Part I : PL Topology

1 Introduction

This book gives an exposition of: the triangulation problem for a topological manifold in dimensions strictly greater than four; the smoothing problem for a piecewise-linear manifold; and, finally, of some of Sullivan's ideas about the topological resolution of singularities.

The book is addressed to readers who, having a command of the basic notions of combinatorial and differential topology, wish to gain an insight into those which we still call the golden years of the topology of manifolds.[1]

With this aim in mind, rather than embarking on a detailed analytical introduction to the contents of the book, I shall confine myself to a historically slanted outline of the triangulation problem, hoping that this may be of help to the reader.

A piecewise-linear manifold, abbreviated PL, is a topological manifold together with a maximal atlas whose transition functions between open sets of $\mathbb{R}^n_+$ admit a graph that is a locally finite union of simplexes.

There is no doubt that the unadorned topological manifold, stripped of all possible additional structures (differentiable, PL, analytic etc) constitutes an object of remarkable charm and that the same is true of the equivalences, namely the homeomorphisms, between topological manifolds. Due to a lack of means at one's disposal, the study of such objects, which define the so called topological category, presents huge and frustrating difficulties compared to the admittedly hard study of the analogous PL category, formed by the PL manifolds and the PL homeomorphisms.

A significant fact, which highlights the kind of pathologies affecting the topological category, is the following. It is not difficult to prove that the group of PL self-homeomorphisms of a connected boundariless PL manifold M^m acts transitively not just on the points of M, but also on the PL m-discs contained in M. On the contrary, the group of topological self-homeomorphisms indeed

[1] The book may also be used as an introduction to A Casson, D Sullivan, M Armstrong, C Rourke, G Cooke, *The Hauptvermutung Book*, K–monographs in Mathematics 1996.

acts transitively on the points of M, but not on the topological m–discs of M. The reason dates back to an example of Antoine's (1920), better known in the version given by Alexander and usually called the Alexander horned sphere. This is a the boundary of a *topological* embedding $h\colon D^3 \to \mathbb{R}^3$ (where D^3 is the standard disc $x^2+y^2+z^2 \le 1$), such that $\pi_1(\mathbb{R}^3 \setminus h(D^3)) \neq 1$. It is clear that there cannot be any automorphism of $\mathbb{R}^3$ taking $h(D^3)$ to D^3, since $\mathbb{R}^3 \setminus D^3$ is simply connected.

As an observation of a different nature, let us recall that people became fairly easily convinced that simplicial homology, the first notion of homology to be formalised, is invariant under PL automorphisms; however its invariance under topological homeomorphisms immediately appeared as an almost intractable problem.

It then makes sense to suppose that the thought occurred of transforming problems related to the topological category into analogous ones to be solved in the framework offered by the PL category. From this attitude two questions naturally emerged: is a given topological manifold homeomorphic to a PL manifold, more generally, is it triangulable? In the affirmative case, is the resulting PL structure unique up to PL homeomorphisms?

The second question is known as *die Hauptvermutung* (the main conjecture), originally formulated by Steinitz and Tietze (1908) and later taken up by Kneser and Alexander. The latter, during his speech at the International Congress of Mathematicians held in Zurich in 1932, stated it as one of the major problems of topology.

The philosophy behind the conjecture is that the relation M_1 topologically equivalent to M_2 should be as close as possible to the relation M_1 combinatorially equivalent to M_2.

We will first discuss the Hauptvermutung, which is, in some sense, more important than the problem of the existence of triangulations, since most known topological manifolds are already triangulated.

Let us restate the conjecture in the form and variations that are currently used. Let Θ_1, Θ_2 be two PL structures on the topological manifold M. Then Θ_1, Θ_2 are said to be *equivalent* if there exists a PL homeomorphism $f\colon M_{\Theta_1} \to M_{\Theta_2}$, they are said to be *isotopy equivalent* if such an f can be chosen to be isotopic to the identity and *homotopy equivalent* if f can be chosen to be homotopic to the identity.

The Hauptvermutung for surfaces and three-dimensinal manifolds was proved by Kerékiárto (1923) and Moise (1952) respectively. We owe to Papakyriakopoulos (1943) the solution to a generalised Haupvermutung, which is valid for any 2-dimensional polyhedron.

We observe, however, that in those same years the topological invariance of homology was being established by other methods.

For the class of C^∞ triangulations of a differentiable manifold, Whitehead proved an isotopy Haupvermutung in 1940, but in 1960 Milnor found a polyhedron of dimension six for which the generalised Hauptvermutung is false. This polyhedron is not a PL manifold and therefore the conjecture remained open for manifolds.

Plenty of water passed under the bridge. Thom suggested that a structure on a manifold should correspond to a section of an appropriate fibration. Milnor introduced microbundles and proved that S^7 supports twenty-eight differentiable structures which are inequivalent from the C^∞ viewpoint, thus refuting the C^∞ Hauptvermutung. The semisimplicial language gained ground, so that the set of PL structures on M could be replaced effectively by a topological space $\mathrm{PL}(M)$ whose path components correspond to the isotopy classes of PL structures on M. Hirsch in the differentiable case and Haefliger and Poenaru in the PL case studied the problem of immersions between manifolds. They conceived an approach to immersion theory which validates Thom's hypothesis and establishes a homotopy equivalence between the space of immersions and the space of monomorphisms of the tangent microbundles. This reduces theorems of this kind to a test of a few precise axioms followed by the classical obstruction theory to the existence and uniqueness of sections of bundles.

Inspired by this approach, Lashof, Rothenberg, Casson, Sullivan and other authors gave significant contributions to the triangulation problem of topological manifolds, until in 1969 Kirby and Siebenmann shocked the mathematical world by providing the following final answer to the problem.

Theorem (Kirby–Siebenmann) *If M^m is an unbounded* PL *manifold and $m \geq 5$, then the whole space* $\mathrm{PL}(M)$ *is homotopically equivalent to the space of maps $K(\mathbb{Z}/2,3)^M$.*

If $m \leq 3$, then $\mathrm{PL}(M)$ *is contractible* (Moise). □

$K(\mathbb{Z}/2,3)$ denotes, as usual, the Eilenberg–MacLane space whose third homotopy group is $\mathbb{Z}/2$. Consequently the isotopy classes of PL structures on M are given by $\pi_0(PL(M)) = [M, K(\mathbb{Z}/2,3)] = H^3(M,\mathbb{Z}/2)$. The isotopy Hauptvermutung was in this way disproved. In fact, there are two isotopy classes of PL structures on $S^3 \times \mathbb{R}^2$ and, moreover, Siebenmann proved that $S^3 \times S^1 \times S^1$ admits two PL structures inequivalent up to isomorphism and, consequently, up to isotopy or homotopy.

The Kirby–Siebenmann theorem reconfirms the validity of the Hauptvermutung for $\mathbb{R}^m$ $(m \neq 4)$ already established by Stallings in 1962.

The homotopy-Hauptvermutung was previously investigated by Casson and Sullivan (1966), who provided a solution which, for the sake of simplicity, we will enunciate in a particular case.

Theorem (Casson–Sullivan) *Let M^m be a compact simply-connected manifold without boundary with $m \geq 5$, such that $H^4(M,\mathbb{Z})$ has no 2-torsion. Then two* PL *structures on M are homotopic.*[2] □

With respect to the existence of PL structures, Kirby and Siebenmann proved, as a part of the above theorem, that: *A boundariless M^m, with $m \geq 5$, admits a* PL *structure if and only if a single obstruction $k(M) \in H^4(M,\mathbb{Z}/2)$ vanishes.*

Just one last comment on the triangulation problem. It is still unknown whether a topological manifold of dimension ≥ 5 can *always* be triangulated by a simplicial complex that is not necessarily a combinatorial manifold. Certainly there exist triangulations that are not combinatorial, since Edwards has shown that the double suspension of a three-dimensional homological sphere is a genuine sphere.

Finally, the reader will have noticed that the four-dimensional case has always been excluded. This is a completely different and more recent story, which, thanks to Freedman and Donaldson, constitutes a revolutionary event in the development of the topology of manifolds. As evidence of the schismatic behaviour of the fourth dimension, here we have room only for two key pieces of information with which to whet the appetite:

(a) $\mathbb{R}^4$ admits uncountably many PL structures.

(b) 'Few' four-dimensional manifolds are triangulable.

[2] This book will not deal with this most important and difficult result. The reader is referred to [Casson, Sullivan, Armstrong, Rourke, Cooke 1996].

2 Problems, conjectures, classical results

This section is devoted to a sketch of the state of play in the field of combinatorial topology, as it presented itself during the sixties. Brief information is included on developments which have occurred since the sixties.

Several of the topics listed here will be taken up again and developed at leisure in the course of the book.

An *embedding* of a topological space X into a topological space Y is a continuous map $\mu\colon X \to Y$, which restricts to a homeomorphism between X and $\mu(X)$.

Two embeddings, μ and ν, of X into Y are *equivalent*, if there exists a homeomorphism $h\colon Y \to Y$ such that $h\mu = \nu$.

2.1 Knots of spheres in spheres

A *topological knot* of *codimension* c in the sphere S^n is an embedding $\nu\colon S^{n-c} \to S^n$. The knot is said to be *trivial* if it is equivalent to the standard knot, that is to say to the natural inclusion of S^{n-c} into S^n.

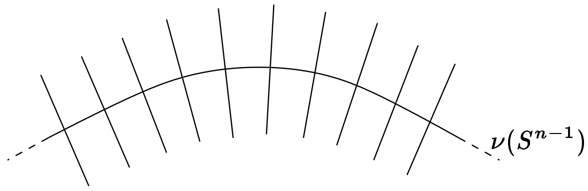
Codimension 1 – the Schoenflies conjecture

Topological Schoenflies conjecture Every knot of codimension one in S^n is trivial.

- The conjecture is true for $n = 2$ (Schoenflies 1908) and plays an essential role in the triangulation of surfaces. The conjecture is false in general, since Antoine and Alexander (1920–24) have knotted S^2 in S^3.

A knot $\nu\colon S^{n-c} \to S^n$ is *locally flat* if there exists a covering of S^{n-c} by open sets such that on each open U of the covering the restriction $\nu\colon U \to S^n$ extends to an embedding of $U \times \mathbb{R}^c$ into S^n.

If $c = 1$, locally flat $=$ locally bicollared:



Weak Schoenflies Conjecture Every locally flat knot is trivial.

- The conjecture is true (Brown and Mazur–Morse 1960).

Canonicalness of the weak Schoenflies problem

The weak Schoenflies problem may be enunciated by saying that any embedding $\mu\colon S^{n-1}\times[-1,1]\to\mathbb{R}^n$ extends to an embedding $\boldsymbol{\mu}\colon D^n\to\mathbb{R}^n$, with $\boldsymbol{\mu}(x)=\mu(x,0)$ for $x\in S^{n-1}$.

Consider μ and $\boldsymbol{\mu}$ as elements of $\mathrm{Emb}(S^{n-1}\times[-1,1],\mathbb{R}^n)$ and $\mathrm{Emb}(D^n,\mathbb{R}^n)$ respectively, ie, of the spaces of embeddings with the compact open topology.

[Huebsch and Morse 1960/1963] proved that it is possible to choose the solution $\boldsymbol{\mu}$ to the Schoenflies problem μ in such a way that the correspondence $\mu\to\boldsymbol{\mu}$ is continuous as a map between the embedding spaces. We describe this by saying that $\boldsymbol{\mu}$ depends *canonically* on μ and that the solution to the Schoenflies problem is *canonical*. Briefly, if the problems μ and μ' are close, their solutions too may be assumed to be close. See also [Gauld 1971] for a far shorter proof.

The definitions and the problems above are immediately transposed into the PL case, but the answers are different.

PL–**Schoenflies Conjecture** Every PL knot of codimension one in S^n is trivial.

- The conjecture is true for $n\le 3$, Alexander (1924) proved the case $n=3$. For $n>3$ the conjecture is still open; if the $n=4$ case is proved, then the higher dimensional cases will follow.

Weak PL–**Schoenflies Conjecture** Every PL knot, of codimension one and locally flat in S^n, is trivial.

- The conjecture is true for $n\ne 4$ (Alexander $n<4$, Smale $n\ge 5$).

Weak Differentiable Schoenflies Conjecture Every differentiable knot of codimension one in S^n is setwise trivial, ie, there is a diffeomorphism of S^n carrying the image to the image of the standard embedding.

- The conjecture is true for $n\ne 4$ (Smale $n>4$, Alexander $n<4$).

The strong Differentiable Schoenflies Conjecture, that every differentiable knot of codimension one in S^n is trivial is false for $n>5$ because of the existence of exotic diffeomorphisms of S^n for $n\ge 6$ [Milnor 1958].

A less strong result than the PL Shoenflies problem is a classical success of the Twenties.

Theorem (Alexander–Newman) *If B^n is a PL disc in S^n then the closure $\overline{S^n-B^n}$ is itself a PL disc.* □

The result holds also in the differentiable case (Munkres).

Higher codimensions

Theorem [Stallings 1963] *Every locally flat knot of codimension $c \neq 2$ is trivial.* □

Theorem [Zeeman 1963] *Every PL knot of codimension $c \geq 3$ is trivial.* □

Zeeman's theorem does not carry over to the differentiable case, since Haefliger (1962) has differentiably knotted S^{4k-1} in S^{6k}; nor it can be transposed into the topological case, where there exist knots (necessarily not locally flat if $c \neq 2$) in all codimensions $0 < c < n$.

2.2 The annulus conjecture

PL annulus theorem [Hudson–Zeeman 1964] *If B_1^n, B_2^n are* PL *discs in S^n, with $B_1 \subset \operatorname{Int} B_2$, then*

$$\overline{B_2 - B_1} \approx_{PL} \dot{B}_1 \times [0,1].$$ □

Topological annulus conjecture Let $\mu, \nu \colon S^{n-1} \to \mathbb{R}^n$ be two locally flat topological embeddings with S_μ contained in the interior of the disc bounded by S_ν. Then there exists an embedding $\lambda \colon S^{n-1} \times I \to \mathbb{R}^n$ such that

$$\lambda(x,0) = \mu(x) \quad \text{and} \quad \lambda(x,1) = \nu(x).$$

- The conjecture is true (Kirby 1968 for $n > 4$, Quinn 1982 for n=4).

The following beautiful result is connected to the annulus conjecture:

Theorem [Cernavskii 1968, Kirby 1969, Edwards–Kirby 1971] *The space $\mathcal{H}(\mathbb{R}^n)$ of homeomorphisms of $\mathbb{R}^n$ with the compact open topology is locally contractible.* □

2.3 The Poincaré conjecture

A *homotopy sphere* is, by definition, a closed manifold of the homotopy type of a sphere.

Topological Poincaré conjecture A homotopy sphere is a topological sphere.

- The conjecture is true for $n \neq 3$ (Newman 1966 for $n > 4$, Freedman 1982 for n=4)

Weak PL–**Poincaré conjecture** A PL homotopy sphere is a topological sphere.

- The conjecture is true for $n \neq 3$. This follows from the topological conjecture above, but was first proved by Smale, Stallings and Zeeman (Smale and Stallings 1960 for $n \geq 7$, Zeeman 1961/2 for $n \geq 5$, Smale and Stallings 1962 for $n \geq 5$).

(Strong) PL–**Poincaré conjecture** A PL homotopy sphere is a PL sphere.

- The conjecture is true for $n \neq 3, 4$, (Smale 1962, for $n \geq 5$).

In the differentiable case the weak Poincaré conjecture is true for $n \neq 3$ (follows from the Top or PL versions) the strong one is false in general (Milnor 1958).

Notes For $n = 3$, the weak and the strong versions are equivalent, due to the theorems on triangulation and smoothing of 3–manifolds. Therefore the Poincaré conjecture, *still open*, assumes a unique form: a homotopy 3–sphere (Top, PL or Diff) is a 3–sphere. For $n = 4$ the strong PL and Diff conjectures are similarly equivalent and are also *still open*. Thus, for $n = 4$, we are today in a similar situation as that in which topologists were during 1960/62 before Smale proved the strong PL high-dimensional Poincaré conjecture.

2.4 Structures on manifolds

Structures on $\mathbb{R}^n$

Theorem [Stallings 1962] *If $n \neq 4$, $\mathbb{R}^n$ admits a unique structure of PL manifold and a unique structure of C^∞ manifold.* □

Theorem (Edwards 1974) *There exist non combinatorial triangulations of $\mathbb{R}^n$, $n \geq 5$.* □

Therefore $\mathbb{R}^n$ does not admit, in general, a unique polyhedral structure.

Theorem *$\mathbb{R}^4$ admits uncountably many* PL *or* C^∞ *structures.*

This is one of the highlights following from the work of Casson, Edwards (1973-75), Freedman (1982), Donaldson (1982), Gompf (1983/85), Taubes (1985). The result stated in the theorem is due to Taubes. An excellent historical and mathematical account can be found in [Kirby 1989].

PL–structures on spheres

Theorem *If $n \neq 4$, S^n admits a unique structure of PL manifold.* □

This result is classical for $n \leq 2$, it is due to Moise (1952) for $n = 3$, and to Smale (1962) for $n > 4$.

Theorem (Edwards 1974) *The double suspension of a PL homology sphere is a topological sphere.* □

Therefore there exist non combinatorial triangulations of spheres. Consequently spheres, like Euclidean spaces, do not admit, in general, a unique polyhedral structure.

Smooth structures on spheres

Let $C(S^n)$ be the set of orientation-preserving diffeomorphism classes of C^∞ structures on S^n. For $n \neq 4$ this can be given a group structure by using connected sum and is the same as the group of differentiable homotopy spheres Γ_n for $n > 4$.

Theorem *Assume $n \neq 4$. Then*

(a) *$C(S^n)$ is finite,*

(b) *$C(S^n)$ is the trivial group for $n \leq 6$ and for some other values of n, while, for instance, $C(S^{4k-1}) \neq \{1\}$ for all $k \geq 2$.* □

The above results are due to Milnor (1958), Smale (1959), Munkres (1960), Kervaire-Milnor (1963).

The 4-dimensional case

It is unknown whether S^4 admits exotic PL and C^∞ structures. The two problems are equivalent and they are also both equivalent to the strong four-dimensional PL and C^∞ Poincaré conjecture. If $C(S^4)$ is a group then the four-dimensional PL and C^∞ Poincaré conjectures reduce to the PL and C^∞ Schoenflies conjectures (all unsolved).

A deep result of Cerf's (1962) implies that there is no C^∞ structure on S^4 which is an effectively twisted sphere, ie, a manifold obtained by glueing two copies of the standard disk through a diffeomorphism between their boundary spheres. Note that the PL analogue of Cerf's result is an easy exercise: effectively twisted PL spheres cannot exist (in any dimension) since there are no exotic PL automorphisms of S^n.

These results fall within the ambit of the problems listed below.

Structure problems for general manifolds

Problem 1 Is a topological manifold of dimension n homeomorphic to a PL manifold?

- Yes for $n \leq 2$ (Radò 1924/26).
- Yes for $n = 3$ (Moise, 1952).
- No for $n = 4$ (Donaldson 1982).
- No for $n > 4$: in each dimension > 4 there are non-triangulable topological manifolds (Kirby–Siebenmann 1969).

Problem 2 Is a topological manifold homeomorphic to a polyhedron?

- Yes if $n \leq 3$ (Radò, Moise).
- No for $n = 4$ (Casson, Donaldson, Taubes, see [Kirby Problems 4.72]).
- Unknown for $n > 5$, see [Kirby op cit].

Problem 3 Is a polyhedron, which is a topological manifold, also a PL manifold?

- Yes if $n \leq 3$.
- Unknown for $n = 4$, see [Kirby op cit]. If the 3-dimensional Poincaré conjecture holds, then the problem can be answered in the affirmative, since the link of a vertex in any triangulation of a 4-manifold is a simply connected 3-manifold.
- No if $n > 4$ (Edwards 1974).

Problem 4 (Hauptvermutung for polyhedra) If two polyhedra are homeomorphic, are they also PL homeomorphic?

- Negative in general (Milnor 1961).

Problem 5 (Hauptvermutung for manifolds) If two PL manifolds are homeomorphic, are they also PL homeomorphic?

- Yes for $n = 1$ (trivial).
- Yes for $n = 2$ (classical).
- Yes for $n = 3$ (Moise).
- No for $n = 4$ (Donaldson 1982).
- No for $n > 4$ (Kirby–Siebenmann–Sullivan 1967–69).

Problem 6 (C^∞ Hauptvermutung) Are two homeomorphic C^∞ manifolds also diffeomorphic?

- For $n \leq 6$ the answers are the same as the last problem.
- No for $n \geq 7$, for example there are 28 C^∞ differential structures on S^7 (Milnor 1958).

Problem 7 Does a C^∞ manifold admit a PL manifold structure which is compatible (according to Whitehead) with the given C^∞ structure?
In the affirmative case is such a PL structure unique?

- The answer is affirmative to both questions, with no dimensional restrictions. This is the venerable Whitehead Theorem (1940).

Note A PL structure being *compatible* with a C^∞ structure means that the transition functions relating the PL atlas and the C^∞ atlas are piecewise–differentiable maps, abbreviated PD.

By exchanging the roles of PL and C^∞ one obtains the so called and much more complicated "smoothing problem".

Problem 8 Does a PL manifold M^n admit a C^∞ structure which is Whitehead compatible?

- Yes for $n \leq 7$ but no in general. There exists an obstruction theory to smoothing, with obstructions $\alpha_i \in H^{i+1}(M;\Gamma_i)$, where Γ_i is the (finite) group of differentiable homotopy spheres (Cairns, Hirsch, Kervaire, Lashof, Mazur, Munkres, Milnor, Rothenberg et al $\sim$ 1965).
- The C^∞ structure is unique for $n \leq 6$.

Problem 9 Does there always exist a C^∞ structure on a PL manifold, possibly not Whitehead–compatibile?

- No in general (Kervaire's counterexample, 1960).

3 Polyhedra and categories of topological manifolds

In this section we will introduce the main categories of geometric topology. These are defined through the concept of supplementary structure on a topological manifold. This structure is usually obtained by imposing the existence of an atlas which is compatible with a pseudogroup of homeomorphisms between open sets in Euclidean spaces.

We will assume the reader to be familiar with the notions of simplicial complex, simplicial map and subdivision. The main references to the literature are [Zeeman 1963], [Stallings 1967], [Hudson 1969], [Glaser 1970], [Rourke and Sanderson 1972].

3.1 The combinatorial category

A *locally finite simplicial complex* K is a collection of simplexes in some Euclidean space E, such that:

(a) $A \in K$ and B is a face of A, written $B < A$, then $B \in K$.

(b) If $A, B \in K$ then $A \cap B$ is a common face, possibly empty, of both A and B.

(c) Each simplex of K has a neighbourhood in E which intersects only a finite number of simplexes of K.

Often it will be convenient to confuse K with its *underlying topological space*

$$|K| = \bigcup_{A \in K} A$$

which is called a *Euclidean polyhedron.*

We say that a map $f\colon K \to L$ is *piecewise linear*, abbreviated PL, if there exists a linear subdivision K' of K such that f sends each simplex of K' linearly into a simplex of L.

It is proved, in a non trivial way, that the locally finite simplicial complexes and the PL maps form a category with respect to composition of maps. This is called the *combinatorial category.*

There are three important points to be highlighted here which are also non trivial to establish:

(a) If $f\colon K \to L$ is PL and K, L are *finite*, then there exist subdivisions $K' \lhd K$ and $L' \lhd L$ such that $f\colon K' \to L'$ is simplicial. Here $\lhd$ is the symbol used to indicate subdivision.

(b) A theorem of Runge ensures that an open set U of a simplicial complex K or, more precisely, of $|K|$, can be *triangulated*, ie, underlies a locally finite simplicial complex, in a way such that the inclusion map $U \subset K$ is PL. Furthermore such a triangulation is unique up to a PL homeomorphism. For a proof see [Alexandroff and Hopf 1935, p. 143].

(c) A PL map, which is a homeomorphism, is a PL isomorphism, ie, the inverse map is automatically PL. This does not happen in the differentiable case as shown by the function $f(x) = x^3$ for $x \in \mathbb{R}$.

As evidence of the little flexibility of PL isomorphisms consider the differentiable map of $\mathbb{R}$ into itself

$$f(x) = \begin{cases} x + \frac{e^{-1/x^2}}{4} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0. \end{cases}$$

This is even a C^∞ diffeomorphism but it can not in any way be well approximated by a PL map, since the origin is an accumulation point of isolated fixed points (Siebenmann).

If $S \subset K$ is a subset made of simplexes, we call the *simplicial closure* of S the smallest subcomplex of K which contains S:

$$\overline{S} := \{B \in K : \exists\, A \in S \text{ with } B < A\}.$$

In other words we add to the simplexes of S all their faces. Since, clearly, $|S| = |\overline{S}|$, we will say that S *generates* $\overline{S}$.

Let v be a vertex of K, then the *star of* v *in* K, written $S(v, K)$, is the subcomplex of K generated by all the simplexes which admit v as a vertex, while the *link* of v in K, written $L(v, K)$, is the subcomplex consisting of all the simplexes of $S(v, K)$ which do not admit v as a vertex. The most important property of the link is the following: if $K' \lhd K$ then $L(v, K) \approx_{\mathrm{PL}} L(v, K')$.

K is called a n–dimensional *combinatorial manifold without boundary*, if the link of each vertex is a PL n–sphere. More generally, K is a combinatorial n–manifold *with boundary* if the link of each vertex is a PL n–sphere or PL n–ball. (PL *spheres* and *balls* will be defined precisely in subsection 3.6 below.) It can be verified that the subcomplex $\dot{K} = \partial K \subset K$ generated by all the $(n-1)$–simplexes which are faces of *exactly one* n–simplex is itself a combinatorial $(n-1)$–manifold without boundary.

3.2 Polyhedra and manifolds

Until now we have dealt with objects such as simplicial complexes which are, by definition, contained in a given Euclidean space. Yet, as happens in the case of differentiable manifolds, it is advisable to introduce the notion of a polyhedron in an intrinsic manner, that is to say independent of an ambient Euclidean space.

Let P be a topological space such that each point in P admits an open neighbourhood U and a homeomorphism

$$\varphi\colon U \to |K|$$

where K is a locally finite simplicial complex. Both U and φ are called a coordinate chart. Two charts are PL *compatible* if they are related by a PL isomorphism on their intersection.

A *polyhedron* is a metrisable topological space endowed with a maximal atlas of PL compatible charts. The atlas is called a *polyhedral structure.* For example, a simplicial complex is a polyhedron in a natural way.

A PL *map* of polyhedra is defined in the obvious manner using charts. Now one can construct the *polyhedral category*, whose objects are the polyhedra and whose morphisms are the PL maps.

It turns out to be a non trivial fact that each polyhedron is PL homeomorphic to a simplicial complex.

A *triangulation* of a polyhedron P is a PL homeomorphism $t\colon |K| \to P$, where $|K|$ is a Euclidean polyhedron. When there is no danger of confusion we will identify, through the map t, the polyhedron P with $|K|$ or even with K.

Alternative definition Firstly we will extend the concept of triangulation. A *triangulation* of a *topological space* X is a homeomorphism $t\colon |K| \to X$, where K is a simplicial complex. A *polyhedron* is a pair $(P, \mathcal{F})$, where P is a topological space and $\mathcal{F}$ is a maximal collection of PL *compatible triangulations*. This means that, if t_1, t_2 are two such triangulations, then $t_2^{-1}t_1$ is a PL map. The reader who is interested in the equivalence of the two definitions of polyhedron, ie, the one formulated using local triangulations and the latter formulated using global triangulations, can find some help in [Hudson 1969, pp. 76–87].

[E C Zeeman 1963] generalised the notion of a *polyhedron* to that of a *polyspace*. As an example, $\mathbb{R}^\infty$ is not a polyhedron but it is a polyspace, and therefore it makes sense to talk about PL maps defined on or with values in $\mathbb{R}^\infty$.

$P_0 \subset P$ is a *closed subpolyhedron* if there exists a triangulation of P which restricts to a triangulation of P_0.

A full subcategory of the polyhedral category of central importance is that consisting of PL *manifolds*. Such a manifold, *of dimension* m, is a polyhedron M whose charts take values in open sets of $\mathbb{R}^m$.

When there is no possibility of misunderstanding, the category of PL manifolds and PL maps is abbreviated to the PL *category*. It is a non trivial fact that every triangulation of a PL manifold is a combinatorial manifold and actually, as happens for the polyhedra, this provides an *alternative definition*: a PL manifold consists of a polyhedron M such that each triangulation of M is a combinatorial manifold. The reader who is interested in the equivalence of the two definitions of PL manifold can refer to [Dedecker 1962].

3.3 Structures on manifolds

The main problem upon which most of the geometric topology is based is that of classifying and comparing the various supplementary structures that can be imposed on a topological manifold, with a particular interest in the piecewise linear and differentiable structures.

The definition of PL manifold by means of an atlas given in the previous subsection is a good example of the more general notion of manifold with structure which we now explain. For the time being we will limit ourselves to the case of manifolds without boundary.

A *pseudogroup* Γ on a Euclidean space E is a category whose objects are the open subsets of E. The morphisms are given by a class of homeomorphisms between open sets, which is closed with respect to composition, restriction, and inversion; furthermore $1_U \in \Gamma$ for each open set U. Finally we require the class to be *locally defined*. This means that if Γ_0 is the set of all the germs of the morphisms of Γ and $f: U \to V$ is a homeomorphism whose germ at every point of U is in Γ_0, then $f \in \Gamma$.

Examples

(a) Γ is trivial, ie, it consists of the identity maps. This is the smallest pseudogroup.

(b) Γ consists of all the homeomorphisms. This is the biggest pseudogroup, which we will denote Top.

(c) Γ consists of all the *stable* homeomorphisms according to [Brown and Gluck 1964]. This is denoted SH. We will return to this important pseudogroup in IV, section 9.

(d) Γ consists of all the C^r homeomorphisms whose inverses are C^r.

(e) Γ consists of all the C^∞ diffeomophisms, denoted by Diff, or all the C^ω diffeomorphisms (real analytic), or all C^Ω diffeomorphisms (complex analytic).

(f) Γ consists of all the Nash homeomorphisms.

(g) Γ consists of all the PL homeomorphisms, denoted by PL.

(h) Γ is a pseudogroup associated to foliations (see below).

(i) E could be a Hilbert space, in which case an example is offered by the Fredholm operators.

Let us recall that a *topological manifold* of dimension m is a metrisable topological space M, such that each point x in M admits an open neighbourhood U and a homeomorphism φ between U and an open set of $\mathbb{R}^m$. Both U and φ are called a *chart around* x. A Γ *structure* Θ on M is a maximal atlas Γ–compatible. This means that, if $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are two charts around x, then $\varphi_\beta \circ \varphi_\alpha^{-1}$ is in Γ,where the composition is defined.

If Γ is the pseudogroup of PL homeomorphisms of open sets of $\mathbb{R}^m$, Θ is nothing but a *PL structure* on the topological manifold M. If Γ is the pseudogroup of the diffeomorphisms of open sets of $\mathbb{R}^m$, then Θ is a C^∞ *structure* on M. If, instead, the diffeomorphisms are C^r, then we have a C^r–structure on M. Finally if Γ = SH, Θ is called a *stable structure* on M. Another interesting example is described below.

Let $\pi\colon \mathbb{R}^m \to \mathbb{R}^p$ be the Cartesian projection onto the first p coordinates and let Γ_m be one of the peudogroups PL, C^∞, Top, on $\mathbb{R}^m$ considered above. We define a new pseudogroup $\mathcal{F}^p_\Gamma \subset \Gamma_m$ by requiring that $f\colon U \to V$ is in $\mathcal{F}^p_\Gamma$ if there is a commutative diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\pi"'] & V \arrow[d, "\pi"] \\ \pi(U) \arrow[r, "g"'] & \pi(V) \end{tikzcd}$$

with $f \in \Gamma_m$, $g \in \Gamma_p$. A $\mathcal{F}^p_\Gamma$–structure on M is called a Γ*–structure with a foliation of codimension* p. Therefore we have the notion of *manifold with foliation*, either topological, PL or differentiable.

A Γ*–manifold* is a pair (M, Θ), where M is a topological manifold and Θ is a Γ–structure on M. We will often write M_Θ, or even M when the Γ–structure Θ is obvious from the context. If $f\colon M' \to M_\Theta$ is a homeomorphism, the Γ *structure* *induced* on M' , $f^*(\Theta)$, is the one which has a composed homeomorphism as a typical chart

$$f^{-1}(U) \xrightarrow{f} U \xrightarrow{\varphi} \varphi(U)$$

where φ is a chart of Θ on M.

From now on we will concentrate only on the pseudogroups $\Gamma =$ Top, PL, Diff.

A homeomorphism $f\colon M_\Theta \to M'_{\Theta'}$ of Γ–manifolds is a Γ–*isomorphism* if $\Theta = f^*(\Theta')$. More generally, a Γ–*map* $f\colon M \to N$ between two Γ–manifolds is a continuous map f of the underlying topological manifolds, such that, written locally in coordinates it is a topological PL or C^∞ map, according to the pseudogroup chosen. Then we have the category of the Γ–manifolds and Γ–maps, in which the isomorphisms are the Γ–isomorphisms described above and usually denoted by the symbol $\approx_\Gamma$, or simply $\approx$.

3.4 Isotopy

In the category of topological spaces and continuous maps, an *isotopy* of X is a homeomorphism $F\colon X \times I \to X \times I$ which respects the levels, ie, $p = pF$, where p is the projection on I.

Such an F determines a continuous set of homeomorphisms $f_t\colon X \to X$ through the formula

$$F(x,t) = (f_t(x), t) \qquad t \in I.$$

Usually, in order to reduce the use of symbols, we write F_t instead of f_t. The isotopy F is said to be *ambient* if $f_0 = 1_X$. We say that F *fixes* $Z \subset X$, or that F is *relative* to Z, if $f_t|Z = 1_Z$ for each $t \in I$; we say that F has *support* in $W \subset X$ if F it fixes $X - W$. Two topological embeddings $\lambda, \mu\colon Y \to X$ are *isotopic* if there exists an embedding $H\colon Y \times I \to X \times I$, which preserves the levels and such that $h_0 = \lambda$ and $h_1 = \mu$. The embeddings are *ambient isotopic* if there exists H which factorises through an ambient isotopy, F, of X:

$$\begin{tikzcd} Y \times I \arrow[rr, "H"] \arrow[dr, "\lambda\times I"'] & & X \times I \\ & X \times I \arrow[ur, "F"'] & \end{tikzcd}$$

and, in this case, we will say that F *extends* H. The embedding H is said to be an *isotopy* between λ and μ.

The language of isotopies can be applied, with some care, to each of the categories Top, PL, Diff.

3.5 Boundary

The notion of Γ*–manifold with boundary* and its main properties do not present any problem. It is sufficient to require that the pseudogroup Γ is defined satisfying the usual conditions, but starting from a class of homeomorphisms of the open sets of the halfspace $\mathbb{R}^m_+ = \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_1 \geq 0\}$. The points in M that correspond, through the coordinate charts, to points in the hyperplane, $\{(x_1, \dots, x_m) \in \mathbb{R}^m_+ : x_1 = 0\}$ define the *boundary* ∂M or $\dot{M}$ of M. This can be proved to be an $(m-1)$-dimensional Γ–manifold without boundary. The complement of ∂M in M is the *interior* of M, denoted either by $\operatorname{Int} M$ or by $\mathring{M}$. A *closed* Γ–manifold is defined as a compact Γ–manifold without boundary. A Γ*–collar* of ∂M in M is a Γ–embedding

$$\gamma \colon \partial M \times I \to M$$

such that $\gamma(x,0) = x$ and $\gamma(\partial M \times [0,1))$ is an open neighbourhood of ∂M in M. The fact that *the boundary of a* Γ*–manifold always admits a* Γ*–collar, which is unique up to* Γ*–ambient isotopy* is very important and non trivial.

3.6 Notation

Now we wish to establish a unified notation for each of the two standard objects which are mentioned most often, ie, the *sphere* S^{m-1} and the *disc* D^m.

In the PL category, D^m means either the cube $I^m = [0,1]^m \subset \mathbb{R}^m$ or the simplex

$$\Delta^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_i \geq 0 \text{ and } \Sigma x_i \leq 1\}.$$

S^{m-1} is either ∂I^m or $\dot{\Delta}^m$, with their standard PL structures.

In the category of differentiable manifolds D^m is the closed unit disc of $\mathbb{R}^m$, with centre the origin and standard differentiable structure, while $S^{m-1} = \partial D^m$.

A PL manifold is said to be a PL m*–disc* if it is PL homeomorphic to D^m. It is a PL m*–sphere* if it is PL homeomorphic S^m. Analogously a C^∞ manifold is said to be a *differentiable* m*–disc* (or *differentiable* m–sphere) if it is diffeomorphic to D^m (or S^m respectively).

3.7 h–cobordism

We will finish by stating two celebrated results of the topology of manifolds: the h–cobordism theorem and the s–cobordism theorem.

Let $\Gamma = PL$ or Diff. A Γ*–cobordism* (V, M_0, M_1) is a compact Γ–manifold V, such that ∂V is the disjoint union of M_0 and M_1. V is said to be an h*–cobordism* if the inclusions $M_0 \subset V$ and $M_1 \subset V$ are homotopy equivalences.

h–cobordism theorem *If an h–cobordism V is simply connected and* $\dim V \geq 6$*, then*

$$V \approx_\Gamma M_0 \times I,$$

and in particular

$$M_0 \approx_\Gamma M_1. \qquad \square$$

In the case of $\Gamma = \text{Diff}$, the theorem was proved by [Smale 1962]. He introduced the idea of attaching a handle to a manifold and proved the result using a difficult procedure of cancelling handles. Nevertheless, for some technical reasons, the handle theory is better suited to the PL case, while in differential topology the equivalent concept of the Morse function is often preferred. This is, for example, the point of view adopted by [Milnor 1965]. The extension of the theorem to the PL case is due mainly to Stallings and Zeeman. For an exposition see [Rourke and Sanderson, 1972]

The strong PL Poincaré conjecture in dim > 5 follows from the h–cobordism theorem (dimension five also follows but the proof is rather more difficult). The differentiable h–cobordism theorem implies the differentiable Poincaré conjecture, necessarily in the weak version, since the strong version has been disproved by Milnor (the group of differentiable homotopy 7–sphere is $\mathbb{Z}/28$): in other words a differentiable homotopy sphere of dim ≥ 5 is a topological sphere.

Weak h–cobordism theorem

(1) *If (V, M_0, M_1) is a PL h–cobordism of dimension five, then*

$$V - M_1 \approx_{\mathrm{PL}} M_0 \times [0, 1).$$

(2) *If (V, M_0, M_1) is a topological h–cobordism of dimension ≥ 5, then*

$$V - M_1 \approx_{\mathrm{Top}} M_0 \times [0, 1). \qquad \square$$

Let $\Gamma = \text{PL}$ or Diff and (V, M_0, M_1) be a connected Γ h–cobordism not necessarily simply connected. There is a well defined element $\tau(V, M_0)$, in the Whitehead group $\mathrm{Wh}(\pi_1(V))$, which is called the *torsion* of the h–cobordism V. The latter is called an *s–cobordism* if $\tau(V, M_0) = 0$.

s–cobordism theorem *If (V, M_0, M_1) is an s–cobordism of* $\dim \geq 6$*, then*

$$V \approx_\Gamma M_0 \times I. \qquad \square$$

This result was proved independently by [Barden 1963], [Mazur 1963] and [Stallings 1967] (1963).

Note If A is a free group of finite type then $\mathrm{Wh}(A) = 0$ [Stallings 1965].

4 Uniqueness of the PL structure on $\mathbb{R}^m$, Poincaré conjecture

In this section we will cover some of the great achievments made by geometric topology during the sixties and, in order to do that, we will need to introduce some more elements of combinatorial topology.

4.1 Stars and links

Recall that *the join* AB of two disjoint simplexes, A and B, in a Euclidean space is the simplex whose vertices are given by the union of the vertices of A and B if those are independent, otherwise the join is undefined. Using joins, we can extend stars and links (defined for verticesin 3.1) to simplexes.

Let A be a simplex of a simplicial complex K, then the *star* and the *link* of A in K are defined as follows:

$$S(A,K) = \{B \in K : A \leq B\} \qquad \text{(here } \{,\} \text{ means simplicial closure)}$$
$$L(A,K) = \{B \in K : AB \in K\}.$$

Then $S(A,K) = AL(A,K)$ (join).

If $A = A'A''$, then

$$L(A,K) = L(A', L(A'',K)).$$

From the above formula it follows that a combinatorial manifold K is characterised by the property that for each $A \in K$:

$$L(A,K) \quad \text{is either a PL sphere or a PL disc.}$$

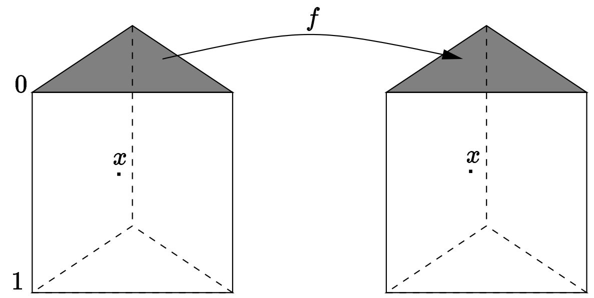
Furthermore $\partial K \equiv \{A \in K : L(A,K) \text{ is a disc}\}$.

4.2 Alexander's trick

This applies to both PL and Top.

Theorem (Alexander) *A homeomorphism of a disc which fixes the boundary sphere is isotopic to the identity, relative to that sphere.*

Proof It will suffice to prove this result for a simplex Δ. Given $f\colon \Delta \to \Delta$, we construct an isotopy $F\colon \Delta \times I \to \Delta \times I$ in the following manner:

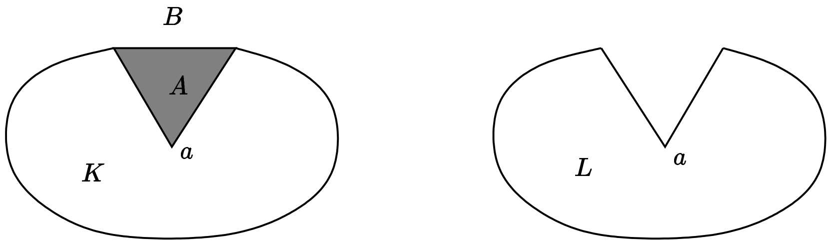


$F \mid \Delta \times \{0\} = f$; $F = 1$ if restricted to any other face of the prism. In this way we have defined F on the boundary of the prism. In order to extend F to its interior we define $F(x) = x$, where x is the centre of the prism, and then we join conically with $F|\partial$. In this way we obtain the required isotopy. □

It is also obvious that each homeomorphism of the boundaries of two discs extends conically to the interior.

4.3 Collapses

If $K \supset L$ are two complexes, we say that there is an *elementary simplicial collapse* of K to L if $K - L$ consists of a principal simplex A, together with a free face. More precisely if $A = aB$, then $K = L \cup A$ and $a\dot{B} = L \cap A$



K collapses simplicially to L, written $K \overset{s}{\searrow} L$, if there is a finite sequence of simplicial elementary collapses which transforms K into L.

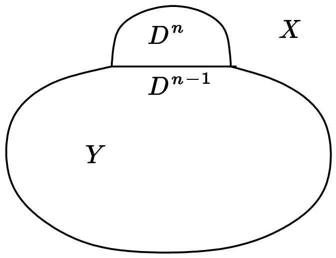
In other words K collapses to L if there exist simplexes $A_1, \ldots, A_q$ of K such that

(a) $K = L \cup A_1 \cup \cdots \cup A_q$

(b) each A_i has one vertex v_i and one face B_i, such that $A_i = v_i B_i$ and
$$(L \cup A_1 \cup \cdots \cup A_{i-1}) \cap A_i = v_i \dot{B}_i.$$

For example, a cone vK collapses to the vertex v and to any subcone.

The definition for polyhedra is entirely analogous. If $X \supset Y$ are two polyhedra we say that there is an *elementary collapse of* X *into* Y if there exist PL discs D^n and D^{n-1}, with $D^{n-1} \subset \partial D^n$, such that $X = Y \cup D^n$ and, also, $D^{n-1} = Y \cap D^n$



X *collapses* to Y, written $X \searrow Y$, if there is a finite sequence of elementary collapses which transforms X into Y.

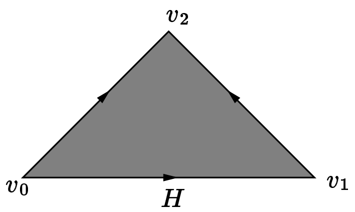
For example, a disc collapses to a point: $D \searrow *$.

Let K and L be triangulations of X and Y respectively and $X \searrow Y$, the reader can prove that there exist subdivisions $K' \triangleleft K$, $L' \triangleleft L$ such that $K' \overset{s}{\searrow} L'$.

Finally, if $K \overset{s}{\searrow} L$, we say that L *expands simplicially* to K. The technique of collapses and of regular neighbourhoods was invented by J H C Whitehead (1939).

The dunce hat Clearly, if $X \searrow *$, then X is contractible, since each elementary collapse defines a deformation retraction, while the converse is false.

For example, consider the so called dunce hat H, defined as a triangle $v_0v_1v_2$, with the sides identified by the rule $v_0v_1 = v_0v_2 = v_1v_2$.



It follows that H is contractible (exercise), but H does not collapse to a point since there are no free faces to start.

It is surprising that $H \times I \searrow *$ [Zeeman, 1964, p. 343].

Zeeman's conjecture If K is a 2-dimensional contractible simplicial complex, then $K \times I \searrow *$.

The conjecture is interesting since it implies a positive answer to the three-dimensional Poincaré conjecture using the following reasoning. Let M^3 be a compact contractible 3–manifold with $\partial M^3 = S^2$. It will suffice to prove that M^3 is a disc. We say that X is a *spine* of M if $M \searrow X$. It is now an easy exercise to prove that M^3 has a 2-dimensional contractible spine K. From the Zeeman conjecture $M^3 \times I \searrow K \times I \searrow *$. PL discs are characterised by the property that they are the only compact PL manifolds that collapse to a point. Therefore $M^3 \times I \approx D^4$ and then $M^3 \subset \dot{D}^4 = S^3$. Since $\partial M^3 \approx S^2$ the manifold M^3 is a disc by the Schoenflies theorem.

For more details see [Glaser 1970, p. 78].

4.4 General position

The *singular set* of a proper map $f\colon X \to Y$ of polyhedra is defined as

$$S(f) = \text{closure}\,\{x \in X : f^{-1}f(x) \neq x\}.$$

Let f be a PL map, then f is *non degenerate* if $f^{-1}(y)$ has dimension 0 for each $y \in f(X)$.

If f is PL, then $S(f)$ is a subpolyhedron.

Let X_0 be a closed subpolyhedron of X^x, with $\overline{X - X_0}$ compact and M^m a PL manifold without boundary, $x \leq m$. Let Y^y be a possibly empty fixed subpolyhedron of M.

A proper continuous map $f\colon X \to M$ is said to be *in general position*, relative to X_0 and with respect to Y, if

(a) f is PL and non degenerate,

(b) $\dim(S(f) - X_0) \leq 2x - m$,

(c) $\dim\,(f(X - X_0) \cap Y) \leq x + y - m$.

Theorem *Let $g\colon X \to M$ be a proper map such that $g|X_0$ is PL and non degenerate. Given $\varepsilon > 0$, there exists a ε–homotopy of g to f, relative to X_0, such that f is in general position.* □

For a proof the following reading is advised [Rourke–Sanderson 1972, p. 61].

In terms of triangulations one may think of general position as follows: $f\colon X \to M$ is *in general position* if there exists a triangulation (K, K_0) of (X, X_0) such that

(1) f embeds each simplex of K piecewise linearly into M,

(2) if A and B are simplexes of $K - K_0$ then

$$\dim\left(f(A) \cap f(B)\right) \leq \dim A + \dim B - m,$$

(3) if A is a simplex of $K - K_0$ then

$$\dim\left((f(A) \cap Y\right) \leq \dim A + \dim Y - m.$$

One can also arrange that the following *double-point* condition be satisfied (see [Zeeman 1963]). Let $d = 2x - m$

(4) $S(f)$ is a subcomplex of K. Moreover, if A is a d–simplex of $S(f) - K_0$, then there is exactly one other d–simplex A_* of $S(f) - K_0$ such that $f(A) = f(A_*)$. If S, S_* are the open stars of A, A_* in K then the restrictions $f \mid S$, $f \mid S_*$ are embeddings, the images $f(S)$, $f(S_*)$ intersect in $f(\mathring{A}) = f(\mathring{A}_*)$ and contain no other points of $f(X)$.

Remark Note that we have described general position of f both as a map and with respect to the subspace Y, which has been dropped from the notation for the sake of simplicity. We will need a full application of general position later in the proof of Stallings' Engulfing theorem.

Proposition *Let X be compact and let $f\colon X \to Z$ be a PL map. Then if $X \supset Y \supset S(f)$ and $X \searrow Y$, then $f(X) \searrow f(Y)$.* □

The proof is left to the reader. The underlying idea of the proof is clear: $X - Y \not\supset S(f)$, the map f is injective on $X - Y$, therefore each elementary collapse corresponds to an analogous elementary collapse in the image of f.

4.5 Regular neighbourhoods

Let X be a polyhedron contained in a PL manifold M^m. A *regular neighbourhood* of X in M is a polyhedron N such that

(a) N is a closed neighbourhood of X in M

(b) N is a PL manifold of dimension m

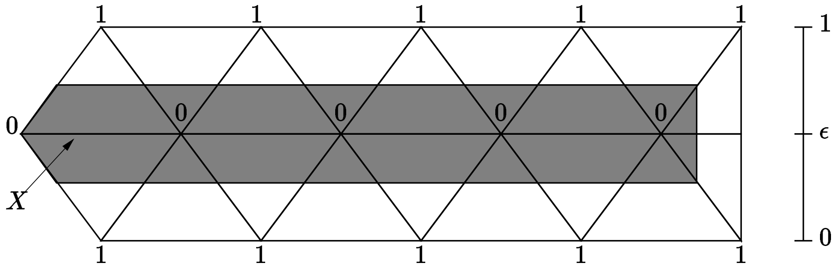
(c) $N \searrow X$.

We will denote by ∂N the frontier of N in M.

We say that the regular neighbourhood N of X in M meets ∂M *transversally* if either $N \cap \partial M$ is a regular neighbourhood of $X \cap \partial M$ in ∂M, or $N \cap \partial M = X \cap \partial M = \emptyset$.

The example of a regular neighbourhood par excellence is the following.

Let (K, L) be a triangulation of (M, X) so that each simplex of K meets L in a (possibly empty) face; let $f\colon K \to I = \Delta^1$ be the unique simplicial map such that $f^{-1}(0) = L$. Then for each $\varepsilon \in (0,1)$ it follows that $f^{-1}[0, \varepsilon]$ is a regular neighbourhood of X in M, which meets ∂M transversally:



Such a neighbourhood is simply called an *ε-neighbourhood.*

Theorem *If X is a polyhedron of a* PL *manifold M^m, then:*

(1) (Existence) *There always exists a regular neighbourhood of X in M.*

(2) (Uniqueness up to PL isomorphism) *If N_1, N_2 are regular neighbourhoods of X in M, then there exists a* PL *isomorphism of N_1 and N_2, which fixes X.*

(3) *If $X \searrow *$, then each regular neighbourhood of X is a* PL *disc.*

(4) (Uniqueness up to isotopy) *If N_1, N_2 are regular neighbourhoods of X in M, which meet ∂M transversally, then there exists an ambient isotopy which takes N_1 to N_2 and fixes X.* □

For a proof see [Hudson 1969, pp. 57–74] or [Rourke–Sanderson 1972, Chapter 3].

The following properties are an easy consequence of the theorem and therefore are left as an exercise.

A) *Let N_1, N_2 be regular neighbourhoods of X in M with $N_1 \subset \mathring{N}_2$. Then if N_1 meets ∂M transversally, there exists a PL homeomorphism*

$$\overline{N_2 - N_1} \approx_{\mathrm{PL}} \partial N_1 \times I.$$

B) **PL annulus theorem** *If* D_1, D_2 *are* $m-$ *discs with* $D_1 \subset \mathring{D}_2$, *then* $\overline{D_2 - D_1} \approx_{\mathrm{PL}} \partial D_1 \times I$.

Corollary *Let* $D_1 \subset \mathring{D}_2 \subset D_2 \subset \mathring{D}_3 \subset \ldots$ *be a chain of* PL m*–discs. Then*

$$\bigcup_1^\infty D_i \approx_{\mathrm{PL}} \mathbb{R}^m. \qquad \square$$

The statement of the corollary is valid also in the *topological* case: a monotonic union of open m–cells is an m–cell (M Brown 1961).

4.6 Introduction to engulfing

At the start of the Sixties a new powerful geometric technique concerning the topology of manifolds arose and developed thanks to the work of J Stallings and E C Zeeman. It was called *Engulfing* and had many applications, of which the most important were the proofs of the PL weak Poincaré conjecture and of the Hauptvermutung for Euclidean spaces of high dimension.

We say that a subset X—most often a closed subpolyhedron—of a PL m–manifold M *may be engulfed* by a given open subset U of M if there exists a PL homeomorphism $h\colon M \to M$ such that $X \subset h(U)$. Generally h is required to be ambient isotopic to the identity relative to the complement of a compact subset of M.

Stallings and Zeeman compared U to a PL amoeba which expands in M until it swallows X, provided that certain conditions of dimension, of connection and of finiteness are satisfied. This is a good intuitive picture of engulfing in spite of a slight inaccuracy due the fact that U may not be contained in $h(U)$. When X^x is fairly big, ie $x = m - 3$, the amoeba needs lots of help in order to be able to swallow X. This kind of help is offered either by Zeeman's sophisticated *piping* technique or by Stallings' equally sophisticated *covering–and–uncovering* procedure. When X is even bigger, ie $x \geq m-2$, then the amoeba might have to give up its dinner, as shown by examples constructed using the Whitehead manifolds (1937) and Mazur manifolds (1961). See [Zeeman 1963].

There are many versions of engulfing according to the authors who formalised them and to the specific objectives to which they were turned to. Our primary purpose is to describe the engulfing technique and give all the necessary proofs, with as little jargon as possible and in a way aimed at the quickest achievement of the two highlights mentioned above. At the end of the section the interested reader will find an appendix outlining the main versions of engulfing together with other applications.

We start here with a sketch of one of the highlights—the Hauptvermutung for high-dimensional Euclidean spaces. Full details will be given later. The uniqueness of the PL structure of $\mathbb{R}^m$ for $m \leq 3$ has been proved by Moise (1952), while the uniqueness of the differentiable structure is due to Munkres (1960). J Stallings (1962) proved the PL and Diff uniqueness of $\mathbb{R}^m$ for $m \geq 5$. Stalling's proof can be summarised as follows: start from a PL manifold, M^m, which is contractible and simply connected at infinity and use engulfing to prove that each compact set $C \subset M$ is contained in an m–cell PL.

Now write M as a countable union $M = \cup_1^\infty C_i$ of compact sets and inductively find m–cells D_i such that $\mathring{D}_i$ engulfs $C_{i-1} \cup D_{i-1}$. Then M is the union $D_1 \subset \mathring{D}_2 \subset D_2 \subset \mathring{D}_3 \subset \cdots \subset D_i \subset \mathring{D}_{i+1} \subset \cdots$ and it follows from Corollary 4.5 that $M \approx_{\mathrm{PL}} \mathbb{R}^m$. If M has also a C^∞ structure which is compatible with the PL structure, then M is even diffeomorphic to $\mathbb{R}^m$.

Exercise Show that PL engulfing is not possible, in general, if M has dimension four.

4.7 Engulfing in codimension 3

Zeeman observed that the idea behind an Engulfing Theorem is to convert a homotopical statement into a geometric statement, in other words to pass from Algebra to Geometry.

The fact that X is *homotopic to zero* in the contractible manifold M, ie, that the inclusion $X \subset M$ is homotopic to a constant is a property which concerns the homotopy groups exclusively. The fact that X is contained in a cell of M is a much stronger property of purely geometrical character.

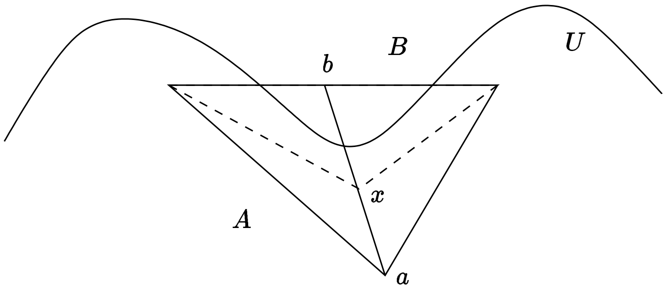
As a first illustratation of engulfing we consider a particular case of Stallings' and Zeeman's theorems.

Theorem *Let M^m be a contractible* PL *manifold without boundary, and let X^x be a compact subpolyhedron of M with $x \leq m-3$. Then X is contained in an m–cell of M.*

We will first prove the theorem for $x < m-3$. The case $x = m-3$ is rather more delicate. We will need two lemmas, the first of which is quite general, as it does not use the hypothesis of contractibility on M.

4.7.1 Lemma *Suppose that $X \searrow Y$ and let U be an open subset of M. Then, if Y may be engulfed by U, X too may be engulfed. In particular, if Y is contained in an m–cell of M, then so is X.*

Proof Without loss of generality, assume $Y \subset U$. The idea of the proof is simple: while Y expands to X, it also pulls U with it.



If we take an appropriate triangulation of (M, X, Y), we can assume that $X \searrow^s Y$. By induction on the number of elementary collapses it will suffice to consider the case when $X \searrow Y$ is an elementary simplicial collapse. Suppose that this collapse happens via the simplex $A = aB$ from the free face B of baricentre b.

Let $L(B, M)$ be the link of B in M, which is a PL sphere so that $bL(B, M)$ is a PL disc D and $S(B, M) = D\dot{B}$. Let $x \in ab$, be such that

$$ax\dot{B} \subset U.$$

There certainly exists a PL homeomorphism $f\colon D \to D$ with $f(x) = b$ and $f|\dot{D} =$ identity.

By joining f with $1_{\dot{B}}$, we obtain a PL homeomorphism

$$h\colon S(B, M) \to S(B, M)$$

which is the identity on $\dot{S}(B, M)$ and therefore it extends to a PL homeomorphism $h_M\colon M \to M$ which takes $ax\dot{B}$ to A. Since

$$U \supset Y \cup ax\dot{B}$$

we will have

$$h_M(U) \supset Y \cup A = X.$$

Since h_M is clearly ambient isotopic to the identity $\mathrm{rel}(M - S(B, M))$,the lemma is proved. □

4.7.2 Lemma *If M^m is contractible, then there exist subpolyhedra Y^y, $Z^z \subset M$ so that $X \subset Y \searrow Z$ and, furthermore:*

$$y \leq x + 1$$
$$z \leq 2x - m + 3.$$

Proof Let us consider a cone vX on X. Since X is homotopic to zero in M, we can extend the inclusion $X \subset M$ to a continuous map $f\colon vX \to M$. By general position we can make f a PL map fixing the restriction $f|_X$. Then we obtain

$$\dim S(f) \leq 2(x+1) - m.$$

If $vS(f)$ is the subcone of vX, it follows that

$$\dim vS(f) \leq 2x - m + 3.$$

Take $Y = f(vX)$ and $Z = f(vS(f))$.

Since a cone collapses onto a subcone we have

$$vX \searrow vS(f)$$

and, since $vS(f) \supset S(f)$, we deduce that $Y \searrow Z$ by Proposition 4.4. Since $f(X) = X$, it follows that

$$X \subset Y \searrow Z\ ,$$

as required. □

Proof of theorem 4.7 in the case $x < m - 3$ We will proceed by induction on x, starting with the trivial case $x = -1$ and assuming the theorem true for the dimensions $< x$.

By Lemma 4.7.2 there exist $Y, Z \subset M$ such that

$$X \subset Y \searrow Z$$

and $z \leq 2x - m + 3 < x$ by the hypothesis $x < m - 3$.

Therefore Z is contained in a cell by the inductive hypothesis; by Lemma 4.7.1 the same happens for Y and, a fortiori, for $X \subset Y$. The theorem is proved. □

▼

Proof of theorem 4.7 in the case $x = m - 3$

This short proof was found by Zeeman in 1966 and communicated to Rourke in a letter [Zeeman, letter 1966].[3] The original proofs of Zeeman and Stallings used techniques which are considerably more delicate. We will discuss them in outline in the appendix.

Let f be a map in general position of the cone on X, CX, into M and let $S = S(f) \subset CX$. Consider the projection $p\colon S \to X$ (projected down the cone lines of CX). Suppose that everything is triangulated. Then the top

[3]The letter is reproduced on Colin Rourke's web page at:
`http://www.maths.warwick.ac.uk/~cpr/Zeeman-letter.jpg`

dimensional simplexes of $p(S)$ have dimension $x-1$ and come in pairs τ_1, τ_2 where $\tau_i = p(\sigma_i)$, $\sigma_i \in S$, $i = 1,2$, with $f(\sigma_1) = f(\sigma_2) = fC(\tau_1) \cap fC(\tau_2)$.

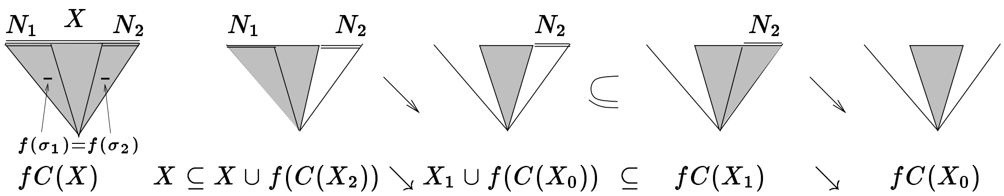
Now let N_i be the union of the open stars of all the τ_i for $i = 1,2$ and let $X_i = X - N_i$ and $X_0 = X_1 \cap X_2$, ie X minus all the stars. Note that S meets $C(X_0)$ in dimension $\le x-2$.

Then $X \subset X \cup f(C(X_2)) \searrow Z = X_1 \cup f(C(X_0))$, by collapsing the cones on the stars of the τ_1's.

But $Z \subset f(C(X_1)) \searrow fC(X_0)$, by collapsing the cones on the stars of the τ_2's.

Finally $fC(X_0) \searrow fC(S \cap C(X_0))$ which has dimension $\le x-1$ where we have abused notation and written $C(S \cap C(X_0))$ for the union of the cone lines through $S \cap C(X_0)$. We are now in codimension 4 and the earlier proof takes over.

▲ □



4.8 Hauptvermuting for $\mathbb{R}^m$ and the weak Poincaré conjecture

A topological space X is *simply connected* (or *1–connected*) *at infinity* if, for each compact subset C of X, there exists a compact set C_1 such that $C \subset C_1 \subset X$ and, furthermore, $X - C_1$ is simply connected.

For example, $\mathbb{R}^m$, with $m > 2$, is 1–connected at infinity, while $\mathbb{R}^2$ is not.

Observation *Let X be 2–connected and 1–connected at infinity. Then for each compact set $C \subset M$ there exists a compact set C_1 such that $C \subset C_1 \subset M$ and, furthermore, $(X, X - C_1)$ is 2–connected.*

Apply the homotopy exact sequence to the pair $(X, X - C_1)$ with $C_1 \supset C$ so that $X - C_1$ is 1–connected. □

Stallings' Engulfing Theorem *Let M^m be a PL manifold without boundary and let U be an open set of M. Let X^x be a closed subpolyhedron of M, such that*

(a) *(M, U) is x–connected,*

(b) *$X \cap (M - U)$ is compact,*

(c) *$x \le m - 3$.*

Then there exist a compact set $G \subset M$ and a PL homeomorphism $h\colon M \to M$, such that

(1) $X \subset h(U)$,

(2) *h is ambient isotopic to the identity rel $M - G$*

Proof Write X as $X_0 \cup Y$ where $X_0 \subset U$ and Y is compact. We argue by induction on the dimension y of Y. The induction starts with $y = -1$ when there is nothing to prove. For the induction step there are two cases.

Case of codim ≥ 4 ie, $y \leq m - 4$

Denote by $Y \times' I$ the result of squeezing $(X_0 \cap Y) \times I$ to $X_0 \cap Y$ fibrewise in $Y \times I$. For $i = 0, 1$, continue to write $Y \times i$ for the image of $Y \times i$ under the projection $Y \times I \to Y \times' I$.

Since $y \leq x$, by hypothesis (a) there is a map $f\colon Y \times' I \to M$ such that $f \mid Y \times 0 = \mathrm{id}$ and $f(Y \times 1) \subset U$. Put f in general position both as a map and with respect to X. Let $\Sigma \subset Y \times' I$ be the preimage of the singular set, which includes the points where the image intersects X_0. Define the *shadow* of Σ, denoted $\mathrm{Sh}(\Sigma)$, to be $\{(y,t) \mid (y,s) \in \Sigma \text{ some } s\}$. Then since Σ has codim at least 3 in $Y \times' I$, $\mathrm{Sh}(\Sigma)$ has codim at least 2 in $Y \times' I$, ie $\dim \leq y - 1$.

Now write $X_0' = X_0 \cup f(Y \times 1)$ and $Y' = f(\mathrm{Sh}(\Sigma))$ and $X' = X_0' \cup Y'$, then we have $\dim(Y') < y$ and

$$X \subset X'' = X \cup f(Y \times' I) \searrow X'$$

where the collapse is induced by cylindrical collapse of $Y \times' I - \mathrm{Sh}(\Sigma)$ from $Y \times 0$ which is embedded by f. But by induction X' can be engulfed and hence by lemma 4.7.1 so can X'' and hence X.

It remains to remark that the engulfing moves are induced by a finite collapse and hence are supported in a compact set G as required.

▼

Case of codim 3 ie, $y = m - 3$

The proof is similar to the proof of theorem 4.7 in the codim 3 case.

Let f and Σ be as in the last case and consider the projection $p\colon Y \times' I \to Y$. Suppose that everything is triangulated so that X is a subcomplex and f and p are simplicial. Then the top dimensional simplexes of $p(\Sigma)$ have dimension $y - 1$ and come in pairs τ_1, τ_2 where $\tau_i = p(\sigma_i)$, $\sigma_i \in \Sigma$, $i = 1, 2$, with $f(\sigma_1) = f(\sigma_2) = f(\tau_1 \times I) \cap f(\tau_2 \times I)$.

Now let N_i be the union of the open stars of all the τ_i for $i = 1, 2$ and let $Y_i = Y - N_i$ and $Y_0 = Y_1 \cap Y_2$, ie Y minus all the stars. Note that Σ meets $Y_0 \times' I$ in dimension $\leq y - 2$.

Then $X \subset X \cup f(Y_2 \times' I \cup Y \times 1) \searrow Z = X_0 \cup f(Y_0 \times' I \cup Y_1 \cup Y \times 1)$, by cylindrically collapsing the cylinders over the stars of the τ_1's from the 0–end. But

$$Z \subset X_0 \cup f(Y_1 \times' I \cup Y \times 1) \searrow T = X_0 \cup f(Y_0 \times' I \cup Y \times 1)$$

by similarly collapsing the $\tau_2 \times I$'s. Finally let $Y' = \mathrm{Sh}(\Sigma) \cap Y_0 \times' I$ which has dimension $< y$ and let $X_0' = X_0 \cup f(Y \times 1)$ and $X' = X_0' \cup Y'$. Then $T \searrow X'$ by cylindrically collapsing $Y_0 \times' I - \mathrm{Sh}(\Sigma)$.

But X' can be engulfed by induction, hence so can T and hence Z and hence X.

▲ □

4.8.1 Note If we apply the theorem with X compact, M contractible and U an open m–cell, we reobtain Theorem 4.7 above.

The following corollary is of crucial importance.

4.8.2 Corollary *Let M^m be a contractible PL manifold, 1–connected at infinity and $C \subset M$ a compact set. Let T be a triangulation of M, and T^2 its 2–skeleton, $m \geq 5$. Then there exists a compact set $G_1 \supset C$ and a PL homeomorphism $h_1\colon M \to M$, such that*

$$T^2 \subset h_1(M - C) \quad \text{and} \quad h_1 \text{ fixes } M - G_1.$$

Proof By Observation 4.8 there exists a compact set C_1, with $C \subset C_1 \subset M$ and $(M, M - C_1)$ 2–connected. We apply the Engulfing Theorem with $U = M - C_1$ and $X = T^2$. The result follows if we take $h_1 = h$ and $G_1 = G \cup C$. The condition $m \geq 5$ ensures that $2 = x \leq m - 3$. □

Note Since $h_1(M) = M$, it follows that $h_1(C) \cap T^2 = \emptyset$. In other words there is a deformation of M so that the 2–skeleton avoids C.

Theorem (PL uniqueness for $\mathbb{R}^m$) *Let M^m be a contractible PL manifold which is 1–connected at infinity and with $m \geq 5$. Then*

$$M^m \approx_{\mathrm{PL}} \mathbb{R}^m.$$

Proof By the discussion in 4.6 it suffices to show that each compact subset of M is contained in an m–cell in M. So let $C \subset M$ be a compact set and T a triangulation of M. First we apply Corollary 4.8.2 to T. Now let $K \subset T$ be the subcomplex

$$K = T^2 \cup \{\text{simplexes of } T \text{ contained in } M - G_1\}.$$

Since $T^2 \subset h_1(M - C)$ and h_1 fixes $M - G_1$, then necessarily

$$K \subset h_1(M - C).$$

Now, if Y is a subcomplex of the simplicial complex X, the *complementary complex* of Y in X, denoted $X \div Y$ by Stallings, is defined as the subcomplex of the barycentric subdivision X' of X which is maximal with respect to the property of not intersecting Y. If Y contains all the vertices of X, then regular neigbourhoods of the two complexes Y and $X \div Y$ cover X. Indded the $\frac{1}{2}$–neighbourhoods of Y and $X \div Y$ in have a common frontier since the 1–simplexes of X' have some vertices in X and the rest in $X \div Y$.

Let $L = T \div K$. Then L is a compact polyhedron of dimension $\leq m - 3$. By Theorem 4.7, or Note 4.8.1, L is contained in an m–cell. Since $K \subset h_1(M - C) \subset M - h_1(C)$, we have $h_1(C) \cap K = \emptyset$, therefore there exists a ε–neighbourhood, N_ε, of L in M such that

$$h_1(C) \subset N_\varepsilon \searrow L.$$

By Lemma 4.7.1 N_ε, and therefore $h_1(C)$, is contained in an m–cell $\mathring{D}$. But then $h_1^{-1}(\mathring{D})$ is an m– cell which contains C, as we wanted to prove. □

Corollary (Weak Poincaré conjecture) *Let M^m be a closed PL manifold homotopically equivalent to S^m, with $m \geq 5$. Then*

$$M^m \approx_{\text{Top}} S^m.$$

Proof If $*$ is a point of M, an argument of Algebraic Topology establishes that $M \setminus *$ is contractible and simply connected at infinity. Therefore M is topologically equivalent to the compactification of $\mathbb{R}^m$ with one point, ie to an m–sphere. □

4.9 The differentiable case

The reader is reminded that each differentiable manifold admits a unique PL manifold structure which is compatible [Whitehead 1940]. We will prove this theorem in the following sections. We also know that two differentiable structures on $\mathbb{R}^m$ are diffeomorphic if they are PL homeomorphic [Munkres 1960].

The following theorem follows from these facts and from what we proved forPL manifolds.

Theorem *Let M^m be a differentiable manifold contractible and 1–connected at infinity. Then if $m \geq 5$,*

$$M^m \approx_{\mathrm{Diff}} \mathbb{R}^m . \qquad \square$$

Corollary (C^∞ uniqueness for $\mathbb{R}^m$) *If $m \geq 5$, $\mathbb{R}^m$ admits a unique differentiable structure.* □

4.10 Remarks

These are wonderful and amazingly powerful theorems, especially so considering the simple tools which formed the basis of the techniques used. It is worth recalling that combinatorial topology was revived from obscurity at the beginning of the Sixties. When, later on, in a much wider, more powerful and sophisticated context, we will reprove that a Euclidean space E, of dimension ≥ 5, admits a unique PL or Diff structure simply because, E being contractible, each bundle over E is trivial, some readers might want to look again at these pages and these pioneers, with due admiration.

4.11 Engulfing in a topological product

We finish this section (apart from the appendix) with a simple engulfing theorem, whose proof does not appear in the literature, which will be used to establish the important fibration theorem III.1.7.

4.11.1 Theorem *Let W^w be a closed topological manifold with $w \neq 3$, let Θ be a* PL *structure on $W \times \mathbb{R}$ and $C \subset W \times \mathbb{R}$ a compact subset. Then there exists a* PL *isotopy G of $(W \times \mathbb{R})_\Theta$ having compact support and such that $G_1(C) \subset W \times (-\infty, 0]$.*

▼

Proof For $w = 2$ the 3–dimensional Hauptvermutung of Moise implies that $(W \times \mathbb{R})_\Theta$ is PL isomorphic to $W \times \mathbb{R}$, where W is a surface with its unique PL structure. Therefore the result is clear.

Let now $Q = (W \times \mathbb{R})_\Theta$ and dim $Q \geq 5$. If (a, b) is an interval in $\mathbb{R}$ we write $Q_{(a,b)}$ for $W \times (a, b)$. Let U be the open set $Q_{(-\infty,0)}$ and assume that C is contained in $Q_{(-r,r)}$. Write V for the open set $Q_{(r,\infty)}$ so that $V \cap C = \emptyset$. We want to engulf C into U.

Let T be a triangulation of Q by small simplexes, and let K be the smallest subcomplex containing a neighbourhood of $Q_{[-r,2r]}$. Let K^2 be the 2–skeleton and L be the complementary complex in K. Then L has codimension three. Now consider $V_0 = Q_{(r,2r)}$ in $Q_0 = Q_{(-\infty,2r)}$ and let $L_0 = L \cap Q_0$. The pair (Q_0, V_0) is ∞-connected. Therefore, by Stallings' engulfing theorem, there exists a PL homeomorphism $j\colon Q_0 \to Q_0$ such that

(a) $L_0 \subset j(V_0)$

(b) there is an isotopy of j to the identity, which is supported by a compact set.

It follows from (b) that j is fixed near level $2r$ and hence extends by the identity to a homeomorphism of Q to itself such that $j(V) \supset L \cup Q_{[2r,\infty]}$.

In exactly the same way there is a PL homeomorphism $h\colon Q \to Q$ such that $h(U) \supset K^2 \cup Q_{[-\infty,-r]}$. Now $h(U)$ and $j(V)$ contain all of Q outside K and also neighbourhoods of complementary conplexes of the first derived of K. By stretching one of these neighbourhoods we can assume that they cover K. Hence we can assume $h(U) \cup j(V) = Q$. Then $j^{-1} \circ h(U) \cup V = Q$ and it follows that $j^{-1} \circ h(U) \supset C$. But each of j^{-1}, h is isotopic to the identity with compact support. Hence there is an isotopy G with compact support finishing with $G_1 = j^{-1} \circ h$ and $G_1(C) \subset U$.

▲ □

Remark If W is compact with boundary the same engulfing theorem holds, provided $C \cap \partial W \subset U$.

4.12 Appendix: other versions of engulfing

This appendix, included for completeness and historical interest, discusses other versions of engulfing and their main applications.

▼

Engulfing à la Zeeman

Instead of Stallings' engulfing *by* or *into* an open subset, Zeeman considers engulfing *from* a closed subpolyhedron of the ambient manifold M.

Precisely, given a closed subpolyhedron C of M, we say that X *may be engulfed from* C if X is contained in a regular neighbourhood of C in M.

Theorem (Zeeman) *Let X^x, C^c be subpolyhedra of the compact manifold M, with C closed and X compact, $X \subset \mathring{M}$, and suppose the following conditions are met:*

(i) *(M, C) is k-connected, $k \geq 0$*

(ii) *there exists a homotopy of X into C which is modulo C*

(iii) $x \leq m-3$; $\quad c \leq m-3$; $\quad c + x \leq m + k - 2$; $\quad 2x \leq m + k - 2$

Then X may be engulfed from C □

Zeeman considers also the cases in which X meets or is completely contained in the boundary of M but we do not state them here and refer the reader to [Zeeman 1963]. The above theorem is probably the most accurate engulfing theorem, in the sense that examples show that its hypotheses cannot be weakened. Thus no significant improvements are possible except, perhaps, for some comments regarding the boundary.

Piping This was invented by Zeeman to prove his engulfing Theorem in codimension three, which enabled him to improve the Poincaré conjecture from the case $n \geq 7$ to the case $n \geq 5$.

A rigorous treatment of the piping construction—not including the preliminary parts—occupies about twenty-five pages of [Zeeman 1963]. Here I will just try to explain the gist of it in an intuitive way, using the terminology of isotopies rather than the more common language of collapsing. As we saw earlier, Zeeman [Letter 1966] found a short proof avoiding this rather delicate construction.

Instead of seeing a ball which expands to engulf X, change your reference system and think of a (magnetized) ball U by which X is homotopically attracted. Let f be the appropriate homotopy. On its way towards U, X will bump into lots of obstacles represented by polyhedra of varying dimensions, that cause X to step backward, curl up and take a different route. This behaviour is encoded by the singular set $S(f)$ of f. Consider the union $T(f)$ of the shadow–lines leading to these singularities.

If $x < m - 3$, then $\dim T(f) < x$. Thus, by induction, $T(f)$ may be engulfed into U. Once this has been done, it is not difficult to view the remaining part of the homotopy as an isotopy f' which takes X into U. Then any ambient isotopy covering f' performs the required engulfing.

If $x = m - 3$, $\dim T(f)$ may be equal to $\dim X$ so that we cannot appeal to induction. Now comes the piping technique. By general position we may obtain that $T(f)$ meets the relevant obstructing polyhedron at single points. Zeeman's procedure consists of piping away these points so as to reduce to the previous easier case. The difficulty lies in the fact that the intersection– points to be eliminated are *essential*, in the sense that they cannot be removed by a local shift. On the contrary, the whole map f needs to be altered, and in a way such that the part of X which is already covered by U be not uncovered during the alteration.

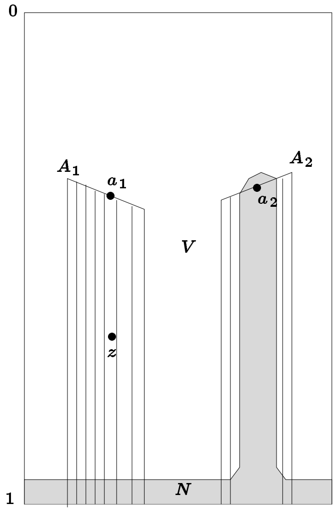
Here is the germ of the construction.

Work in the homotopy cylinder on which f is defined. Let z be a *bad* point, ie, a point of $T(f)$ that gives rise to an intersection which we want to eliminate. Once general position has been fully exploited, we may assume–to fix ideas–that

(a) z lies above the barycenter a_1 of a top–dimensional simplex $A_1 \in S(f)$ such that there is exactly one other simplex A_2 with $f(\mathring{A}_1) = f(\mathring{A}_2)$; moreover f is non degenerate and $f(a_1) = f(a_2)$.

(b) no bad points lie above the barycenter a_2.

Run a thin pipe from the top of the cylinder so as to pierce a hole around the barycenter a_2. More precisely, take a small regular neighbourhood N of the union with $X \times I$ with the shadow–line starting at a_2. Then consider the closure V of $X \times [0,1] - N$ in $X \times [0,1]$. Clearly V is a collar on $X \times 0$. Identify V with $X \times [0,1]$ by a vertical stretch. This produces a new homotopy $\overline{f}$ which takes X off the obstructing polyhedron. Now note that z is still there, but, thanks to the pipe, it has magically ceased to be a bad point. In fact a_1 is not in $S(\overline{f})$ because its brother a_2 has been removed by the pipe, so z does not belong to the shadow–lines leading to $S(\overline{f})$ and the easier case takes over.



We have skated over many things: one or both of A_1, A_2 could belong to $X \times 0$, A_2 could be a vertical simplex, in general there will be many pipes to be constructed simultaneously, et cetera. But these constitute technical complications which can be dealt with and the core of the piping argument is the one described above.

The original proof of Stallings did not use piping but a careful inductive collapsing procedure which has the following subtle implication: when the open set U tries to expand to finally engulf the interior of the $m-3$ simplexes of X, it is forced to *uncover* the interior of some superfluous $(m-2)$–simplexes of M which had been previously covered.

To sum up, while in codimension > 3 one is able to engulf more than it is necessary, in the critical codimension one can barely engulf just what is necessary, and only after a lot of padding has been eliminated.

Engulfing à la Bing or Radial Engulfing

Sometimes one wants that the engulfing isotopy moves each point of X along a *prescribed direction.*

Theorem (Bing) *Let $\{A_\alpha\}$ be a collection of sets in a boundariless* PL *manifold M^m, let $X^x \subset M$ be a closed subpolyhedron, $x \leq m-4$, U an open subset of M with $X \cap (M-U)$ compact. Suppose that for each compact y–dimensional polyhedron Y, $y \leq x$, there exists a homotopy F of Y into U such that, for each point $y \in Y$, $F(y \times [0,1])$ lies in one element of $\{A_\alpha\}$.*

Then, for each $\varepsilon > 0$, there is an ambient engulfing isotopy H of M satisfying the condition that, for each point $p \in M$, there are $x+1$ elements of $\{A_\alpha\}$ such that the track $H(p \times [0,1])$ lies in an ε–neighbourhood of the union of these $x+1$ elements. □

For a proof see [Bing 1967].

There is also a Radial Engulfing Theorem for the codimension three, but it is more complicated and we omit it [Bing op. cit.].

Engulfing by handle-moves

This idea is due to [Rourke–Sanderson 1972]. It does not lead to a different engulfing theorem, but rather to an alternative method for proving the classical engulfing theorems. The approach consists of using the basic constructions of Smale's handle–theory (originally aimed at the proof of the h–cobordism theorem), namely the elementary handle–moves, in order to engulf a given subpolyhedron of a PL manifold. Consequently it is an easy guess that the language of cobordism turns out to be the most appropriate here.

Given a compact PL cobordism (V^v, M_0, M_1), and a compact subpolyhedron X of W, we say that X *may be engulfed from the end M_0 of V* if X is contained in a collar of M_0.

Theorem *Assume $X \cap M_1 = \emptyset$, and suppose that the following conditions are met:*

(i) *there is a homotopy of X into a collar of M_0 relative to $X \cap M_0$*

(ii) *(V, M_0) is k–connected*

(iii) $2x \leq v + k - 2$ and $x \leq v - 3$

Then X may be engulfed from M_0 □

It could be shown that the main engulfing theorems previously stated, including radial engulfing, may be obtained using handle–moves,with tiny improvements here and there, but this is hardly worth our time here.

Topological engulfing

This was worked out by M Newman (1966) in order to prove the topological Poincarè conjecture. E Connell (1967) also proved topological engulfing independently, using PL techniques, and applied it to establish the weak topological h–cobordism theorem.

The statement of Newman's theorem is completely analogous to Stallings' engulfing, once some basic notions have been extended from the PL to the topological context. We keep the notations of Stallings' theorem. The concept of p–connectivity for (M, U) must be replaced by that of monotonic connectivity. The pair (M, U) is *monotonically p–connected*, if, given any compact subset C of U, there exists a closed subset D of U containing C and such that $(M - D, U - D)$ is p–connected.

Assume that X is a polyhedron contained in the topological boundariless manifold M. We say that X is *tame* in M if around each point x of X there is a chart to $\mathbb{R}^m$ whose restriction to X is PL.

Theorem *If (M, U) is monotonically x–connected and X is tame in M, then there is an ambient compactly supported topological isotopy which engulfs X into U.* □

See [Newman 1966] and [Connell 1967].

Applications

We conclude this appendix by giving a short list of the main applications of engulfing.

- The Hauptvermutung for $\mathbb{R}^m$ $(n \geq 5)$ (Theorem 4.8) (Stallings' or Zeeman's engulfing)
- Weak PL Poincarè conjecture for $n \geq 5$ (Corollary 4.8) (Stallings' or Zeeman's engulfing)
- Topological Poincarè conjecture for $n \geq 5$ (Newman's engulfing)
- Weak PL h–cobordism theorem for $n \geq 5$ (Stallings' engulfing)
- Weak topological h–cobordism theorem for $n \geq 5$ (Newman's or Connell's engulfing)
- Any stable homeomorphism of $\mathbb{R}^m$ can be ε–approximated by a PL homeomorphism (Radial engulfing)
- (Irwin's embedding theorem) Let $f\colon M^m \to Q^q$ be a map of unbounded PL manifolds with M compact, and assume that the following conditions are met:

  (i) $q - m \geq 3$

  (ii) M is $(2m - q)$–connected

  (iii) Q is $(2m - q + 1)$–connected

  Then f is homotopic to a PL embedding. □

  In particular:

  (a) any element of $\pi_m(Q)$ may be represented by an embedding of an m–sphere

  (b) a closed k–connected m–manifold embeds in $\mathbb{R}^{2m-k}$, provided $m-k \geq 3$.

  The theorem may be proved using Zeeman's engulfing

  See [Irwin 1965],and also [Zeeman 1963] and [Rourke–Sanderson 1972].

▲