϕ of $Z *_Z D$ determined by $\phi(t) = xt$ and $\phi(x) = x$ we may assume that $w_1(t) = 0$ in this case.

In all cases, to each choice of orientation character there corresponds a unique action of π on $\pi_2(M)$, by Lemma 10.3. However the homomorphism from π to $Z/2Z$ determining the action may differ from $w_1(M)$. (Note also that elements of order 2 must act nontrivially, by Theorem 10.1).

We shall need the following lemma about plane bundles over RP^2 in order to calculate self intersections here and in Chapter 12.

Lemma 10.14 *The total space of the R^2-bundle p over RP^2 with $w_1(p) = 0$ and $w_2(p) \neq 0$ is $S^2 \times R^2/\langle g \rangle$, where $g(s,v) = (-s,-v)$ for all $(s,v) \in S^2 \times R^2$.*

Proof Let $[s]$ and $[s,v]$ be the images of s in RP^2 and of (s,v) in $N = S^2 \times R^2/\langle g \rangle$, respectively, and let $p([s,v]) = [s]$, for $s \in S^2$ and $v \in R^2$. Then $p : N \to RP^2$ is an R^2-bundle projection, and $w_1(N) = p^* w_1(RP^2)$, so $w_1(p) = 0$. Let $\sigma_t([s]) = [s, t(x,y)]$, where $s = (x,y,z) \in S^2$ and $t \in R$. The embedding $\sigma_t : RP^2 \to N$ is isotopic to the 0-section σ_0, and $\sigma_t(RP^2)$ meets $\sigma_0(RP^2)$ transversally in one point, if $t > 0$. Hence $w_2(p) \neq 0$. □

Lemma 10.15 *Let M be the $\mathbb{S}^2 \times \mathbb{E}^2$-manifold with $\pi_1(M) \cong Z *_Z D$ generated by the isometries $(-I, \begin{pmatrix} 0 \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix})$ and $(-I, \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, -I)$. Then $v_2(M) = U^2$ and $U^4 = 0$ in $H^4(M; \mathbb{F}_2)$.*

Proof This manifold is fibred over RP^2 with fibre Kb. As $\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}$ is a fixed point of the involution $(\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, -I)$ of R^2 there is a cross-section given by $\sigma([s]) = [s, \begin{pmatrix} \frac{1}{4} \\ \frac{1}{4} \end{pmatrix}]$. Hence $H_2(M; \mathbb{F}_2)$ has a basis represented by embedded copies of Kb and RP^2, with self-intersection numbers 0 and 1, respectively. (See Lemma 10.14). Thus the characteristic element for the intersection pairing is $[Kb]$, and $v_2(M)$ is the Poincaré dual to $[Kb]$. The cohomology class $U \in H^1(M; \mathbb{F}_2)$ is induced from the generator of $H^1(RP^2; \mathbb{F}_2)$. The projection formula gives $p_*(U^2 \cap \sigma_*[RP^2]) = 1$ and $p_*(U^2 \cap [Kb]) = 0$. Hence we have also $v_2(M) = U^2$ and so $U^4 = 0$. □

This lemma is used below to compute some products in $H^*(Z *_Z D; \mathbb{F}_2)$. Ideally, we would have a purely algebraic argument.

Theorem 10.16 *Let M be a closed 4-manifold such that $\pi_2(M) \cong Z$ and $\beta_1(M) = \chi(M) = 0$, and let $\pi = \pi_1(M)$. Let U be the cohomology class in $H^1(\pi; \mathbb{F}_2)$ corresponding to the action $u : \pi \to Aut(\pi_2(M))$. Then*

(1) *if $\pi \cong D *_Z D$ or $D *_Z (Z \oplus (Z/2Z))$ then*

$$H^*(M; \mathbb{F}_2) \cong \mathbb{F}_2[S, T, U]/(S^2 + SU, T^2 + TU, U^3),$$

where S, T and U have degree 1;

(2) *if $\pi \cong Z *_Z D$ then*

$$H^*(M; \mathbb{F}_2) \cong \mathbb{F}_2[S, U, V, W]/I,$$

where S, U have degree 1, V has degree 2 and W has degree 3, and $I = (S^2, SU, SV, U^3, UV, UW, VW, V^2 + U^2V, V^2 + SW, W^2)$;

(3) *$v_2(M) = U^2$ and $k_1(M) = \beta^u(U^2) \in H^3(\pi; Z^u)$, in all cases.*

Proof We shall consider the three possible fundamental groups in turn.

$D *_Z D$ **:** Since x, y and xs have order 2 in $D *_Z D$ they act nontrivially, and so $K = \langle s, t \rangle \cong Z^2$. Let S, T, U be the basis for $H^1(\pi; \mathbb{F}_2)$ determined by the equations $S(t) = S(x) = T(s) = T(x) = U(s) = U(t) = 0$. It follows easily from the LHSSS for π as an extension of $Z/2Z$ by K that $H^2(\pi; \mathbb{F}_2)$ has dimension ≤ 4. We may check that the classes $\{U^2, US, UT, ST\}$ are linearly independent, by restriction to the cyclic subgroups generated by x, xs, xt and xst. Therefore they form a basis of $H^2(\pi; \mathbb{F}_2)$. The squares S^2 and T^2 must be linear combinations of the above basis elements. On restricting such linear combinations to subgroups as before we find that $S^2 = US$ and $T^2 = UT$. Now $H^s(\pi; \mathbb{F}_2) \cong H^s(M; \mathbb{F}_2)$ for $s \leq 2$, by Lemma 10.4. It follows easily from the nondegeneracy of Poincaré duality that $U^2ST \neq 0$ in $H^4(M; \mathbb{F}_2)$, while $U^3S = U^3T = U^4 = 0$, so $U^3 = 0$. Hence the cohomology ring $H^*(M; \mathbb{F}_2)$ is isomorphic to the ring $\mathbb{F}_2[S, T, U]/(S^2 + SU, T^2 + TU, U^3)$. Moreover $v_2(M) = U^2$, since $(US)^2 = USU^2$, $(UT)^2 = UTU^2$ and $(ST)^2 = STU^2$. An element of π has order 2 if and only if it is of the form xs^mt^n for some $(m, n) \in Z^2$. It is easy to check that the only linear combination $aU^2 + bUS + cUT + dST$ which has nonzero restriction to all subgroups of order 2 is U^2. Hence $k_1(M) = \beta^u(U^2)$.

$D *_Z (Z \oplus (Z/2Z))$ **:** Since x, y and xs have order 2 in $D *_Z (Z \oplus (Z/2Z))$ they act nontrivially, and so $K = \langle s, t\rangle \cong Z \times_{-1} Z$. Let S, T, U be the basis for $H^1(\pi; \mathbb{F}_2)$ determined by the equations $S(t) = S(x) = T(s) = T(x) = U(s) = U(t) = 0$. We again find that $\{U^2, US, UT, ST\}$ forms a basis for $H^2(\pi; \mathbb{F}_2) \cong H^2(M; \mathbb{F}_2)$, and may check that $S^2 = US$ and $T^2 = UT$, by restriction to the subgroups generated by $\{x, xs\}$, $\{x, xt\}$ and K. As before, the nondegeneracy of Poincaré duality implies that $H^*(M; \mathbb{F}_2)$ is isomorphic to the ring $\mathbb{F}_2[S, T, U]/(S^2 + SU, T^2 + TU, U^3)$, while $v_2(M) = U^2$. An element of π has order 2 if and only if it is of the form $xs^m t^n$ for some $(m, n) \in Z^2$, with either $m = 0$ or n even. Hence U^2 and $U^2 + ST$ are the only elements of $H^2(\pi; \mathbb{F}_2)$ with nonzero restriction to all subgroups of order 2. Now $H^1(\pi; Z^u) \cong Z \oplus (Z/2Z)$ and $H^1(\pi; \mathbb{F}_2) \cong (Z/2Z)^3$. Since $\pi/K = Z/2Z$ acts nontrivially on $H^1(K; \mathbb{Z})$ it follows from the LHSSS with coefficients Z^u that $H^2(\pi; Z^u) \le E_2^{0,2} = Z/2Z$. As the functions $f(x^a s^m t^n) = (-1)^a n$ and $g(x^a s^m t^n) = (1 - (-1)^a)/2$ define crossed homomorphisms from G to Z^u (i.e., $f(wz) = u(w)f(z) + f(w)$ for all w, z in G) which reduce modulo (2) to T and U, respectively, $H^2(\pi; Z^u)$ is generated by $\beta^u(S)$ and has order 2. We may check that $\overline{\beta^u}(S) = ST$, by restriction to the subgroups generated by $\{x, xs\}$, $\{x, xt\}$ and K. Hence $k_1(M) = \beta^u(U^2) = \beta^u(U^2 + ST)$.

$Z *_Z D$ **:** If $\pi \cong Z *_Z D$ then $\pi/\pi' \cong (Z/4Z) \oplus (Z/2Z)$ and we may assume that $K \cong Z \times_{-1} Z$ is generated by r and s. Let S, U be the basis for $H^1(\pi; \mathbb{F}_2)$ determined by the equations $S(x) = U(s) = 0$. Note that S is in fact the *mod*-2 reduction of the homomorphism $\tilde{S} : \pi \to Z/4Z$ given by $\tilde{S}(s) = 1$ and $\tilde{S}(x) = 0$. Therefore $S^2 = Sq^1 S = 0$. Let $f : \pi \to \mathbb{F}_2$ be the function defined by $f(k) = f(rsk) = f(xrk) = f(xrsk) = 0$ and $f(rk) = f(sk) = f(xk) = f(xsk) = 1$ for all $k \in K'$. Then $U(g)S(h) = f(g) + f(h) + f(gh) = \delta f(g, h)$ for all $g, h \in \pi$, and so $US = 0$ in $H^2(\pi; \mathbb{F}_2)$. In the LHSSS all differentials ending on the bottom row must be 0, since π is a semidirect product of $Z/2Z$ with the normal subgroup K. Since $H^p(Z/2Z; H^1(K; \mathbb{F}_2)) = 0$ for all $p > 0$, it follows that $H^n(\pi; \mathbb{F}_2)$ has dimension 2, for all $n \ge 1$.

In particular, $H^2(M; \mathbb{F}_2) \cong H^2(\pi; \mathbb{F}_2)$ has a basis $\{U^2, V\}$, where $V|_K$ generates $H^2(K; \mathbb{F}_2)$. Moreover $H^4(M; \mathbb{F}_2)$ is a quotient of $H^4(\pi; \mathbb{F}_2)$. It follows from Lemma 10.15 that $\{U^4, U^2 V\}$ is a basis for $H^4(\pi; \mathbb{F}_2)$ and $V^2 = U^2 V + mU^4$ in $H^4(\pi; \mathbb{F}_2)$, for some $m = 0$ or 1. Let $\sigma : Z/2Z \to \pi$ be the inclusion of the subgroup $\langle x\rangle$, which splits the projection onto π/K. Then $\sigma^*(V) = \sigma^*(U^2)$ or 0, while $\sigma^* U^4 \neq 0$ in $H^4(\pi; \mathbb{F}_2)$. Hence $m\sigma^*(U^4) = \sigma^*(V^2 + U^2 V) = 0$ and so $V^2 = U^2 V$ in $H^4(\pi; \mathbb{F}_2)$. Therefore if M is *any* closed 4-manifold with $\pi_1(M) \cong Z *_Z D$ and $\chi(M) = 0$ the image of U^4 in $H^4(M; \mathbb{F}_2)$ must be 0, and hence $v_2(M) = U^2$, by Poincaré duality. Moreover $H^*(M;\mathbb{F}_2)$ is generated by S, U (in degree 1), V (in degree 2) and an element W in degree 3 such that $SW \neq 0$ and $UW = 0$. Hence $H^*(M;\mathbb{F}_2)$ is isomorphic to the quotient of $\mathbb{F}_2[S,U,V,W]$ by the ideal $(S^2, SU, SV, U^3, UV, UW, VW, V^2+U^2V, V^2+SW, W^2)$.

Since $U^3U = U^3S = 0$ in $H^4(M;\mathbb{F}_2)$ the image of U^3 in $H^3(M;\mathbb{F}_2)$ must also be 0, by Poincaré duality. Now $k_1(M)$ has image 0 in $H^3(M;\mathbb{F}_2)$ and nonzero restriction to subgroups of order 2. Therefore $k_1(M) = \beta^u(U^2)$, as reduction modulo (2) is injective, by Lemma 10.4. □

The example $M = RP^2 \times T$ has $v_2(M) = 0$ and $k_1(M) \neq 0$, and so in general $k_1(M)$ need not equal $\beta^u(v_2(M))$. Is it always $\beta^u(U^2)$?

Corollary 10.16.1 *The covering space associated to $K = \mathrm{Ker}(u)$ is homeomorphic to $S^2 \times T$ if $\pi \cong D *_Z D$ and to $S^2 \times Kb$ if $\pi \cong D *_Z (Z \oplus (Z/2Z))$ or $Z *_Z D$.*

Proof Since ρ is Z^2 or $Z \times_{-1} Z$ these assertions follow from Theorem 6.11, on computing the Stiefel-Whitney classes of the double cover. Since $D *_Z D$ acts orientably on the euclidean plane R^2 we have $w_1(M) = U$, by Lemma 10.3, and so $w_2(M) = v_2(M) + w_1(M)^2 = 0$. Hence the double cover is $S^2 \times T$. If $\pi \cong D *_Z (Z \oplus (Z/2Z))$ or $Z *_Z D$ then $w_1(M)|_K = w_1(K)$, while $w_2(M)|_K = 0$, so the double cover is $S^2 \times Kb$, in both cases. □

The $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds with groups $D *_Z D$ and $D *_Z (Z \oplus (Z/2Z))$ are unique up to affine diffeomorphism. In each case there is at most one other homotopy type of closed 4-manifold with this fundamental group and Euler characteristic 0, by Theorems 10.5 and 10.16 and the remark following Theorem 10.13. Are the two affine diffeomorphism classes of $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds with $\pi \cong Z *_Z D$ homotopy equivalent? There are again at most 2 homotopy types. In summary, there are 22 affine diffeomorphism classes of closed $\mathbb{S}^2 \times \mathbb{E}^2$-manifolds (representing at least 21 homotopy types) and between 21 and 24 homotopy types of closed 4-manifolds covered by $S^2 \times R^2$ and with Euler characteristic 0.

10.6 Some remarks on the homeomorphism types

In Chapter 6 we showed that if π is Z^2 or $Z \times_{-1} Z$ then M must be homeomorphic to the total space of an S^2-bundle over the torus or Klein bottle, and we were able to estimate the size of the structure sets when π has $Z/2Z$ as a direct factor. The other groups of Theorem 10.11 are not "square-root closed accessible" and we have not been able to compute the surgery obstruction groups completely. However the Mayer-Vietoris sequences of [Ca73] are exact modulo 2-torsion, and we may compare the ranks of $[SM; G/TOP]$ and $L_5(\pi, w_1)$. This is sufficient in some cases to show that the structure set is infinite. For instance, the rank of $L_5(D \times Z)$ is 3 and that of $L_5(D \tilde{\times} Z)$ is 2, while the rank of $L_5(D *_Z (Z \oplus (Z/2Z)), w_1)$ is 2. (The groups $L_*(\pi) \otimes \mathbb{Z}[\frac{1}{2}]$ have been computed for all cocompact planar groups π [LS00]). If M is orientable and $\pi \cong D \times Z$ or $D \times_\tau Z$ then $[SM; G/TOP] \cong H^3(M; \mathbb{Z}) \oplus H^1(M; \mathbb{F}_2) \cong H_1(M; \mathbb{Z}) \oplus H^1(M; \mathbb{F}_2)$ has rank 1. Therefore $S_{TOP}(M)$ is infinite. If $\pi \cong D *_Z (Z \oplus (Z/2Z))$ then $H_1(M; \mathbb{Q}) = 0$, $H_2(M; \mathbb{Q}) = H_2(\pi; \mathbb{Q}) = 0$ and $H_4(M; \mathbb{Q}) = 0$, since M is nonorientable. Hence $H^3(M; \mathbb{Q}) \cong Q$, since $\chi(M) = 0$. Therefore $[SM; G/TOP]$ again has rank 1 and $S_{TOP}(M)$ is infinite. These estimates do not suffice to decide whether there are infinitely many homeomorphism classes in the homotopy type of M. To decide this we need to study the action of the group $E(M)$ on $S_{TOP}(M)$. A scheme for analyzing $E(M)$ as a tower of extensions involving actions of cohomology groups with coefficients determined by Whitehead's Γ-functors is outlined on page 52 of [Ba'].

Chapter 11

Manifolds covered by $S^3 \times R$

In this chapter we shall show that a closed 4-manifold M is covered by $S^3 \times R$ if and only if $\pi = \pi_1(M)$ has two ends and $\chi(M) = 0$. Its homotopy type is then determined by π and the first k-invariant $k_1(M)$. The maximal finite normal subgroup of π is either the group of a $\mathbb{S}^3$-manifold or one of the groups $Q(8a,b,c) \times Z/dZ$ with a,b,c and d odd. (There are examples of the latter type, and no such M is homotopy equivalent to a $\mathbb{S}^3 \times \mathbb{E}^1$-manifold.) The possibilities for π are not yet known even when F is a $\mathbb{S}^3$-manifold group and $\pi/F \cong Z$. Solving this problem may involve first determining which k-invariants are realizable when F is cyclic; this is also not yet known.

Manifolds which fibre over RP^2 with fibre T or Kb and $\partial \neq 0$ have universal cover $S^3 \times R$. In §6 we determine the possible fundamental groups, and show that an orientable 4-manifold M with such a group and with $\chi(M) = 0$ must be homotopy equivalent to a $\mathbb{S}^3 \times \mathbb{E}^1$-manifold which fibres over RP^2.

As groups with two ends are virtually solvable, surgery techniques may be used to study manifolds covered by $S^3 \times R$. However computing $Wh(\pi)$ and $L_*(\pi; w_1)$ is a major task. Simple estimates suggest that there are usually infinitely many nonhomeomorphic manifolds within a given homotopy type.

11.1 Invariants for the homotopy type

The determination of the closed 4-manifolds with universal covering space homotopy equivalent to S^3 is based on the structure of groups with two ends.

Theorem 11.1 *Let M be a closed 4-manifold with fundamental group π. Then $\widetilde{M} \simeq S^3$ if and only if π has two ends and $\chi(M) = 0$. If so*

(1) *M is finitely covered by $S^3 \times S^1$ and so $\widetilde{M} \cong S^3 \times R \cong R^4 \backslash \{0\}$;*

(2) *the maximal finite normal subgroup F of π has cohomological period dividing 4, acts trivially on $\pi_3(M) \cong Z$ and the corresponding covering space M_F has the homotopy type of an orientable finite PD_3-complex;*

(3) *the homotopy type of M is determined by π and the orbit of the first nontrivial k-invariant $k(M) \in H^4(\pi; Z^w)$ under $Out(\pi) \times \{\pm 1\}$; and*

(4) *the restriction of $k(M)$ to $H^4(F;Z)$ is a generator.*

Proof If $\widetilde{M} \simeq S^3$ then $H^1(\pi;\mathbb{Z}[\pi]) \cong Z$ and so π has two ends. Hence π is virtually Z. The covering space M_A corresponding to an infinite cyclic subgroup A is homotopy equivalent to the mapping torus of a self homotopy equivalence of $S^3 \simeq \widetilde{M}$, and so $\chi(M_A) = 0$. As $[\pi : A] < \infty$ it follows that $\chi(M) = 0$ also.

Suppose conversely that $\chi(M) = 0$ and π is virtually Z. Then $H_3(\widetilde{M};\mathbb{Z}) \cong Z$ and $H_4(\widetilde{M};\mathbb{Z}) = 0$. Let M_Z be an orientable finite covering space with fundamental group Z. Then $\chi(M_Z) = 0$ and so $H_2(M_Z;\mathbb{Z}) = 0$. The homology groups of $\widetilde{M} = \widetilde{M}_Z$ may be regarded as modules over $\mathbb{Z}[t,t^{-1}] \cong \mathbb{Z}[Z]$. Multiplication by $t-1$ maps $H_2(\widetilde{M};\mathbb{Z})$ onto itself, by the Wang sequence for the projection of $\widetilde{M}$ onto M_Z. Therefore $Hom_{\mathbb{Z}[Z]}(H_2(\widetilde{M};\mathbb{Z}),\mathbb{Z}[Z]) = 0$ and so $\pi_2(M) = \pi_2(M_Z) = 0$, by Lemma 3.3. Therefore the map from S^3 to $\widetilde{M}$ representing a generator of $\pi_3(M)$ is a homotopy equivalence. Since M_Z is orientable the generator of the group of covering translations $Aut(\widetilde{M}/M_Z) \cong Z$ is homotopic to the identity, and so $M_Z \simeq \widetilde{M} \times S^1 \simeq S^3 \times S^1$. Therefore $M_Z \cong S^3 \times S^1$, by surgery over Z. Hence $\widetilde{M} \cong S^3 \times R$.

Let F be the maximal finite normal subgroup of π. Since F acts freely on $\widetilde{M} \simeq S^3$ it has cohomological period dividing 4 and $M_F = \widetilde{M}/F$ is a PD_3-complex. In particular, M_F is orientable and F acts trivially on $\pi_3(M)$. The image of the finiteness obstruction for M_F under the "geometrically significant injection" of $K_0(\mathbb{Z}[F])$ into $Wh(F \times Z)$ of [Rn86] is the obstruction to $M_F \times S^1$ being a simple PD-complex. If $f : M_F \to M_F$ is a self homotopy equivalence which induces the identity on $\pi_1(M_F) \cong F$ and on $\pi_3(M_F) \cong Z$ then f is homotopic to the identity, by obstruction theory. (See [Pl82].) Therefore $\pi_0(E(M_F))$ is finite and so M has a finite cover which is homotopy equivalent to $M_F \times S^1$. Since manifolds are simple PD_n-complexes M_F must be finite.

The first nonzero k-invariant lies in $H^4(\pi;Z^w)$, since $\pi_2(M) = 0$ and π acts on $\pi_3(M) \cong Z$ via the orientation character. As it restricts to the k-invariant for M_F in $H^4(F;Z^w)$ it generates this group, and (4) follows as in Theorem 2.9. □

The list of finite groups with cohomological period dividing 4 is well known (see [DM85]). There are the generalized quaternionic groups $Q(2^n a,b,c)$ (with $n \geq 3$ and a,b,c odd), the extended binary tetrahedral groups T_k^*, the extended binary octahedral groups O_k^*, the binary icosahedral group I^*, the dihedral groups $A(m,e)$ (with m odd >1), and the direct products of any one of these with a cyclic group Z/dZ of relatively prime order. (In particular, a p-group with periodic cohomology is cyclic if p is odd and cyclic or quaternionic if $p=2$.) We shall give presentations for these groups in §2.

Each such group F is the fundamental group of some PD_3-complex [Sw60]. Such *Swan complexes* for F are orientable, and are determined up to homotopy equivalence by their k-invariants, which are generators of $H^3(F;\mathbb{Z})\cong Z/|F|Z$, by Theorem 2.9. Thus they are parametrized up to homotopy by the quotient of $(Z/|F|Z)^\times$ under the action of $Out(F)\times\{\pm1\}$. The set of finiteness obstructions for all such complexes forms a coset of the "Swan subgroup" of $\tilde{K}_0(\mathbb{Z}[F])$ and there is a finite complex of this type if and only if the coset contains 0. (This condition fails if F has a subgroup isomorphic to $Q(16,3,1)$ and hence if $F\cong O^*_k\times(Z/dZ)$ for some $k>1$, by Corollary 3.16 of [DM85].) If X is a Swan complex for F then $X\times S^1$ is a finite PD_4^+-complex with $\pi_1(X\times S^1)\cong F\times Z$ and $\chi(X\times S^1)=0$.

If $\pi/F\cong Z$ then $k(M)$ is a generator of $H^4(\pi;\pi_3(M))\cong H^4(F;\mathbb{Z})\cong Z/|F|Z$. If $\pi/F\cong D$ then $\pi\cong G*_F H$, where $[G:F]=[H:F]=2$, and $H^4(\pi;\mathbb{Z})\cong \{(\zeta,\xi)\in(Z/|G|Z)\oplus(Z/|H|Z)\mid \zeta\equiv\xi \bmod (|F|)\}$, which is isomorphic to $(Z/2|F|Z)\oplus(Z/2Z)$, and the k-invariant restricts to a generator of each of the groups $H^4(G;\mathbb{Z})$ and $H^4(H;\mathbb{Z})$. In particular, if $\pi\cong D$ the k-invariant is unique, and so any closed 4-manifold M with $\pi_1(M)\cong D$ and $\chi(M)=0$ is homotopy equivalent to $RP^4\sharp RP^4$.

Theorem 11.2 *Let M be a closed 4-manifold such that $\pi=\pi_1(M)$ has two ends and with $\chi(M)=0$. Then the group of unbased homotopy classes of self homotopy equivalences of M is finite.*

Proof We may assume that M has a finite cell structure with a single 4-cell. Suppose that $f:M\to M$ is a self homotopy equivalence which fixes a base point and induces the identity on π and on $\pi_3(M)\cong Z$. Then there are no obstructions to constructing a homotopy from f to $id_{\widetilde{M}}$ on the 3-skeleton $M_0=M\backslash intD^4$, and since $\pi_4(M)=\pi_4(S^3)=Z/2Z$ there are just two possibilities for f. It is easily seen that $Out(\pi)$ is finite. Since every self map is homotopic to one which fixes a basepoint the group of unbased homotopy classes of self homotopy equivalences of M is finite. □

If π is a semidirect product $F\times_\theta Z$ then $Aut(\pi)$ is finite and the group of *based* homotopy classes of based self homotopy equivalences is also finite.

11.2 The action of π/F on F

Let F be a finite group with cohomological period dividing 4. Automorphisms of F act on $H_*(F;\mathbb{Z})$ and $H^*(F;\mathbb{Z})$ through $Out(F)$, since inner automorphisms induce the identity on (co)homology. Let $J_+(F)$ be the kernel of the action on $H_3(F;\mathbb{Z})$, and let $J(F)$ be the subgroup of $Out(F)$ which acts by ± 1.

An outer automorphism class induces a well defined action on $H^4(S;\mathbb{Z})$ for each Sylow subgroup S of F, since all p-Sylow subgroups are conjugate in F and the inclusion of such a subgroup induces an isomorphism from the p-torsion of $H^4(F;\mathbb{Z}) \cong Z/|F|Z$ to $H^4(S;\mathbb{Z}) \cong Z/|S|Z$, by Shapiro's Lemma. Therefore an outer automorphism class of F induces multiplication by r on $H^4(F;\mathbb{Z})$ if and only if it does so for each Sylow subgroup of F, by the Chinese Remainder Theorem.

The map sending a self homotopy equivalence h of a Swan complex X_F for F to the induced outer automorphism class determines a homomorphism from the group of (unbased) homotopy classes of self homotopy equivalences $E(X_F)$ to $Out(F)$. The image of this homomorphism is $J(F)$, and it is a monomorphism if $|F| > 2$, by Corollary 1.3 of [Pl82]. (Note that [Pl82] works with *based* homotopies.) If $F = 1$ or $Z/2Z$ the orientation reversing involution of X_F ($\simeq S^3$ or RP^3, respectively) induces the identity on F.

Lemma 11.3 *Let M be a closed 4-manifold with universal cover $S^3 \times R$, and let F be the maximal finite normal subgroup of $\pi = \pi_1(M)$. The quotient π/F acts on $\pi_3(M)$ and $H^4(F;\mathbb{Z})$ through multiplication by ± 1. It acts trivially if the order of F is divisible by 4 or by any prime congruent to 3 mod (4).*

Proof The group π/F must act through ± 1 on the infinite cyclic groups $\pi_3(M)$ and $H_3(M_F;\mathbb{Z})$. By the universal coefficient theorem $H^4(F;\mathbb{Z})$ is isomorphic to $H_3(F;\mathbb{Z})$, which is the cokernel of the Hurewicz homomorphism from $\pi_3(M)$ to $H_3(M_F;\mathbb{Z})$. This implies the first assertion.

To prove the second assertion we may pass to the Sylow subgroups of F, by Shapiro's Lemma. Since the p-Sylow subgroups of F also have cohomological period 4 they are cyclic if p is an odd prime and are cyclic or quaternionic ($Q(2^n)$) if $p = 2$. In all cases an automorphism induces multiplication by a square on the third homology [Sw60]. But -1 is not a square modulo 4 nor modulo any prime $p = 4n + 3$. □

Thus the groups $\pi\cong F\rtimes Z$ realized by such 4-manifolds correspond to outer automorphisms in $J(F)$ or $J_+(F)$. We shall next determine these subgroups of $Out(F)$ for F a group of cohomological period dividing 4. If m is an integer let $l(m)$ be the number of odd prime divisors of m.

$Z/dZ=\langle x\mid x^d=1\rangle$.

$Out(Z/dZ)=Aut(Z/dZ)=(Z/dZ)^\times$.

Hence $J(Z/dZ)=\{s\in(Z/dZ)^\times\mid s^2=\pm1\}$. $J_+(Z/dZ)=(Z/2Z)^{l(d)}$ if $d\not\equiv 0\,(4)$, $(Z/2Z)^{l(d)+1}$ if $d\equiv 4\,(8)$, and $(Z/2Z)^{l(d)+2}$ if $d\equiv 0\,(8)$.

$Q(8)=\langle x,y\mid x^2=y^2=(xy)^2\rangle$.

An automorphism of $Q=Q(8)$ induces the identity on Q/Q' if and only if it is inner, and every automorphism of Q/Q' lifts to one of Q. In fact $Aut(Q)$ is the semidirect product of $Out(Q)\cong Aut(Q/Q')\cong SL(2,\mathbb{F}_2)$ with the normal subgroup $Inn(Q)=Q/Q'\cong(Z/2Z)^2$. Moreover $J(Q)=Out(Q)$, generated by the images of the automorphisms σ and τ, where σ sends x and y to y and xy, respectively, and τ interchanges x and y.

$Q(8k)=\langle x,y\mid x^{4k}=1,\ x^{2k}=y^2,\ yxy^{-1}=x^{-1}\rangle$, where $k>1$.

All automorphisms of $Q(8k)$ are of the form $[i,s]$, where $(s,2k)=1$, $[i,s](x)=x^s$ and $[i,s](y)=x^iy$, and $Aut(Q(8k))$ is the semidirect product of $(Z/4kZ)^\times$ with the normal subgroup $\langle[1,1]\rangle\cong Z/4kZ$.

$Out(Q(8k))=(Z/2Z)\oplus((Z/4kZ)^\times/(\pm1))$, generated by the images of the $[0,s]$ and [1,1]. The automorphism $[i,s]$ induces multiplication by s^2 on $H^4(Q(2^n);\mathbb{Z})$ [Sw60]. Hence $J(Q(8k))=(Z/2Z)^{l(k)+1}$ if k is odd and $(Z/2Z)^{l(k)+2}$ if k is even.

$T_k^*=\langle Q(8),z\mid z^{3^k}=1,\ zxz^{-1}=y,\ zyz^{-1}=xy\rangle$, where $k\geq1$.

Let ρ be the automorphism which sends x, y and z to y^{-1}, x^{-1} and z^2 respectively. Let ξ, η and ζ be the inner automorphisms determined by conjugation by x, y and z, respectively (i.e., $\xi(g)=xgx^{-1}$, and so on). Then $Aut(T_k^*)$ has the presentation

$$\langle\rho,\xi,\eta,\zeta\mid\rho^{2.3^{k-1}}=\eta^2=\zeta^3=(\eta\zeta)^3=1,\ \rho\zeta\rho^{-1}=\zeta^2,\ \rho\eta\rho^{-1}=\zeta^{-1}\eta\zeta=\xi\rangle.$$

An induction on k gives $4^{3^k}=1+m3^{k+1}$ for some $m\equiv1$ mod (3). Hence the image of ρ generates $Aut(T_k^*/T_k^{*\prime})\cong(Z/3^kZ)^\times$, and so $Out(T_k^*)\cong(Z/3^kZ)^\times$. The 3-Sylow subgroup generated by z is preserved by ρ, and it follows that $J(T_k^*)=Z/2Z$ (generated by the image of $\rho^{3^{k-1}}$).

$O_k^* = \langle T_k^*, w \mid w^2 = x^2,\ wxw^{-1} = yx,\ wzw^{-1} = z^{-1}\rangle$, where $k \geq 1$.

(Note that the relations imply $wyw^{-1} = y^{-1}$.) As we may extend ρ to an automorphism of O_k^* via $\rho(w) = w^{-1}z^2$ the restriction from $Aut(O_k^*)$ to $Aut(T_k^*)$ is onto. An automorphism in the kernel sends w to wv for some $v \in T_k^*$, and the relations for O_k^* imply that v must be central in T_k^*. Hence the kernel is generated by the involution α which sends w, x, y, z to $w^{-1} = wx^2, x, y, z$, respectively. Now $\rho^{3^{k-1}} = \sigma\alpha$, where σ is conjugation by wz in O_k^*, and so the image of ρ generates $Out(O_k^*)$. The subgroup $\langle u, x\rangle$ generated by $u = xw$ and x is isomorphic to $Q(16)$, and is a 2-Sylow subgroup. As $\alpha(u) = u^5$ and $\alpha(x) = x$ it is preserved by α, and $H^4(\alpha|_{\langle u,x\rangle}; \mathbb{Z})$ is multiplication by 25. As $H^4(\rho|_{\langle z\rangle}; \mathbb{Z})$ is multiplication by 4 it follows that $J(O_k^*) = 1$.

$I^* = \langle x, y \mid x^2 = y^3 = (xy)^5\rangle$.

The map sending the generators x, y to $\left(\begin{smallmatrix} 2 & 0 \\ 1 & 3 \end{smallmatrix}\right)$ and $y = \left(\begin{smallmatrix} 2 & 2 \\ 1 & 4 \end{smallmatrix}\right)$, respectively, induces an isomorphism from I^* to $SL(2, \mathbb{F}_5)$. Conjugation in $GL(2, \mathbb{F}_5)$ induces a monomorphism from $PGL(2, \mathbb{F}_5)$ to $Aut(I^*)$. The natural map from $Aut(I^*)$ to $Aut(I^*/\zeta I^*)$ is injective, since I^* is perfect. Now $I^*/\zeta I^* \cong PSL(2, \mathbb{F}_5) \cong A_5$. The alternating group A_5 is generated by 3-cycles, and has ten 3-Sylow subgroups, each of order 3. It has five subgroups isomorphic to A_4 generated by pairs of such 3-Sylow subgroups. The intersection of any two of them has order 3, and is invariant under any automorphism of A_5 which leaves invariant each of these subgroups. It is not hard to see that such an automorphism must fix the 3-cycles. Thus $Aut(A_5)$ embeds in the group S_5 of permutations of these subgroups. Since $|PGL(2, \mathbb{F}_5)| = |S_5| = 120$ it follows that $Aut(I^*) \cong S_5$ and $Out(I^*) = Z/2Z$. The outer automorphism class is represented by the matrix $\omega = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in $GL(2, \mathbb{F}_5)$.

Lemma 11.4 [Pl83] $J(I^*) = 1$.

Proof The element $\gamma = x^3y = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ generates a 5-Sylow subgroup of I^*. It is easily seen that $\omega\gamma\omega^{-1} = \gamma^2$, and so ω induces multiplication by 2 on $H^2(Z/5Z; \mathbb{Z}) \cong H_1(Z/5Z; \mathbb{Z}) = Z/5Z$. Since $H^4(Z/5Z; \mathbb{Z}) \cong Z/5Z$ is generated by the square of a generator for $H^2(Z/5Z; \mathbb{Z})$ we see that $H^4(\omega; \mathbb{Z})$ is multiplication by $4 = -1$ on 5-torsion. Hence $J(I^*) = 1$. □

In fact $H^4(\omega; \mathbb{Z})$ is multiplication by 49 [Pl83].

$A(m, e) = \langle x, y \mid x^m = y^{2^e} = 1,\ yxy^{-1} = x^{-1}\rangle$, where $e \geq 1$ and $m > 1$ is odd.

All automorphisms of $A(m,e)$ are of the form $[s,t,u]$, where $(s,m)=(t,2)=1$, $[s,t,u](x) = x^s$ and $[s,t,u](y) = x^u y^t$. $Out(A(m,e))$ is generated by the images of $[s,1,0]$ and $[1,t,0]$ and is isomorphic to $(Z/2^e)^\times \oplus ((Z/mZ)^\times/(\pm 1))$. $J(A(m,1)) = \{s \in (Z/mZ)^\times \mid s^2 = \pm 1\}/(\pm 1)$,

$J(A(m,2)) = (Z/2Z)^{l(m)}$, $J(A(m,e)) = (Z/2Z)^{l(m)+1}$ if $e > 2$.

$Q(2^n a,b,c) = \langle Q(2^n), u \mid u^{abc} = 1,\ xu^{ab} = u^{ab}x,\ xu^c x^{-1} = u^{-c},\ yu^{ac} = u^{ac}y,\ yu^b y^{-1} = u^{-b}\rangle$, where a, b and c are odd and relatively prime, and either $n = 3$ and at most one of a, b and c is 1 or $n > 3$ and $bc > 1$.

An automorphism of $G = Q(2^n a,b,c)$ must induce the identity on G/G'. If it induces the identity on the characteristic subgroup $\langle u\rangle \cong Z/abcZ$ and on $G/\langle u\rangle \cong Q(2^n)$ it is inner, and so $Out(Q(2^n a,b,c))$ is a subquotient of $Out(Q(2^n)) \times (Z/abcZ)^\times$. In particular, $Out(Q(8a,b,c)) \cong (Z/abcZ)^\times$, and $J(Q(8a,b,c)) \cong (Z/2Z)^{l(abc)}$. (We need only consider $n = 3$, by §5 below.)

As $Aut(G\times H) = Aut(G)\times Aut(H)$ and $Out(G\times H) = Out(G)\times Out(H)$ if G and H are finite groups of relatively prime order, we have $J_+(G\times Z/dZ) = J_+(G)\times J_+(Z/dZ)$. In particular, if G is not cyclic or dihedral $J(G\times Z/dZ) = J_+(G\times Z/dZ) = J(G)\times J_+(Z/dZ)$. In all cases except when F is cyclic or $Q(8)\times Z/dZ$ the group $J(F)$ has exponent 2 and hence π has a subgroup of index at most 4 which is isomorphic to $F\times Z$.

11.3 Extensions of D

We shall now assume that $\pi/F \cong D$. Let $u,v \in D$ be a pair of involutions which generate D and let $s = uv$. Then $s^{-n}us^n = us^{2n}$, and any involution in D is conjugate to u or to $v = us$. Hence any pair of involutions $\{u',v'\}$ which generates D is conjugate to the pair $\{u,v\}$, up to change of order.

Theorem 11.5 *Let M be a closed 4-manifold with $\chi(M) = 0$, and such that there is an epimorphism $p : \pi = \pi_1(M) \to D$ with finite kernel F. Let $\hat{u}$ and $\hat{v}$ be a pair of elements of π whose images $u = p(\hat{u})$ and $v = p(\hat{v})$ in D are involutions which together generate D. Then*

(1) *M is nonorientable and $\hat{u},\hat{v}$ each represent orientation reversing loops;*

(2) *the subgroups G and H generated by F and $\hat{u}$ and by F and $\hat{v}$, respectively, each have cohomological period dividing 4, and the unordered pair $\{G,H\}$ of groups is determined up to isomorphisms by π alone;*

(3) *conversely, π is determined up to isomorphism by the unordered pair $\{G, H\}$ of groups with index 2 subgroups isomorphic to F as the free product with amalgamation $\pi = G *_F H$;*

(4) *π acts trivially on $\pi_3(M)$.*

Proof Let $\hat{s} = \hat{u}\hat{v}$. Suppose that $\hat{u}$ is orientation preserving. Then the subgroup σ generated by $\hat{u}$ and $\hat{s}^2$ is orientation preserving so the corresponding covering space M_σ is orientable. As σ has finite index in π and σ/σ' is finite this contradicts Lemma 3.14. Similarly, $\hat{v}$ must be orientation reversing.

By assumption, $\hat{u}^2$ and $\hat{v}^2$ are in F, and $[G : F] = [H : F] = 2$. If F is not isomorphic to $Q \times Z/dZ$ then $J(F)$ is abelian and so the (normal) subgroup generated by F and $\hat{s}^2$ is isomorphic to $F \times Z$. In any case the subgroup generated by F and $\hat{s}^k$ is normal, and is isomorphic to $F \times Z$ if k is a nonzero multiple of 12. The uniqueness up to isomorphisms of the pair $\{G, H\}$ follows from the uniqueness up to conjugation and order of the pair of generating involutions for D. Since G and H act freely on $\widetilde{M}$ they also have cohomological period dividing 4. On examining the list above we see that F must be cyclic or the product of $Q(8k)$, $T(v)$ or $A(m,e)$ with a cyclic group of relatively prime order, as it is the kernel of a map from G to $Z/2Z$. It is easily verified that in all such cases every automorphism of F is the restriction of automorphisms of G and H. Hence π is determined up to isomorphism as the amalgamated free product $G *_F H$ by the unordered pair $\{G, H\}$ of groups with index 2 subgroups isomorphic to F (i.e., it is unnecessary to specify the identifications of F with these subgroups).

The final assertion follows because each of the spaces $M_G = \widetilde{M}/G$ and $M_H = \widetilde{M}/H$ are PD_3-complexes with finite fundamental group and therefore are orientable, and π is generated by G and H. □

Must the spaces M_G and M_H be homotopy equivalent to finite complexes?

11.4 $\mathbb{S}^3 \times \mathbb{E}^1$-manifolds

With the exception of O_k^* (with $k > 1$), $A(m,1)$ and $Q(2^n a, b, c)$ (with either $n = 3$ and at most one of a, b and c is 1 or $n > 3$ and $bc > 1$) and their products with cyclic groups, all of the groups listed in §2 have fixed point free representations in $SO(4)$ and so act linearly on S^3. (Cyclic groups, the binary dihedral groups $D_{4m}^* = A(m,2)$, with m odd, and $D_{8k}^* = Q(8k,1,1)$, with $k \geq 1$ and the three binary polyhedral groups T_1^*, O_1^* and I^* are subgroups of S^3.) We shall call such groups $\mathbb{S}^3$-*groups*. If F is cyclic then every Swan complex for F is homotopy equivalent to a lens space. If $F = Q(2^k)$ or T_k^* for some $k > 1$ then S^3/F is the unique finite Swan complex for F [Th80]. For the other noncyclic $\mathbb{S}^3$-groups the corresponding $\mathbb{S}^3$-manifold is unique, but in general there may be other finite Swan complexes. (In particular, there are exotic finite Swan complexes for T_1.)

Let N be a $\mathbb{S}^3$-manifold with $\pi_1(N) = F$. Then the projection of $Isom(N)$ onto its group of path components splits, and the inclusion of $Isom(N)$ into $Diff(N)$ induces an isomorphism on path components. Moreover if $|F| > 2$ then an isometry which induces the identity outer automorphism is isotopic to the identity, and so $\pi_0(Isom(M))$ maps injectively to $Out(F)$. (See [Mc02].)

Theorem 11.6 *Let M be a closed 4-manifold with $\chi(M) = 0$ and $\pi = \pi_1(M) \cong F \times_\theta Z$, where F is finite. Then M is homeomorphic to a $\mathbb{S}^3 \times \mathbb{E}^1$-manifold if and only if M is the mapping torus of a self homeomorphism of a $\mathbb{S}^3$-manifold with fundamental group F, and such manifolds are determined up to homeomorphism by their homotopy type.*

Proof Let p_1 and p_2 be the projections of $Isom(\mathbb{S}^3 \times \mathbb{E}^1) = O(4) \times E(1)$ onto $O(4)$ and $E(1)$ respectively. If π is a discrete subgroup of $Isom(\mathbb{S}^3 \times \mathbb{E}^1)$ which acts freely on $S^3 \times R$ then p_1 maps F monomorphically and $p_1(F)$ acts freely on S^3, since every isometry of R of finite order has nonempty fixed point set. Moreover $p_2(\pi)$ is a discrete subgroup of $E(1)$ which acts cocompactly on R, and so has no nontrivial finite normal subgroup. Hence $F = \pi \cap (O(4) \times \{1\})$. If $\pi/F \cong Z$ and $t \in \pi$ represents a generator of π/F then conjugation by t induces an isometry θ of S^3/F, and $M \cong M(\theta)$. Conversely any self homeomorphism of a $\mathbb{S}^3$-manifold is isotopic to an isometry of finite order, and so the mapping torus is homeomorphic to a $\mathbb{S}^3 \times \mathbb{E}^1$-manifold. The final assertion follows from Theorem 3 of [Oh90]. $\square$

If s is an integer such that $s^2 \equiv \pm 1$ modulo (d) then there is an isometry of the lens space $L(d,s)$ inducing multiplication by s, and the mapping torus has fundamental group $(Z/dZ) \times_s Z$. (This group may also be realized by mapping tori of self homotopy equivalences of other lens spaces.) If $d > 2$ a closed 4-manifold with this group and with Euler characteristic 0 is orientable if and only if $s^2 \equiv 1\,(d)$.

If F is a noncyclic $\mathbb{S}^3$-group there is a unique linear k-invariant, and so for each $\theta \in Aut(F)$ there is at most one homeomorphism class of $\mathbb{S}^3 \times \mathbb{E}^1$-manifolds with fundamental group $\pi = F \times_\theta Z$. Every class in $J(F)$ is realizable by an orientation preserving isometry of S^3/F, if $F = Q(8)$, T_k^*, O_1^*, I^*, $A(p^i, e)$, $Q(8) \times Z/q^jZ$ or $A(p^i, 2) \times Z/q^jZ$, where p and q are odd primes and $e > 1$. For the other $\mathbb{S}^3$-groups the subgroup of $J(F)$ realizable by homeomorphisms of S^3/F is usually quite small. (See [Mc02].)

Suppose now that G and H are $\mathbb{S}^3$-groups with index 2 subgroups isomorphic to F. If F, G and H are each noncyclic then the corresponding $\mathbb{S}^3$-manifolds are uniquely determined, and we may construct a nonorientable $\mathbb{S}^3 \times \mathbb{E}^1$-manifold with fundamental group $\pi = G *_F H$ as follows. Let u and $v : S^3/F \to S^3/F$ be the covering involutions with quotient spaces S^3/G and S^3/H, respectively, and let $\phi = uv$. (Note that u and v are isometries of S^3/F.) Then $U([x,t]) = [u(x), 1-t]$ defines a fixed point free involution on the mapping torus $M(\phi)$ and the quotient space has fundamental group π. A similar construction works if F is cyclic and $G \cong H$ or if G is cyclic.

11.5 Realization of the groups

Let F be a finite group with cohomological period dividing 4, and let X_F denote a finite Swan complex for F. If θ is an automorphism of F which induces ± 1 on $H_3(F;\mathbb{Z})$ there is a self homotopy equivalence h of X_F which induces $[\theta] \in J(F)$. The mapping torus $M(h)$ is a finite PD_4-complex with $\pi_1(M) \cong F \times_\theta Z$ and $\chi(M(h)) = 0$. Conversely, every PD_4-complex M with $\chi(M) = 0$ and such that $\pi_1(M)$ is an extension of Z by a finite normal subgroup F is homotopy equivalent to such a mapping torus. Moreover, if $\pi \cong F \times Z$ and $|F| > 2$ then h is homotopic to the identity and so $M(h)$ is homotopy equivalent to $X_F \times S^1$.

Since every PD_n-complex may be obtained by attaching an n-cell to a complex which is homologically of dimension $< n$, the exotic characteristic class of the Spivak normal fibration of a PD_3-complex X in $H^3(X;\mathbb{F}_2)$ is trivial. Hence every 3-dimensional Swan complex X_F has a TOP reduction, i.e., there are normal maps $(f,b) : N^3 \to X_F$. Such a map has a "proper surgery" obstruction $\lambda^p(f,b)$ in $L_3^p(F)$, which is 0 if and only if $(f,b) \times id_{S^1}$ is normally cobordant to a simple homotopy equivalence. In particular, a surgery semicharacteristic must be 0. Hence all subgroups of F of order $2p$ (with p prime) are cyclic, and $Q(2^na, b, c)$ (with $n > 3$ and b or $c > 1$) cannot occur [HM86]. As the $2p$ condition excludes groups with subgroups isomorphic to $A(m,1)$ (with $m > 1$) the cases remaining to be decided are when $F \cong Q(8a, b, c) \times Z/dZ$, where a, b and c are odd and at most one of them is 1. The main result of [HM86] is that in such a case $F \times Z$ acts freely and properly "with almost linear k-invariant" if and only if some arithmetical conditions depending on subgroups of F of the form $Q(8a,b,1)$ hold. (Here "almost linear" means that all covering spaces corresponding to subgroups isomorphic to $A(m,e)\times Z/dZ$ or $Q(8k)\times Z/dZ$ must be homotopy equivalent to $\mathbb{S}^3$-manifolds. The constructive part of the argument may be extended to the 4-dimensional case by reference to [FQ].)

The following more direct argument for the existence of a free proper action of $F\times Z$ on $S^3\times R$ was outlined in [KS88], for the cases when F acts freely on an homology 3-sphere Σ. Let Σ and its universal covering space $\widetilde{\Sigma}$ have equivariant cellular decompositions lifted from a cellular decomposition of Σ/F, and let $\Pi=\pi_1(\Sigma/F)$. Then $C_*(\Sigma)=\mathbb{Z}[F]\otimes_\Pi C_*(\widetilde{\Sigma})$ is a finitely generated free $\mathbb{Z}[F]$-complex, and may be realized by a finite Swan complex X. The chain map (over the epimorphism : $\Pi\to F$) from $C_*(\widetilde{\Sigma})$ to $C_*(\widetilde{X})$ may be realized by a map $h:\Sigma/F\to X$, since these spaces are 3-dimensional. As $h\times id_{S^1}$ is a simple $\mathbb{Z}[F\times Z]$-homology equivalence it has surgery obstruction 0 in $L_4^s(F\times Z)$, and so is normally cobordant to a simple homotopy equivalence. For example, the group $Q(24,313,1)$ acts freely on an homology 3-sphere (see §6 of [DM85]). Is there an explicit action on some Brieskorn homology 3-sphere? Is $Q(24,313,1)$ a 3-manifold group? (This seems unlikely.)

Although $Q(24,13,1)$ cannot act freely on any homology 3-sphere [DM85], there is a closed orientable 4-manifold with fundamental group $Q(24,13,1)\times Z$, by the argument of [HM86]. No such 4-manifold can fibre over S^1, since $Q(24,13,1)$ is not a 3-manifold group. Thus such a manifold is a counter example to a 4-dimensional analogue of the Farrell fibration theorem (of a different kind from that of [We87]), and is not geometric.

If $F=T_k^*$, $Q(8k)$ or $A(m,2)$ then $F\times Z$ can only act freely and properly on $R^4\backslash\{0\}$ with the k-invariant corresponding to the free linear action of F on S^3. (For the group $A(m,2)$, this follows from Corollary C of [HM86'], which also implies that the restriction of the k-invariant to $A(k,r+1)$ and hence to the odd-Sylow subgroup of $Q(2^nk)$ is linear. The nonlinear k-invariants for $Q(2^n)$ have nonzero finiteness obstruction. As the k-invariants of free linear representations of $Q(2^nk)$ are given by elements in $H^4(Q(2^nk);\mathbb{Z})$ whose restrictions to Z/kZ are squares and whose restrictions to $Q(2^n)$ are squares times the basic generator (see page 120 of [Wl78], only the linear k-invariant is realizable in this case also). However in general it is not known which k-invariants are realizable. Every group of the form $Q(8a,b,c)\times Z/dZ\times Z$ admits an "almost linear" k-invariant, but there may be other actions. (See [HM86, 86'] for more on this issue.)

In considering the realization of more general extensions of Z or D by finite normal subgroups the following question seems central. If M is a closed 4-manifold with $\pi=\pi_1(M)\cong(Z/dZ)\times_s Z$ where $s^2\equiv 1$ but $s\not\equiv\pm1\ (d)$

and $\chi(M) = 0$ is M homotopy equivalent to the $\mathbb{S}^3 \times \mathbb{E}^1$-manifold with this fundamental group? Since M is homotopy equivalent to the mapping torus of a self homotopy equivalence $[s] : L(d,r) \to L(d,r)$ (for some r determined by $k(M)$), it would suffice to show that if $r \neq \pm s$ or $\pm s^{-1}$ the Whitehead torsion of the duality homomorphism of $M([s])$ is nonzero. Proposition 4.1 of [Rn86] gives a formula for the Whitehead torsion of such mapping tori. Unfortunately the associated Reidemeister-Franz torsion appears to be 0 in all cases. For other groups F can one use the fact that a closed 4-manifold is a *simple* PD_4-complex to bound the realizable subgroup of $J(F)$?

A positive answer to this question would enhance the interest of the following subsidiary question. If F is a noncyclic $\mathbb{S}^3$-group must an automorphism of F whose restrictions to (characteristic) cyclic subgroups $C < F$ are realized by isometries of the corresponding covering spaces of S^3/F be realized by an isometry of S^3/F? (In particular, is this so for $F = Q(2^t)$ or $A(m,2)$ with m composite?.)

If F is cyclic but neither G nor H is cyclic there may be no geometric manifold with fundamental group $\pi = G *_F H$. If the double covers of $G\backslash S^3$ and $H\backslash S^3$ are homotopy equivalent then π is realised by the union of two twisted I-bundles via a homotopy equivalence, which is a finite (but possibly nonsimple?) PD_4-complex with $\chi = 0$. For instance, the spherical space forms corresponding to $G = Q(40)$ and $H = Q(8) \times (Z/5Z)$ are doubly covered by forms doubly covered by $L(20,1)$ and $L(20,9)$, respectively, which are homotopy equivalent but not homeomorphic. The spherical space forms corresponding to $G = Q(24)$ and $H = Q(8) \times (Z/3Z)$ are doubly covered by $L(12,1)$ and $L(12,5)$, respectively, which are not homotopy equivalent.

11.6 T- and Kb-bundles over RP^2 with $\partial \neq 0$

Let $p : E \to RP^2$ be a bundle with fibre T or Kb. Then $\pi = \pi_1(E)$ is an extension of $Z/2Z$ by $G/\partial Z$, where G is the fundamental group of the fibre and ∂ is the connecting homomorphism. If $\partial \neq 0$ then π has two ends, F is cyclic and central in $G/\partial Z$ and π acts on it by inversion, since π acts nontrivially on $Z = \pi_2(RP^2)$.

If the fibre is T then π has a presentation of the form

$$\langle t, u, v \mid uv = vu,\ u^n = 1,\ tut^{-1} = u^{-1},\ tvt^{-1} = u^a v^\epsilon,\ t^2 = u^b v^c\rangle,$$

where $n > 0$ and $\epsilon = \pm 1$. Either

(1) F is cyclic, $\pi \cong (Z/nZ) \times_{-1} Z$ and $\pi/F \cong Z$; or

(2) $F = \langle s, u \mid s^2 = u^m,\ sus^{-1} = u^{-1}\rangle$; or (if $\epsilon = -1$)

(3) F is cyclic, $\pi = \langle s, t, u \mid s^2 = t^2 = u^b,\ sus^{-1} = tut^{-1} = u^{-1}\rangle$ and $\pi/F \cong D$.

In case (2) F cannot be dihedral. If m is odd $F \cong A(m, 2)$ while if $m = 2^r k$ with $r \geq 1$ and k odd $F \cong Q(2^{r+2}k)$. On replacing v by $u^{[a/2]}v$, if necessary, we may arrange that $a = 0$, in which case $\pi \cong F \times Z$, or $a = 1$, in which case

$$\pi = \langle t, u, v \mid t^2 = u^m,\ tut^{-1} = u^{-1},\ vtv^{-1} = tu,\ uv = vu\rangle,$$

so $\pi/F \cong Z$.

If the fibre is Kb then π has a presentation of the form

$$\langle t, u, w \mid uwu^{-1} = w^{-1},\ u^n = 1,\ tut^{-1} = u^{-1},\ twt^{-1} = u^a w^\epsilon,\ t^2 = u^b w^c\rangle,$$

where $n > 0$ is even (since $\mathrm{Im}(\partial) \leq \zeta\pi_1(Kb)$) and $\epsilon = \pm 1$. On replacing t by ut, if necessary, we may assume that $\epsilon = 1$. Moreover, $tw^2t^{-1} = w^{\pm 2}$ since w^2 generates the commutator subgroup of $G/\partial Z$, so a is even and $2a \equiv 0\ mod\ (n)$, $t^2u = ut^2$ implies that $c = 0$, and $t.t^2.t^{-1} = t^2$ implies that $2b \equiv 0\ mod\ (n)$. As F is generated by t and u^2, and cannot be dihedral, we must have $n = 2b$. Moreover b must be even, as w has infinite order and $t^2w = wt^2$. Therefore

(4) $F \cong Q(8k)$, $\pi/F \cong D$ and

$\pi = \langle t, u, w \mid uwu^{-1} = w^{-1},\ tut^{-1} = u^{-1},\ tw = u^a wt,\ t^2 = u^{2k}\rangle$.

In all cases π has a subgroup of index at most 2 which is isomorphic to $F \times Z$.

Each of these groups is the fundamental group of such a bundle space. (This may be seen by using the description of such bundle spaces given in §5 of Chapter 5.) Orientable 4-manifolds which fibre over RP^2 with fibre T and $\partial \neq 0$ are mapping tori of involutions of $\mathbb{S}^3$-manifolds, and if F is not cyclic two such bundle spaces with the same group are diffeomorphic [Ue91].

Theorem 11.7 *Let M be a closed orientable 4-manifold with fundamental group π. Then M is homotopy equivalent to an $\mathbb{S}^3 \times \mathbb{E}^1$-manifold which fibres over RP^2 if and only $\chi(M) = 0$ and π is of type (1) or (2) above.*

Proof If M is an orientable $\mathbb{S}^3 \times \mathbb{E}^1$-manifold then $\chi(M) = 0$ and $\pi/F \cong Z$, by Theorem 11.1 and Lemma 3.14. Moreover π must be of type (1) or (2) if M fibres over RP^2, and so the conditions are necessary.

Suppose that they hold. Then $\widetilde{M} \cong R^4\setminus\{0\}$ and the homotopy type of M is determined by π and $k(M)$, by Theorem 11.1. If $F \cong Z/nZ$ then $M_F = \widetilde{M}/F$

is homotopy equivalent to some lens space $L(n,s)$. As the involution of Z/nZ which inverts a generator can be realized by an isometry of $L(n,s)$, M is homotopy equivalent to an $\mathbb{S}^3 \times \mathbb{E}^1$-manifold which fibres over S^1.

If $F \cong Q(2^{r+2}k)$ or $A(m,2)$ then $F \times Z$ can only act freely and properly on $R^4 \setminus \{0\}$ with the "linear" k-invariant [HM86]. Therefore M_F is homotopy equivalent to a spherical space form S^3/F. The class in $Out(Q(2^{r+2}k))$ represented by the automorphism which sends the generator t to tu and fixes u is induced by conjugation in $Q(2^{r+3}k)$ and so can be realized by a (fixed point free) isometry θ of $S^3/Q(2^{r+2}k)$. Hence M is homotopy equivalent to a bundle space $(S^3/Q(2^{r+2}k)) \times S^1$ or $(S^3/Q(2^{r+2}k)) \times_\theta S^1$ if $F \cong Q(2^{r+2}k)$. A similar conclusion holds when $F \cong A(m,2)$ as the corresponding automorphism is induced by conjugation in $Q(2^3 d)$.

With the results of [Ue91] it follows in all cases that M is homotopy equivalent to the total space of a torus bundle over RP^2. $\square$

Theorem 11.7 makes no assumption that there be a homomorphism $u : \pi \to Z/2Z$ such that $u^*(x)^3 = 0$ (as in §5 of Chapter 5). If F is cyclic or $A(m,2)$ this condition is a purely algebraic consequence of the other hypotheses. For let C be a cyclic normal subgroup of maximal order in F. (There is an unique such subgroup, except when $F = Q(8)$.) The centralizer $C_\pi(C)$ has index 2 in π and so there is a homomorphism $u : \pi \to Z/2Z$ with kernel $C_\pi(C)$.

When F is cyclic u factors through Z and so the induced map on cohomology factors through $H^3(Z;\tilde{Z}) = 0$.

When $F \cong A(m,2)$ the 2-Sylow subgroup is cyclic of order 4, and the inclusion of $Z/4Z$ into τ induces isomorphisms on cohomology with 2-local coefficients. In particular, $H^q(F;\tilde{Z}_{(2)}) = 0$ or $Z/2Z$ according as q is even or odd. It follows easily that the restriction from $H^3(\pi;\tilde{Z}_{(2)})$ to $H^3(Z/4Z;\tilde{Z}_{(2)})$ is an isomorphism. Let y be the image of $u^*(x)$ in $H^1(Z/4Z;\tilde{Z}_{(2)}) = Z/2Z$. Then y^2 is an element of order 2 in $H^2(Z/4Z;\tilde{Z}_{(2)} \otimes \tilde{Z}_{(2)}) = H^2(Z/4Z;Z_{(2)}) \cong Z/4Z$, and so $y^2 = 2z$ for some $z \in H^2(Z/4Z;Z_{(2)})$. But then $y^3 = 2yz = 0$ in $H^3(Z/4Z;\tilde{Z}_{(2)}) = Z/2Z$, and so $u^*(x)^3$ has image 0 in $H^3(\pi;\tilde{Z}_{(2)}) = Z/2Z$. Since x is a 2-torsion class this implies that $u^*(x)^3 = 0$.

Is there a similar argument when F is a generalized quaternionic group?

If M is nonorientable, $\chi(M) = 0$ and has fundamental group π of type (1) or (2) then M is homotopy equivalent to the mapping torus of the orientation reversing self homeomorphism of S^3 or of RP^3, and does not fibre over RP^2.

If π is of type (3) or (4) then the 2-fold covering space with fundamental group $F \times Z$ is homotopy equivalent to a product $L(n,s) \times S^1$. However we do not know which k-invariants give total spaces of bundles over RP^2.

11.7 Some remarks on the homeomorphism types

In this brief section we shall assume that M is orientable and that $\pi \cong F \times_\theta Z$. In contrast to the situation for the other geometries, the Whitehead groups of fundamental groups of $\mathbb{S}^3 \times \mathbb{E}^1$-manifolds are usually nontrivial. Computation of $Wh(\pi)$ is difficult as the Nil groups occuring in the Waldhausen exact sequence relating $Wh(\pi)$ to the algebraic K-theory of F seem intractable.

We can however compute the relevant surgery obstruction groups modulo 2-torsion and show that the structure sets are usually infinite. There is a Mayer-Vietoris sequence $L_5^s(F) \to L_5^s(\pi) \to L_4^u(F) \to L_4^s(F)$, where the superscript u signifies that the torsion must lie in a certain subgroup of $Wh(F)$ [Ca73]. The right hand map is (essentially) $\theta_* - 1$. Now $L_5^s(F)$ is a finite 2-group and $L_4^u(F) \sim L_4^s(F) \sim Z^R$ modulo 2-torsion, where R is the set of irreducible real representations of F (see Chapter 13A of [Wl]). The latter correspond to the conjugacy classes of F, up to inversion. (See §12.4 of [Se].) In particular, if $\pi \cong F \times Z$ then $L_5^s(\pi) \sim Z^R$ modulo 2-torsion, and so has rank at least 2 if $F \neq 1$. As $[\Sigma M, G/TOP] \cong Z$ modulo 2-torsion and the group of self homotopy equivalences of such a manifold is finite, by Theorem 11.3, there are infinitely many distinct topological 4-manifolds simple homotopy equivalent to M. For instance, as $Wh(Z \oplus (Z/2Z)) = 0$ [Kw86] and $L_5(Z \oplus (Z/2Z), +) \cong Z^2$, by Theorem 13A.8 of [Wl], the set $S_{TOP}(RP^3 \times S^1)$ is infinite. Although all of the manifolds in this homotopy type are doubly covered by $S^3 \times S^1$ only $RP^3 \times S^1$ is itself geometric. Similar estimates hold for the other manifolds covered by $S^3 \times R$ (if $\pi \neq Z$).

Chapter 12

Geometries with compact models

There are three geometries with compact models, namely $\mathbb{S}^4$, $\mathbb{CP}^2$ and $\mathbb{S}^2 \times \mathbb{S}^2$. The first two of these are easily dealt with, as there is only one other geometric manifold, namely RP^4, and for each of the two projective spaces there is one other (nonsmoothable) manifold of the same homotopy type. With the geometry $\mathbb{S}^2 \times \mathbb{S}^2$ we shall consider also the bundle space $S^2\tilde{\times}S^2$. There are eight $\mathbb{S}^2 \times \mathbb{S}^2$-manifolds, seven of which are total spaces of bundles with base and fibre each S^2 or RP^2, and there are two other such bundle spaces covered by $S^2\tilde{\times}S^2$.

The universal covering space $\widetilde{M}$ of a closed 4-manifold M is homeomorphic to $S^2 \times S^2$ if and only if $\pi = \pi_1(M)$ is finite, $\chi(M)|\pi| = 4$ and $w_2(\widetilde{M}) = 0$. (The condition $w_2(\widetilde{M}) = 0$ may be restated entirely in terms of M, but at somewhat greater length.) If these conditions hold and π is cyclic then M is homotopy equivalent to an $\mathbb{S}^2 \times \mathbb{S}^2$-manifold, except when $\pi = Z/2Z$ and M is nonorientable, in which case there is one other homotopy type. The $\mathbb{F}_2$-cohomology ring, Stiefel-Whitney classes and k-invariants must agree with those of bundle spaces when $\pi \cong (Z/2Z)^2$. However there remains an ambiguity of order at most 4 in determining the homotopy type. If $\chi(M)|\pi| = 4$ and $w_2(\widetilde{M}) \neq 0$ then either $\pi = 1$, in which case $M \simeq S^2\tilde{\times}S^2$ or $CP^2\sharp CP^2$, or M is nonorientable and $\pi = Z/2Z$; in the latter case $M \simeq RP^4\sharp CP^2$, the nontrivial RP^2-bundle over S^2, and $\widetilde{M} \simeq S^2\tilde{\times}S^2$.

The number of homeomorphism classes within each homotopy type is at most two if $\pi = Z/2Z$ and M is orientable, two if $\pi = Z/2Z$, M is nonorientable and $w_2(\widetilde{M}) = 0$, four if $\pi = Z/2Z$ and $w_2(\widetilde{M}) \neq 0$, at most four if $\pi \cong Z/4Z$, and at most eight if $\pi \cong (Z/2Z)^2$. We do not know whether there are enough exotic self homotopy equivalences to account for all the normal invariants with trivial surgery obstruction. However a PL 4-manifold with the same homotopy type as a geometric manifold or $S^2\tilde{\times}S^2$ is homeomorphic to it, in (at least) nine of the 13 cases. (In seven of these cases the homotopy type is determined by the Euler characteristic, fundamental group and Stiefel-Whitney classes.)

For the full details of some of the arguments in the cases $\pi \cong Z/2Z$ we refer to the papers [KKR92], [HKT94] and [Te95].

12.1 The geometries $\mathbb{S}^4$ and $\mathbb{CP}^2$

The unique element of $Isom(\mathbb{S}^4) = O(5)$ of order 2 which acts freely on S^4 is $-I$. Therefore S^4 and RP^4 are the only $\mathbb{S}^4$-manifolds. The manifold S^4 is determined up to homeomorphism by the conditions $\chi(S^4) = 2$ and $\pi_1(S^4) = 1$ [FQ].

Lemma 12.1 *A closed 4-manifold M is homotopy equivalent to RP^4 if and only if $\chi(M) = 1$ and $\pi_1(M) = Z/2Z$.*

Proof The conditions are clearly necessary. Suppose that they hold. Then $\widetilde{M} \simeq S^4$ and $w_1(M) = w_1(RP^4) = w$, say, since any orientation preserving self homeomorphism of $\widetilde{M}$ has Lefshetz number 2. Since $RP^\infty = K(Z/2Z, 1)$ may be obtained from RP^4 by adjoining cells of dimension at least 5 we may assume $c_M = c_{RP^4} f$, where $f : M \to RP^4$. Since c_{RP^4} and c_M are each 4-connected f induces isomorphisms on homology with coefficients $Z/2Z$. Considering the exact sequence of homology corresponding to the short exact sequence of coefficients

$$0 \to Z^w \to Z^w \to Z/2Z \to 0,$$

we see that f has odd degree. By modifying f on a 4-cell $D^4 \subset M$ we may arrange that f has degree 1, and the lemma then follows from Theorem 3.2. □

This lemma may also be proven by comparison of the k-invariants of M and RP^4, as in Theorem 4.3 of [Wl67].

By Theorems 13.A.1 and 13.B.5 of [Wl] the surgery obstruction homomorphism is determined by an Arf invariant and maps $[RP^4; G/TOP]$ onto $Z/2Z$, and hence the structure set $S_{TOP}(RP^4)$ has two elements. (See the discussion of nonorientable manifolds with fundamental group $Z/2Z$ in Section 6 below for more details.) As every self homotopy equivalence of RP^4 is homotopic to the identity [Ol53] there is one fake RP^4. The fake RP^4 is denoted $*RP^4$ and is not smoothable [Ru84].

There is a similar characterization of the homotopy type of the complex projective plane.

Lemma 12.2 *A closed 4-manifold M is homotopy equivalent to CP^2 if and only if $\chi(M) = 3$ and $\pi_1(M) = 1$.*

Proof The conditions are clearly necessary. Suppose that they hold. Then $H^2(M;\mathbb{Z})$ is infinite cyclic and so there is a map $f_M : M \to CP^\infty = K(Z,2)$ which induces an isomorphism on H^2. Since CP^∞ may be obtained from CP^2 by adjoining cells of dimension at least 6 we may assume $f_M = f_{CP^2}g$, where $g : M \to CP^2$ and $f_{CP^2} : CP^2 \to CP^\infty$ is the natural inclusion. As $H^4(M;\mathbb{Z})$ is generated by $H^2(M;\mathbb{Z})$, by Poincaré duality, g induces an isomorphism on cohomology and hence is a homotopy equivalence. □

In this case the surgery obstruction homomorphism is determined by the difference of signatures and maps $[CP^2; G/TOP]$ onto Z. The structure set $S_{TOP}(CP^2)$ again has two elements. Since $[CP^2, CP^2] \cong [CP^2, CP^\infty] \cong H^2(CP^2;\mathbb{Z})$, by obstruction theory, there are two homotopy classes of self homotopy equivalences, represented by the identity and by complex conjugation. Thus every self homotopy equivalence of CP^2 is homotopic to a homeomorphism, and so there is one fake CP^2. The fake CP^2 is also known as the Chern manifold Ch or $*CP^2$, and is not smoothable [FQ]. Neither of these manifolds admits a nontrivial fixed point free action, as any self map of CP^2 or $*CP^2$ has nonzero Lefshetz number, and so CP^2 is the only $\mathbb{CP}^2$-manifold.

12.2 The geometry $\mathbb{S}^2 \times \mathbb{S}^2$

The manifold $S^2 \times S^2$ is determined up to homotopy equivalence by the conditions $\chi(S^2 \times S^2) = 4$, $\pi_1(S^2 \times S^2) = 1$ and $w_2(S^2 \times S^2) = 0$, by Theorem 5.19. These conditions in fact determine $S^2 \times S^2$ up to homeomorphism [FQ]. Hence if M is an $\mathbb{S}^2 \times \mathbb{S}^2$-manifold its fundamental group π is finite, $\chi(M)|\pi| = 4$ and $w_2(\widetilde{M}) = 0$.

The isometry group of $\mathbb{S}^2 \times \mathbb{S}^2$ is a semidirect product $(O(3) \times O(3)) \tilde{\times} (Z/2Z)$. The $Z/2Z$ subgroup is generated by the involution τ which switches the factors ($\tau(x,y) = (y,x)$), and acts on $O(3) \times O(3)$ by $\tau(A,B)\tau = (B,A)$ for $A, B \in O(3)$. In particular, $(\tau(A,B))^2 = id$ if and only if $AB = I$, and so such an involution fixes (x, Ax), for any $x \in S^2$. Thus there are no free $Z/2Z$-actions in which the factors are switched. The element (A,B) generates a free $Z/2Z$-action if and only if $A^2 = B^2 = I$ and at least one of A, B acts freely, i.e. if A or $B = -I$. After conjugation with τ if necessary we may assume that $B = -I$, and so there are four conjugacy classes in $Isom(\mathbb{S}^2 \times \mathbb{S}^2)$ of free $Z/2Z$-actions. (The conjugacy classes may be distinguished by the multiplicity (0, 1, 2 or 3) of 1 as an eigenvalue of A.) In each case the projection onto the second factor gives rise to a fibre bundle projection from the orbit space to RP^2, with fibre S^2.

If the involutions (A,B) and (C,D) generate a free $(Z/2Z)^2$-action (AC,BD) is also a free involution. By the above paragraph, one element of each of these ordered pairs must be $-I$. It follows easily that (after conjugation with τ if necessary) the $(Z/2Z)^2$-actions are generated by pairs $(A,-I)$ and $(-I,I)$, where $A^2=I$. Since A and $-A$ give rise to the same subgroup, there are two free $(Z/2Z)^2$-actions. The orbit spaces are the total spaces of RP^2-bundles over RP^2.

If $(\tau(A,B))^4=id$ then (BA,AB) is a fixed point free involution and so $BA=AB=-I$. Since $(A,I)\tau(A,-A^{-1})(A,I)^{-1}=\tau(I,-I)$ every free $Z/4Z$-action is conjugate to the one generated by $\tau(I,-I)$. The orbit space does not fibre over a surface. (See below.)

In the next section we shall see that these eight geometric manifolds may be distinguished by their fundamental group and Stiefel-Whitney classes. Note that if F is a finite group then $q(F)\geq 2/|F|>0$, while $q^{SG}(F)\geq 2$. Thus S^4, RP^4 and the geometric manifolds with $|\pi|=4$ have minimal Euler characteristic for their fundamental groups (i.e., $\chi(M)=q(\pi)$), while $S^2\times S^2/(-I,-I)$ has minimal Euler characteristic among PD_4^+-complexes realizing $Z/2Z$.

12.3 Bundle spaces

There are two S^2-bundles over S^2, since $\pi_1(SO(3))=Z/2Z$. The total space $S^2\tilde{\times}S^2$ of the nontrivial S^2-bundle over S^2 is determined up to homotopy equivalence by the conditions $\chi(S^2\tilde{\times}S^2)=4$, $\pi_1(S^2\tilde{\times}S^2)=1$, $w_2(S^2\tilde{\times}S^2)\neq 0$ and $\sigma(S^2\tilde{\times}S^2)=0$, by Theorem 5.19. However there is one fake $S^2\tilde{\times}S^2$. The bundle space is homeomorphic to the connected sum $CP^2\sharp-CP^2$, while the fake version is homeomorphic to $CP^2\sharp-*CP^2$ and is not smoothable [FQ]. The manifolds $CP^2\sharp CP^2$ and $CP^2\sharp*CP^2$ also have $\pi_1=0$ and $\chi=4$. However it is easily seen that any self homotopy equivalence of either of these manifolds has nonzero Lefshetz number, and so they do not properly cover any other 4-manifold.

Since the Kirby-Siebenmann obstruction of a closed 4-manifold is natural with respect to covering maps and dies on passage to 2-fold coverings, the nonsmoothable manifold $CP^2\sharp-*CP^2$ admits no nontrivial free involution. The following lemma implies that $S^2\tilde{\times}S^2$ admits no orientation preserving free involution, and hence no free action of $Z/4Z$ or $(Z/2Z)^2$.

Lemma 12.3 *Let M be a closed 4-manifold with fundamental group $\pi=Z/2Z$ and universal covering space $\widetilde{M}$. Then*

(1) $w_2(\widetilde{M}) = 0$ *if and only if* $w_2(M) = u^2$ *for some* $u \in H^1(M;\mathbb{F}_2)$*; and*

(2) *if* M *is orientable and* $\chi(M) = 2$ *then* $w_2(\widetilde{M}) = 0$ *and so* $\widetilde{M} \cong S^2 \times S^2$.

Proof The Cartan-Leray cohomology spectral sequence (with coefficients $\mathbb{F}_2$) for the projection $p : \widetilde{M} \to M$ gives an exact sequence

$$0 \to H^2(\pi;\mathbb{F}_2) \to H^2(M;\mathbb{F}_2) \to H^2(\widetilde{M};\mathbb{F}_2),$$

in which the right hand map is induced by p and has image in the subgroup fixed under the action of π. Hence $w_2(\widetilde{M}) = p^*w_2(M)$ is 0 if and only if $w_2(M)$ is in the image of $H^2(\pi;\mathbb{F}_2)$. Since $\pi = Z/2Z$ this is so if and only if $w_2(M) = u^2$ for some $u \in H^1(M;\mathbb{F}_2)$.

Suppose that M is orientable and $\chi(M) = 2$. Then $H^2(\pi;\mathbb{Z}) = H^2(M;\mathbb{Z}) = Z/2Z$. Let x generate $H^2(M;\mathbb{Z})$ and let $\bar{x}$ be its image under reduction modulo (2) in $H^2(M;\mathbb{F}_2)$. Then $\bar{x} \cup \bar{x} = 0$ in $H^4(M;\mathbb{F}_2)$ since $x \cup x = 0$ in $H^4(M;\mathbb{Z})$. Moreover as M is orientable $w_2(M) = v_2(M)$ and so $w_2(M) \cup \bar{x} = \bar{x} \cup \bar{x} = 0$. Since the cup product pairing on $H^2(M;\mathbb{F}_2) \cong (Z/2Z)^2$ is nondegenerate it follows that $w_2(M) = \bar{x}$ or 0. Hence $w_2(\widetilde{M})$ is the reduction of p^*x or is 0. The integral analogue of the above exact sequence implies that the natural map from $H^2(\pi;\mathbb{Z})$ to $H^2(M;\mathbb{Z})$ is an isomorphism and so $p^*(H^2(M;\mathbb{Z})) = 0$. Hence $w_2(\widetilde{M}) = 0$ and so $\widetilde{M} \cong S^2 \times S^2$. □

Since $\pi_1(BO(3)) = Z/2Z$ there are two S^2-bundles over the Möbius band Mb and each restricts to a trivial bundle over ∂Mb. Moreover a map from ∂Mb to $O(3)$ extends across Mb if and only if it homotopic to a constant map, since $\pi_1(O(3)) = Z/2Z$, and so there are four S^2-bundles over $RP^2 = Mb \cup D^2$. (See also Theorem 5.10.)

The orbit space $M = (S^2 \times S^2)/(A, -I)$ is orientable if and only if $det(A) = -1$. If A has a fixed point $P \in S^2$ then the image of $\{P\} \times S^2$ in M is an embedded projective plane which represents a nonzero class in $H_2(M;\mathbb{F}_2)$. If $A = I$ or is a reflection across a plane the fixed point set has dimension > 0 and so this projective plane has self intersection 0. As the fibre S^2 intersects this projective plane in one point and has self intersection 0 it follows that $v_2(M) = 0$ and so $w_2(M) = w_1(M)^2$ in these two cases. If A is a rotation about an axis then the projective plane has self intersection 1, by Lemma 10.14. Finally, if $A = -I$ then the image of the diagonal $\{(x,x) | x \in S^2\}$ is a projective plane in M with self intersection 1. Thus in these two cases $v_2(M) \neq 0$. Therefore, by part (1) of the lemma, $w_2(M)$ is the square of the nonzero element of $H^1(M;\mathbb{F}_2)$ if $A = -I$ and is 0 if A is a rotation. Thus these bundle spaces may be distinguished by their Stiefel-Whitney classes, and every S^2-bundle over RP^2 is geometric.

The group $E(RP^2)$ of self homotopy equivalences of RP^2 is connected and the natural map from $SO(3)$ to $E(RP^2)$ induces an isomorphism on π_1, by Lemma 5.15. Hence there are two RP^2-bundles over S^2, up to fibre homotopy equivalence. The total space of the nontrivial RP^2-bundle over S^2 is the quotient of $S^2\tilde{\times}S^2$ by the bundle involution which is the antipodal map on each fibre. If we observe that $S^2\tilde{\times}S^2 \cong CP^2\sharp - CP^2$ is the union of two copies of the D^2-bundle which is the mapping cone of the Hopf fibration and that this involution interchanges the hemispheres we see that this space is homeomorphic to $RP^4\sharp CP^2$.

There are two RP^2-bundles over RP^2. (The total spaces of each of the latter bundles have fundamental group $(Z/2Z)^2$, since $w_1 : \pi \to \pi_1(RP^2) = Z/2Z$ restricts nontrivially to the fibre, and so is a splitting homomorphism for the homomorphism induced by the inclusion of the fibre.) They may be distinguished by their orientation double covers, and each is geometric.

12.4 Cohomology and Stiefel-Whitney classes

We shall show that if M is a closed connected 4-manifold with finite fundamental group π such that $\chi(M)|\pi| = 4$ then $H^*(M;\mathbb{F}_2)$ is isomorphic to the cohomology ring of one of the above bundle spaces, as a module over the Steenrod algebra $\mathcal{A}_2$. (In other words, there is an isomorphism which preserves Stiefel-Whitney classes.) This is an elementary exercise in Poincaré duality and the Wu formulae.

The classifying map induces an isomorphism $H^1(\pi;\mathbb{F}_2) \cong H^1(M;\mathbb{F}_2)$ and a monomorphism $H^2(\pi;\mathbb{F}_2) \to H^2(M;\mathbb{F}_2)$. If $\pi = 1$ then M is homotopy equivalent to $S^2 \times S^2$, $S^2\tilde{\times}S^2$ or $CP^2\sharp CP^2$, and the result is clear.

$\pi = Z/2Z$ In this case $\beta_2(M;\mathbb{F}_2) = 2$. Let x generate $H^1(M;\mathbb{F}_2)$. Then $x^2 \neq 0$, so $H^2(M;\mathbb{F}_2)$ has a basis $\{x^2, u\}$. If $x^4 = 0$ then $x^2u \neq 0$, by Poincaré duality, and so $H^3(M;\mathbb{F}_2)$ is generated by xu. Hence $x^3 = 0$, for otherwise $x^3 = xu$ and $x^4 = x^2u \neq 0$. Therefore $v_2(M) = 0$ or x^2, and clearly $v_1(M) = 0$ or x. Since x restricts to 0 in $\widetilde{M}$ we must have $w_2(\widetilde{M}) = v_2(\tilde{M}) = 0$. (The four possibilities are realized by the four S^2-bundles over RP^2.)

If $x^4 \neq 0$ then we may assume that $x^2u = 0$ and that $H^3(M;\mathbb{F}_2)$ is generated by x^3. In this case $xu = 0$. Since $Sq^1(x^3) = x^4$ we have $v_1(M) = x$, and

$v_2(M) = u + x^2$. In this case $w_2(\widetilde{M}) \neq 0$, since $w_2(M)$ is not a square. (This possibility is realized by the nontrivial RP^2-bundle over S^2.)

$\pi \cong (Z/2Z)^2$ In this case $\beta_2(M;\mathbb{F}_2) = 3$ and $w_1(M) \neq 0$. Fix a basis $\{x, y\}$ for $H^1(M;\mathbb{F}_2)$. Then $\{x^2, xy, y^2\}$ is a basis for $H^2(M;\mathbb{F}_2)$, since $H^2(\pi;\mathbb{F}_2)$ and $H^2(M;\mathbb{F}_2)$ both have dimension 3.

If $x^3 = y^3$ then $x^4 = Sq^1(x^3) = Sq^1(y^3) = y^4$. Hence $x^4 = y^4 = 0$ and $x^2y^2 \neq 0$, by the nondegeneracy of cup product on $H^2(M;\mathbb{F}_2)$. Hence $x^3 = y^3 = 0$ and so $H^3(M;\mathbb{F}_2)$ is generated by $\{x^2y, xy^2\}$. Now $Sq^1(x^2y) = x^2y^2$ and $Sq^1(xy^2) = x^2y^2$, so $v_1(M) = x + y$. Also $Sq^2(x^2) = 0 = x^2xy$, $Sq^2(y^2) = 0 = y^2xy$ and $Sq^2(xy) = x^2y^2$, so $v_2(M) = xy$. Since the restrictions of x and y to the orientation cover M^+ agree we have $w_2(M^+) = x^2 \neq 0$. (This possibility is realized by $RP^2 \times RP^2$.)

If x^3, y^3 and $(x + y)^3$ are all distinct then we may assume that (say) y^3 and $(x+y)^3$ generate $H^3(M;\mathbb{F}_2)$. If $x^3 \neq 0$ then $x^3 = y^3 + (x+y)^3 = x^3 + x^2y + xy^2$ and so $x^2y = xy^2$. But then we must have $x^4 = y^4 = 0$, by the nondegeneracy of cup product on $H^2(M;\mathbb{F}_2)$. Hence $Sq^1(y^3) = y^4 = 0$ and $Sq^1((x + y)^3) = (x + y)^4 = x^4 + y^4 = 0$, and so $v_1(M) = 0$, which is impossible, as M is nonorientable. Therefore $x^3 = 0$ and so $x^2y^2 \neq 0$. After replacing y by $x + y$, if necessary, we may assume $xy^3 = 0$ (and hence $y^4 \neq 0$). Poincaré duality and the Wu relations then give $v_1(M) = x + y$, $v_2(M) = xy + x^2$ and hence $w_2(M^+) = 0$. (This possibility is realized by the nontrivial RP^2-bundle over RP^2.)

Note that if $\pi \cong (Z/2Z)^2$ then $H^*(M;\mathbb{F}_2)$ is generated by $H^1(M;\mathbb{F}_2)$ and so the image of $[M]$ in $H_4(\pi;\mathbb{F}_2)$ is uniquely determined.

In all cases, a class $x \in H^1(M;\mathbb{F}_2)$ such that $x^3 = 0$ may be realized by a map from M to $K(Z/2Z, 1) = RP^\infty$ which factors through $P_2(RP^2)$. However there are such 4-manifolds which do not fibre over RP^2.

12.5 The action of π on $\pi_2(M)$

Let M be a closed 4-manifold with finite fundamental group π and orientation character $w = w_1(M)$. The intersection form $S(\widetilde{M})$ on $\Pi = \pi_2(M) = H_2(\widetilde{M};\mathbb{Z})$ is unimodular and symmetric, and π acts w-isometrically (that is, $S(ga, gb) = w(g)S(a, b)$ for all $g \in \pi$ and a, $b \in \Pi$).

The two inclusions of S^2 as factors of $S^2 \times S^2$ determine the standard basis for $\pi_2(S^2 \times S^2)$. Let $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ be the matrix of the intersection form $\bullet$ on

$\pi_2(S^2\times S^2)$, with respect to this basis. The group $Aut(\pm\bullet)$ of automorphisms of $\pi_2(S^2\times S^2)$ which preserve this intersection form up to sign is the dihedral group of order eight, and is generated by the diagonal matrices and J or $K=\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$. The subgroup of strict isometries has order four, and is generated by $-I$ and J. (Note that the isometry J is induced by the involution τ.)

Let f be a self homeomorphism of $S^2\times S^2$ and let f_* be the induced automorphism of $\pi_2(S^2\times S^2)$. The Lefshetz number of f is $2+trace(f_*)$ if f is orientation preserving and $trace(f_*)$ if f is orientation reversing. As any self homotopy equivalence which induces the identity on π_2 has nonzero Lefshetz number the natural representation of a group π of fixed point free self homeomorphisms of $S^2\times S^2$ into $Aut(\pm\bullet)$ is faithful.

Suppose first that f is a free involution, so $f_*^2=I$. If f is orientation preserving then $trace(f_*)=-2$ so $f_*=-I$. If f is orientation reversing then $trace(f_*)=0$, so $f_*=\pm JK=\pm\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$. Note that if $f'=\tau f\tau$ then $f'_*=-f_*$, so after conjugation by τ, if necessary, we may assume that $f_*=JK$.

If f generates a free $Z/4Z$-action the induced automorphism must be $\pm K$. Note that if $f'=\tau f\tau$ then $f'_*=-f_*$, so after conjugation by τ, if necessary, we may assume that $f_*=K$.

Since the orbit space of a fixed point free action of $(Z/2Z)^2$ on $S^2\times S^2$ has Euler characteristic 1 it is nonorientable, and so the action is generated by two commuting involutions, one of which is orientation preserving and one of which is not. Since the orientation preserving involution must act via $-I$ and the orientation reversing involutions must act via $\pm JK$ the action of $(Z/2Z)^2$ is essentially unique.

The standard inclusions of $S^2=CP^1$ into the summands of $CP^2\sharp-CP^2\cong S^2\tilde{\times}S^2$ determine a basis for $\pi_2(S^2\tilde{\times}S^2)\cong Z^2$. Let $\tilde{J}=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)$ be the matrix of the intersection form $\tilde{\bullet}$ on $\pi_2(S^2\tilde{\times}S^2)$ with respect to this basis. The group $Aut(\pm\tilde{\bullet})$ of automorphisms of $\pi_2(S^2\tilde{\times}S^2)$ which preserve this intersection form up to sign is the dihedral group of order eight, and is also generated by the diagonal matrices and $J=\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. The subgroup of strict isometries has order four, and consists of the diagonal matrices. A nontrivial group of fixed point free self homeomorphisms of $S^2\tilde{\times}S^2$ must have order 2, since $S^2\tilde{\times}S^2$ admits no fixed point free orientation preserving involution. If f is an orientation reversing free involution of $S^2\tilde{\times}S^2$ then $f_*=\pm J$. Since the involution of CP^2 given by complex conjugation is orientation preserving it is isotopic to a selfhomeomorphism c which fixes a 4-disc. Let $g=c\sharp id_{CP^2}$. Then $g_*=\left(\begin{smallmatrix}-1&0\\0&1\end{smallmatrix}\right)$ and so $g_*Jg_*^{-1}=-J$. Thus after conjugating f by g, if necessary, we may assume that $f_*=J$.

All self homeomorphisms of $CP^2\sharp CP^2$ preserve the sign of the intersection form, and thus are orientation preserving. With part (2) of Lemma 12.3, this implies that no manifold in this homotopy type admits a free involution.

12.6 Homotopy type

The *quadratic 2-type* of M is the quadruple $[\pi,\pi_2(M),k_1(M),S(\widetilde{M})]$. Two such quadruples $[\pi,\Pi,\kappa,S]$ and $[\pi',\Pi',\kappa',S']$ with π a finite group, Π a finitely generated, $\mathbb{Z}$-torsion free $\mathbb{Z}[\pi]$-module, $\kappa\in H^3(\pi;\Pi)$ and $S:\Pi\times\Pi\to\mathbb{Z}$ a unimodular symmetric bilinear pairing on which π acts $\pm$-isometrically are *equivalent* if there is an isomorphism $\alpha:\pi\to\pi'$ and an (anti)isometry $\beta:(\Pi,S)\to(\Pi',(\pm)S')$ which is α-equivariant (i.e., such that $\beta(gm)=\alpha(g)\beta(m)$ for all $g\in\pi$ and $m\in\Pi$) and $\beta_*\kappa=\alpha^*\kappa'$ in $H^3(\pi,\alpha^*\Pi')$. Such a quadratic 2-type determines homomorphisms $w:\pi\to\mathbb{Z}^\times=Z/2Z$ and $v:\Pi\to Z/2Z$ by the equations $S(ga,gb)=w(g)S(a,b)$ and $v(a)\equiv S(a,a)\ mod\ (2)$, for all $g\in\pi$ and a, $b\in\Pi$. (These correspond to the orientation character $w_1(M)$ and the Wu class $v_2(\widetilde{M})=w_2(\widetilde{M})$, of course.)

Let $\gamma:A\to\Gamma(A)$ be the universal quadratic functor of Whitehead. Then the pairing S may be identified with an indivisible element of $\Gamma(Hom_{\mathbb{Z}}(\Pi,\mathbb{Z}))$. Via duality, this corresponds to an element $\widehat{S}$ of $\Gamma(\Pi)$, and the subgroup generated by the image of $\widehat{S}$ is a $\mathbb{Z}[\pi]$-submodule. Hence $\pi_3=\Gamma(\Pi)/\langle\widehat{S}\rangle$ is again a finitely generated, $\mathbb{Z}$-torsion free $\mathbb{Z}[\pi]$-module. Let B be the Postnikov 2-stage corresponding to the algebraic 2-type $[\pi,\Pi,\kappa]$. A *PD_4-polarization* of the quadratic 2-type $q=[\pi,\Pi,\kappa,S]$ is a 3-connected map $f:X\to B$, where X is a PD_4-complex, $w_1(X)=w\pi_1(f)$ and $\tilde{f}_*(\widehat{S}_{\widetilde{X}})=\widehat{S}$ in $\Gamma(\Pi)$. Let $S_4^{PD}(q)$ be the set of equivalence classes of PD_4-polarizations of q, where $f:X\to B\sim g:Y\to B$ if there is a map $h:X\to Y$ such that $f\simeq gh$.

Theorem 12.4 [Te] *There is an effective, transitive action of the torsion subgroup of $\Gamma(\Pi)\otimes_{\mathbb{Z}[\pi]}Z^w$ on $S_4^{PD}(q)$.*

Proof (We shall only sketch the proof.) Let $f:X\to B$ be a fixed PD_4-polarization of q. We may assume that $X=K\cup_g e^4$, where $K=X^{[3]}$ is the 3-skeleton and $g\in\pi_3(K)$ is the attaching map. Given an element α in $\Gamma(\Pi)$ whose image in $\Gamma(\Pi)\otimes_{\mathbb{Z}[\pi]}Z^w$ lies in the torsion subgroup, let $X_\alpha=K\cup_{g+\alpha}e^4$. Since $\pi_3(B)=0$ the map $f|_K$ extends to a map $f_\alpha:X_\alpha\to B$, which is again a PD_4-polarization of q. The equivalence class of f_α depends only on the image of α in $\Gamma(\Pi)\otimes_{\mathbb{Z}[\pi]}Z^w$. Conversely, if $g:Y\to B$ is another PD_4-polarization of q then $f_*[X]-g_*[Y]$ lies in the image of $Tors(\Gamma(\Pi)\otimes_{\mathbb{Z}[\pi]}Z^w)$ in $H_4(B;Z^w)$. See [Te] for the full details. ∎

Corollary 12.4.1 *If X and Y are PD_4-complexes with the same quadratic 2-type then each may be obtained by adding a single 4-cell to $X^{[3]} = Y^{[3]}$.* $\square$

If $w = 0$ and the Sylow 2-subgroup of π has cohomological period dividing 4 then $Tors(\Gamma(\Pi) \otimes_{\mathbb{Z}[\pi]} Z^w) = 0$ [Ba88]. In particular, if M is orientable and π is finite cyclic then the equivalence class of the quadratic 2-type determines the homotopy type [HK88]. Thus in all cases considered here the quadratic 2-type determines the homotopy type of the orientation cover.

The group $Aut(B) = Aut([\pi, \Pi, \kappa])$ acts on $S_4^{PD}(q)$ and the orbits of this action correspond to the homotopy types of PD_4-complexes X admitting such polarizations f. When q is the quadratic 2-type of $RP^2 \times RP^2$ this action is nontrivial. (See below in this paragraph. Compare also Theorem 10.5.)

The next lemma shall enable us to determine the possible k-invariants.

Lemma 12.5 *Let M be a closed 4-manifold with fundamental group $\pi = Z/2Z$ and universal covering space $S^2 \times S^2$. Then the first k-invariant of M is a nonzero element of $H^3(\pi; \pi_2(M))$.*

Proof The first k-invariant is the primary obstruction to the existence of a cross-section to the classifying map $c_M : M \to K(Z/2Z, 1) = RP^\infty$ and is the only obstruction to the existence of such a cross-section for $c_{P_2(M)}$. The only nonzero differentials in the Cartan-Leray cohomology spectral sequence (with coefficients $Z/2Z$) for the projection $p : \widetilde{M} \to M$ are at the E_3^{**} level. By the results of Section 4, π acts trivially on $H^2(\widetilde{M}; \mathbb{F}_2)$, since $\widetilde{M} = S^2 \times S^2$. Therefore $E_3^{22} = E_2^{22} \cong (Z/2Z)^2$ and $E_3^{50} = E_2^{50} = Z/2Z$. Hence $E_\infty^{22} \neq 0$, so E_∞^{22} maps onto $H^4(M; \mathbb{F}_2) = Z/2Z$ and $d_3^{12} : H^1(\pi; H^2(\widetilde{M}; \mathbb{F}_2)) \to H^4(\pi; \mathbb{F}_2)$ must be onto. But in this region the spectral sequence is identical with the corresponding spectral sequence for $P_2(M)$. It follows that the image of $H^4(\pi; \mathbb{F}_2) = Z/2Z$ in $H^4(P_2(M); \mathbb{F}_2)$ is 0, and so $c_{P_2(M)}$ does not admit a cross-section. Thus $k_1(M) \neq 0$. $\square$

If $\pi = Z/2Z$ and M is orientable then π acts via $-I$ on Z^2 and the k-invariant is a nonzero element of $H^3(Z/2Z; \pi_2(M)) = (Z/2Z)^2$. The isometry which transposes the standard generators of Z^2 is π-linear, and so there are just two equivalence classes of quadratic 2-types to consider. The k-invariant which is invariant under transposition is realised by $(S^2 \times S^2)/(-I, -I)$, while the other k-invariant is realized by the orientable bundle space with $w_2 = 0$. Thus M must be homotopy equivalent to one of these spaces.

If $\pi = Z/2Z$, M is nonorientable and $w_2(\widetilde{M}) = 0$ then $H^3(\pi;\pi_2(M)) = Z/2Z$ and there is only one quadratic 2-type to consider. There are four equivalence classes of PD_4-polarizations, as $Tors(\Gamma(\Pi)\otimes_{\mathbb{Z}[\pi]} Z^w) \cong (Z/2Z)^2$. The corresponding PD_4-complexes are all of the form $K\cup_f e^4$, where $K = (S^2\times RP^2) - intD^4$ is the 3-skeleton of $S^2\times RP^2$ and $f\in\pi_3(K)$. (In all cases $H^1(M;\mathbb{F}_2)$ is generated by an element x such that $x^3 = 0$.) Two choices for f give total spaces of S^2-bundles over RP^2, while a third choice gives $RP^4\sharp_{S^1}RP^4$, which is the union of two disc bundles over RP^2, but is not a bundle space and is not geometric. There is a fourth homotopy type which has nontrivial Browder-Livesay invariant, and so is not realizable by a closed manifold [HM78]. The product space $S^2\times RP^2$ is characterized by the additional conditions that $w_2(M) = w_1(M)^2 \neq 0$ (i.e., $v_2(M) = 0$) and that there is an element $u\in H^2(M;\mathbb{Z})$ which generates an infinite cyclic direct summand and is such that $u\cup u = 0$. (See Theorem 5.19.) The nontrivial nonorientable S^2-bundle over RP^2 has $w_2(M) = 0$. The manifold $RP^4\sharp_{S^1}RP^4$ also has $w_2(M) = 0$, but it may be distinguished from the bundle space by the $Z/4Z$-valued quadratic function on $\pi_2(M)\otimes(Z/2Z)$ introduced in [KKR92].

If $\pi = Z/2Z$ and $w_2(\widetilde{M}) \neq 0$ then $H^3(\pi_1;\pi_2(M)) = 0$, and the quadratic 2-type is unique. (Note that the argument of Lemma 12.5 breaks down here because $E_\infty^{22} = 0$.) There are two equivalence classes of PD_4-polarizations, as $Tors(\Gamma(\Pi)\otimes_{\mathbb{Z}[\pi]} Z^w) = Z/2Z$. They are each of the form $K\cup_f e^4$, where $K = (RP^4\sharp CP^2) - intD^4$ is the 3-skeleton of $RP^4\sharp CP^2$ and $f\in\pi_3(K)$. The bundle space $RP^4\sharp CP^2$ is characterized by the additional condition that there is an element $u\in H^2(M;\mathbb{Z})$ which generates an infinite cyclic direct summand and such that $u\cup u = 0$. (See Theorem 5.19.) In [HKT94] it is shown that any closed 4-manifold M with $\pi = Z/2Z$, $\chi(M) = 2$ and $w_2(\widetilde{M}) \neq 0$ is homotopy equivalent to $RP^4\sharp CP^2$.

If $\pi\cong Z/4Z$ then $H^3(\pi;\pi_2(M)) \cong Z^2/(I-K)Z^2 = Z/2Z$, since $\Sigma_{k=1}^{k=4} f_*^k = \Sigma_{k=1}^{k=4} K^k = 0$. The k-invariant is nonzero, since it restricts to the k-invariant of the orientation double cover. In this case $Tors(\Gamma(\Pi)\otimes_{\mathbb{Z}[\pi]} Z^w) = 0$ and so M is homotopy equivalent to $(S^2\times S^2)/\tau(I,-I)$.

Finally, let $\pi\cong(Z/2Z)^2$ be the diagonal subgroup of $Aut(\pm\bullet) < GL(2,\mathbb{Z})$, and let α be the automorphism induced by conjugation by J. The standard generators of $\pi_2(M) = Z^2$ generate complementary π-submodules, so that $\pi_2(M)$ is the direct sum $\tilde{Z}\oplus\alpha^*\tilde{Z}$ of two infinite cyclic modules. The isometry $\beta = J$ which transposes the factors is α-equivariant, and π and $V = \{\pm I\}$ act nontrivially on each summand. If ρ is the kernel of the action of π on $\tilde{Z}$ then $\alpha(\rho)$ is the kernel of the action on $\alpha^*\tilde{Z}$, and $\rho\cap\alpha(\rho) = 1$. Let $j_V : V\to\pi$ be the inclusion. As the projection of $\pi = \rho \oplus V$ onto V is compatible with the action, $H^*(j_V;\tilde{Z})$ is a split epimorphism and so $H^*(V;\tilde{Z})$ is a direct summand of $H^*(\pi;\tilde{Z})$. This implies in particular that the differentials in the LHSSS $H^p(V;H^q(\rho;\tilde{Z})) \Rightarrow H^{p+q}(\pi;\tilde{Z})$ which end on the row $q=0$ are all 0. Hence $H^3(\pi;\tilde{Z}) \cong H^1(V;\mathbb{F}_2) \oplus H^3(V;\tilde{Z}) \cong (Z/2Z)^2$. Similarly $H^3(\pi;\alpha^*\tilde{Z}) \cong (Z/2Z)^2$, and so $H^3(\pi;\pi_2(M)) \cong (Z/2Z)^4$. The k-invariant must restrict to the k-invariant of each double cover, which must be nonzero, by Lemma 12.5. Let K_V, K_ρ and $K_{\alpha(\rho)}$ be the kernels of the restriction homomorphisms from $H^3(\pi;\pi_2(M))$ to $H^3(V;\pi_2(M))$, $H^3(\rho;\pi_2(M))$ and $H^3(\alpha(\rho);\pi_2(M))$, respectively. Now $H^3(\rho;\tilde{Z}) = H^3(\alpha(\rho);\alpha^*\tilde{Z}) = 0$, $H^3(\rho;\alpha^*\tilde{Z}) = H^3(\alpha(\rho);\tilde{Z}) = Z/2Z$ and $H^3(V;\tilde{Z}) = H^3(V;\alpha^*\tilde{Z}) = Z/2Z$. Since the restrictions are epimorphisms $|K_V| = 4$ and $|K_\rho| = |K_{\alpha(\rho)}| = 8$. It is easily seen that $|K_\rho \cap K_{\alpha(\rho)}| = 4$. Moreover $\mathrm{Ker}(H^3(j_V;\tilde{Z})) \cong H^1(V;H^2(\rho;\tilde{Z})) \cong H^1(V;H^2(\rho;\mathbb{F}_2))$ restricts nontrivially to $H^3(\alpha(\rho);\tilde{Z}) \cong H^3(\alpha(\rho);\mathbb{F}_2)$, as can be seen by reduction modulo (2), and similarly $\mathrm{Ker}(H^3(j_V;\alpha^*\tilde{Z}))$ restricts nontrivially to $H^3(\rho;\alpha^*\tilde{Z})$. Hence $|K_V \cap K_\rho| = |K_V \cap K_\rho| = 2$ and $K_V \cap K_\rho \cap K_{\alpha(\rho)} = 0$. Thus $|K_V \cup K_\rho \cup K_{\alpha(\rho)}| = 8+8+4-4-2-2+1 = 13$ and so there are at most three possible k-invariants. Moreover the automorphism α and the isometry $\beta = J$ act on the k-invariants by transposing the factors. The k-invariant of $RP^2 \times RP^2$ is invariant under this transposition, while that of the nontrivial RP^2 bundle over RP^2 is not, for the k-invariant of its orientation cover is not invariant. Thus there are two equivalence classes of quadratic 2-types to be considered. Since $Tors(\Gamma(\Pi) \otimes_{\mathbb{Z}[\pi]} Z^w) \cong (Z/2Z)^2$ there are four equivalence classes of PD_4-polarizations of each of these quadratic 2-types. In each case the quadratic 2-type determines the cohomology ring, since it determines the orientation cover (see §4). The canonical involution of the direct product interchanges two of these polarizations in the $RP^2 \times RP^2$ case, and so there are seven homotopy types of PD_4-complexes X with $\pi \cong (Z/2Z)^2$ and $\chi(X) = 1$. Can the Browder-Livesay arguments of [HM78] be adapted to show that the two bundle spaces are the only such 4-manifolds?

12.7 Surgery

We may assume that M is a proper quotient of $S^2 \times S^2$ or of $S^2 \tilde{\times} S^2$, so $|\pi|\chi(M) = 4$ and $\pi \neq 1$. In the present context every homotopy equivalence is simple since $Wh(\pi) = 0$ for all groups π of order ≤ 4 [Hg40].

Suppose first that $\pi = Z/2Z$. Then $H^1(M;\mathbb{F}_2) = Z/2Z$ and $\chi(M) = 2$, so $H^2(M;\mathbb{F}_2) \cong (Z/2Z)^2$. The $\mathbb{F}_2$-Hurewicz homomorphism from $\pi_2(M)$ to $H_2(M;\mathbb{F}_2)$ has cokernel $H_2(\pi;\mathbb{F}_2) = Z/2Z$. Hence there is a map $\beta : S^2 \to M$

such that $\beta_*[S^2] \neq 0$ in $H_2(M;\mathbb{F}_2)$. If moreover $w_2(\widetilde{M}) = 0$ then $\beta^* w_2(M) = 0$, since β factors through $\widetilde{M}$. Then there is a self homotopy equivalence f_β of M with nontrivial normal invariant in $[M;G/TOP]$, by Lemma 6.5. Note also that M is homotopy equivalent to a PL 4-manifold (see §6 above).

If M is orientable $[M;G/TOP] \cong Z\oplus(Z/2Z)^2$. The surgery obstruction groups are $L_5(Z/2Z,+) = 0$ and $L_4(Z/2Z,+) \cong Z^2$, where the surgery obstructions are determined by the signature and the signature of the double cover, by Theorem 13.A.1 of [Wl]. Hence it follows from the surgery exact sequence that $S_{TOP}(M) \cong (Z/2Z)^2$. Since $w_2(\widetilde{M}) = 0$ (by Lemma 12.3) there is a self homotopy equivalence f_β of M with nontrivial normal invariant and so there are at most two homeomorphism classes within the homotopy type of M. Any $\alpha \in H^2(M;\mathbb{F}_2)$ is the codimension-2 Kervaire invariant of some homotopy equivalence $f : N \to M$. We then have $KS(N) = f^*(KS(M)+\alpha^2)$, by Lemma 15.5 of [Si71]. We may assume that M is PL. If $w_2(M) = 0$ then $KS(N) = f^*(KS(M)) = 0$, and so N is homeomorphic to M [Te97]. On the other hand if $w_2(M) \neq 0$ there is an $\alpha \in H^2(M;\mathbb{F}_2)$ such that $\alpha^2 \neq 0$ and then $KS(N) \neq 0$. Thus there are three homeomorphism classes of orientable closed 4-manifolds with $\pi = Z/2Z$ and $\chi = 2$. One of these is a fake $(S^2 \times S^2)/(-I,-I)$ and is not smoothable.

Nonorientable closed 4-manifolds with fundamental group $Z/2Z$ have been classified in [HKT94]. If M is nonorientable then $[M;G/TOP] \cong (Z/2Z)^3$, the surgery obstruction groups are $L_5(Z/2Z,-) = 0$ and $L_4(Z/2Z,-) = Z/2Z$, and $\sigma_4(\hat{g}) = c(\hat{g})$ for any normal map $\hat{g} : M \to G/TOP$, by Theorem 13.A.1 of [Wl]. Therefore $\sigma_4(\hat{g}) = (w_1(M)^2 \cup \hat{g}^*(k_2))[M]$, by Theorem 13.B.5 of [Wl]. (See also §2 of Chapter 6 above.) As $w_1(M)$ is not the reduction of a class in $H^1(M;\mathbb{Z}/4\mathbb{Z})$ its square is nonzero and so there is an element $\hat{g}^*(k_2)$ in $H^2(M;\mathbb{F}_2)$ such that this cup product is nonzero. Hence $S_{TOP}(M) \cong (Z/2Z)^2$. There are two homeomorphism types within each homotopy type if $w_2(\widetilde{M}) = 0$; if $w_2(\widetilde{M}) \neq 0$ (i.e., if $M \simeq RP^4 \sharp CP^2$) there are four corresponding homeomorphism types [HKT94]. Thus there are eight homeomorphism classes of nonorientable closed 4-manifolds with $\pi = Z/2Z$ and $\chi = 2$.

The image of $[M;G/PL]$ in $[M;G/TOP]$ is a subgroup of index 2 (see Section 15 of [Si71]). It follows that if M is the total space of an S^2-bundle over RP^2 any homotopy equivalence $f : N \to M$ where N is also PL is homotopic to a homeomorphism. (For then $S_{TOP}(M)$ has order 4, and the nontrivial element of the image of $S_{PL}(M)$ is represented by an exotic self homotopy equivalence of M. The case $M = S^2 \times RP^2$ was treated in [Ma79]. See also [Te97] for the cases with $\pi = Z/2Z$ and $w_1(M) = 0$.) This is also true if $M = S^4$, RP^4, CP^2,

$S^2 \times S^2$ or $S^2 \tilde{\times} S^2$. The exotic homeomorphism types within the homotopy type of $RP^4 \sharp CP^2$ (the nontrivial RP^2-bundle over S^2) are $RP^4 \sharp * CP^2$, $*RP^4 \sharp CP^2$, which have nontrivial Kirby-Siebenmann invariant, and $(*RP^4) \sharp * CP^2$, which is smoothable [RS97]. Moreover $(*RP^4 \sharp * CP^2) \sharp (S^2 \times S^2) \cong (RP^4 \sharp CP^2) \sharp (S^2 \times S^2)$ [HKT94].

When $\pi \cong Z/4Z$ or $(Z/2Z)^2$ the manifold M is nonorientable, since $\chi(M) = 1$. As the $\mathbb{F}_2$-Hurewicz homomorphism is 0 in these cases Lemma 6.5 does not apply to give any exotic self homotopy equivalences.

If $\pi \cong Z/4Z$ then $[M; G/TOP] \cong (Z/2Z)^2$ and the surgery obstruction groups $L_4(Z/4Z, -)$ and $L_5(Z/4Z, -)$ are both 0, by Theorem 3.4.5 of [Wl76]. Hence $S_{TOP}(M) \cong (Z/2Z)^2$. Since $w_2(M) \neq 0$ there is a homotopy equivalence $f : N \to M$ where $KS(N) \neq KS(M)$. An argument of Fang using [Da95] shows that there is such a manifold N with $KS(N) = 0$ which is not homeomorphic to the geometric example. Thus there are either three or four homeomorphism classes of closed 4-manifolds with $\pi \cong Z/4Z$ and $\chi = 1$. In all cases the orientable double covering space has trivial Kirby-Siebenmann invariant and so is homeomorphic to $(S^2 \times S^2)/(-I, -I)$.

If $\pi \cong (Z/2Z)^2$ then $[M; G/TOP] \cong (Z/2Z)^4$ and the surgery obstruction groups are $L_5((Z/2Z)^2, -) = 0$ and $L_4((Z/2Z)^2, -) = Z/2Z$, by Theorem 3.5.1 of [Wl76]. Since $w_1(M)$ is a split epimorphism $L_4(w_1(M))$ is an isomorphism, so the surgery obstruction is detected by the Kervaire-Arf invariant. As $w_1(M)^2 \neq 0$ we find that $S_{TOP}(M) \cong (Z/2Z)^3$. Thus there are at most 56 homeomorphism classes of closed 4-manifolds with $\pi \cong (Z/2Z)^2$ and $\chi = 1$.

Chapter 13

Geometric decompositions of bundle spaces

We begin by considering which closed 4-manifolds with geometries of euclidean factor type are mapping tori of homeomorphisms of 3-manifolds. We also show that (as an easy consequence of the Kodaira classification of surfaces) a complex surface is diffeomorphic to a mapping torus if and only if its Euler characteristic is 0 and its fundamental group maps onto Z with finitely generated kernel, and we determine the relevant 3-manifolds and diffeomorphisms. In §2 we consider when an aspherical 4-manifold which is the total space of a surface bundle is geometric or admits a geometric decomposition. If the base and fibre are hyperbolic the only known examples are virtually products. In §3 we shall give some examples of torus bundles over closed surfaces which are not geometric, some of which admit geometric decompositions of type $\mathbb{F}^4$ and some of which do not. In §4 we apply some of our earlier results to the characterization of certain complex surfaces. In particular, we show that a complex surfaces fibres smoothly over an aspherical orientable 2-manifold if and only if it is homotopy equivalent to the total space of a surface bundle. In the final two sections we consider first S^1-bundles over geometric 3-manifolds and then the existence of symplectic structures on geometric 4-manifolds.

13.1 Mapping tori

In §3-5 of Chapter 8 and §3 of Chapter 9 we used 3-manifold theory to characterize mapping tori of homeomorphisms of geometric 3-manifolds which have product geometries. Here we shall consider instead which 4-manifolds with product geometries or complex structures are mapping tori.

Theorem 13.1 *Let M be a closed geometric 4-manifold with $\chi(M) = 0$ and such that $\pi = \pi_1(M)$ is an extension of Z by a finitely generated normal subgroup K. Then K is the fundamental group of a geometric 3-manifold.*

Proof Since $\chi(M) = 0$ the geometry must be either an infrasolvmanifold geometry or a product geometry $\mathbb{X}^3 \times \mathbb{E}^1$, where $\mathbb{X}^3$ is one of the 3-dimensional geometries $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{E}^1$ or $\widetilde{\mathbb{SL}}$. If M is an infrasolvmanifold then π is torsion free and virtually poly-Z of Hirsch length 4, so K is torsion free and virtually poly-Z of Hirsch length 3, and the result is clear.

If $\mathbb{X}^3 = \mathbb{S}^3$ then π is a discrete cocompact subgroup of $O(4) \times E(1)$. Since π maps onto Z it must in fact be a subgroup of $O(4) \times R$, and K is a finite subgroup of $O(4)$. Since π acts freely on $S^3 \times R$ the subgroup K acts freely on S^3, and so K is the fundamental group of an $\mathbb{S}^3$-manifold. If $\mathbb{X}^3 = \mathbb{S}^2 \times \mathbb{E}^1$ it follows from Corollary 4.5.3 that $K \cong Z$, $Z \oplus (Z/2Z)$ or D, and so K is the fundamental group of an $\mathbb{S}^2 \times \mathbb{E}^1$-manifold.

In the remaining cases $\mathbb{X}^3$ is of aspherical type. The key point here is that a discrete cocompact subgroup of the Lie group $Isom(\mathbb{X}^3 \times \mathbb{E}^1)$ must meet the radical of this group in a lattice subgroup. Suppose first that $\mathbb{X}^3 = \mathbb{H}^3$. After passing to a subgroup of finite index if necessary, we may assume that $\pi \cong H \times Z < PSL(2,\mathbb{C}) \times R$, where H is a discrete cocompact subgroup of $PSL(2,\mathbb{C})$. If $K \cap (\{1\} \times R) = 1$ then K is commensurate with H, and hence is the fundamental group of an X-manifold. Otherwise the subgroup generated by $K \cap H = K \cap PSL(2,\mathbb{C})$ and $K \cap (\{1\} \times R)$ has finite index in K and is isomorphic to $(K \cap H) \times Z$. Since K is finitely generated so is $K \cap H$, and hence it is finitely presentable, since H is a 3-manifold group. Therefore $K \cap H$ is a PD_2-group and so K is the fundamental group of a $\mathbb{H}^2 \times \mathbb{E}^1$-manifold.

If $\mathbb{X}^3 = \mathbb{H}^2 \times \mathbb{E}^1$ then we may assume that $\pi \cong H \times Z^2 < PSL(2,\mathbb{R}) \times R^2$, where H is a discrete cocompact subgroup of $PSL(2,\mathbb{R})$. Since such groups do not admit nontrivial maps to Z with finitely generated kernel $K \cap H$ must be commensurate with H, and we again see that K is the fundamental group of an $\mathbb{H}^2 \times \mathbb{E}^1$-manifold.

A similar argument applies if $\mathbb{X}^3 = \widetilde{\mathbb{SL}}$. We may assume that $\pi \cong H \times Z$ where H is a discrete cocompact subgroup of $Isom(\widetilde{\mathbb{SL}})$. Since such groups H do not admit nontrivial maps to Z with finitely generated kernel K must be commensurate with H and so is the fundamental group of a $\widetilde{\mathbb{SL}}$-manifold. $\square$

Corollary 13.1.1 *Suppose that M has a product geometry $X \times E^1$. If $\mathbb{X}^3 = \mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{E}^1$, $\widetilde{\mathbb{SL}}$ or $\mathbb{H}^2 \times \mathbb{E}^1$ then M is the mapping torus of an isometry of an $\mathbb{X}^3$-manifold with fundamental group K. (In the latter case we must assume that M is orientable.) If $\mathbb{X}^3 = \mathbb{N}il^3$ or $\mathbb{S}ol^3$ then K is the fundamental group of an $\mathbb{X}^3$-manifold or of a $\mathbb{E}^3$-manifold. If $\mathbb{X}^3 = \mathbb{H}^3$ then K is the fundamental group of a $\mathbb{H}^3$- or $\mathbb{H}^2 \times \mathbb{E}^1$-manifold.*

Proof In all cases π is a semidirect product $K \times_\theta Z$ and may be realised by the mapping torus of a self homeomorphism of a closed 3-manifold with fundamental group K. If this manifold is an $\mathbb{X}^3$-manifold then the outer automorphism class of θ is finite (see Chapter 8) and θ may then be realized by an isometry of an $\mathbb{X}^3$-manifold. Infrasolvmanifolds are determined up to diffeomorphism by their fundamental groups. This is also true of $\mathbb{S}^2\times\mathbb{E}^2$- and $\mathbb{S}^3\times\mathbb{E}^1$-manifolds [Oh90], provided K is not finite cyclic, and $\widetilde{\mathbb{SL}}\times\mathbb{E}^1$- and orientable $\mathbb{H}^2\times\mathbb{E}^2$-manifolds [Ue90, 91]. (Note that $\widetilde{\mathbb{SL}}$-manifolds are orientable and self homeomorphisms of such manifolds are orientation preserving [NR78].) When K is finite cyclic it is still true that every such $\mathbb{S}^3\times\mathbb{E}^1$-manifold is a mapping torus of an isometry of a suitable lens space [Oh90]. Thus if M is an $\mathbb{X}^3\times\mathbb{E}^1$-manifold and K is the fundamental group of an $\mathbb{X}^3$-manifold M is the mapping torus of an isometry of an $\mathbb{X}^3$-manifold with fundamental group K. □

Does the Corollary remain true for nonorientable $\mathbb{H}^2\times\mathbb{E}^2$-manifolds?

There are (orientable) $\mathbb{N}il^3\times\mathbb{E}^1$- and $\mathbb{S}ol^3\times\mathbb{E}^1$-manifolds which are mapping tori of self homeomorphisms of flat 3-manifolds, but which are not mapping tori of self homeomorphisms of $\mathbb{N}il^3$- or $\mathbb{S}ol^3$-manifolds. (See Chapter 8.) There are analogous examples when $\mathbb{X}^3=\mathbb{H}^3$. (See §4 of Chapter 9.)

We may now improve upon the characterization of mapping tori up to homotopy equivalence from Chapter 4.

Theorem 13.2 *Let M be a closed 4-manifold with fundamental group π. Then M is homotopy equivalent to the mapping torus $M(\Theta)$ of a self homeomorphism of a closed 3-manifold with one of the geometries $\mathbb{E}^3$, $\mathbb{N}il^3$, $\mathbb{S}ol^3$, $\mathbb{H}^2\times\mathbb{E}^1$, $\widetilde{\mathbb{SL}}$ or $\mathbb{S}^2\times\mathbb{E}^1$ if and only if*

(1) $\chi(M)=0$;

(2) *π is an extension of Z by an FP_2 normal subgroup K; and*

(3) *K has a nontrivial torsion free abelian normal subgroup A.*

If π is torsion free M is s-cobordant to $M(\Theta)$, while if moreover π is solvable M is homeomorphic to $M(\Theta)$.

Proof The conditions are clearly necessary. Since K has an infinite abelian normal subgroup it has one or two ends. If K has one end then M is aspherical and so K is a PD_3-group by Theorem 4.1. Condition (3) then implies that M' is homotopy equivalent to a closed 3-manifold with one of the first five of the geometries listed above, by Theorem 2.14. If K has two ends then M' is homotopy equivalent to $S^2\times S^1$, $S^2\tilde{\times}S^1$, $RP^2\times S^1$ or $RP^3\sharp RP^3$, by Corollary 4.5.3.

In all cases K is isomorphic to the fundamental group of a closed 3-manifold N which is either Seifert fibred or a $\mathbb{S}ol^3$-manifold, and the outer automorphism class $[\theta]$ determined by the extension may be realised by a self homeomorphism Θ of N. The manifold M is homotopy equivalent to the mapping torus $M(\Theta)$. Since $Wh(\pi)=0$, by Theorems 6.1 and 6.3, any such homotopy equivalence is simple.

If K is torsion free and solvable then π is virtually poly-Z, and so M is homeomorphic to $M(\Theta)$, by Theorem 6.11. Otherwise N is a closed $\mathbb{H}^2\times\mathbb{E}^1$- or $\widetilde{\mathbb{SL}}$-manifold. As $\mathbb{H}^2\times\mathbb{E}^1$ has a metric of nonpositive sectional curvature, the surgery obstruction homomorphisms σ_i^N are isomorphisms for i large in this case, by [FJ93']. This holds also for any irreducible, orientable 3-manifold N such that $\beta_1(N)>0$ [Ro00], and therefore also for all $\widetilde{\mathbb{SL}}$-manifolds, by the Dress induction argument of [NS85]. Comparison of the Mayer-Vietoris sequences for $\mathbb{L}_0$-homology and L-theory (as in Proposition 2.6 of [St84]) shows that σ_i^M and $\sigma_i^{M\times S^1}$ are also isomorphisms for i large, and so $S_{TOP}(M(\Theta)\times S^1)$ has just one element. Therefore M is s-cobordant to $M(\Theta)$. □

Mapping tori of self homeomorphisms of $\mathbb{H}^3$- and $\mathbb{S}^3$-manifolds satisfy conditions (1) and (2). In the hyperbolic case there is the additional condition

(3-H) *K has one end and no noncyclic abelian subgroup.*

If every PD_3-group is a 3-manifold group and the geometrization conjecture for atoroidal 3-manifolds is true then the fundamental groups of closed hyperbolic 3-manifolds may be characterized as PD_3-groups having no noncyclic abelian subgroup. Assuming this, and assuming also that group rings of such hyperbolic groups are regular coherent, Theorem 13.2 could be extended to show that a closed 4-manifold M with fundamental group π is s-cobordant to the mapping torus of a self homeomorphism of a hyperbolic 3-manifold if and only these three conditions hold.

In the spherical case the appropriate additional conditions are

(3-S) *K is a fixed point free finite subgroup of $SO(4)$ and (if K is not cyclic) the characteristic automorphism of K determining π is realized by an isometry of S^3/K*; and

(4-S) *the first nontrivial k-invariant of M is "linear".*

The list of fixed point free finite subgroups of $SO(4)$ is well known. (See Chapter 11.) If K is cyclic or $Q\times Z/p^jZ$ for some odd prime p or T_k^* then the second part of (3-S) and (4-S) are redundant, but the general picture is not yet clear [HM86].

The classification of complex surfaces leads easily to a complete characterization of the 3-manifolds and diffeomorphisms such that the corresponding mapping tori admit complex structures. (Since $\chi(M) = 0$ for any mapping torus M we do not need to enter the imperfectly charted realm of surfaces of general type.)

Theorem 13.3 *Let N be a closed orientable 3-manifold with $\pi_1(N) = \nu$ and let $\theta : N \to N$ be an orientation preserving self diffeomorphism. Then the mapping torus $M(\theta)$ admits a complex structure if and only if one of the following holds:*

(1) *$N = S^3/G$ where G is a fixed point free finite subgroup of $U(2)$ and the monodromy is as described in* [Kt75]*;*

(2) *$N = S^2 \times S^1$ (with no restriction on θ);*

(3) *$N = S^1 \times S^1 \times S^1$ and the image of θ in $SL(3,\mathbb{Z})$ either has finite order or satisfies the equation $(\theta^2 - I)^2 = 0$;*

(4) *N is the flat 3-manifold with holonomy of order 2, θ induces the identity on ν/ν' and the absolute value of the trace of the induced automorphism of $\nu' \cong Z^2$ is at most 2;*

(5) *N is one of the flat 3-manifolds with holonomy cyclic of order 3, 4 or 6 and θ induces the identity on $H_1(N;\mathbb{Q})$;*

(6) *N is a $\mathbb{N}il^3$-manifold and either the image of θ in $Out(\nu)$ has finite order or $M(\theta)$ is a $\mathbb{S}ol_1^4$-manifold;*

(7) *N is a $\mathbb{H}^2 \times \mathbb{E}^1$- or $\widetilde{\mathbb{SL}}$-manifold and the image of θ in $Out(\nu)$ has finite order.*

Proof The mapping tori of these diffeomorphisms admit 4-dimensional geometries, and it is easy to read off which admit complex structures from [Wl86]. In cases (3), (4) and (5) note that a complex surface is Kähler if and only if its first Betti number is even, and so the parity of this Betti number is invariant under passage to finite covers. (See Proposition 4.4 of [Wl86].)

The necessity of these conditions follows from examining the list of complex surfaces X with $\chi(X) = 0$ on page 188 of [BPV], in conjunction with Bogomolov's theorem on surfaces of class VII_0. (See [Tl94] for a clear account of the latter result.) □

In particular, N must be Seifert fibred and most orientable Seifert fibred 3-manifolds (excepting only $RP^3\sharp RP^3$ and the Hantzsche-Wendt flat 3-manifold) occur. Moreover, in most cases (with exceptions as in (3), (4) and (6)) the image of θ in $Out(\nu)$ must have finite order. Some of the resulting 4-manifolds arise as mapping tori in several distinct ways. The corresponding result for complex surfaces of the form $N\times S^1$ for which the obvious smooth S^1-action is holomorphic was given in [GG95]. In [EO94] it is shown that if N is a rational homology 3-sphere then $N\times S^1$ admits a complex structure if and only if N is Seifert fibred, and the possible complex structures on such products are determined.

Conversely, the following result is very satisfactory from the 4-dimensional point of view.

Theorem 13.4 *Let X be a complex surface. Then X is diffeomorphic to the mapping torus of a self diffeomorphism of a closed 3-manifold if and only if $\chi(X)=0$ and $\pi=\pi_1(X)$ is an extension of Z by a finitely generated normal subgroup.*

Proof The conditions are clearly necessary. Sufficiency of these conditions again follows from the classification of complex surfaces, as in Theorem 13.3. □

13.2 Surface bundles and geometries

Let $p:E\to B$ be a bundle with base B and fibre F aspherical closed surfaces. Then p is determined up to bundle isomorphism by the group $\pi=\pi_1(E)$. If $\chi(B)=\chi(F)=0$ then E has geometry $\mathbb{E}^4$, $\mathbb{N}il^3\times\mathbb{E}^1$, $\mathbb{N}il^4$ or $\mathbb{S}ol^3\times\mathbb{E}^1$, by Ue's Theorem. When the fibre is Kb the geometry must be $\mathbb{E}^4$ or $\mathbb{N}il^3\times\mathbb{E}^1$, for then π has a normal chain $\zeta\pi_1(Kb)\cong Z<\sqrt{\pi_1(Kb)}\cong Z^2$, so $\zeta\sqrt{\pi}$ has rank at least 2. Hence a $\mathbb{S}ol^3\times\mathbb{E}^1$- or $\mathbb{N}il^4$-manifold M is the total space of a T-bundle over T if and only if $\beta_1(\pi)=2$. If $\chi(F)=0$ but $\chi(B)<0$ then E need not be geometric. (See Chapter 7 and §3 below.)

We shall assume henceforth that F is hyperbolic, i.e. that $\chi(F)<0$. Then $\zeta\pi_1(F)=1$ and so the characteristic homomorphism $\theta:\pi_1(B)\to Out(\pi_1(F))$ determines π up to isomorphism, by Theorem 5.2.

Theorem 13.5 *Let B and F be closed surfaces with $\chi(B)=0$ and $\chi(F)<0$. Let E be the total space of the F-bundle over B corresponding to a homomorphism $\theta:\pi_1(B)\to Out(\pi_1(F))$. Then E virtually has a geometric decomposition if and only if* $\mathrm{Ker}(\theta)\neq 1$. *Moreover*

(1) *E admits the geometry $\mathbb{H}^2 \times \mathbb{E}^2$ if and only if θ has finite image;*

(2) *E admits the geometry $\mathbb{H}^3 \times \mathbb{E}^1$ if and only if* $\mathrm{Ker}(\theta) \cong Z$ *and* $\mathrm{Im}(\theta)$ *contains the class of a pseudo-Anasov homeomorphism of F;*

(3) *otherwise E is not geometric.*

Proof Let $\pi = \pi_1(E)$. Since E is aspherical, $\chi(E) = 0$ and π is not solvable the only possible geometries are $\mathbb{H}^2 \times \mathbb{E}^2$, $\mathbb{H}^3 \times \mathbb{E}^1$ and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$. If E has a proper geometric decomposition the pieces must all have $\chi = 0$, and the only other geometry that may arise is $\mathbb{F}^4$. In all cases the fundamental group of each piece has a nontrivial abelian normal subgroup.

If $\mathrm{Ker}(\theta) \neq 1$ then E is virtually a cartesian product $N \times S^1$, where N is the mapping torus of a self diffeomorphism ψ of F whose isotopy class in $\pi_0(Diff(F)) \cong Out(\pi_1(F))$ generates a subgroup of finite index in $\mathrm{Im}(\theta)$. Since N is a Haken 3-manifold it has a geometric decomposition and hence so does E. The mapping torus N is an $\mathbb{H}^3$-manifold if and only if ψ is pseudo-Anasov. In that case the action of $\pi_1(N) \cong \pi_1(F) \times_\psi Z$ on H^3 extends to an embedding $p : \pi/\sqrt{\pi} \to Isom(\mathbb{H}^3)$, by Mostow rigidity. Since $\sqrt{\pi} \neq 1$ we may also find a homomorphism $\lambda : \pi \to D < Isom(E^1)$ such that $\lambda(\sqrt{\pi}) \cong Z$. Then $\mathrm{Ker}(\lambda)$ is an extension of Z by F and is commensurate with $\pi_1(N)$, so is the fundamental group of a Haken $\mathbb{H}^3$-manifold, $\widehat{N}$ say. Together these homomorphisms determine a free cocompact action of π on $H^3 \times E^1$. If $\lambda(\pi) \cong Z$ then $M = \pi\backslash(H^3 \times E^1)$ is the mapping torus of a self homeomorphism of $\widehat{N}$; otherwise it is the union of two twisted I-bundles over $\widehat{N}$. In either case it follows from standard 3-manifold theory that since E has a similar structure E and M are diffeomorphic.

If θ has finite image then $\pi/C_\pi(\pi_1(F))$ is a finite extension of $\pi_1(F)$ and so acts properly and cocompactly on $\mathbb{H}^2$. We may therefore construct an $\mathbb{H}^2 \times \mathbb{E}^2$-manifold with group π and which fibres over B as in Theorems 7.3 and 9.8. Since such bundles are determined up to diffeomorphism by their fundamental groups E admits this geometry.

Conversely, if a finite cover of E has a geometric decomposition then we may assume that the cover is itself the total space of a surface bundle over the torus, and so we may assume that E has a geometric decomposition and that $B \cong S^1 \times S^1$. Let $\phi = \pi_1(F)$. Suppose first that E has a proper geometric decomposition. Then $\pi = \pi_1(E) \cong A *_C B$ or $A *_C$, where C is solvable and of Hirsch length 3, and where A is the fundamental group of one of the pieces of E. Note that $\sqrt{A} \neq 1$. Let $\bar{A} = A/A \cap \phi$, $\bar{B} = B/B \cap \phi$ and $\bar{C} = C/C \cap \phi$. Then $\bar{\pi} = \pi/\phi \cong Z^2$ has a similar decomposition as $\bar{A} *_{\bar{C}} \bar{B}$ or $\bar{A} *_{\bar{C}}$. Now

$C\cap\phi=1$ or Z, since $\chi(F)<0$. Hence $\bar{C}\cong Z^2$ and so $\bar{A}=\bar{C}=\bar{B}$. In particular, $\mathrm{Im}(\theta)=\theta(A)$. But as $\sqrt{A}\cap\phi\leq\sqrt{\phi}=1$ and $\sqrt{A}$ and $A\cap\phi$ are normal subgroups of A it follows that $\sqrt{A}$ and $A\cap\phi$ commute. Hence $\theta(A)$ is a quotient of $A/\sqrt{A}.(A\cap\phi)$, which is abelian of rank at most 1, and so $\mathrm{Ker}(\theta)\neq 1$.

If E admits the geometry $\mathbb{H}^2\times\mathbb{E}^2$ then $\sqrt{\pi}=\pi\cap Rad(Isom(\mathbb{H}^2\times\mathbb{E}^2))=\pi\cap(\{1\}\times R^2)\cong Z^2$, by Proposition 8.27 of [Rg]. Hence θ has finite image.

If E admits the geometry $\mathbb{H}^3\times\mathbb{E}^1$ then $\sqrt{\pi}=\pi\cap(\{1\}\times R)\cong Z$, by Proposition 8.27 of [Rg]. Hence $\mathrm{Ker}(\theta)\cong Z$ and E is finitely covered a cartesian product $N\times S^1$, where N is a hyperbolic 3-manifold which is also an F-bundle over S^1. The geometric monodromy of the latter bundle is a pseudo-Anasov diffeomorphism of F whose isotopy class is in $\mathrm{Im}(\theta)$.

If ρ is the group of a $\widetilde{\mathbb{SL}}\times\mathbb{E}^1$-manifold then $\sqrt{\rho}\cong Z^2$ and $\sqrt{\rho}\cap K'\neq 1$ for all subgroups K of finite index, and so E cannot admit this geometry. □

In particular, if $\chi(B)=0$ and θ is injective E admits no geometric decomposition.

We shall assume henceforth that B is also hyperbolic. Then $\chi(E)>0$ and $\pi_1(E)$ has no solvable subgroups of Hirsch length 3. Hence the only possible geometries on E are $\mathbb{H}^2\times\mathbb{H}^2$, $\mathbb{H}^4$ and $\mathbb{H}^2(\mathbb{C})$. (These are the least well understood geometries, and little is known about the possible fundamental groups of the corresponding 4-manifolds.)

Theorem 13.6 *Let B and F be closed hyperbolic surfaces, and let E be the total space of the F-bundle over B corresponding to a homomorphism $\theta:\pi_1(B)\to Out(\pi_1(F))$. Then the following are equivalent:*

(1) *E admits the geometry $\mathbb{H}^2\times\mathbb{H}^2$;*

(2) *E is finitely covered by a cartesian product of surfaces;*

(3) *θ has finite image.*

If $\mathrm{Ker}(\theta)\neq 1$ then E does not admit either of the geometries $\mathbb{H}^4$ or $\mathbb{H}^2(\mathbb{C})$.

Proof Let $\pi=\pi_1(E)$ and $\phi=\pi_1(F)$. If E admits the geometry $\mathbb{H}^2\times\mathbb{H}^2$ it is virtually a cartesian product, by Corollary 9.8.1, and so (1) implies (2).

If π is virtually a direct product of PD_2-groups then $[\pi:C_\pi(\phi)]<\infty$, by Theorem 5.4. Therefore the image of θ is finite and so (2) implies (3).

If θ has finite image then $\mathrm{Ker}(\theta) \neq 1$ and $\pi/C_\pi(\phi)$ is a finite extension of ϕ. Hence there is a homomorphism $p : \pi \to Isom(\mathbb{H}^2)$ with kernel $C_\pi(\phi)$ and with image a discrete cocompact subgroup. Let $q : \pi \to \pi_1(B) < Isom(\mathbb{H}^2)$. Then (p,q) embeds π as a discrete cocompact subgroup of $Isom(\mathbb{H}^2 \times \mathbb{H}^2)$, and the closed 4-manifold $M = \pi\backslash(H^2 \times H^2)$ clearly fibres over B. Such bundles are determined up to diffeomorphism by the corresponding extensions of fundamental groups, by Theorem 5.2. Therefore E admits the geometry $\mathbb{H}^2 \times \mathbb{H}^2$ and so (3) implies (1).

If θ is not injective $Z^2 < \pi$ and so E cannot admit either of the geometries $\mathbb{H}^4$ or $\mathbb{H}^2(\mathbb{C})$, by Theorem 9 of [Pr43]. $\square$

The mapping class group of a closed orientable surface has only finitely many conjugacy classes of finite groups [Ha71]. With the finiteness result for $\mathbb{H}^4$- and $\mathbb{H}^2(\mathbb{C})$-manifolds of [Wa72], this implies that only finitely many orientable bundle spaces with given Euler characteristic are geometric. In Corollary 13.7.2 we shall show that no such bundle space is homotopy equivalent to a $\mathbb{H}^2(\mathbb{C})$-manifold. Is there one which admits the geometry $\mathbb{H}^4$? If $\mathrm{Im}(\theta)$ contains the outer automorphism class determined by a Dehn twist on F then E admits no metric of nonpositive sectional curvature [KL96].

If E has a proper geometric decomposition the pieces are reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds and the inclusions of the cusps induce monomorphisms on π_1. Must E be a $\mathbb{H}^2 \times \mathbb{H}^2$-manifold?

Every closed orientable $\mathbb{H}^2 \times \mathbb{H}^2$-manifold has a 2-fold cover which is a complex surface, and has signature 0. Conversely, if E is a complex surface and p is a holomorphic submersion then $\sigma(E) = 0$ implies that the fibres are isomorphic, and so E is an $\mathbb{H}^2 \times \mathbb{H}^2$-manifold [Ko99]. This is also so if p is a holomorphic fibre bundle (see §V.6 of [BPV]). Any holomorphic submersion with base of genus at most 1 or fibre of genus at most 2 is a holomorphic fibre bundle [Ks68]. There are such holomorphic submersions in which $\sigma(E) \neq 0$ and so which are not virtually products. (See §V.14 of [BPV].) The image of θ must contain the outer automorphism class determined by a pseudo-Anasov homeomorphism and not be virtually abelian [Sh97].

Orientable $\mathbb{H}^4$-manifolds also have signature 0, but no closed $\mathbb{H}^4$-manifold admits a complex structure.

If B and E are orientable $\sigma(E) = -\theta^*\tau \cap [B]$, where $\tau \in H^2(Out(\pi_1(F)); \mathbb{Z})$ is induced from a universal class in $H^2(Sp_{2g}(\mathbb{Z}); \mathbb{Z})$ via the natural representation of $Out(\pi_1(F))$ as symplectic isometries of the intersection form on $H_1(F; \mathbb{Z}) \cong Z^{2g}$ [Me73]. In particular, if $g = 2$ then $\sigma(E) = 0$. Does the genus 2 mapping class group contain any subgroups which are hyperbolic PD_2-groups?

13.3 Geometric decompositions of torus bundles

In this section we shall give some examples of torus bundles over closed surfaces which are not geometric, some of which admit geometric decompositions of type $\mathbb{F}^4$ and some of which do not. If M is a compact manifold with boundary whose interior is an $\mathbb{F}^4$-manifold of finite volume then $\pi_1(M)$ is a semidirect product $Z^2 \times_\theta F$ where $\theta : F \to GL(2,\mathbb{Z})$ is a monomorphism with image of finite index. The double $DM = M \cup_\partial M$ is fibred over a hyperbolic base but is not geometric, since $\sqrt{\pi} \cong Z^2$ but $[\pi : C_\pi(\sqrt{\pi})]$ is infinite. The orientable surface of genus 2 can be represented as a double in two distinct ways; we shall give corresponding examples of nongeometric torus bundles which admit geometric decompositions of type $\mathbb{F}^4$. (Note that $\mathbb{F}^4$-manifolds are Seifert fibred with base a punctured hyperbolic orbifold.)

1. Let $F(2)$ be the free group of rank two and let $\gamma : F(2) \to SL(2,\mathbb{Z})$ have image the commutator subgroup $SL(2,\mathbb{Z})'$, which is freely generated by $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$. The natural surjection from $SL(2,\mathbb{Z})$ to $PSL(2,\mathbb{Z})$ induces an isomorphism of commutator subgroups. (See §2 of Chapter 1.) The parabolic subgroup $PSL(2,\mathbb{Z})' \cap Stab(0)$ is generated by the image of $\left(\begin{smallmatrix} -1 & 0 \\ -6 & -1 \end{smallmatrix}\right)$. Hence $[Stab(0) : PSL(2,\mathbb{Z})' \cap Stab(0)] = 6 = [PSL(2,\mathbb{Z}) : PSL(2,\mathbb{Z})']$, and so $PSL(2,\mathbb{Z})'$ has a single cusp at 0. The quotient space $PSL(2,\mathbb{Z})'\backslash H^2$ is the once-punctured torus. Let $N \subset PSL(2,\mathbb{Z})'\backslash H^2$ be the complement of an open horocyclic neighbourhood of the cusp. The double DN is the closed orientable surface of genus 2. The semidirect product $\Gamma = Z^2 \times_\gamma F(2)$ is a lattice in $Isom(\mathbb{F}^4)$, and the double of the bounded manifold with interior $\Gamma\backslash F^4$ is a torus bundle over DN.

2. Let $\delta : F(2) \to SL(2,\mathbb{Z})$ have image the subgroup which is freely generated by $U = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$ and $V = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$. Let $\bar\delta : F(2) \to PSL(2,\mathbb{Z})$ be the composed map. Then $\bar\delta$ is injective and $[PSL(2,\mathbb{Z}) : \bar\delta(F(2))] = 6$. (Note that $\delta(F(2))$ and $-I$ together generate the level 2 congruence subgroup.) Moreover $[Stab(0) : \bar\delta(F(2)) \cap Stab(0)] = 2$. Hence $\bar\delta(F(2))$ has three cusps, at 0, ∞ and 1, and $\bar\delta(F(2))\backslash H^2$ is the thrice-punctured sphere. The corresponding parabolic subgroups are generated by U, V and VU^{-1}, respectively. Doubling the complement N of disjoint horocyclic neighbourhoods of the cusps in $\bar\delta(F(2))\backslash H^2$ again gives a closed orientable surface of genus 2. The presentation for $\pi_1(DN)$ derived from this construction is

$$\langle U, V, U_1, V_1, s, t \mid s^{-1}Us = U_1, t^{-1}Vt = V_1, VU^{-1} = V_1U_1^{-1}\rangle,$$

which simplifies to the usual presentation $\langle U, V, s, t \mid s^{-1}V^{-1}sV = t^{-1}U^{-1}tU\rangle$. The semidirect product $\Delta = Z^2 \times_\delta F(2)$ is a lattice in $Isom(\mathbb{F}^4)$, and the double of the bounded manifold with interior $\Delta\backslash F^4$ is again a torus bundle over DN.

3. If G is an orientable PD_2-group which is not virtually Z^2 and $\lambda : G \to SL(2,\mathbb{Z})$ is a homomorphism whose image is infinite cyclic then $\pi = Z^2 \times_\lambda G$ is the fundamental group of a closed orientable 4-manifold which is fibred over an orientable hyperbolic surface but which has no geometric decomposition at all. (The only possible geometries are $\mathbb{F}^4$, $\mathbb{H}^2 \times \mathbb{E}^2$ and $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$. We may exclude pieces of type $\mathbb{F}^4$ as $Im(\lambda)$ has infinite index in $SL(2,\mathbb{Z})$, and we may exclude pieces of type $\mathbb{H}^2 \times \mathbb{E}^2$ or $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$ as $Im(\lambda) \cong Z$ is not generated by finite subgroups.)

13.4 Complex surfaces and fibrations

It is an easy consequence of the classification of surfaces that a minimal compact complex surface S is ruled over a curve C of genus ≥ 2 if and only if $\pi_1(S) \cong \pi_1(C)$ and $\chi(S) = 2\chi(C)$. (See Chapter VI of [BPV].) We shall give a similar characterization of the complex surfaces which admit holomorphic submersions to complex curves of genus ≥ 2, and more generally of quotients of such surfaces by free actions of finite groups. However we shall use the classification only to handle the cases of non-Kähler surfaces.

Theorem 13.7 *Let S be a complex surface. Then S has a finite covering space which admits a holomorphic submersion onto a complex curve, with base and fibre of genus ≥ 2, if and only if $\pi = \pi_1(S)$ has normal subgroups $K < \hat{\pi}$ such that K and $\hat{\pi}/K$ are PD_2^+-groups, $[\pi : \hat{\pi}] < \infty$ and $[\pi : \hat{\pi}]\chi(S) = \chi(K)\chi(\hat{\pi}/K) > 0$.*

Proof The conditions are clearly necessary. Suppose that they hold. Then S is aspherical, by Theorem 5.2. In particular, π is torsion free and $\pi_2(S) = 0$, so S is minimal. After enlarging K if necessary we may assume that π/K has no nontrivial finite normal subgroup. Let $\widehat{S}$ be the finite covering space corresponding to $\hat{\pi}$. Then $\beta_1(\widehat{S}) \geq 4$. If $\beta_1(\widehat{S})$ were odd then $\widehat{S}$ would be minimal properly elliptic, by the classification of surfaces. But then either $\chi(S) = 0$ or $\widehat{S}$ would have a singular fibre and the projection of $\widehat{S}$ to the base curve would induce an isomorphism on fundamental groups [CZ79]. Hence $\beta_1(\widehat{S})$ is even and so $\widehat{S}$ and S are Kähler (see Theorem 4.3 of [Wl86]). Since π/K is not virtually Z^2 it is isomorphic to a discrete group of isometries of the upper half plane $\mathbb{H}^2$ and $\beta_1^{(2)}(\pi/K) \neq 0$. Hence there is a properly discontinuous holomorphic action of π/K on $\mathbb{H}^2$ and a π/K-equivariant holomorphic map from the covering space S_K to $\mathbb{H}^2$, with connected fibres, by Theorems 4.1 and 4.2 of [ABR92]. Let B and $\widehat{B}$ be the complex curves $\mathbb{H}^2/(\pi/K)$ and $\mathbb{H}^2/(\hat{\pi}/K)$, respectively, and let $h : S \to B$ and $\hat{h} : \widehat{S} \to \widehat{B}$ be the induced maps. The quotient map from $\mathbb{H}^2$ to $\hat{B}$ is a covering projection, since $\hat{\pi}/K$ is torsion free, and so $\pi_1(\hat{h})$ is an epimorphism with kernel K.

The map h is a submersion away from the preimage of a finite subset $D \subset B$. Let F be the general fibre and F_d the fibre over $d \in D$. Fix small disjoint discs $\Delta_d \subset B$ about each point of D, and let $B^* = B - \cup_{d\in D}\Delta_d$, $S^* = h^{-1}(B^*)$ and $S_d = h^{-1}(\Delta_d)$. Since $h|_{S^*}$ is a submersion $\pi_1(S^*)$ is an extension of $\pi_1(B^*)$ by $\pi_1(F)$. The inclusion of ∂S_d into $S_d - F_d$ is a homotopy equivalence. Since F_d has real codimension 2 in S_d the inclusion of $S_d - F_d$ into S_d is 2-connected. Hence $\pi_1(\partial S_d)$ maps onto $\pi_1(S_d)$.

Let $m_d = [\pi_1(F_d)] : Im(\pi_1(F))]$. After blowing up S at singular points of F_d we may assume that it has only normal crossings. We may then pull $h|_{S_d}$ back over a suitable branched covering of Δ_d to obtain a singular fibre $\widetilde{F}_d$ with no multiple components and only normal crossing singularities. In that case $\widetilde{F}_d$ is obtained from F by shrinking vanishing cycles, and so $\pi_1(F)$ maps onto $\pi_1(\widetilde{F}_d)$. Since blowing up a point on a curve does not change the fundamental group it follows from §9 of Chapter III of [BPV] that in general m_d is finite.

We may regard B as an orbifold with cone singularities of order m_d at $d \in D$. By the Van Kampen theorem (applied to the space S and the orbifold B) the image of $\pi_1(F)$ in π is a normal subgroup and h induces an isomorphism from $\pi/\pi_1(F)$ to $\pi_1^{orb}(B)$. Therefore the kernel of the canonical map from $\pi_1^{orb}(B)$ to $\pi_1(B)$ is isomorphic to $K/\mathrm{Im}(\pi_1(F))$. But this is a finitely generated normal subgroup of infinite index in $\pi_1^{orb}(B)$, and so must be trivial. Hence $\pi_1(F)$ maps onto K, and so $\chi(F) \leq \chi(K)$.

Let $\widehat{D}$ be the preimage of D in $\widehat{B}$. The general fibre of $\hat{h}$ is again F. Let $\widehat{F}_d$ denote the fibre over $d \in \widehat{D}$. Then $\chi(\widehat{S}) = \chi(F)\chi(B)+\Sigma_{d\in\widehat{D}}(\chi(\widehat{F}_d)-\chi(F))$ and $\chi(\widehat{F}_d) \geq \chi(F)$, by Proposition III.11.4 of [BPV]. Moreover $\chi(\widehat{F}_d) > \chi(F)$ unless $\chi(\widehat{F}_d) = \chi(F) = 0$, by Remark III.11.5 of [BPV]. Since $\chi(\widehat{B}) = \chi(\hat{\pi}/K) < 0$, $\chi(\widehat{S}) = \chi(K)\chi(\hat{\pi}/K)$ and $\chi(F) \leq \chi(K)$ it follows that $\chi(F) = \chi(K) < 0$ and $\chi(\widehat{F}_d) = \chi(F)$ for all $d \in \widehat{D}$. Therefore $\widehat{F}_d \cong F$ for all $d \in \widehat{D}$ and so $\hat{h}$ is a holomorphic submersion. □

Similar results have been found independently by Kapovich and Kotschick [Ka98, Ko99]. Kapovich assumes instead that K is FP_2 and S is aspherical. As these hypotheses imply that K is a PD_2-group, by Theorem 1.19, the above theorem applies.

We may construct examples of such surfaces as follows. Let $n > 1$ and C_1 and C_2 be two curves such that Z/nZ acts freely on C_1 and with isolated fixed points on C_2. Then the quotient S of $C_1 \times C_2$ under the induced action is a complex surface and the projection from $C_1 \times C_2$ to C_2 induces a surjective holomorphic mappping from S to $C_2/(Z/nZ)$ with critical values corresponding to the fixed points.

Corollary 13.7.1 *The surface S admits such a holomorphic submersion onto a complex curve if and only if π/K is a PD_2^+-group.* □

Corollary 13.7.2 *No bundle space E is homotopy equivalent to a closed $\mathbb{H}^2(\mathbb{C})$-manifold.*

Proof Since $\mathbb{H}^2(\mathbb{C})$-manifolds have 2-fold coverings which are complex surfaces, we may assume that E is homotopy equivalent to a complex surface S. By the theorem, S admits a holomorphic submersion onto a complex curve. But then $\chi(S) > 3\sigma(S)$ [Li96], and so S cannot be a $\mathbb{H}^2(\mathbb{C})$-manifold. □

The relevance of Liu's work was observed by Kapovich, who has also found a cocompact $\mathbb{H}^2(\mathbb{C})$-lattice which is an extension of a PD_2^+-group by a finitely generated normal subgroup, but which is not almost coherent [Ka98].

Similar arguments may be used to show that a Kähler surface S is a minimal properly elliptic surface with no singular fibres if and only if $\chi(S) = 0$ and $\pi = \pi_1(S)$ has a normal subgroup $A \cong Z^2$ such that π/A is virtually torsion free and indicable, but is not virtually abelian. (This holds also in the non-Kähler case as a consequence of the classification of surfaces.) Moreover, if S is not a ruled surface then it is a complex torus, a hyperelliptic surface, an Inoue surface, a Kodaira surface or a minimal elliptic surface if and only if $\chi(S) = 0$ and $\pi_1(S)$ has a normal subgroup A which is poly-Z and not cyclic, and such that π/A is infinite and virtually torsion free indicable. (See Theorem X.5 of [H2].)

We may combine Theorem 13.7 with some observations deriving from the classification of surfaces for our second result.

Theorem 13.8 *Let S be a complex surface such that $\pi = \pi_1(S) \neq 1$. If S is homotopy equivalent to the total space E of a bundle over a closed orientable 2-manifold then S is diffeomorphic to E.*

Proof Let B and F be the base and fibre of the bundle, respectively. Suppose first that $\chi(F)=2$. Then $\chi(B)\leq 0$, for otherwise S would be simply-connected. Hence $\pi_2(S)$ is generated by an embedded S^2 with self-intersection 0, and so S is minimal. Therefore S is ruled over a curve diffeomorphic to B, by the classification of surfaces.

Suppose next that $\chi(B)=2$. If $\chi(F)=0$ and $\pi\not\cong Z^2$ then $\pi\cong Z\oplus(Z/nZ)$ for some $n>0$. Then S is a Hopf surface and so is determined up to diffeomorphism by its homotopy type, by Theorem 12 of [Kt75]. If $\chi(F)=0$ and $\pi\cong Z^2$ or if $\chi(F)<0$ then S is homotopy equivalent to $S^2\times F$, so $\chi(S)<0$, $w_1(S)=w_2(S)=0$ and S is ruled over a curve diffeomorphic to F. Hence E and S are diffeomorphic to $S^2\times F$.

In the remaining cases E and F are both aspherical. If $\chi(F)=0$ and $\chi(B)\leq 0$ then $\chi(S)=0$ and π has one end. Therefore S is a complex torus, a hyperelliptic surface, an Inoue surface, a Kodaira surface or a minimal properly elliptic surface. (This uses Bogomolov's theorem on class VII_0 surfaces [Te94].) The Inoue surfaces are mapping tori of self-diffeomorphisms of $S^1\times S^1\times S^1$, and their fundamental groups are not extensions of Z^2 by Z^2, so S cannot be an Inoue surface. As the other surfaces are Seifert fibred 4-manifolds E and S are diffeomorphic, by [Ue91].

If $\chi(F)<0$ and $\chi(B)=0$ then S is a minimal properly elliptic surface. Let A be the normal subgroup of the general fibre in an elliptic fibration. Then $A\cap\pi_1(F)=1$ (since $\pi_1(F)$ has no nontrivial abelian normal subgroup) and so $[\pi:A.\pi_1(F)]<\infty$. Therefore E is finitely covered by a cartesian product $T\times F$, and so is Seifert fibred. Hence E and S are diffeomorphic, by [Ue].

The remaining case ($\chi(B)<0$ and $\chi(F)<0$) is an immediate consequence of Theorem 13.7, since such bundles are determined by the corresponding extensions of fundamental groups (see Theorem 5.2). □

A simply-connected smooth 4-manifold which fibres over a 2-manifold must be homeomorphic to $CP^1\times CP^1$ or $CP^2\sharp\overline{CP^2}$. (See Chapter 12.) Is there such a surface of general type? (No surface of general type is diffeomorphic to $CP^1\times CP^1$ or $CP^2\sharp\overline{CP^2}$ [Qi93].)

Corollary 13.8.1 *If moreover the base has genus* 0 *or* 1 *or the fibre has genus* 2 *then* S *is finitely covered by a cartesian product.*

Proof A holomorphic submersion with fibre of genus 2 is the projection of a holomorphic fibre bundle and hence S is virtually a product, by [Ks68]. □

Up to deformation there are only finitely many algebraic surfaces with given Euler characteristic > 0 which admit holomorphic submersions onto curves [Pa68]. By the argument of the first part of Theorem 13.1 this remains true without the hypothesis of algebraicity, for any such complex surface must be Kähler, and Kähler surfaces are deformations of algebraic surfaces (see Theorem 4.3 of [Wl86]). Thus the class of bundles realized by complex surfaces is very restricted. Which extensions of PD_2^+-groups by PD_2^+-groups are realized by complex surfaces (i.e., not necessarily aspherical)?

The equivalence of the conditions "S is ruled over a complex curve of genus ≥ 2", "$\pi = \pi_1(S)$ is a PD_2^+-group and $\chi(S) = 2\chi(\pi) < 0$" and "$\pi_2(S) \cong Z$, π acts trivially on $\pi_2(S)$ and $\chi(S) < 0$" also follows by an argument similar to that used in Theorems 13.7 and 13.8. (See Theorem X.6 of [H2].)

If $\pi_2(S) \cong Z$ and $\chi(S) = 0$ then π is virtually Z^2. The finite covering space with fundamental group Z^2 is Kähler, and therefore so is S. Since $\beta_1(S) > 0$ and is even, we must have $\pi \cong Z^2$, and so S is either ruled over an elliptic curve or is a minimal properly elliptic surface, by the classification of complex surfaces. In the latter case the base of the elliptic fibration is CP^1, there are no singular fibres and there are at most 3 multiple fibres. (See [Ue91].) Thus S may be obtained from a cartesian product $CP^1 \times E$ by logarithmic transformations. (See §V.13 of [BPV].) Must S in fact be ruled?

If $\pi_2(S) \cong Z$ and $\chi(S) > 0$ then $\pi = 1$, by Theorem 10.1. Hence $S \simeq CP^2$ and so S is analytically isomorphic to CP^2, by a result of Yau (see Theorem I.1 of [BPV]).

13.5 S^1-Actions and foliations by circles

For each of the geometries $\mathbb{X}^4 = \mathbb{S}^3 \times \mathbb{E}^1$, $\mathbb{H}^3 \times \mathbb{E}^1$, $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$, $\mathbb{N}il^3 \times \mathbb{E}^1$, $\mathbb{S}ol^3 \times \mathbb{E}^1$, $\mathbb{N}il^4$ and $\mathbb{S}ol_1^4$ the real line R is a characteristic subgroup of the radical of $Isom(\mathbb{X}^4)$. (However the translation subgroup of the euclidean factor is *not* characteristic if $\mathbb{X}^4 = \widetilde{\mathbb{SL}} \times \mathbb{E}^1$ or $\mathbb{N}il^3 \times \mathbb{E}^1$.) The corresponding closed geometric 4-manifolds are foliated by circles, and the leaf space is a geometric 3-orbifold, with geometry $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{E}^1$, $\mathbb{E}^3$, $\mathbb{S}ol^3$, $\mathbb{N}il^3$ and $\mathbb{S}ol^3$, respectively. In each case it may be verified that if π is a lattice in $Isom(\mathbb{X}^4)$ then $\pi \cap R \cong Z$. As this characteristic subgroup is central in the identity component of the isometry group such manifolds have double coverings which admit S^1-actions without fixed points. These actions lift to principal S^1-actions (without exceptional orbits) on suitable finite covering spaces. (This does not hold for all S^1-actions. For instance, S^3 admits non-principal S^1-actions without fixed points.)

Closed $\mathbb{E}^4$-, $\mathbb{S}^2\times\mathbb{E}^2$- or $\mathbb{H}^2\times\mathbb{E}^2$-manifolds all have finite covering spaces which are cartesian products with S^1, and thus admit principal S^1-actions. However these actions are not canonical. (There are also non-canonical S^1-actions on many $\widetilde{\mathbb{SL}}\times\mathbb{E}^1$- and $\mathbb{N}il^3\times\mathbb{E}^1$-manifolds.) No other closed geometric 4-manifold is finitely covered by the total space of an S^1-bundle. For if a closed manifold M is foliated by circles then $\chi(M)=0$. This excludes all other geometries except $\mathbb{S}ol^4_{m,n}$ and $\mathbb{S}ol^4_0$. If moreover M is the total space of an S^1-bundle and is aspherical then $\pi_1(M)$ has an infinite cyclic normal subgroup. As lattices in $Isom(\mathbb{S}ol^4_{m,n})$ or $Isom(\mathbb{S}ol^4_0)$ do not have such subgroups these geometries are excluded also. Does every geometric 4-manifold M with $\chi(M)=0$ nevertheless admit a foliation by circles?

In particular, a complex surface has a foliation by circles if and only if it admits one of the above geometries. Thus it must be Hopf, hyperelliptic, Inoue of type $S^{\pm}_{N\dots}$, Kodaira, minimal properly elliptic, ruled over an elliptic curve or a torus. With the exception of some algebraic minimal properly elliptic surfaces and the ruled surfaces over elliptic curves with $w_2\neq 0$ all such surfaces admit S^1-actions without fixed points.

Conversely, the total space E of an S^1-orbifold bundle ξ over a geometric 3-orbifold is geometric, except when the base B has geometry $\mathbb{H}^3$ or $\widetilde{\mathbb{SL}}$ and the characteristic class $c(\xi)$ has infinite order. More generally, E has a (proper) geometric decomposition if and only if B is a $\widetilde{\mathbb{SL}}$-orbifold and $c(\xi)$ has finite order or B has a (proper) geometric decomposition and the restrictions of $c(\xi)$ to the hyperbolic pieces of B each have finite order.

Total spaces of circle bundles over aspherical Seifert fibred 3-manifolds and $\mathbb{S}ol^3$-manifolds have a characterization parallel to that of Theorem 13.2.

Theorem 13.9 *Let M be a closed 4-manifold with fundamental group π. Then:*

(1) *M is simple homotopy equivalent to the total space E of an S^1-bundle over an aspherical closed Seifert fibred 3-manifold or a $\mathbb{S}ol^3$-manifold if and only if $\chi(M)=0$ and π has normal subgroups $A<B$ such that $A\cong Z$, π/A is torsion free and B/A is abelian.*
If $B/A\cong Z$ and is central in π/A then M is s-cobordant to E. If B/A has rank at least 2 then M is homeomorphic to E.

(2) *M is s-cobordant to the total space E of an S^1-bundle over the mapping torus of a self homeomorphism of an aspherical surface if and only if $\chi(M)=0$ and π has normal subgroups $A<B$ such that $A\cong Z$, π/A is torsion free, B is FP_2 and $\pi/B\cong Z$.*

Proof (1) The conditions are clearly necessary. If they hold then $h(\sqrt{\pi}) \geq h(B/A) + 1 \geq 2$, and so M is aspherical. If $h(\sqrt{\pi}) = 2$ then $\sqrt{\pi} \cong Z^2$, by Theorem 9.2. Hence $B/A \cong Z$ and $H^2(\pi/B; \mathbb{Z}[\pi/B]) \cong Z$, so π/B is virtually a PD_2-group, by Bowditch's Theorem. Since π/A is torsion free it is a PD_3-group, and so is the fundamental group of a closed Seifert fibred 3-manifold, N say, by Theorem 2.14. As $Wh(\pi) = 0$, by Theorem 6.4, M is simple homotopy equivalent to the total space E of an S^1-bundle over N. If moreover B/A is central in π/A then N admits an effective S^1-action, and $E \times S^1$ is an $S^1 \times S^1$-bundle over N. Hence $M \times S^1$ is homeomorphic to $E \times S^1$ (see Remark 3.4 of [NS85]), and so M is s-cobordant to E.

If B/A has rank at least 2 then $h(\sqrt{\pi}) > 2$ and so π is virtually poly-Z. Hence π/A is the fundamental group of a $\mathbb{E}^3$-, $\mathbb{N}il^3$- or $\mathbb{S}ol^3$-manifold and M is homeomorphic to such a bundle space E, by Theorem 6.11.

(2) The conditions are again necessary. If they hold then B/A is infinite, so B has one end and hence is a PD_3-group, by Theorem 4.1. Since B/A is torsion free it is a PD_2-group, by Bowditch's Theorem, and so π/A is the fundamental group of a mapping torus, N say. As $Wh(\pi) = 0$, by Theorem 6.4, M is simple homotopy equivalent to the total space E of an S^1-bundle over N. Since $\pi \times Z$ is square root closed accessible $M \times S^1$ is homeomorphic to $E \times S^1$ [Ca73], and so M is s-cobordant to E. □

If $B/A \cong Z$ and π/B acts nontrivially on B/A is M s-cobordant to E?

Simple homotopy equivalence implies s-cobordism for such bundles over other Haken bases (with square root closed accessible fundamental group or with $\beta_1 > 0$ and orientable) using [Ca73] or [Ro00]. However we do not yet have good intrinsic characterizations of the fundamental groups of such 3-manifolds.

If M fibres over a hyperbolic 3-manifold N then $\chi(M) = 0$, $\sqrt{\pi} \cong Z$ and $\pi/\sqrt{\pi}$ has one end, finite cohomological dimension and no noncyclic abelian subgroups. Conversely if π satisfies these conditions then $\rho = \pi/\sqrt{\pi}$ is a PD_3-group, by Theorem 4.12, and $\sqrt{\rho} = 1$. It may be conjectured that every such PD_3-group (with no nocyclic abelian subgroups and trivial Hirsch-Plotkin radical) is the fundamental group of a closed hyperbolic 3-manifold. If so, Theorem 13.9 may be extended to a characterization of such 4-manifolds up to s-cobordism, using Theorem 10.7 of [FJ89] instead of [NS85].

13.6 Symplectic structures

If M is a closed orientable 4-manifold which fibres over an orientable surface and the image of the fibre in $H_2(M; \mathbb{R})$ is nonzero then M has a symplectic structure [Th76]. The homological condition is automatic unless the fibre is a torus; some such condition is needed, as $S^3 \times S^1$ is the total space of a T-bundle over S^2 but $H^2(S^3 \times S^1; \mathbb{R}) = 0$, so it has no symplectic structure. If the base is also a torus then M admits a symplectic structure [Ge92]. Closed Kähler manifolds have natural symplectic structures. Using these facts, it is easy to show for most geometries that either every closed geometric manifold is finitely covered by one admitting a symplectic structure or no closed geometric manifold admits any symplectic structure.

If M is orientable and admits one of the geometries $\mathbb{CP}^2$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{E}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{E}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$ or $\mathbb{H}^2(\mathbb{C})$ then it has a 2-fold cover which is Kähler, and therefore symplectic. If it admits $\mathbb{E}^4$, $\mathbb{N}il^4$, $\mathbb{N}il^3 \times \mathbb{E}^1$ or $\mathbb{S}ol^3 \times \mathbb{E}^1$ then it has a finite cover which fibres over the torus, and therefore is symplectic. If all $\mathbb{H}^3$-manifolds are virtually mapping tori then $\mathbb{H}^3 \times \mathbb{E}^1$-manifolds would also be virtually symplectic. However, the question is not settled for this geometry.

As any closed orientable manifold with one of the geometries $\mathbb{S}^4$, $\mathbb{S}^3 \times \mathbb{E}^1$, $\mathbb{S}ol^4_{m,n}$ (with $m \neq n$), $\mathbb{S}ol^4_0$ or $\mathbb{S}ol^4_1$ has $\beta_2 = 0$ no such manifold can be symplectic. Nor are closed $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifolds [Et01]. The question appears open for the geometry $\mathbb{H}^4$, as is the related question about bundles. (Note that symplectic 4-manifolds with index 0 have Euler characteristic divisible by 4, by Corollary 10.1.10 of [GS]. Hence covering spaces of odd degree of the Davis 120-cell space provide many examples of nonsymplectic $\mathbb{H}^4$-manifolds.)

If N is a 3-manifold which is a mapping torus then $S^1 \times N$ fibres over T, and so admits a symplectic structure. Taubes has asked whether the converse is true; if $S^1 \times N$ admits a symplectic structure must N fibre over S^1? More generally, one might ask which 4-dimensional mapping tori and S^1-bundles are symplectic?

Which manifolds with geometric decompositions are symplectic?