Part III : The differential

1 Submersions

In this section we will introduce topological and PL submersions and we will prove that each closed submersion with compact fibres is a locally trivial fibration.

We will use Γ to stand for either Top or PL and we will suppose that we are in the category of Γ–manifolds without boundary.

1.1 A Γ–map $p\colon E^k \to X^l$ between Γ–manifolds is a Γ*–submersion* if p is locally the projection $\mathbb{R}^k \xrightarrow{\pi_l} \mathbb{R}^l$ on the first l–coordinates. More precisely, $p\colon E \to X$ is a Γ–submersion if there exists a commutative diagram

$$\begin{tikzcd} E \arrow[r, "p"] & X \\ U_y \arrow[u, "\phi_y"] & U_x \arrow[u, "\phi_x"'] \\[-20pt] \cap & \cap \\[-20pt] \mathbb{R}^k \arrow[r, "\pi_l"] & \mathbb{R}^l \end{tikzcd}$$

where $x = p(y)$, U_y and U_x are open sets in $\mathbb{R}^k$ and $\mathbb{R}^l$ respectively and φ_y, φ_x are charts around x and y respectively.

It follows from the definition that, for each $x \in X$, the *fibre* $p^{-1}(x)$ is a Γ–manifold.

1.2 The link between the notion of submersions and that of bundles is very straightforward. A Γ–map $p\colon E \to X$ is a *trivial* Γ*–bundle* if there exists a Γ–manifold Y and a Γ–isomorphism $f\colon Y \times X \to E$, such that $pf = \pi_2$, where π_2 is the projection on X.

More generally, $p\colon E \to X$ is a *locally trivial* Γ*–bundle* if each point $x \in X$ has an open neighbourhood restricted to which p is a trivial Γ–bundle.

Even more generally, $p\colon E \to X$ is a Γ*–submersion* if each point y of E has an open neighbourhood A, such that $p(A)$ is open in X and the restriction $A \to p(A)$ is a trivial Γ–bundle.

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Note A submersion is not, in general, a bundle. For example consider $E = \mathbb{R}^2 - \{0\}$, $X = \mathbb{R}$ and p projection on the first coordinate.

1.3 We will now introduce the notion of a product chart for a submersion. If $p\colon E \to X$ is a Γ–submersion, then for each point y in E, there exist a Γ–manifold U, and an open neighbourhood S of $x = p(y)$ in X and a Γ–embedding

$$\varphi\colon U \times S \to E$$

such that $\operatorname{Im}\varphi$ is a neighbourhood of y in E and, also, $p \circ \varphi$ is the projection $U \times S \to S \subset E$. Therefore, as we have already observed, $p^{-1}(x)$ is a Γ–manifold. Let us now assume that φ satisfies further properties:

(a) $U \subset p^{-1}(x)$

(b) $\varphi(u, x) = u$ for each $u \in U$.

Then we can use interchangeably the following terminology:

(i) the embedding φ is *normalised*

(ii) φ is a *product chart* around U for the submersion p

(iii) φ is a *tubular neighbourhood* of U in E with *fibre* S with respect to the submersion p.

The second is the most suitable and most commonly used.

With this terminology, $p\colon E \to X$ is a Γ–bundle if, for each $x \in X$, there exists a product chart $\varphi\colon p^{-1}(x) \times S \to E$ around the fibre $p^{-1}(x)$, such that the image of φ coincides with $p^{-1}(S)$.

1.4 The fact that many submersions are fibrations is a consequence of the fundamental isotopy extension theorem, which we will state here in the version that is more suited to the problem that we are tackling.

Let V be an open set in the Γ–manifold X, Q another Γ–manifold which acts as the *parameter space* and let us consider an isoptopy of Γ–embeddings

$$G\colon V \times Q \to X \times Q.$$

Given a compact subset C of V and a point q in Q, we are faced with the problem of establishing if and when there exists a neighbourhood S of q in Q and an ambient isotopy $G'\colon X \times S \to X \times S$, which extends G on C, ie $G' \mid C \times S = G \mid C \times S$.

Isotopy extension theorem *Let $C \subset V \subset X$ and $G\colon V \times Q \to X \times Q$ be defined as above. Then there exists a compact neighbourhood C_+ of C in V and an extension G' of G on C, such that the restriction of G' to $(X - C_+) \times S$ is the identity.* □

This remarkable result for the case $\Gamma = \mathrm{Top}$ is due to [Černavskii 1968], [Lees 1969], [Edwards and Kirby 1971], [Siebenmann 1972].

For the case $\Gamma = \mathrm{PL}$ instead we have to thank [Hudson and Zeeman 1964] and [Hudson 1966]. A useful bibliographical reference is [Hudson 1969].

Note In general, there is no way to obtain an extension of G to the whole open set V. Consider, for instance, $V = \overset{\circ}{D}{}^m$, $X = \mathbb{R}^m$, $Q = \mathbb{R}$ and

$$G(v,t) = \left(\frac{v}{1 - t\|v\|}\,,t\right)$$

for $t \in Q$ and $v \in \overset{\circ}{D}{}^m$ and $t \in [0,1]$, and $G(v,t)$ stationary outside $[0,1]$. For $t = 1$, we have

$$G_1(\overset{\circ}{D}{}^m) = \mathbb{R}^m.$$

Therefore G_1 does not extend to any homeomorphism $G_1'\colon \mathbb{R}^m \to \mathbb{R}^m$, and therefore G does not admit any extension on V.

1.5 Let us now go back to submersions. We have to establish two lemmas, of which the first is a direct consequence of the isotopy extension theorem.

Lemma *Let $p\colon Y \times X \to X$ be the product Γ–bundle and let $x \in X$. Further let $U \subset Y_x = p^{-1}(x)$ be a bounded open set and $C \subset U$ a compact set. Finally, let*

$$\varphi\colon U \times S \to Y_x \times X$$

be a product chart for p around U. Then there exists a product chart

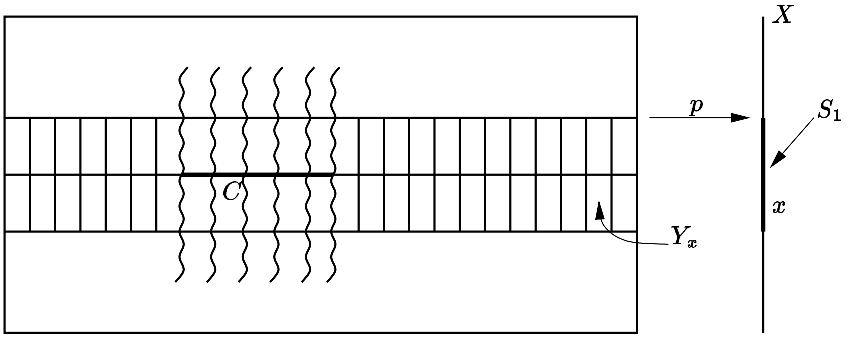
$$\varphi_1\colon Y_x \times S_1 \to Y_x \times X$$

for the submersion p around the whole of Y_x, such that

(a) *$\varphi = \varphi_1$ on $C \times S_1$*

(b) *$\varphi_1 =$ the identity outside $C_+ \times S_1$, where, as usual, C_+ is a compact neighbourhood of C in U.*

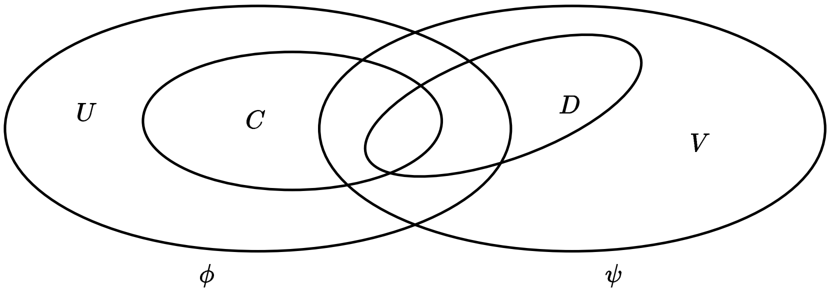
Proof Apply the isotopy extension theorem with X, or better still S, as the space of the parameters and Y_x as ambient manifold. □



Glueing Lemma *Let $p\colon E \to X$ be a submersion, $x \in X$, with C and D compact in $p^{-1}(x)$. Let U, V be open neighbourhoods of C, D in $p^{-1}(x)$; let $\varphi\colon U \times S \to E$ and $\psi\colon V \times S \to E$ be products charts. Then there exists a product chart $\omega\colon M \times T \to E$, where M is an open neighbourhood of $C \cup D$ in $p^{-1}(x)$. Furthermore, we can chose ω such that $\omega = \varphi$ on $C \times T$ and $\omega = \psi$ on $(D - U) \times T$.*

▼

Proof Let $C_+ \subset U$ and $D_+ \subset V$ be compact neighbourhoods of C, D in $p^{-1}(x)$.



Applying the lemma above to $V \times X \to X$ we deduce that there exists a product chart for p around V

$$\psi_1\colon V \times S_1 \to E$$

such that

(a) $\psi_1 = \psi$ on $(V - U) \times S_1$

(b) $\psi_1 = \varphi$ on $(C_+ \cap D_+) \times S_1$

Let $M_1 = \mathring{C}_+ \cup \mathring{D}_+$ and $T_1 = S \cap S_1$ and define

$$\omega\colon M_1 \times T_1 \to E$$

by putting

$$\omega \mid \mathring{C}_+ \times T_1 = \varphi \mid \mathring{C}_+ \times T_1 \quad \text{and} \quad \omega \mid \mathring{D}_+ \times T_1 = \psi_1 \mid \mathring{D}_+ \times T_1$$

Essentially, this is the required product chart. Since ω is obtained by glueing two product charts, it suffices to ensure that ω is injective. It may not be injective but it is locally injective by definition and furthermore, $\omega|M_1$ is injective, being equal to the inclusion $M_1 \subset p^{-1}(x)$. Now we restrict ω firstly to the interior of a compact neighbourhood of $C \cup D$ in M_1, let us say M. Once this has been done it will suffice to show that there exists a neighbourhood T of x in X, contained in T_1, such that $\omega \mid M \times T_1$ is injective. The existence of such a T_1 follows from a standard argument, see below. This completes the proof. □

The standard argument which we just used is the same as the familiar one which establishes that, if $N \subset A$ are differential manifolds, with N compact and $E(\varepsilon)$ is a small ε–neighbourhood of the zero–section of the normal vector bundle of N in A, then a diffeomorphism between $E(\varepsilon)$ and a tubular neighbourhood of N in A is given by the exponential function, which is locally injective on $E(\varepsilon)$.

▲

Theorem (Siebenmann) *Let $p\colon E \to X$ be a closed Γ–submersion, with compact fibres. Then p is a locally trivial Γ–bundle.*

Proof The glueing lemma, together with a finite induction, ensures that, if $x \in X$, then there exists a product chart

$$\varphi\colon p^{-1}(x) \times S \to E$$

around $p^{-1}(x)$. The set $N = p(E - \operatorname{Im}\varphi)$ is closed in X, since p is a closed map. Furthermore N does not contain x. If $S_1 = S - (X - N)$, then the restricted chart $p^{-1}(x) \times S_1 \to E$ has image equal to $p^{-1}(S_1)$. In fact, when $p(y) \in S_1$, we have that $p(y) \notin N$ and therefore $y \in \operatorname{Im}\varphi$. This ends the proof of the theorem. □

We recall that a continuous map between metric spaces and with compact fibres, is closed if and only if it is proper, ie, if the preimage of each compact set is compact.

1.6 Submersions $p\colon E \to X$ between manifolds with boundary

Submersions between manifolds with boundary are defined in the same way and the theory is developed in an analogous way to that for manifolds without boundary. The following changes apply:

(a) for $i = k, l$ in 1.1, we substitute $\mathbb{R}^i_+ \equiv \{x_1 \geq 0\}$ for $\mathbb{R}^i$

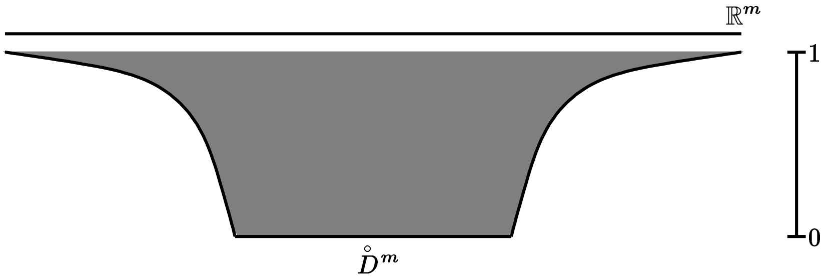
(b) in 1.4 the isotopy $G_t\colon V \to X$ must be *proper*, ie, formed by embeddings onto open subsets of X (briefly, G_t must be an isotopy of *open embeddings*).

Addendum to the isotopy extension lemma 1.4 *If $Q = I^n$, then we can take S to be the whole of Q.*

Note Even in the classical case $Q = [0,1]$ the extension of the isotopy cannot, in general, be on the whole of V. For example the isotopy $G(v,t)\colon \mathring{D}^m \times I \to \mathbb{R}^m \times I$ of note 1.4, ie,

$$G(v,t) = \left(\frac{v}{1 - t\|v\|} \,, t \right),$$

with $t \in [0,1]$, connects the inclusion $\mathring{D}^m \subset \mathbb{R}^m (t = 0)$ with G_1, which cannot be extended. A fortiori, G cannot be extended.



1.7 Differentiable submersions

These are much more familiar objects than the topological ones. Changing the notation slightly, a differentiable map $f\colon X \to Y$ between manifolds without boundary is a *submersion* if it verifies the conditions in 1.1 and 1.2, taking now $\Gamma = \mathrm{Diff}$. However the following *alternative definition* is often used: f is a submersion if its differential is surjective for each point in X.

Theorem *A proper submersion, with compact fibres, is a differentiable bundle.*

Proof For each $y \in Y$, a sufficiently small tubular neighbourhood of $p^{-1}(y)$ is the required product chart. □

1.8 As we saw in 1.2 there are simple examples of submersions with non-compact fibres which are not fibrations.

We now wish to discuss a case which is remarkable for its content and difficulty. This is a case where a submersion with non-compact fibres is a submersion. This result has a central role in the theorem of classification of PL structures on a topological manifold.

Let Δ be a simplex or a cube and let M^m be a topological manifold without boundary which is not necessarily compact and let also Θ be a PL structure on $\Delta \times M$ such that the projection

$$p\colon (\Delta \times M)_\Theta \to \Delta$$

is a PL submersion.

Fibration theorem (Kirby–Siebenmann 1969) *If $m \neq 4$, then p is a* PL *bundle (necessarily trivial).*

Before starting to explain the theorem's intricate line of the proof we observe that in some sense it might appear obvious. It is therefore symbolic for the hidden dangers and the possibilities of making a blunder found in the study of the interaction between the combinatorial and the topological aspects of manifolds. Better than any of my efforts to represent, with inept arguments, the uneasiness caused by certain idiosyncrasies is an outburst of L Siebenmann, which is contained in a small note of [Kirby–Siebenmann 1977, p. 217], which is referring exactly to the fibration theorem:

> *"This modest result may be our largest contribution to the final classification theorem; we worked it out in 1969 in the face of a widespread belief that it was irrelevant and/or obvious and/or provable for all dimensions (cf [Mor$_3$], [Ro$_2$] and the 1969 version of [Mor$_4$]). Such a belief was not so unreasonable since 0.1 is obvious in case M is compact: every proper* CAT *submersion is a locally trivial bundle". (L Siebenmann)*

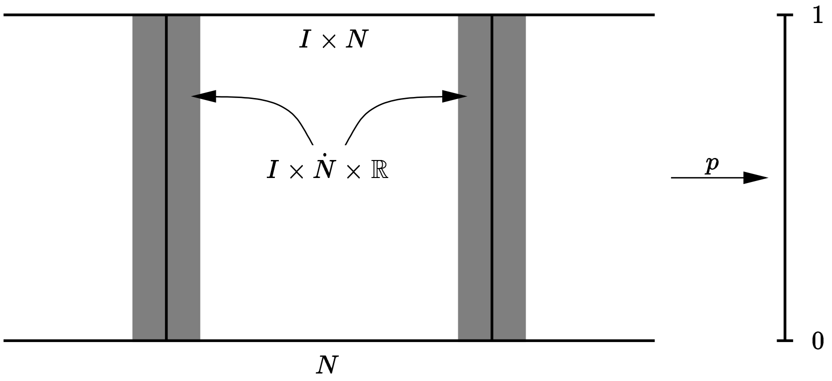
Proof We will assume $\Delta = I$. The general case is then analogous with some more technical detail. We identify M with $0 \times M$ and observe that, since p is a submersion, then Θ restricts to a PL structure on $M = p^{-1}(0)$. This enables us to assume that M is a PL manifold. We filter M by means of an ascending chain

$$M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_i \subset \cdots$$

of PL compact m–submanifolds, such that each M_i is in a regular neighbourhood of some polyhedron contained in M and, furthermore, $M_i \subset \mathring{M}_{i+1}$ and $M = \cup_i M_i$. Such a chain certainly exists. Furthermore, since M_i is a regular neighbourhood, its frontier $\dot{M}_i$ is PL bicollared in M and we can take open disjoint PL bicollars $V_i \approx \dot{M}_i \times \mathbb{R}$, such that $V_i \cap M_i = M_i \times (-\infty, 0]$. Let us fix an index i and, for the sake simplicity, we will write N instead of M_i. We will work in $E = (I \times \dot{N} \times \mathbb{R})_\Theta$, equipped with Cartesian projections.

The reader can observe that, even if $I \times \dot{N}$ is a PL manifold with the PL manifold structure coming from M, it is not, a priori, a PL submanifold of E. It is exactly this situation that creates some difficulties which will force us to avoid the dimension $m = 4$.

$$\begin{array}{ccc} E & \xrightarrow{p} & I \\ {\scriptstyle\pi}\downarrow & & \\ \mathbb{R} & & \end{array}$$



1.8.1 First step

We start by recalling the engulfing theorem proved in I.4.11:

Theorem *Let W^w be a closed topological manifold with $w \neq 3$, let Θ be a* PL *structure on $W \times \mathbb{R}$ and $C \subset W \times \mathbb{R}$ a compact subset. Then there exists a* PL *isotopy G of $(W \times \mathbb{R})_\Theta$ having compact support and such that $G_1(C) \subset W \times (-\infty, 0]$.*

The theorem tells us that the tide, which rises in a PL way, swamps every compact subset of $(W \times \mathbb{R})_\Theta$, even if W is not a PL manifold.

Corollary (Engulfing from below) *For each $\lambda \in I$ and for each pair of integers $a < b$, there exists a PL isotopy with compact support*

$$G_t \colon (\lambda \times \dot{N} \times \mathbb{R})_\Theta \to (\lambda \times \dot{N} \times \mathbb{R})_\Theta$$

such that

$$G_1(\lambda \times \dot{N} \times (-\infty, a)) \supset \lambda \times \dot{N} \times (-\infty, b]$$

provided that $m \neq 4$.

The proof is immediate. □

1.8.2 Second step (Local version of engulfing from below)

By theorem 1.5 each compact subset of the fibre of a submersion is contained in a product chart. Therefore, for each integer r and each point λ of I, there exists a product chart

$$\varphi \colon \lambda \times \dot{N} \times (-r, r) \times I_\lambda \to E$$

for the submersion p, where I_λ indicates a suitable open neighbourhood of λ in I. If $a \leq b$ are *any* two integers, then *Corollary* 1.8.1 ensures that r can be chosen such that $[a, b] \subset (-r, r)$ and also that there exists a PL isotopy,

$$G_t \colon \lambda \times \dot{N} \times (-r, r) \to \lambda \times \dot{N} \times (-r, r),$$

which engulfs level b inside level a and also has a compact support. Now let $f \colon I \to I$ be a PL map, whose support is contained in I_λ and is 1 on a neighbourhood of λ. We define a PL isotopy

$$H_t \colon E \to E$$

in the following way:

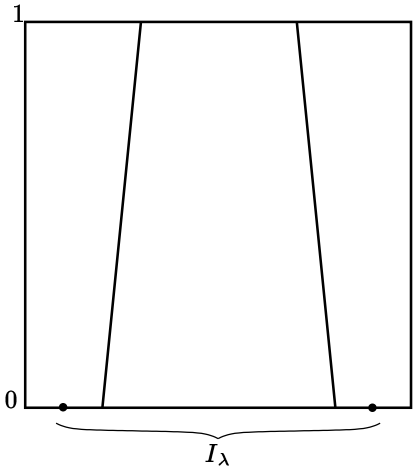
(a) $H_t|\mathrm{Im}\,\varphi$ is determined by the formula

$$H_t(\varphi(x,\mu)) = \varphi(G_{f(\mu)t}(x),\mu)$$

where $x \in \lambda \times \dot{N} \times (-r,r)$ and $\mu \in I_\lambda$.

(b) H_t is the identity outside $\mathrm{Im}\,\varphi$.

It results that H_t is an isotopy of all of E which *commutes* with the projection p, ie, H_t is a *spike isotopy.*



A possibilty for $f\colon I \to I$

The effect of H_t is that of including level b inside level a, at least as far as small a neighbourhood of λ.

1.8.3 Third step (A global spike version of the Engulfing form below)

For each pair of integers $a < b$, there exists a PL isotopy

$$H_t\colon E \to E,$$

which commutes with the projection p, has compact support and engulfs the level b inside the level a, ie,

$$H_1(I \times \dot{N} \times (-\infty, a)) \supset I \times \dot{N} \times (-\infty, b].$$

The proof of this claim is an instructive exercise and is therefore left to the reader. Note that I will have to be divided into a finite number of sufficiently small intervals, and that the isotopies of local spike engulfing provided by the step 1.8.2 above will have to be wisely composed.

1.8.4 Fourth step (The action of $\mathbb{Z}$)

For each pair of integers $a < b$, there exists an open set $E(a,b)$ of E, which contains $\pi^{-1}[a,b]$ and is such that

$$p\colon E(a,b) \to I$$

is a PL bundle.

Proof Let $H_1\colon E\to E$ be the PL homeomorphism constructed in 1.8.3. Let us consider the compact set

$$C(a,b)=H_1(\pi^{-1}(-\infty,a])\setminus\pi^{-1}(-\infty,a)$$

and the open set

$$E(a,b)=\bigcup_{n\in\mathbb{Z}}H_1^n(C(a,b)).$$

There is a PL action of $\mathbb{Z}$ on $E(a,b)$, given by

$$\begin{aligned}q\colon\mathbb{Z}\times E(a,b)&\to E(a,b)\\(1,x)&\mapsto H_1(x)\end{aligned}$$

This action commutes with p.

If $B=E(a,b)/\mathbb{Z}$ is the space of the orbits then we have a commutative diagram

$$\begin{tikzcd} E(a,b) \arrow[rr, "q"] \arrow[dr, "p"'] & & B \arrow[dl, "p'"] \\ & I & \end{tikzcd}$$

Since H_1 is PL, then B inherits a PL structure which makes q into a PL covering; therefore since p is a PL submersion, then p' also is a submersion. Furthermore each fibre of p' is compact, since it is the quotient of a compact set, and p' is closed. So p' is a PL bundle, and from that it follows that p also is such a bundle (some details have been omitted).

1.8.5 Fifth step (Construction of product charts around the manifolds M_i)

Until now we have worked with a given manifold $M_i\subset M$ and denoted it with N. Now we want to vary the index i. Step 1.8.4 ensures the existence of an open subset

$$E_i'\subset E_i=(I\times\dot{M}_i\times\mathbb{R})_\Theta$$

which contains $I\times\dot{M}_i\times 0$ such that it is a locally trivial PL bundle on I. We chose PL trivialisations

$$\begin{tikzcd} h_i\colon I\times Y_i' \arrow[rr, "\approx"] \arrow[dr] & & E_i'\subset I\times V_i \arrow[dl] \\ & I & \end{tikzcd}$$

and we write M_i' for $Y_i'\cap M_i=Y_i'\cap(\dot{M}_i\times(-\infty,0])$.

We define a PL submanifold X_i of $(I\times M)_\Theta$, by putting

$$X_i=\{(I\times M_i-E_i')\cup h_i(I\times M_i')\}_\Theta$$

and observe that $X_i\subset\mathring{X}_{i+1}$ and $\bigcup_i X_i=(I\times M)_\Theta$.

The projection $p_i\colon X_i \to I$ is a PL submersion and we can say that the whole proof of the theorem developed until now has only one aim: ensure for i the existence of a PL submersion of type p_i.

Now, since X_i is compact, the projection p_i is a locally trivial PL bundle and therefore we have trivialisations

$$\begin{tikzcd} I\times M_i \arrow[rr, "g_i"] \arrow[dr] & & X_i \arrow[dl, "p_i"] \\ & I & \end{tikzcd}$$

1.8.6 Sixth step (Compatibility of the trivialisations)

In general we cannot expect that g_i coincides with g_{i+1} on $I \times M_i$. However it is possible to alter g_{i+1} in order to obtain a new chart g'_{i+1} which is compatible with g_i. To this end let us consider the following commutative diagram

$$\begin{tikzcd} I \times M_{i+1} \arrow[dr, "\Gamma_i"'] & \supset \quad I \times M_i \arrow[r, "g_i", "\approx"'] \arrow[d, "\gamma_i"'] & X_i \subset (I \times M)_\Theta \arrow[d, phantom, "\cap"] \arrow[d, equal, shift left=8ex] \\ & I \times M_{i+1} \arrow[r, "g_{i+1}"] & X_{i+1} \subset (I \times M)_\Theta \end{tikzcd}$$

where all the maps are intended to be PL and they also commute with the projection on I. The map γ_i is defined by commutativity and Γ_i exists by the isotopy extension theorem of Hudson and Zeeman. It follows that

$$g'_{i+1} := g_{i+1}\Gamma_i$$

is the required compatible chart.

1.8.7 Conclusion

In light of 1.8.6. and of an infinite inductive procedure we can assume that the trivialisations $\{g_i\}$ are compatible with each other. Then

$$g := \bigcup_i g_i$$

is a PL isomorphism $I \times M_\Theta \approx (I \times M)_\Theta$, which proves the theorem. □

▲

Note I advise the interested reader who wishes to study submersions in more depth, including also the case of submersions of stratified topological spaces, as well as other difficult topics related to the spaces of homomorphisms, to consult [Siebenmann, 1972].

To the reader who wishes to study in more depth the theorem of fibrations for submersions with non compact fibres, including extension theorems of sliced concordances, I suggest [Kirby–Siebemann 1977 Essay II].

2 The space of the PL structures on a topological manifold M

Let M^m be a topological manifold without boundary, which is not necessarily compact.

2.1 The complex $\mathrm{PL}(M)$

The space $PL(M)$ of PL structures on M is the SS–set which has as typical k–simplex σ a PL structure Θ on $\Delta^k \times M$, such that the projection

$$(\Delta^k \times M)_\Theta \xrightarrow{\pi_1} \Delta^k$$

is a PL submersion. The semisimplicial operators are defined using fibred products. More precisely, if $\lambda\colon \Delta^l \to \Delta^k$ is in Δ^*, then $\lambda^\#(\sigma)$ is the PL structure on $\Delta^l \times M$, which is obtained by pulling back π_1 by λ:

$$\lambda^\#(\sigma)\left\{\begin{array}{ccc} (\Delta^l \times M)_{\lambda^*\Theta} & \longrightarrow & (\Delta^k \times M)_\Theta \\ \pi_1 \downarrow & & \downarrow \pi_1 \\ \Delta^l & \xrightarrow{\lambda} & \Delta^k \end{array}\right.$$

An equivalent definition is that a k–simplex of $PL(M)$ is an equivalence class of commutative diagrams

$$\begin{tikzcd} \Delta^k \times M \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & Q \arrow[dl, "p"] \\ & \Delta^k & \end{tikzcd}$$

where Q is a PL manifold, p a PL submersion, f a topological homeomorphism and the two diagrams are equivalent if $f' = \varphi \circ f$, where $\varphi\colon Q \to Q'$ is a PL isomorphism.

Under this definition a k–simplex of $\mathrm{PL}(M)$ is a *sliced concordance* of PL structures on M.

▼

In order to show the equivalence of these two definitions, let temporarily $\mathrm{PL}'(M)$ (respectively $\mathrm{PL}''(M)$) be the SS–set obtained by using the first (respectively the second) definition. We will show that there is a canonical semisimplicial isomorphism $\alpha\colon \mathrm{PL}'(M) \to \mathrm{PL}''(M)$. Define $\alpha(\Delta \times M)_\Theta$ to be the equivalence class of $Id\colon \Delta \times M \to (\Delta \times M)_\Theta$ where $\Delta = \Delta^k$. Now let $\beta\colon \mathrm{PL}''(M) \to$

$\mathrm{PL}'(M)$ be constructed as follows. Given $f\colon \Delta \times M \to Q_{PL}$, let Θ be a maximal PL atlas on Q_{PL}. Then set $\beta(f) := (\Delta \times M)_{f^*(\Theta)}$. The map β is well defined since, if f' is equivalent to f in $\mathrm{PL}''(M)$, then

$$(\Delta \times M)_{f'^*\Theta'} = (\Delta \times M)_{(\phi\circ f)^*\Theta'} = (\Delta \times M)_{f^*\phi^*\Theta'} = (\Delta \times M)_{f^*\Theta}.$$

The last equality follows from the fact that ϕ is PL, hence $\phi^*\Theta' = \Theta$. Now let us prove that each of α and β is the inverse of the other. It is clear that $\beta \circ \alpha = Id_{\mathrm{PL}'(M)}$. Moreover

$$\alpha \circ \beta(\Delta \times M \xrightarrow{f} Q) = \alpha(\Delta \times M)_{f^*\Theta} = (\Delta \times M \xrightarrow{Id} (\Delta \times M)_{f^*\Theta}).$$

But $f \circ Id = f\colon (\Delta \times M)_{f^*\Theta} \to Q$ is PL by construction, therefore $\alpha \circ \beta$ is the identity.

Since the submersion condition plays no relevant role in the proof, we have established that $\mathrm{PL}'(M)$ and $\mathrm{PL}''(M)$ are canonically isomorphic.

▲

Observations (a) If M is compact, we know that the submersion π_1 is a trivial PL bundle. In this case a k–simplex is a *k–isotopy of structures* on M. See also the next observation.

(b) (Exercise) If M is compact then the set $\pi_0(\mathrm{PL}(M))$ of path components of $\mathrm{PL}(M)$ has a precise geometrical meaning: two PL structures Θ, Θ' on M are in the same path component if and only if there exists a topological isotopy $h_t\colon M \to M$, with $h_0 = 1_M$ and $h_1\colon M_\Theta \to M_{\Theta'}$ a PL isomorphism. This is also true if M is non-compact and the dimension is not 4 (hint: use the fibration theorem).

(c) $\mathrm{PL}(M) \neq \emptyset$ if and only if M admits a PL structure.

(d) If $\mathrm{PL}(M)$ is contractible then M admits a PL structure and such a structure is strongly unique. This means that two structures Θ, Θ' on M are isotopic (or concordant). Furthermore any two isotopies (concordances) between Θ and Θ' can be connected through an isotopy (concordance respectively) with two parameters, and so on.

(e) If $m \leq 3$, Kerékjárto (1923) and Moise (1952, 1954) have proved that $\mathrm{PL}(M)$ is contractible. See [Moise 1977].

2.2 The SS–set $\mathrm{PL}(TM)$

Now we wish to define the space of PL structures on the tangent microbundle on M. In this case it will be easier to take as TM the microbundle

$$M \xrightarrow{\Delta} M \times M \xrightarrow{\pi_2} M\ ,$$

where π_2 is the projection on the second factor. Hirsch calls this the *second* tangent bundle. This is obviously a notational convention since if we swap the factors we obtain a canonical isomorphism between the first and the second tangent bundle.

More generally, let, $\xi\colon X \xrightarrow{i} E(\xi) \xrightarrow{p} X$ be a topological m–microbundle on a topological manifold X. A *PL structure* Θ on ξ is a PL manifold structure on an open neighbourhood U of $i(X)$ in $E(\xi)$, such that $p\colon U_\Theta \to X$ is a PL submersion.

If Θ' is another PL structure on ξ, we say that Θ is *equal* to Θ' if Θ and Θ' define the same germ around the zero-section, ie, if $\Theta = \Theta'$ in some open neighbourhood of $i(X)$ in $E(\xi)$. Then Θ really represents an equivalence class.

Note A PL structure Θ on ξ is different from a PL microbundle structure on ξ, namely a PL_μ–structure, as it was defined in II.4.1. The former does *not* require that the zero–section $i\colon X \to U_\Theta$ is a PL map. Consequently $i(X)$ does not have to be a PL submanifold of U_Θ, even if it is, obviously, a topological submanifold.

The *space of the PL structures* on ξ, namely $\mathrm{PL}(\xi)$, is the SS–set, whose typical k–simplex is the germ around $\Delta^k \times X$ of a PL structure on the product microbundle $\Delta^k \times \xi$. The semisimplicial operators are defined using the construction of the induced bundle.

Later we shall see that as far as the classification theorem is concerned the concepts of PL structures and PL_μ–structures on a topological microbundle are effectively the same, namely we shall prove (fairly easily) that the SS–sets $\mathrm{PL}(\xi)$ and $\mathrm{PL}_\mu(\xi)$ have the same homotopy type (proposition 4.8). However the former space adapts naturally to the case of smoothings (Part V) when there is no fixed PL structure on M.

Lemma $\mathrm{PL}(M)$ *and* $\mathrm{PL}(TM)$ *are* KSS*–sets.*

Proof This follows by pulling back over the PL retraction $\Delta^k \to \Lambda^k$. □

3 Relation between $\mathrm{PL}(M)$ and $\mathrm{PL}(TM)$

From now on, unless otherwise stated, we will introduce a hypothesis, which is only apparently arbitrary, on our initial topological manifold M.

(*) We will assume that there is a PL structure *fixed* on M.

The arbitrariness of this assumption is in the fact that it is our intention to tackle jointly the two problems of *existence* and of the *classification* of the PL structures on M. However this preliminary hypothesis simplifies the exposition and makes the technique more clear, without invalidating the problem of the classification. Later we will explain how to avoid using (*), see section 5.

3.1 The differential

Firstly we define an SS–map

$$d\colon \mathrm{PL}(M) \to \mathrm{PL}(TM),$$

namely the *differential*, by setting, for $\Theta \in PL(M)^{(k)}$, $d\Theta$ to be equal to the PL structure $\Theta \times M$ on $E(\Delta^k \times TM) = \Delta^k \times M \times M$.

Our aim is to prove that the differential is a homotopy equivalence, except in dimension $m = 4$.

Classification theorem *$d\colon \mathrm{PL}(M) \to \mathrm{PL}(TM)$ is a homotopy equivalence for $m \neq 4$.*

The philosophy behind this result is that *infinitesimal* information contained in TM can be *integrated* in order to solve the classification problem on M. In other words d is used to *linearise* the classification problem.

The theorem also holds for $m = 4$ if none of the components of M are compact. However the proof uses results of [Gromov 1968] which are beyond the scope of this book.

We now set the stage for the proof of theorem 3.1.

3.2 The Mayer–Vietoris property

Let U be an open set of M. Consider the PL structure induced on U by the one fixed on M. The correspondences $U \to \mathrm{PL}(U)$ and $U \to \mathrm{PL}(TU)$ define contravariant functors from the category of the inclusions between open sets of M, with values in the category of SS–sets. Note that $TU = TM|_U$.

Notation We write $F(U)$ to denote either $\mathrm{PL}(U)$ or $\mathrm{PL}(TU)$ without distinction.

Lemma (Mayer–Vietoris property) *The functor F transforms unions into pullbacks, ie, the following diagram*

$$\begin{tikzcd} F(U \cup V) \arrow[r] \arrow[d] & F(U) \arrow[d] \\ F(V) \arrow[r] & F(U \cap V) \end{tikzcd}$$

is a pull back for each pair of open sets $U, V \subset M$.

The proof is an easy exercise. □

3.3 Germs of structures

Let A be any subset of M. The functor F can then be extended to A using the germs. More precisely, we set

$$\mathrm{PL}(A \subset M) := \lim_{\rightarrow} \{\mathrm{PL}(U) : A \subset U \text{ open in } M\}$$

$$\mathrm{PL}(TM|_A) := \lim_{\rightarrow} \{\mathrm{PL}(TU) : A \subset U \text{ open in } M\}.$$

The differential can also be extended to an SS–map

$$d_A \colon \mathrm{PL}(A \subset M) \to \mathrm{PL}(TM|_A)$$

which is still defined using the rule $\Theta \to \Theta \times U$.

Finally, the Mayer–Vietoris property 3.2 is still valid if, instead of open sets we consider closed subsets. This implies that, when we write $F(A)$ for either $\mathrm{PL}(A \subset M)$ or $\mathrm{PL}(TM|A)$, then the diagram of restrictions

$$\begin{tikzcd} F(A \cup B) \arrow[r] \arrow[d] & F(A) \arrow[d] \\ F(B) \arrow[r] & F(A \cap B) \end{tikzcd} \tag{3.3.1}$$

is a pullback for closed $A, B \subset M$. □

3.4 Note about base points

If $\Theta \in \mathrm{PL}(M)^{(0)}$, ie, Θ is a PL structure on M, there is a canonical base point $*$ for the SS–set $\mathrm{PL}(M)$, such that

$$*_k = \Delta^k \times \Theta.$$

In this way we can point each path component of $\mathrm{PL}(M)$ and correspondingly of $\mathrm{PL}(TM)$. Furthermore we can assume that d is a pointed map on each path component. The same thing applies more generally for $\mathrm{PL}(A \subset M)$ and its related differential. In other words we can always assume that the diagram 3.3.1 is made up of SS–maps which are pointed on each path component.

4 Proof of the classification theorem

The method of the proof is based on immersion theory as viewed by Haefliger and Poenaru (1964) et al. Among the specialists, this method of proof has been named the *Haefliger and Poenaru machine* or the *immersion theory machine*. Various authors have worked on this topic. Among these we cite [Gromov 1968], [Kirby and Siebenmann 1969], [Lashof 1970] and [Rourke 1972].

There are several versions of the immersion machine tailored to the particular theorem to be proved. All versions have a common theme. We wish to prove that a certain (differential) map d connecting functors defined on manifolds, or more generally on germs, is a homotopy equivalence. We prove:

(1) The functors satisfy a Mayer–Vietoris property (see for example 3.2 above).

(2) The differential is a homotopy equivalence when the manifold is $\mathbb{R}^n$.

(3) Restrictions to certain subsets are Kan fibrations.

Once these are established there is a transparent and automatic procedure which leads to the conclusion that d is a homotopy equivalence. This procedure could even be decribed with axioms in terms of categories. We shall not axiomatise the machine. Rather we shall illustrate it by example.

The versions differ according to the precise conditions and subsets used. In this section we apply the machine to prove theorem 3.1. We are working in the topological category and we shall establish (3) for arbitrary compact subsets. The Mayer–Vietoris property was established in 3.2. We shall prove (2) in sections 4.1–4.4 and (3) in section 4.5 and 4.6. The machine proof itself comes in section 4.7.

In the next part (IV.1) we shall use the machine for its original purpose, namely immersion theory. In this version, (3) is established for the restricion of X to X_0 where X is obtained from X_0 attaching one handle of index $< \dim X$.

The classification theorem for $M = \mathbb{R}^m$

4.1 The following proposition states that the function which restricts the PL structures to their germs in the origin is a homotopy equivalence in $\mathbb{R}^m$.

Proposition *If $M = \mathbb{R}^m$ with the standard PL structure, then the restriction $r\colon \mathrm{PL}(\mathbb{R}^m) \to \mathrm{PL}(0 \subset \mathbb{R}^m)$ is a homotopy equivalence.*

Proof We start by stating that, given an open neighbourhood U of 0 in $\mathbb{R}^m$, there always exist a homeomorphism ρ between $\mathbb{R}^m$ and a neighbourhood of 0 contained in U, which is the identity on a neighbourhood of 0. There also exists an isotopy $H\colon I \times \mathbb{R}^m \to \mathbb{R}^m$, such that $H(0,x) = x$, $H(1,x) = \rho(x)$ for each $x \in \mathbb{R}^m$ and $H(t,x) = x$ for each $t \in I$ and for each x in some neighbourhood of 0.

In order to prove that r is a homotopy equivalence we will show that r induces an isomorphism between the homotopy groups.

(a) Consider a SS–map $S^i \to PL(0 \subset \mathbb{R}^m)$. This is nothing but an i–sphere of structures on an open neighbourhood U of 0, ie, a diagram:

$$\begin{tikzcd} S^i \times U \arrow[rr, "\phi"] \arrow[dr, "\pi_1"'] & & (S^i \times U)_\Theta \arrow[dl, "p"] \\ & S^i & \end{tikzcd}$$

where Θ is a PL structure, p is a PL submersion and φ is a homeomorphism. Then the composed map

$$S^i \times \mathbb{R}^m \xrightarrow{f} S^i \times U \xrightarrow{\varphi} (S^i \times U)_\Theta,$$

where $f(\tau, x) = (\tau, \rho(x))$, gives us a sphere of structures on the whole of $\mathbb{R}^m$. The germ of this structure is represented by φ. This proves that r induces an epimorphism between the homotopy groups.

(b) Let

$$f_0\colon S^i \times \mathbb{R}^m \to (S^i \times \mathbb{R}^m)_{\Theta_0}$$

and

$$f_1\colon S^i \times \mathbb{R}^m \to (S^i \times \mathbb{R}^m)_{\Theta_1}$$

be two spheres of structures on $\mathbb{R}^m$ and assume that their germs in $S^i \times 0$ define homotopic maps of S^i in $\mathrm{PL}(0 \subset \mathbb{R}^m)$. This implies that there exists a PL structure Θ and a homeomorphism

$$G\colon I \times S^i \times U \to (I \times S^i \times U)_\Theta$$

which represents a map of $I \times S^i$ in PL $(0 \subset \mathbb{R}^m)$ and which is such that

$$G(0,\tau,x) = f_0(\tau,x) \qquad G(1,\tau,x) = f_1(\tau,x)$$

for $\tau \in S^i$, $x \in U$.

We can assume that $G(t,\tau,x)$ is independent of t for $0 \le t \le \varepsilon$ and $1 - \varepsilon \le t \le 1$. Also consider, in the topological manifold $I \times S^i \times \mathbb{R}^m$, the structure $\overline{\overline{\Theta}}$ given by

$$\Theta_0 \times [0,\varepsilon) \cup \Theta \cup (1-\varepsilon, 1] \times \Theta_1.$$

The three structures coincide since Θ restricts to Θ_i on the overlaps, and therefore $\overline{\Theta}$ is defined on a topological submanifold Q of $I \times S^i \times \mathbb{R}^m$.

Finally we define a homeomorphism

$$F\colon I \times S^i \times \mathbb{R}^m \to Q_{\overline{\Theta}}$$

with the formula

$$F(t,\tau,x) = \begin{cases} G\left(t,\tau,H\left(\frac{t}{\varepsilon},x\right)\right) & 0 \le t \le \varepsilon \\ G\left(t,\tau,\rho(x)\right) & \varepsilon \le t \le 1-\varepsilon \\ G\left(t,\tau,H\left(\frac{1-t}{\varepsilon},x\right)\right) & 1-\varepsilon \le t \le 1. \end{cases}$$

($x \in \mathbb{R}^m$). The map F is a homotopy of Θ_0 and Θ_1, and then r induces a monomorphism between the homotopy groups which ends the proof of the proposition. □

4.2 The following result states that a similar property applies to the structures on the tangent bundle $\mathbb{R}^m$.

Proposition *The restriction map*

$$r\colon \mathrm{PL}(T\mathbb{R}^m) \to \mathrm{PL}(T\mathbb{R}^m|0)$$

is a homotopy equivalence.

Proof We observe that $T\mathbb{R}^m$ is trivial and therefore we will write it as

$$\mathbb{R}^m \times X \xrightarrow{\pi_X} X$$

with zero–section $0 \times X$, where X is a copy of $\mathbb{R}^m$ with the standard PL structure.

Given any neighbourhood U of 0, let $H\colon I \times X \to X$ be the isotopy considered at the beginning of the proof of 4.1. We remember that a PL structure on $T\mathbb{R}^m$ is a PL structure of manifolds around the zero- -section. Furthermore π_X is submersive with respect to this structure. The same applies for the PL structures on TU, where U is a neighbourhood of 0 in X. It is then clear that by using the isotopy H, or even only its final value $\rho\colon X \to U$, each PL structure on TU expands to a PL structure on the whole of $T\mathbb{R}^m$. The same thing happens for each sphere of structures on TU. This tells us that r induces an epimorphism between the homotopy groups. The injectivity is proved in a similar way, by using the whole isotopy H. It is not even necessary for H to be an isotopy, and in fact a homotopy would work just as well. □

Summarising we can say that proposition 4.1 is established by expanding isotopically a typical neighbourhood of the origin to the whole of $\mathbb{R}^m$, while proposition 4.2 follows from the fact that 0 is a deformation retract of $\mathbb{R}^m$.

4.3 We will now prove that, still in $\mathbb{R}^m$, if we pass from the structures to their germs in 0, the differential becomes in fact an isomorphism of ss–sets (in particular a homotopy equivalence).

Proposition *$d_0\colon \mathrm{PL}(0 \subset \mathbb{R}^m) \to \mathrm{PL}(T\mathbb{R}^m|0)$ is an isomorphism of complexes.*

Proof As above, we write

$$T\mathbb{R}^m : \mathbb{R}^m \times X \xrightarrow{\pi_X} X \qquad (X = \mathbb{R}^m)$$

and we observe that a germ of a structure in $T\mathbb{R}^m|0$ is locally a product in the following way. Given a PL structure Θ near U in $\mathbb{R}^m \times U$, where U is a neighbourhood of 0 in X, then, since π_X is a PL submersion, there exists a neighbourhood $V \subset U$ of 0 in X and a PL isomorphism between $\Theta|TV$ and $\Theta_V \times U$, where Θ_V is a PL structure on V, which defines an element of $\mathrm{PL}(0 \subset \mathbb{R}^m)$. Since the differential $\underline{d} = d_0$ puts a PL structure around 0 in the fibre of $T\mathbb{R}^m$, then it is clear that d_0 is nothing but another way to view the same object. □

4.4 The following theorem is the first important result we were aiming for. It states that the differential is a homotopy equivalence for $M = \mathbb{R}^m$.

In other words, the classification theorem 3.1 holds for $M = \mathbb{R}^m$.

Theorem *$d\colon \mathrm{PL}(\mathbb{R}^m) \to \mathrm{PL}(T\mathbb{R}^m)$ is a homotopy equivalence.*

Proof Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{PL}(\mathbb{R}^m) & \xrightarrow{d} & \mathrm{PL}(T\mathbb{R}^m) \\ \downarrow{\scriptstyle r} & & \downarrow{\scriptstyle r} \\ \mathrm{PL}(0 \subset \mathbb{R}^m) & \xrightarrow[d_0]{} & \mathrm{PL}(T\mathbb{R}^m|0) \end{array}$$

By 4.1 and 4.2 the vertical restrictions are homotopy equivalences. Also by 4.3 d_0 is a homeomorphism and therefore d is a homotopy equivalence. □

The two fundamental fibrations

4.5 The following results which prepare for the proof the classification theorem have a different tone. In a word, they establish that the majority of the restriction maps in the PL structure spaces are Kan fibrations.

Theorem *For each compact pair $C_1 \subset C_2$ of M the natural restriction*

$$r\colon \mathrm{PL}(TM|C_2) \to \mathrm{PL}(TM|C_1)$$

is a Kan fibration.

Proof We need to prove that each commutative diagram

$$\begin{array}{ccc} \Lambda^k & \longrightarrow & \mathrm{PL}(TM|C_2) \\ \big\downarrow & & \big\downarrow{\scriptstyle r} \\ \Delta^k & \longrightarrow & \mathrm{PL}(TM|C_1) \end{array}$$

can be completed by a map

$$\Delta^k \to \mathrm{PL}(TM|C_2)$$

which preserves commutativity.

In order to make the explanation easier we will assume $C_2 = M$ and we will write $C = C_1$. The general case is completely analogous, the only difference being that the are more "germs" (To those in C_1 we need to add those in C_2).

We will give details only for the lifting of paths when $(k = 1)$, the general case being identical.

We start with a simple observation. If ξ/X is a topological m–microbundle on the PL manifold X, if Θ is a PL structure on ξ and if $r\colon Y \to X$ is a PL map between PL manifolds, then Θ gives the induced bundle $r^*\xi$ a PL structure in a natural way using pullback. We will denote this structure by $r^*\Theta$. This has already been used (implicitly) to define the degeneracy operators $\Delta^{i+1} \to \Delta^i$ in $\mathrm{PL}(\xi)$, in the particular case of elementary simplicial maps cf 2.2.

Consider a path in $\mathrm{PL}(TM|C)$, ie, a PL structure Θ' on $I \times TU = I \times (TM|U)$, with U an open neighbourhood of C. A lifting of the starting point of this path to $\mathrm{PL}(TM)$ gives us a PL structure Θ'' on TM, such that $\Theta' \cup \Theta''$ is a PL structure Θ on the microbundle $0 \times TM \cup I \times TU$. Without asking for apologies we will ignore the inconsistency caused by the fact that the base of the last microbundle is not a PL manifold but a polyhedron given by the union of two PL manifolds along $0 \times U$. This inconsistency could be eliminated with some effort. We want to extend Θ to the whole of $I \times TM$. We choose a PL map $r\colon I \times M \to 0 \times M \cup I \times U$ which fixes $0 \times M$ and some neighbourhood of $I \times C$. Then $r^*\Theta$ is the required PL structure.

This ends the proof of the theorem. □

4.6 It is much more difficult to establish the property analogous to 4.5 for the PL structures on the manifold M, rather than on its tangent bundle:

Theorem *For each compact pair $C_1 \subset C_2 \subset M^m$ the natural restriction*

$$r\colon PL(C_2 \subset M) \to \mathrm{PL}(C_1 \subset M)$$

is a Kan fibration, if $m \neq 4$.

Proof If we use cubes instead of simplices we need to prove that each commutative diagram

$$\begin{tikzcd} I^k \arrow[r] \arrow[d] & \mathrm{PL}(C_2 \subset M) \arrow[d, "r"] \\ I^{k+1} \arrow[r] & \mathrm{PL}(C_1 \subset M) \end{tikzcd}$$

can be completed by a map

$$I^{k+1} \to \mathrm{PL}(C_2 \subset M)$$

which preserves commutativity.

We will assume again that $C_2 = M$ and we will write $C_1 = C$.

We have a PL k–cube of PL structures on M and an extension to a $(k+1)$–cube near C. This implies that we have a structure Θ on $I^k \times M$ and a structure Θ' on $I^{(k+1)} \times U$, where U is some open neighbourhood of C. By hypothesis the two structures coincide on the overlap, ie, $\Theta \mid I^k \times U = \Theta' \mid 0 \times I^k \times U$.

We want to extend $\Theta \cup \Theta'$ to a structure $\overline{\Theta}$ over the whole of $I^{k+1} \times M$, such that $\overline{\Theta}$ coincides with Θ' on $I^{k+1}\times$ some neighbourhood of C which is possibly smaller than U.

We will consider first the case $k = 0$, ie, the lifting of paths.

By the fibration theorem 1.8, if $m \neq 4$ there exists a sliced PL isomorphism over I

$$h\colon (I \times U)_{\Theta'} \to I \times U_{\Theta}$$

(recall that $\Theta'|0 = \Theta$). There is the natural topological inclusion $j\colon I \times U \subset I \times M$ so that the composition

$$j \circ h\colon I \times U \to I \times M$$

gives a topological isotopy of U in M and thus also of W in M, where W is the interior of a compact neighbourhood of C in U. From the topological isotopy extension theorem we deduce that the isotopy of W in M given by $(j \circ h)\,|_W$

extends to an ambient topological isotopy $F\colon I \times M \to I \times M$. Now endow the range of F with the structure $I \times M_\Theta$.

Since it preserves projection to I, the map F provides a 1-simplex of PL(M), ie a PL structure $\overline{\Theta}$ on $I \times M$. It is clear that $\overline{\Theta}$ coincides with Θ' at least on $I \times W$. In fact $F \mid I \times W$ is the composition of PL maps

$$(I \times W)_{\Theta'} \subset (I \times U)_{\Theta'} \xrightarrow{h} I \times U_\Theta \subset I \times M_\Theta$$

and therefore is PL, which is the same as saying that $\overline{\Theta} = \Theta'$ on $I \times W$.

In the general case of two cubes (I^{k+1}, I^k) write X^* for $I^k \times X$ and apply the above argument to M^*, U^*,W^*. □

4.7 The immersion theory machine

Notation We write $F(X)$, $G(X)$ for PL($X \subset M$) and PL($TM|X$) respectively.

We can now complete the proof of the classification theorem 4.1 under hypothesis (*).

Proof of 4.1 All the charts on M are intended to be PL homeomorphic images of $\mathbb{R}^m$ and the simplicial complexes are intended to be PL embedded in some of those charts.

(1) *The theorem is true for each simplex A, linearly embedded in a chart of M.*

Proof We can suppose that $A \subset \mathbb{R}^m$ and observe that A has a base of neighbourhoods which are canonically PL isomorphic to $\mathbb{R}^m$. The result follows from 4.4 taking the direct limits.

More precisely, A is the intersection of a nested countable family $V_1 \supset V_2 \supset \cdots V_i \supset \cdots$ of open neighbourhoods each of which is considered as a copy of $\mathbb{R}^m$. Then

$$F(A) = \lim_{\to} F(V_i) \qquad G(A) = \lim_{\to} G(V_i) \qquad d_A = \lim_{\to} d_{V_i}$$

Since each d_{V_i} is a weak homotopy equivalence by 4.4, then d_A is also a weak homotopy equivalence and hence a homotopy equivalence.

(2) *If the theorem is true for the compact sets $A, B, A \cap B$, then it is true for $A \cup B$.*

Proof We have a commutative diagram.

$$\begin{tikzcd}
F(A\cup B) \arrow[rrr] \arrow[ddd, "r_1"] \arrow[dr, "d_{A\cup B}"] & & & F(B) \arrow[ddd, "r_3"] \arrow[dl, "d_B"'] \\
& G(A\cup B) \arrow[r] \arrow[d, "r_2"] & G(B) \arrow[d, "r_4"] & \\
& G(A) \arrow[r] & G(A\cap B) & \\
F(A) \arrow[rrr] \arrow[ur, "d_A"] & & & F(A\cap B) \arrow[ul, "d_{A\cap B}"']
\end{tikzcd}$$

where the r_i are fibrations, by 4.5 and 4.6, and d_A, d_B, and $d_{A\cap B}$ are homotopy equivalences by hypothesis. It follows that d is a homotopy equivalence between each of the fibres of r_3 and the corresponding fibre of r_4 (by the Five Lemma). By 3.3.1 each of the squares is a pullback, therefore each fibre of r_1 is isomorphic to the corresponding fibre of r_3 and similarly for r_2, r_4. Therefore d induces a homotopy equivalence between each fibre of r_1 and the corresponding fibre of r_2. Since d_A is a homotopy equivalence, it follows from the Five Lemma that $d_{A\cup B}$ is a homotopy equivalence. In a word, we have done nothing but appy proposition II.1.7 several times.

(3) *The theorem is true for each simplicial complex (which is contained in a chart of M).* With this we are saying that if $K\subset\mathbb{R}^m$ is a simplicial complex, then

$$d_K\colon \mathrm{PL}(K\subset\mathbb{R}^m)\to\mathrm{PL}(T\mathbb{R}^m|K)$$

is a homotopy equivalence.

Proof This follows by induction on the number of simplices of K, using (1) and (2).

(4) *The theorem is true for each compact set C which is contained in a chart.* With this we are saying that if C is a compact set of $\mathbb{R}^m$, then

$$d_C\colon \mathrm{PL}(C\subset\mathbb{R}^m)\to\mathrm{PL}(T\mathbb{R}^m|C)$$

is a homotopy equivalence.

Proof C is certainly an intersection of finite simplicial complexes. Then the result follows using (3) and passing to the limit.

(5) *The theorem is true for any compact set $C\subset M$.*

Proof C can be decomposed into a finite union of compact sets, each of which is contained in a chart of M. The result follows applying (2) repeatedly.

(6) *The theorem is true for M.*

Proof M is the union of an ascending chain of compact sets $C_1 \subset C_2 \subset \cdots$ with $C_i \subset \mathring{C}_{i+1}$.

From definitions we have

$$F(M) = \lim_{\leftarrow} F(C_i) \qquad G(M) = \lim_{\leftarrow} G(C_i) \qquad d_M = \lim_{\leftarrow} d_{C_i}$$

Each d_{C_i} is a weak homotopy equivalence by (5), hence d_M is a weak homotopy equivalence.

This conclude the proof of (6) and the theorem □

To extend theorem 3.1 to the case $m = 4$ we would need to prove that, if M is a PL manifold and none of whose components is compact, then the differential

$$d\colon \mathrm{PL}(M) \to \mathrm{PL}(TM)$$

is a homotopy equivalence without any restrictions on the dimension.

We will omit the proof of this result, which is established using similar techniques to those used for the case $m \neq 4$. For $m = 4$ one will need to use a weaker version of the fibration property 4.6 which forces the hypothesis of non-compactness (Gromov 1968).

However it is worth observing that in 4.4 we have already established the result in the particular case of $M^m = \mathbb{R}^m$ which is of importance. Therefore the classification theorem also holds for $\mathbb{R}^4$, the Euclidean space which astounded mathematicians in the 1980's because of its unpredictable anomalies.

Finally, we must not forget that we still have to prove the classification theorem when M^m is a topological manifold upon which no PL structure has been fixed. We will do this in the next section.

The proof of the classification theorem gives us a stronger result: *if $C \subset M$ is closed, then*

$$d_C\colon \mathrm{PL}(C \subset M) \to \mathrm{PL}(TM|C) \tag{4.7.1}$$

is a homotopy equivalence.

Proof C is the intersection of a nested sequence $V_1 \supset \dots V_i \supset$ of open neighbourhoods in M. Each d_{V_i} is a weak homotopy equivalence by the theorem applied with $M = V_i$. Taking direct limits we obtain that d_C is also a weak homotopy equivalence. □

Classification via sections

4.8 In order to make the result 4.6 usable and to arrive at a real structure theorem for $\mathrm{PL}(M)$ we need to analyse the complex $\mathrm{PL}(TM)$ in terms of classifying spaces. For this purpose we wish to finish the section by clarifying the notion of PL structure on a microbundle ξ/X.

As we saw in 2.2 when Θ defines a PL structure on ξ/X we do not need to require that $i\colon X \to U_\Theta$ is a PL map. When this happens, as in II.4.1, we say that a PL_μ–structure is fixed on ξ. In this case

$$X \xrightarrow{i} U_\Theta \xrightarrow{p} X$$

is a PL microbundle, which is topologically micro–isomorphic to ξ/X.

Alternatively, we can say that a PL_μ–structure on ξ is an equivalence class of topological micro–isomorphisms $f\colon \xi \to \eta$, where η/X is a PL microbundle and $f \sim f'$ if $f' = h \circ f$, and $h\colon \eta \to \eta'$ is a PL micro–isomorphism.

In II.4.1 we defined the SS–set $\mathrm{PL}_\mu(\xi)$, whose typical k–simplex is an equivalence class of commutative diagrams

$$\begin{tikzcd} \Delta^k \times \xi \arrow[rr, "f"] \arrow[dr] & & \eta \arrow[dl] \\ & \Delta^k & \end{tikzcd}$$

where f is a topological micro–isomorphism and η is a PL microbundle.

Clearly

$$\mathrm{PL}_\mu(\xi) \subset \mathrm{PL}(\xi).$$

Proposition *The inclusion $PL_\mu(\xi) \subset \mathrm{PL}(\xi)$ is a homotopy equivalence.*

▼

Proof We will prove that

$$\pi_k(\mathrm{PL}(\xi), \mathrm{PL}_\mu(\xi)) = 0.$$

Let $k = 0$ and $\Theta \in \mathrm{PL}(\xi)^{(0)}$. In the microbundle

$$I \times \xi\colon I \times X \xrightarrow{1\times i} I \times E(\xi) \xrightarrow{p} I \times X$$

we approximate the zero–section $1 \times i$ using a zero–section j which is PL on $0 \times X$ (with respect to the PL structure $I \times \Theta$) and which is i on $1 \times X$. This can be done by the simplicial ε–approximation theorem of Zeeman. This way we have a *new* topological microbundle ξ' on $I \times X$, whose zero–section is j. To this topological microbundle we can apply the homotopy theorem for

microbundles in order to obtain a topological micro–isomorphism $I \times \xi \xrightarrow{h} \xi'$. If we identify $I \times \xi$ with ξ' through h, we can say that the PL structure $I \times \Theta$ gives us a PL structure on ξ'. This structure coincides with Θ on $1 \times X$ and is, by construction, a PL_μ–structure on $0 \times X$. This proves that each PL structure can be connected to a PL_μ–structure using a path of PL structures.
An analogous reasoning establishes the theorem for the case $k > 0$ starting from a sphere of PL structures on ξ/X. □

▲

4.9 Let $\xi = TM$ and let

$$\begin{tikzcd} TM \arrow[r, "\mathbf{f}"] \arrow[d] & \gamma^m_{\mathrm{Top}} \arrow[d] \\ M \arrow[r, "f"'] & B\mathrm{Top} \end{tikzcd}$$

be a fixed classifying map. We will recall here some objects which have been defined previously. Let

$$\mathcal{B}\colon \mathrm{Top}_m/\mathrm{PL}_m \longrightarrow B\mathrm{PL}_m \xrightarrow{p_m} B\mathrm{Top}_m$$

be the fibration II.3.15; let

$$TM_f = f^*(\mathcal{B}) = TM[\mathrm{Top}_m/\mathrm{PL}_m]$$

be the bundle associated to TM with fibre $\mathrm{Top}_m/\mathrm{PL}_m$, and let

$$\mathrm{Lift}(f)$$

be the space of the liftings of f to $B\mathrm{PL}_m$.

Since there is a fixed PL structure on M, we can assume that f is precisely a map with values in $B\mathrm{PL}_m$ composed with p_m.

Classification theorem via sections *Assuming the hypothesis of theorem 3.1 we have homotopy equivalences*

$$\mathrm{PL}(M) \simeq \mathrm{Lift}(f) \simeq \mathrm{Sect}\, TM[\mathrm{Top}_m/\mathrm{PL}_m].$$

Proof Apply 3.1, 4.8, II.4.1, II.4.1.1. □

The theorem above translates the problem of determining $\mathrm{PL}(TM)$ to an obstruction theory with coefficients in the homotopy groups $\pi_k(\mathrm{Top}_m/\mathrm{PL}_m)$.

5 Classification of PL–structures on a topological manifold M. Relative versions

We will now abandon the hypothesis (*) of section 3, ie, we do not assume that there is a PL structure fixed on M and we look for a classification theorem for this general case. Choose a topological embedding of M in an open set N of an Euclidean space and a deformation retraction $r\colon N \to M \subset N$. Consider the induced microbundle r^*TM whose base is the PL manifold N. The reader is reminded that

$$r^*TM\colon N \xrightarrow{j} M \times N \xrightarrow{p_2} N$$

where p_2 is the projection and $j(y) = (r(y), y)$. Since N is PL, then the space $\mathrm{PL}(r^*TM)$ is defined and it will allow us to introduce a *new differential*

$$d\colon \mathrm{PL}(M) \to \mathrm{PL}(r^*TM)$$

by setting $d\Theta := \Theta \times N$.

5.1 Classification theorem

$d\colon \mathrm{PL}(M) \to \mathrm{PL}(r^*TM)$ is a homotopy equivalence provided that $m \neq 4$.

The proof follows the same lines as that of Theorem 3.1, with some technical details added and is therefore omitted. □

5.2 Theorem *Let $f\colon M \to B\mathrm{Top}_m$ be a classifying map for TM. Then*

$$\mathrm{PL}(M) \simeq \mathrm{Sect}(TM_f).$$

Proof Consider the following diagram of maps of microbundles

$$\begin{tikzcd}
TM \arrow[r, "\mathbf{i}"] \arrow[d] & r^*TM \arrow[r] \arrow[d] & TM \arrow[r, "\mathbf{f}"] \arrow[d] & \gamma^m_{\mathrm{Top}} \arrow[d] \\
M \arrow[r, "i"] & N \arrow[r, "r"] & M \arrow[r, "f"] & B\mathrm{Top}_m.
\end{tikzcd}$$

Passing to the bundles induced by the fibration

$$\mathcal{B}\colon \mathrm{Top}_m/\mathrm{PL}_m \to B\mathrm{PL}_m \to B\mathrm{Top}_m$$

we have

$$\mathrm{PL}(r^*TM) \simeq \mathrm{Sect}((r^*TM)_{for}) \overset{i^*}{\simeq} \mathrm{Sect}(TM_f).$$ □

Therefore $\mathrm{PL}(M)$ is homotopically equivalent to the space of sections of the $\mathrm{Top}_m/\mathrm{PL}_m$–bundle associated to TM.

It follows that in this case as well the problem is translated to an obstruction theory with coefficients in $\pi_k(\mathrm{Top}_m/\mathrm{PL}_m)$.

5.3 Relative version

Let M be a topological manifold with the usual hypothesis on the dimension, and let C be a closed set in M. Also let $\mathrm{PL}(M \,\mathrm{rel}\, C)$ be the *space of* PL *structures of M, which restrict to a given structure, Θ_0, near C, and let* $\mathrm{PL}(TM$ rel $C)$ be defined analogously.

Theorem *$d\colon \mathrm{PL}(M \text{ rel } C) \to \mathrm{PL}(TM \text{ rel } C)$ is a homotopy equivalence.*

Proof Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{PL}(M) & \xrightarrow{d} & \mathrm{PL}(M) \\ {\scriptstyle r_1}\downarrow & & \downarrow{\scriptstyle r_2} \\ \mathrm{PL}(C \subset M) & \xrightarrow[d]{} & \mathrm{PL}(TM|C) \end{array}$$

where we have written TM for r^*TM and $TM|C$ for $r^*TM|_{r^{-1}(C)}$; Θ_0 defines basepoints of both the spaces in the lower part of the diagram and r_1, r_2 are Kan fibrations. The complexes $\mathrm{PL}(M \text{ rel } C)$, and $\mathrm{PL}(TM \text{ rel } C)$ are the fibres of r_1 and r_2 respectively. The result follows from 4.7.1 and the Five lemma. □

Corollary *$\mathrm{PL}(M \,\mathrm{rel}\, C)$ is homotopically equivalent to the space of those sections of the $\mathrm{Top}_m/\mathrm{PL}_m$– bundle associated to TM which coincide with a section near C (precisely the section corresponding to Θ_0).* □

5.4 Version for manifolds with boundary

The idea is to reduce to the case of manifolds without boundary. If M^m is a topological manifold with boundary ∂M, we attach to M an external open collar, thus obtaining

$$M_+ = M \cup_\partial \partial M \times [0,1)$$

and we define $TM := TM_+|M$.

If ξ is a microbundle on M, we define $\xi \oplus \mathbb{R}^q$ (or even better $\xi \oplus \varepsilon^q$) as the microbundle with total space $E(\xi) \times \mathbb{R}^q$ and projection

$$E(\xi) \times \mathbb{R}^q \to E(\xi) \xrightarrow{p_\xi} M.$$

This is, obviously, a particular case of the notion of direct sum of locally trivial microbundles which the reader can formulate.

Once a collar $(-\infty, 0] \times \partial M \subset M$ is fixed we have a canonical isomorphism

$$TM_+|\partial M \approx T(\partial M) \oplus \mathbb{R} \tag{5.4.1}$$

and we *require* that a PL structure on TM is always so that it can be desospended according to (5.4.1) on the boundary ∂M. We can then define a *differential*

$$d\colon \mathrm{PL}(M) \to \mathrm{PL}(TM)$$

and we have:

Theorem *If $m \neq 4, 5$, then d is a homotopy equivalence .*

Proof (Hint) Consider the diagram of fibrations

$$\begin{tikzcd}
\mathrm{PL}(M_+ \mathrm{rel}\, \partial M \times [0,1)) \arrow[r, "d"] \arrow[d] & \mathrm{PL}(TM_+ \mathrm{rel}\, \partial M \times [0,1)) \arrow[d] \\
\mathrm{PL}(M) \arrow[r, "d"] \arrow[d, "r_1"] & \mathrm{PL}(TM) \arrow[d, "r_2"] \\
\mathrm{PL}(TM) \arrow[r, "d"] & \mathrm{PL}(T\partial M)
\end{tikzcd}$$

The reader can verify that the restrictions r_1, r_2 exist and are Kan fibrations whose fibres are homotopically equivalent to the upper spaces and that d is a morphism of fibrations. The differential at the bottom is a homotopy equivalence as we have seen in the case of manifolds without boundary, the one at the top is a homotopy equivalence by the relative version 5.3 Therefore the result follows from the Five lemma. □

5.5 The version for manifolds with boundary can be combined with the relative version. In at least one case, the most used one, this admits a good interpretation in terms of sections.

Theorem *If $\partial M \subset C$ and $m \neq 4$, (giving the symbols the obvious meanings) then there is a homotopy equivalence:*

$$\mathrm{PL}(M \mathrm{rel}\, C) \simeq \mathrm{Sect}\,(TM_f \mathrm{rel}\, C)$$

where $f\colon M \to B\mathrm{Top}_m$ is a classifying map which extends such a map already defined near C. □

Note If $\partial M \not\subset C$, then $\mathrm{Sect}(TM_f)$ has to be substituted by a more complicated complex, which takes into account the sections on ∂M with values in $\mathrm{Top}_{m-1}/\mathrm{PL}_{m-1}$. However it can be proved, *in a non trivial way*, that, if $m \geq 6$, then there is an equivalence analogous to that expressed by the theorem.

Corollary *If M is parallelizable, then M admits a* PL *structure.*

Proof $(TM_+)_f$ is trivial and therefore there is a section. □

Proposition *Each closed compact topological manifold has the same homotopy type of a finite CW complex.*

Proof [Hirsch 1966] established that, if we embed M in a big Euclidean space $\mathbb{R}^N$, then M admits a normal disk bundle E.

E is a compact manifold, which has the homotopy type of M and whose tangent microbundle is trivial. Therefore the result follows from the Corollary. □

5.6 We now have to tackle the most difficult part, ie, the calculation of the coefficients $\pi_k(\mathrm{Top}_m/\mathrm{PL}_m)$ of the obstructions. For this purpose we need to recall some important results of the immersion theory and this will be done in the next part.

Meanwhile we observe that, since

$$\mathrm{PL}_m \subset \mathrm{Top}_m \to \mathrm{Top}_m/\mathrm{PL}_m$$

is a Kan fibration, we have:

$$\pi_k(\mathrm{Top}_m/\mathrm{PL}_m) \approx \pi_k(\mathrm{Top}_m, \mathrm{PL}_m).$$