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Poincaré duality in dimension 3

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Abstract The paper gives a review of progress towards extending the Thurston programme to the Poincaré duality case. In the first section, we fix notation and terminology for Poincaré complexes X (with fundamental group G) and pairs, and discuss finiteness conditions.
For the case where there is no boundary, π_2 is non-zero if and only if G has at least 2 ends: here one would expect X to split as a connected sum. In fact, Crisp has shown that either G is a free product, in which case Turaev has shown that X indeed splits, or G is virtually free. However very recently Hillman has constructed a Poincaré complex with fundamental group the free product of two dihedral groups of order 6, amalgamated along a subgroup of order 2.
In general it is convenient to separate the problem of making the boundary incompressible from that of splitting boundary-incompressible complexes. In the case of manifolds, cutting along a properly embedded disc is equivalent to attaching a handle along its boundary and then splitting along a 2–sphere. Thus if an analogue of the Loop Theorem is known (which at present seems to be the case only if either G is torsion-free or the boundary is already incompressible) we can attach handles to make the boundary incompressible. A very recent result of Bleile extends Turaev's arguments to the boundary-incompressible case, and leads to the result that if also G is a free product, X splits as a connected sum.
The case of irreducible objects with incompressible boundary can be formulated in purely group theoretic terms; here we can use the recently established JSJ type decompositions. In the case of empty boundary the conclusion in the Poincaré duality case is closely analogous to that for manifolds; there seems no reason to expect that the general case will be significantly different.
Finally we discuss geometrising the pieces. Satisfactory results follow from the JSJ theorems except in the atoroidal, acylindrical case, where there are a number of interesting papers but the results are still far from conclusive.
The latter two sections are adapted from the final chapter of my survey article on group splittings.

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Dedicated to Andrew J Casson on the occasion of his 60th birthday

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1 Preliminaries

In [27] I defined Poincaré complexes as CW–complexes satisfying the strongest (global) form of the Poincaré duality theorem that holds for manifolds.

Suppose given a connected complex Y, with fundamental group $G := \pi_1(Y)$ having group ring $R := \mathbb{Z}G$ (we fix these notations throughout this article); and a homomorphism $w\colon G \to \{\pm 1\}$. The ring $R = \mathbb{Z}G$ admits the involutory anti-automorphism $(\sum a_g g)^* = \sum a_g g^{-1}$. This allows us to regard any left R–module as a right module and vice-versa; by default, we use 'R–module' for right R–module. Write $C_*(Y)$ for the chain complex of the universal cover $\widetilde{Y}$, regarded as a complex of free R–modules. Then for any (right) R–module B, we set $H^*(Y;B) := H^*(\mathrm{Hom}_R(C_*(Y),B))$ and ${}^tH_*(Y;B) := H_*(C_*(Y)\otimes_R B)$. Here the affix t is to emphasise that we used the homomorphism w to transfer the given right module structure on B to a left module structure.

If we are given a class $[Y] \in {}^tH_n(Y;\mathbb{Z})$ such that, for all $r \in \mathbb{Z}$, cap product with $[Y]$ induces an isomorphism

$$[Y] \frown\colon\ H^r(Y;R) \longrightarrow {}^tH_{n-r}(Y;R\otimes\mathbb{Z}^t);$$

then we call Y a connected PD^n complex with fundamental class $[Y]$. According to [27, Lemma 1.1], it follows that $[Y]$ is unique (up to sign) and that for any $r \in \mathbb{Z}$ and any R–module B, we have an isomorphism $[Y] \frown\colon H^r(Y;B) \to {}^tH_{n-r}(Y;B)$. We say that Y is orientable if w is trivial; $[Y]$ then defines an orientation. We will usually assume orientability.

The above definition contains no explicit finiteness condition. However, following [4, page 222], we may argue as follows. The homology and cohomology groups are those of a complex C_*, say, of R–modules. The functors $B \to {}^tH_k(Y;B)$ commute with direct limits. Since $[Y] \frown$ defines a natural equivalence, the functors $B \to H^k(Y;B)$ also commute with direct limits. Hence by [3, Theorem 1], $C_*(Y)$ is homotopy equivalent to a complex C' of f.g. (finitely generated) projective modules. Since moreover the cohomology groups all vanish in dimensions exceeding n, we may suppose by [26, Theorem E] that $C'_r = 0$ except when $0 \le r \le n$.

Proposition 1.1 *Let Y be a connected PD^n complex. Then*

(i) *The chain complex $C_*(Y)$ is chain homotopy equivalent to a complex of f.g. projective R–modules, vanishing except in dimensions r with $0 \le r \le n$.*

(ii) *The fundamental group G is f.g. and a.f.p. (almost finitely presented).*

(iii) *Y is dominated by a finite complex if and only if G is f.p.*

We have already proved (i). Since we can attach cells of dimension ≥ 3 to Y to obtain a classifying space for G, we can take $C'_2 \to C'_1 \to C'_0$ as the beginning of a resolution of $\mathbb{Z}$ over R. Since C'_1 is f.g., so is G; since C'_2 is, G is a.f.p. Now (iii) follows from [26, Theorem A].

A PD^n complex (or Poincaré complex) in general is a complex with a finite number of components, each of which is a connected PD^n complex. We say that the group G is a PD^n group if $K(G,1)$ is a PD^n complex: we then have $\operatorname{cd} G = n$.

Corresponding to a manifold with boundary, a connected PD^n pair is a CW pair (Y,X) with Y connected, with a homomorphism $w\colon G \to \{\pm 1\}$ and a class $[Y] \in {}^tH_n(Y,X;\mathbb{Z})$ such that cap product with $[Y]$ induces isomorphisms $H^r(Y;R) \longrightarrow {}^tH_{n-r}(Y,X;R)$ and X is a PD^{n-1} complex with fundamental class $\partial_*[Y]$ (so in particular, w induces the homomorphisms w for the components of X). It thus follows from the five lemma that we have induced isomorphisms $H^r(Y,X;R) \longrightarrow {}^tH_{n-r}(Y;R)$, and now as before that $[Y]\frown$ induces isomorphisms $H^i(Y;B) \to {}^tH_{n-i}(Y,X;B)$, $H^i(Y,X;B) \to {}^tH_{n-i}(Y;B)$, for any R-module B. If $X = \emptyset$ this reduces to the definition of PD^n complex. The same arguments as above give:

Addendum 1.2 *The conclusions of Proposition 1.1 apply also if (Y,X) is a PD^n pair.*

Moreover, since we can find a chain complex for $(\widetilde{Y},\widetilde{X})$ with no 0–cells, it follows from duality that there is a chain complex for $\widetilde{Y}$ with no n–cells.

Attaching two PD^n pairs by identifying some components of the boundary yields another such pair. Conversely, there are also results about cutting along an embedded PD^{n-1} complex. Here we only need the following [27, Theorem 2.4].

Proposition 1.3 *Let (Y,X) be a PD^n pair with $n \geq 3$. Then there exists a pair (Y',X) with Y' dominated by an $(n-1)$–dimensional complex, a map $f\colon S^{n-1} \to Y'$, and a homotopy equivalence $Y' \cup_f e^n \to Y$ (rel X). The triple (Y',X,f) is unique up to homotopy and orientation. If we suppose (as we may) f an inclusion, $(Y', X \cap S^{n-1})$ is a PD^n pair.*

Proof In the case when X is empty, this agrees with the result quoted. In general we first apply the same result to obtain a homotopy equivalence

$h\colon X' \cup_g e^{n-1} \to X$. Attach two copies of the mapping cylinder of h in turn to Y along X. The result contains a cell of the form $e^{n-1} \times e^1$, and it suffices to remove an embedded n–cell from the interior of this to give the existence statement.

If $n = 3$ we cannot apply the result to the boundary, but we can use Theorem 1.5 below instead.

The same argument as in the case X empty establishes uniqueness in the general case also. □

Corollary 1.4 *The connected sum operation is well defined on connected PD^n pairs (in the sense that if both are orientable there is a unique connected sum preserving orientation).*

A connected PD^n complex or pair with $n \leq 1$ is easily shown to be homotopy equivalent to a manifold pair: a point, circle or interval. The same is also known in dimension 2.

Theorem 1.5 *A connected PD^2 complex or pair is homotopy equivalent to a compact manifold pair.*

The original result was obtained by Eckmann, Müller and Linnell in [11] and [10]. Even stronger results — in particular, an analysis of the case when Poincaré duality holds over a ring of coefficients other than $\mathbb{Z}$ — are obtained by Bowditch [2], without assuming G f.p.

2 Decompositions by spheres

To simplify the discussion (and the notation), we restrict from now on to the orientable case, though the characterisation and splitting results below were obtained without this restriction. We also assume throughout that G is f.p., though much is valid without needing this.

Let (Y, X) be a connected PD^3 pair. We can define $\widehat{Y}$ by attaching a 3–disc to Y along each 2–sphere boundary component. Conversely, we can regard Y as the connected sum of $\widehat{Y}$ with a collection of discs D^3. We thus suppose from now on that no component of X is a 2–sphere.

Following 3–manifold terminology, we call a component X_r of X incompressible if the natural map $\pi_1(X_r) \to \pi_1(Y)$ is injective; X is incompressible if each component is.

Denote the universal cover of Y, and the induced coverings of X and its components by adding a tilde.

Lemma 2.1 *If Y is orientable and G is finite, $X=\emptyset$ and $\widetilde{Y}\sim S^3$.*

Proof The exact sequence $H_2(Y,X;\mathbb{Q})\to H_1(X;\mathbb{Q})\to H_1(Y;\mathbb{Q})$ is self-dual, so the image of the first map is a Lagrangian subspace, of half the dimension. Since $H_1(Y;\mathbb{Q})=0$, each component of X has vanishing first Betti number, so is a sphere; hence there are no components. Now $H_2(\widetilde{Y};\mathbb{Z})=H^1(\widetilde{Y};\mathbb{Z})=0$. The result follows. □

In the nonorientable case one may also have $P^2(\mathbb{R})\times I$, for example.

Proposition 2.2 *If Y is orientable, the following are equivalent:*

(i) $H^1(G;\mathbb{Z}G)=0$;
(ii) $e(G)\le 1$;
(iii) $e(\widetilde{Y})\le 1$;
(iv) $\pi_2(Y)=0$ *and X is incompressible.*

Proof Here $\mathbb{Z}$ can be replaced by a field k.

Since we can obtain a $K(G,1)$ by attaching cells of dimension ≥ 3 to Y, $H^1(G;\mathbb{Z}G)=H^1(Y;\mathbb{Z}G)=H^1_c(\widetilde{Y})$ is the cohomology of $\widetilde{Y}$ calculated with finite cochains. Denote by $C^*(\widetilde{Y})$, $C^*_c(\widetilde{Y})$ the chain complex of cochains of $\widetilde{Y}$ and the subcomplex of finite cochains; write $C^*_e(\widetilde{Y})$ for the quotient, and $H^*_e(\widetilde{Y})$ for its cohomology groups. From the exact sequence

$$H^0_c(\widetilde{Y})\to H^0(\widetilde{Y})\to H^0_e(\widetilde{Y})\to H^1_c(\widetilde{Y})\to H^1(\widetilde{Y}), \tag{1}$$

where $e(G)=e(\widetilde{Y})$ is the rank of the middle term, we see first that if G is finite $e(G)=0$ and $H^1_c(\widetilde{Y})=0$; thus by the lemma, all of (i)–(iv) hold. From now on, assume G infinite. Then the extreme terms of (1) vanish and $H^0(\widetilde{Y})=\mathbb{Z}$, so $e(G)\ge 1$, with equality holding only if $H^1_c(\widetilde{Y})=0$. Thus (i)–(iii) are equivalent.

Observe that, for each component X_r of X, the image of $\pi_1(X_r)\to\pi_1(Y)$ is infinite. For, if not, the map $H_1(X_r;\mathbb{Q})\to H_1(Y;\mathbb{Q})$ would be zero. This contradicts the facts that the kernel of $H_1(X;\mathbb{Q})\to H_1(Y;\mathbb{Q})$ is a Lagrangian subspace and, since X_r is not a sphere, $H_1(X_r;\mathbb{Q})$ has non-vanishing intersection numbers. Hence the kernel, J_r say, of $\pi_1(X_r)\to\pi_1(Y)$ has infinite index in $\pi_1(X_r)$, so is a free group. Then each component of $\widetilde{X_r}$ is non-compact, and $H_1(\widetilde{X_r};\mathbb{Z})$ is a sum of copies of J_r^{ab}.

Now by duality $H^1_c(\widetilde{Y};\mathbb{Z}) \cong H_2(\widetilde{Y},\widetilde{X};\mathbb{Z})$, and in the exact sequence

$$H_2(\widetilde{X};\mathbb{Z}) \to H_2(\widetilde{Y};\mathbb{Z}) \to H_2(\widetilde{Y},\widetilde{X};\mathbb{Z}) \to H_1(\widetilde{X};\mathbb{Z}) \to H_1(\widetilde{Y};\mathbb{Z}),$$

the extreme terms vanish: the left hand one since each component of $\widetilde{X}$ is non-compact, the right since we have the universal cover. Hence $H_2(\widetilde{Y},\widetilde{X};\mathbb{Z})$ vanishes if and only if (a) $\pi_2(Y) \cong H_2(\widetilde{Y};\mathbb{Z})$ vanishes and (b) $H_1(\widetilde{X};\mathbb{Z})$ vanishes, ie each J_r^{ab} does. But since J_r is free, this implies that J_r is trivial, hence X_r incompressible. □

In the case when Y is a closed orientable 3–manifold, the sphere theorem states that the following are equivalent:

(i) $e(G) \geq 2$ and $G \ncong \mathbb{Z}$,

(ii) G is a free product,

(iii) Y splits non-trivially as a connected sum.

Moreover, if $G \cong \mathbb{Z}$, Y is homeomorphic to $S^2 \times S^1$.

If $G \ncong \mathbb{Z}$ and $e(G) \geq 2$, G is a free product, and we can decompose Y as a connected sum. By Gruško's Theorem, the process of consecutive decompositions must terminate. We end with G being a free product of free factors each of which is either ($e = 2$) $\mathbb{Z}$, ($e = 0$) finite, or ($e = 1$) the fundamental group of a 3–manifold which, by Proposition 2.2, is aspherical, so has torsion-free fundamental group. Thus G is a free product of finite groups and a torsion-free group. It follows that any finite subgroup of G is contained in a free factor.

It is natural to hope for corresponding results for orientable PD^3 complexes: we shall see that this is too optimistic. We still know, by [27], that if $G \cong \mathbb{Z}$, Y is homotopy equivalent to $S^2 \times S^1$. Next there is a recent result of Crisp.

Theorem 2.3 [8] *If Y is an orientable PD^3 complex and $e(G) \geq 2$, then either (a) G is a free product or (b) G is virtually free.*

Since $e(G) \geq 2$, it follows from Stallings' theorem that there is a non-trivial action of G on a tree T with finite edge groups G_e; moreover, as G is accessible, we may suppose that each vertex group G_v has at most 1 end. In the exact sequence

$$0 \to H^0(G;M) \to \oplus_v H^0(G_v;M) \to$$
$$\oplus_e H^0(G_e;M) \to H^1(G;M) \to \oplus_v H^1(G_v;M)$$

take $M = \mathbb{Z}G$. Since, for any subgroup $K \subset G$, $H^i(K;\mathbb{Z}G) = H^i(K;\mathbb{Z}K) \otimes_{\mathbb{Z}K} \mathbb{Z}G$, and for K infinite and 1–ended, $H^0(K;\mathbb{Z}K) = H^1(K;\mathbb{Z}K) = 0$, while for K finite $H^0(K;\mathbb{Z}K) = \mathbb{Z}$ and $H^1(K;\mathbb{Z}K) = 0$, the sequence reduces to

$$0 \to \oplus_{G_v \,\mathrm{finite}}(\mathbb{Z} \otimes_{\mathbb{Z}G_v} \mathbb{Z}G) \to \oplus_e(\mathbb{Z} \otimes_{\mathbb{Z}G_e} \mathbb{Z}G) \to H^1(G;\mathbb{Z}G) \to 0.$$

Now Crisp regards $H^1(G;\mathbb{Z}G)$ as a modification $\Pi(T)$ of $H^1_c(T)$ in which the vertices with infinite stabilisers are omitted from the calculation, and proves that $\Pi(T)$ is free over $\mathbb{Z}$ with rank $\max(0, e(T) + \infty(T) - 1)$, where $e(T)$ is the number of ends of T and $\infty(T)$ is the number of vertices with infinite stabilisers.

If any edge group is trivial, G splits as a free product, and if all vertex groups are finite, G is virtually free. The key idea of the proof is to consider a finite cyclic subgroup C of an edge group and compare a calculation $H_s(C;H^1(G;\mathbb{Z}G)) \cong H_{s+3}(C;\mathbb{Z})$ using Poincaré duality with the above.

It is natural to expect that case (b) cannot occur unless G itself is free. An analysis by Hillman showed that the first case which cannot be resolved by easy arguments is the free product of two copies of the dihedral group of order 6, amalgamated along a subgroup of order 2. In a recent preprint he established the following, thus showing that case (b) does indeed occur.

Theorem 2.4 [14] *There is an orientable PD^3 complex whose fundamental group is the amalgamated free product $D_6 *_{\mathbb{Z}_2} D_6$.*

The proof depends on the criterion of Turaev to be discussed below (Theorem 2.7).

The first result indicating a purely algebraic treatment of PD^3 complexes was the following theorem of Hendriks:

Proposition 2.5 [13] *Two Poincaré 3–complexes with the same fundamental group G are homotopy equivalent if and only if the images of their fundamental classes in $H_3(G;\mathbb{Z})$ coincide.*

Next Turaev in [23] obtained a characterisation of which pairs (G, z) with $z \in H_3(G;\mathbb{Z})$ correspond to PD^3 complexes. In [24] he used this, together with Hendriks' theorem, so show that if the fundamental group of a PD^3 complex splits as a free product, there is a corresponding split of the complex as a connected sum. In a subsequent paper [25] he gave an improved and unified version of all three results.

We now review Turaev's argument in our own notation. Let Mod_R denote the category of (f.g. right) R–modules; define a morphism to be nullhomotopic if it factors through a projective module. Form the quotient category $PMod_R$ by these morphisms, and write $[M,N]$ for the group of morphisms from M to N in $PMod_R$. Equivalence in $PMod_R$ is known as *stable equivalence*; R–modules M, M' are stably equivalent if and only if there exist f.g. projective modules P, P' such that $M \oplus P$ and $M' \oplus P'$ are isomorphic.

We introduce abbreviated notation as follows. Write $K := K(G,1)$: consider spaces mapped to K and for each such space W, write $\widetilde{W}$ for the covering space of W induced from the universal cover of K, and $C_*(W)$, $C^*(W)$ for the chain and finite cochain groups of $\widetilde{W}$, considered as free modules over $R := \mathbb{Z}G$. We use a corresponding notation for pairs. Also where coefficients for homology or cohomology are unspecified, they should be understood as R. We write $\mathfrak{G} := \mathrm{Ker}\,(R \to \mathbb{Z})$ for the augmentation ideal of G.

The following algebraic construction is the key to the argument. For any projective chain complex C_* over $R = \mathbb{Z}G$, write IC_* for $\mathfrak{G} \otimes_R C_*$ and set $F^rC := \mathrm{Coker}\,(\delta^{r-1}\colon C^{r-1} \to C^r)$. Then there is a homomorphism $\nu_{C,r}\colon H_r(IC_*) \to [F^rC, \mathfrak{G}]$ induced by evaluating a representative cocycle in C^r (for an element of F^rC) on a representative cycle in $\mathfrak{G} \otimes_R C_r$ (for an element in $H_r(IC_*)$). This construction is natural with respect to maps of chain complexes C_*. Moreover, Turaev shows that:

Lemma 2.6 *For any projective chain complex C_*, $\nu_{C,r}\colon H_r(IC_*) \to [F^rC, \mathfrak{G}]$ is an isomorphism.*

If C_* is a positive complex with $H_0(C) \cong \mathbb{Z}$, then the image of $d_1\colon C_1 \to C_0$ is stably equivalent to $\mathfrak{G} \lhd \mathbb{Z}G$ so that if also $H_1(C) = 0$, $\mathrm{Coker}\, d_2\colon C_2 \to C_1$ is stably equivalent to $\mathfrak{G}$. A class $\mu \in H_{r+1}(C;\mathbb{Z})$ induces a chain mapping $C^{r+1-*}(K) \to C_*(K)$, which is determined up to chain homotopy. There is an induced map of the cokernel F^rC of $\delta\colon C^{r-1}(K) \to C^r(K)$ to the cokernel of $d\colon C_2(K) \to C_1(K)$, and hence to the augmentation ideal $\mathfrak{G}$. Denote the composite by $\nu(\mu) \in [F^r(C), \mathfrak{G}]$.

We also have an exact sequence

$$H_{r+1}(C) \to H_{r+1}(C;\mathbb{Z}) \xrightarrow{\partial_*} H_r(IC_*) \to H_r(C).$$

Turaev also shows that $\nu(\mu) = \nu_{C,r}(\partial_*\mu)$. In particular, if also $H_{r+1}(C) = H_r(C) = 0$, then ∂_* is an isomorphism, so $\nu_{C,r}$ induces an isomorphism $\nu\colon H_{r+1}(C;\mathbb{Z}) \to [F^rC, \mathfrak{G}]$.

For our purposes we need only the case $r = 2$, so omit r from the notation.

If Y is an oriented PD^3 complex with fundamental class $[Y] \in H_3(Y;\mathbb{Z})$, we have a classifying map $i\colon Y \to K$ and hence a class $i_*(Y) \in H_3(G;\mathbb{Z})$. Now Turaev's main theorem (in the orientable case) is:

Theorem 2.7 *Given a triple (G,μ) with G an (f.p.) group and $\mu \in H_3(G;\mathbb{Z})$, there is an oriented PD^3 complex Y with fundamental group G and $i_*[Y] = \mu$ if and only if $\nu(\mu)$ is a stable equivalence.*

Moreover, if this holds, Y is unique up to oriented homotopy equivalence.

Necessity of the condition follows easily from the definition of ν. The key step in the proof of sufficiency is the construction of the complex Y.

Let Z be a finite connected 2–complex with $\pi_1(Z) = G$, eg the 2–skeleton of K. Our hypothesis gives a preferred stable equivalence from the cokernel $F^2C(Z) = C^2(Z)/\delta(C^1(Z)$ to $\mathfrak{G}$. Replacing Z, if necessary, by its bouquet with a number of 2–spheres, we may suppose given an isomorphism ϕ of the cokernel $F^2C(Z)$ to $\mathfrak{G} \oplus P$ for some f.g. projective R–module P.

We can make P free as follows. By a lemma of Kaplansky, for any f.g. projective module P' there exist (infinitely generated) free modules F, F' with $F' \cong F \oplus P'$. We can thus attach 2–spheres to Z corresponding to generators of F' and 3–cells to kill the generators of F: this has the same effect as adding a copy of P' to $C_2(Z)$, while the new complex Z still has the homological properties of a 2–dimensional complex. If we choose P' appropriately, this will make P free and f.g., of rank t, say.

Write d^3 for the map $C^2(Z) \to R^{t+1}$ defined by composing ϕ with the projection. Dualising gives a homomorphism $d_3\colon R^{t+1} \to C_2(Z)$ such that $d_2 \circ d_3 = 0$. Thus the image of d_3 is contained in $Z_2(Z) = \pi_2(\widetilde{Z}) = \pi_2(Z)$. Attach 3–cells to Z by maps $S^2 \to Z$ corresponding to the images of the generators of R^{t+1}: this gives a complex Y.

We now show that this complex Y is a Poincaré 3–complex: first we calculate $H_3(Y;\mathbb{Z})$. The map d^3 is the composite $C^2(Z) \twoheadrightarrow \mathfrak{G} \oplus R^t \hookrightarrow R^{t+1}$, which is the direct sum of an isomorphism on R^t and a composite $A \twoheadrightarrow \mathfrak{G} \hookrightarrow R$. When we tensor over R with $\mathbb{Z}$, the map $\mathfrak{G} \hookrightarrow R$ gives 0. Dually, d_3 with coefficients $\mathbb{Z}$ is the direct sum of an isomorphism of $\mathbb{Z}^t$ on itself and a zero map of $\mathbb{Z}$. Thus $H_3(Y;\mathbb{Z}) \cong \mathbb{Z}$. Also, there is a preferred generator, giving a class $[Y]$.

We next show that $i_*[Y]$ is equal to the given class μ. For this we apply Turaev's Lemma 2.6 to $C_*(K)$. Since $\widetilde{K}$ is contractible, the hypothesis $H_3(K,X) =$

$H_2(K,X)=0$ is satisfied. By construction, $\nu[Y]$, which by naturality is equal to $\nu i_*[Y]$, is the stable isomorphism $\nu(\mu)$. It follows that indeed $i_*[Y]=\mu$.

The map ϕ_* is part of an exact triangle $C^{3-*}(Y)\to C_*(Y)\to D_*\to C^{4-*}(Y)$ in the derived category. Since ϕ_* is self-dual, so is D_*. By construction, $H^2(Y)=0$ and $H^3(Y)\cong\mathbb{Z}$, and so ϕ_* induces isomorphisms $H^2(Y)\to H_1(Y)$ and $H^3(Y)\to H_0(Y)$. In particular, $H_0(D)\cong H_1(D)=0$, so D is chain equivalent to a complex $0\to D_4\to D_3\to D_2\to 0$. As D_* is self-dual, we may similarly eliminate D_4 and D_3, leaving only a projective module D_2', say. Thus $C_2\to D_2'$ is a split surjection and $H_2(Y)\cong D_2'\oplus H^1(Y)$. By duality, $D_2'\to C^2$ is a split injection and $H^2(Y)\cong D_2'\oplus H_1(Y)$. As $H^2(Y)=0$, $D_2'=0$ and D_* is acyclic.

As to uniqueness, we may construct a $K(G,1)$ complex K by attaching cells of dimension ≥ 3 to Y. If Y' is another PD^3 complex with the same fundamental group G, there is no obstruction to deforming a map $Y'\to K$ inducing an isomorphism of G to a map into Y; moreover if we assume that the fundamental classes have the same image in $H_3(K;\mathbb{Z})$, a careful argument shows that we may suppose $Y'\to Y$ of degree 1. It is now easy to see that we have a homotopy equivalence.

Theorem 2.8 [24] *If Y is a PD^3 complex such that $G=\pi_1(Y)=G'*G''$ is a free product, Y is homotopy equivalent to the connected sum of PD^3 complexes Y' and Y'', with $\pi_1(Y')\cong G'$ and $\pi_1(Y'')\cong G''$.*

The image of the fundamental class of Y gives an element $\mu\in H_3(G;\mathbb{Z})\cong H_3(G';\mathbb{Z})\oplus H_3(G'';\mathbb{Z})$, and hence classes $\mu'\in H_3(G';\mathbb{Z})$, $\mu''\in H_3(G'';\mathbb{Z})$. It will suffice by Theorem 2.7 to show that $\nu(\mu')$ is a stable equivalence.

Choose Eilenberg–MacLane 2–complexes K',K'', with respective fundamental groups G',G'' and group rings R',R''; then $K=K'\vee K''$ is a $K(G,1)$. We have $C_2(K')\to C_1(K')\to\mathfrak{G}'\to 0$, and similarly for G''. Tensor these over R',R'' with R and add. We obtain

$$C_2(K)\to C_1(K)\to(\mathfrak{G}'\otimes_{R'}R)\oplus(\mathfrak{G}''\otimes_{R''}R)\to 0.$$

Thus $(\mathfrak{G}'\otimes_{R'}R)\oplus(\mathfrak{G}''\otimes_{R''}R)\cong\mathfrak{G}$.

We have $\nu(\mu)\in[F^2C(K),\mathfrak{G}]$, and similarly for G', G''. Since $C^*(K)\cong(C^*(K')\otimes_{R'}R)\oplus(C^*(K'')\otimes_{R''}R)$, we can identify $F^2C(K)$ with $(F^2C(K')\otimes_{R'}R)\oplus(F^2C(K'')\otimes_{R''}R)$. Indeed, with the obvious interpretation, we can write $\nu(\mu)=(\nu(\mu')\otimes_{R'}R)\oplus(\nu(\mu'')\otimes_{R''}R)$.

We now apply $\otimes_R R'$. If M denotes either $\mathfrak{G}$ or $F^2C(K)$, we can write $M \cong (M' \otimes_{R'} R) \oplus (M'' \otimes_{R''} R)$. The first summand yields $(M' \otimes_{R'} R) \otimes_R R' = M'$; the second gives

$$(M'' \otimes_{R''} R) \otimes_R R' = (M'' \otimes_{R''} \mathbb{Z}) \otimes_{\mathbb{Z}} R'.$$

In each case, M'' is f.g. over R'', so $M'' \otimes_{R''} \mathbb{Z}$ is an f.g. abelian group, which we can express as $F \oplus T$ with F free abelian and T finite abelian. The summands $F \otimes_{\mathbb{Z}} R'$ can be ignored, as we are only concerned with stable isomorphism.

It follows that the given stable isomorphism $\nu(\mu)$ gives, on tensoring over R with R', the direct sum of $\nu(\mu')$, which is a map of torsion-free modules, and a map of torsion modules $T_1 \otimes_{\mathbb{Z}} R' \to T_2 \otimes_{\mathbb{Z}} R'$, which must thus be an isomorphism. Hence indeed $\nu(\mu')$ is a stable isomorphism.

3 Decomposition by spheres and discs

In the case $X \neq \emptyset$, matters are distinctly more complicated. Lemma 2.1, Proposition 2.2 and Theorem 2.3 were already framed to include the general case. We now seek to decompose (Y, X) in some way (some analogue of a decomposition of a 3–manifold by embedded spheres and discs) until the conditions of Proposition 2.2 hold for each piece (Y, X) of the decomposition. Then for a piece with fundamental group G, G is not a free product and either

(i) $e(G) = 0$, G is finite, Y is finitely covered by a homotopy S^3,

(ii) $e(G) = 2$ and (Y, X) is one of few possibilities,

(iii) $e(G) = 1$, X is incompressible and Y aspherical.

Although this is overoptimistic, we will investigate how far one can go towards such a result: our overall conclusion is that results corresponding to the manifold case can be proved if either G is torsion free or X is incompressible.

If $e(G) < 2$, we can apply Proposition 2.2, so $\pi_2(Y) = 0$. If $e(G) = 1$, Y is aspherical: we treat this case in the next section. If $e(G) = 0$, G is finite, so $X = \emptyset$, and $\widetilde{Y}$ is homotopy equivalent to S^3. For the case $e(G) = 2$, we have:

Proposition 3.1 *An orientable PD^3 pair (Y, X) such that $e(G) = 2$ is homotopy equivalent to one of $(P^3(\mathbb{R})\#P^3(\mathbb{R}), \emptyset)$, $(S^2 \times S^1, \emptyset)$ and $(D^2 \times S^1, S^1 \times S^1)$.*

Proof If $X = \emptyset$, then it follows from [27, Theorem 4.4] that either $Y \simeq P^3(\mathbb{R})\#P^3(\mathbb{R})$ (so G is a free product) or $Y \simeq S^2 \times S^1$. In general G has a finite normal subgroup F with quotient either $\mathbb{Z}_2 * \mathbb{Z}_2$ or $\mathbb{Z}$, so $H_1(Y;\mathbb{Z}) \cong G^{ab}$ has rank 0 or 1. Thus $H_1(X;\mathbb{Z})$ has rank at most 0 resp. 2. We have already dealt with the case $X = \emptyset$, so may suppose $G/F \cong \mathbb{Z}$ and X a torus.

It follows from (1) that $H^1_c(\widetilde{Y};\mathbb{Z}) \cong \mathbb{Z}$, so by duality $H_2(\widetilde{Y},\widetilde{X};\mathbb{Z}) \cong \mathbb{Z}$. Since the image of $H_1(X;\mathbb{Z}) \to H_1(Y;\mathbb{Z})$ has rank 1, each component of $\widetilde{X}$ is homotopy equivalent to S^1. It now follows from the exact sequence

$$H_2(\widetilde{X};\mathbb{Z}) \to H_2(\widetilde{Y};\mathbb{Z}) \to H_2(\widetilde{Y},\widetilde{X};\mathbb{Z}) \to H_1(\widetilde{X};\mathbb{Z}) \to H_1(\widetilde{Y};\mathbb{Z}),$$

whose end terms vanish, that $H_2(\widetilde{Y};\mathbb{Z}) = 0$, so Y is aspherical. Hence G is torsion free, so $G \cong \mathbb{Z}$. Moreover, $\widetilde{X}$ is connected, so $\pi_1(X) \to \pi_1(Y)$ is surjective. Thus indeed $(Y,X) \simeq (D^2 \times S^1, S^1 \times S^1)$. □

Since $e(G) \geq 2$ did not imply G a free product in the preceding section, we cannot hope to do better here. In some sense, things are now no worse. For suppose (Y,X) an orientable PD^3 pair. Form the double DY, with fundamental group $\widehat{G}$, say. Suppose $\widehat{G}$ is a free product of finite groups and a torsion free group. By Kuroš' subgroup theorem, the same follows for any subgroup of $\widehat{G}$. But the natural map $G \to \widehat{G}$ is injective since there is a retraction by folding the factors of the double. Thus any finite subgroup of G is contained in a free factor, and Theorem 2.3 now shows that G is a free product of the desired type.

Crisp, in [8], gives an extension of Theorem 2.3 to the case of Poincaré pairs (Y,X). First he shows that his argument remains valid if X is incompressible.

Theorem 3.2 *If (Y,X) is an orientable PD^3 pair with X incompressible and $e(G) \geq 2$, then either (a) G is a free product or (b) G is virtually free and $X = \emptyset$.*

It remains only to observe in the second case that a non-trivial fundamental group of a closed surface is never virtually free.

Next Crisp observes that if the loop theorem is applicable, he can reduce to the incompressible case by attaching 2–handles to Y. We would thus like a version of the loop theorem, and next digress to discuss this.

The original Loop Theorem for 3–manifolds was proved by Papakyriakopoulos [19], further proofs were given by Stallings [21] and Maskit [18]. The result we would like (shorn of refinements) is:

Hope 3.1 *Let (Y,X) be a PD^3 pair with X a 2–manifold; let F be a component of X such that $\pi_1(F) \to \pi_1(Y)$ is not injective. Then there exists a simple loop u in F which is nullhomotopic in Y but not in F.*

This result is claimed in [22], but the proposed proof appears to have gaps. The only other relevant reference known to the author is Casson–Gordon [7], but (despite the assertion in the review MR0722728 that it "proves, via a geometric argument on planar coverings, that the loop theorem of Papakyriakopoulos is true for surfaces that bound mere 3–dimensional duality spaces") the result obtained in that paper is of a somewhat different nature, and works only modulo the intersection of the dimension subgroups of G.

We obtain a partial result by following Maskit's proof of the theorem for 3–manifolds.

Lemma 3.3 *Let (Y,X) be a PD^3 pair with X a 2–manifold, with universal covering $\widetilde{Y}$ and induced covering $\widetilde{X}$. Then every component of $\widetilde{X}$ is planar (ie can be embedded in a plane).*

Proof Suppose, if possible, there is a non-planar component $\widetilde{F}$. Since $\widetilde{Y}$ and hence $\widetilde{X}$ is orientable, it follows from the standard theory of surfaces that there exist simple loops $A, B \subset \widetilde{F}$ which meet transversely in just one point. As $\widetilde{Y}$ is simply-connected, B bounds a 2–cycle, which thus defines a homology class in $H_2(\widetilde{Y},\widetilde{X};\mathbb{Z})$. This has a dual cohomology class $\beta \in H^1_c(\widetilde{Y};\mathbb{Z})$.

Write $i\colon \widetilde{X} \to \widetilde{Y}$ for the inclusion, and α for the class of A in $H_1(\widetilde{X};\mathbb{Z})$. It follows from the definition of β that $i^*\beta$ is the cohomology class dual to the cycle B. From the choice of A and B, $\langle i^*\beta, \alpha\rangle = 1$. Hence

$$\langle \beta, i_*\alpha\rangle = \langle i^*\beta, \alpha\rangle = 1.$$

Thus $0 \neq i_*\alpha \in H_1(\widetilde{Y};\mathbb{Z})$, contradicting simple connectivity of $\widetilde{Y}$. □

Proposition 3.4 [18, Theorem 3] *Let $p\colon \widetilde{F} \to F$ be a regular covering with F a compact surface and $\widetilde{F}$ planar. Then there exist a finite disjoint set of simple, orientation-preserving loops $u_1, \ldots, u_s$ on F and positive integers $n_1, \ldots, n_s$ such that $\pi_1(\widetilde{F})$ is the normal subgroup of $\pi_1(F)$ generated by $u_1^{n_1}, \ldots, u_s^{n_s}$.*

Applying this to the covering of F induced by the universal covering of Y in Hope 3.1, which is planar by the lemma, gives simple loops u_i in F with $u_i^{n_i}$ nullhomotopic in Y. In the manifold case it can now be shown (see [19, p 287] that for each such simple loop u_i, supposed orientation-preserving, u_i is already nullhomotopic in Y. Here, at least if $\pi_1(Y)$ is torsion free, it follows that each u_i is already nullhomotopic in Y. Thus:

Theorem 3.5 *Hope 3.1 is true if G is torsion free.*

This is of no use for applying Crisp's theorem since if $e(G) \geq 2$ and G is torsion free, it follows anyway that either $G \cong \mathbb{Z}$ or G is a free product.

Now suppose that G splits as a free product: can one establish the existence of some kind of splitting of Y? Let us begin by considering what happens in the case of manifolds. Suppose given a splitting $G = G' * G''$. Join the base points of K' and K'' by an arc to form the Stallings wedge K (which is indeed a $K(G,1)$), take the induced map $h\colon Y \to K$, and make h transverse to the centre point of the arc. The pre-image is an orientable surface V with each boundary loop null-homotopic in Y. For each component V_i of V, $\pi_1(V_i)$ maps to 0 in G, so by the loop theorem, if V_i is not simply connected, there is a compressing disc. An argument of Stallings now shows that any surgery on V performed using a compressing disc can be induced by a homotopy of h. We thus reduce to the case when each V_i is simply-connected, hence a sphere or disc.

Consider a single $V_i = V$, cut Y along it to give Y^0, and attach 3–discs to the new boundary components to obtain $\widehat{Y^0}$. First suppose V is a sphere. If V separates Y, Y is a connected sum of the two components of $\widehat{Y^0}$. If one of these has trivial fundamental group, it is a homotopy sphere and we can deform h to remove the component V. If V fails to separate Y, the union Y' of a collar neighbourhood of V and the neighbourhood in Y of an embedded circle meeting V transversely in one point has boundary a disc and closed complement Y'', say, so Y is a connected sum of $\widehat{Y''}$ and $\widehat{Y'} \cong S^2 \times S^1$.

If V is a disc, and separates Y, then Y is a boundary-connected sum of the components of Y^0; ie it is formed by identifying 2–discs embedded in the boundaries of these components. If either of these components is simply-connected, it is contractible, and we can deform to remove the component V.

If V is a disc which fails to separate Y, and ∂V fails to separate the relevant component X_r of X, let Y' be a regular neighbourhood of the union of V and an embedded circle in X_r meeting V transversely in one point. Then Y' is homeomorphic to $S^1 \times D^2$ and has relative boundary a 2–disc, so Y is a boundary-connected sum of $S^1 \times D^2$ with some Y''; if Y'' is simply-connected, Y itself is homotopy equivalent to $S^1 \times D^2$. Finally if ∂V separates X_r and V doesn't separate Y, then Y is formed from Y^0 by identifying discs embedded in distinct boundary components — a sort of boundary-connected sum of Y^0 with itself.

We may thus speculate that if (Y, X) is a PD^3 pair and $G = \pi_1(Y)$ is a non-trivial free product, there is a splitting of (Y, X) as either connected sum or boundary-connected sum corresponding to a non-trivial splitting of G, or self-boundary-connected sum. This seems somewhat complicated, so we look for an alternative approach.

In the case when Y is a manifold, the result of cutting Y along a properly embedded disc $(D^2, S^1) \subset (Y, X)$ is homeomorphic to that obtained by removing from Y the (relative) interior of a collar neighbourhood $(D^2, S^1) \times D^1$. This is homeomorphic to the result of the following sequence of operations:

(i) attach a 2–handle $(D^2, S^1) \times D^1$ to Y using the same embedding $S^1 \times D^1 \to X$;

(ii) cut along the copy of S^2 which is the union of the copy of $D^2 \times 0$ embedded in Y and the copy we have just attached;

(iii) attach a 3–disc to each of the copies of S^2 on the boundary of the result.

Thus instead of cutting along discs we will consider a sequence of operations of attaching 2–handles, and then splitting along embedded spheres.

If this sequence of operations leads to a manifold with incompressible boundary, then this property must already hold at the stage when we have done the handle attachments. Thus our new plan is: first attach 2–handles to the PD^3 pair (Y, X) to make the boundary incompressible; then split by embedded 2–spheres as long as the fundamental group is a free product. To attach 2–handles, we require simple loops in X. Since we wish to apply the loop theorem, let us assume $\pi_1(Y)$ torsion free. Then if X_r is compressible, there is an essential embedding of S^1 in X_r which is null-homotopic in Y. Let X_b be a regular neighbourhood of the image of S^1 (we think of this part of X as coloured black).

Suppose inductively that we have a compact 2–dimensional submanifold X_b of X such that the composite $X_b \subset X \subset Y$ is nullhomotopic. Write X_w for the closure of $X - X_b$ (the 'white part'). If, for some component Z of X_w, $\pi_1(Z) \to \pi_1(Y)$ is not injective, the loop theorem provides an essential embedding of S^1 in Z which is nullhomotopic in Y, and we add a neighbourhoood of this loop to X_b. If Z has a boundary, we also add to X_b an arc joining this neighbourhood to the boundary. We can also attach a 2–handle to Y by the chosen curve at each stage of the construction. These attachments do not affect $\pi_1(Y)$ since the curves were nullhomotopic. The boundary of the result is obtained from X_w by attaching a 2–disc to each boundary component. No component of this can be a 2–sphere, since this would mean we had used an inessential curve.

At each step we are doing a surgery on the boundary. Thus the set of genera of the components is changed either by decreasing one by 1 or by replacing $g_1 + g_2$ by g_1 and g_2. As all genera are positive, the procedure must terminate. Thus there is a disjoint union of embedded copies of S^1 in X such that, for each component X_r of X, the classes of the circles embedded in X_r generate the kernel of $\pi_1(X_r) \to \pi_1(Y)$ as a normal subgroup.

At each stage each $X_b \cap X_r$ is connected. Recall that the kernel of $H_1(X;\mathbb{Q}) \to H_1(Y;\mathbb{Q})$ is a Lagrangian subspace with respect to the intersection product. Thus for each component X_r of X, $Ker(H_1(X_r;\mathbb{Q}) \to H_1(Y;\mathbb{Q}))$ is isotropic. Hence the image of $H_1(X_b \cap X_r;\mathbb{Q})$ in $H_1(X_r;\mathbb{Q})$ is isotropic. It follows that the surface $X_b \cap X_r$ is planar. Suppose $X_b \cap X_r$ has $m+1$ boundary components. Then our procedure involved m steps acting on X_r. These can be performed along any m of the $m+1$ boundary curves of $X_b \cap X_r$.

If the result of this construction were unique, it would follow that we had lost no information by doing all the handle additions first, since there was only one way to make the boundary incompressible. But it is not clear that any simple loop in X_r which is nullhomotopic in Y can be isotoped to lie in $X_b \cap X_r$. Or we can consider taking X, collapsing X_b to a point, and take the map $\pi_1(X/X_b) \to \pi_1(Y)$. Now X/X_b is a bouquet of closed surfaces, and the fundamental group of each injects in G, but it does not follow that $\pi_1(X/X_b)$ (which is their free product) does. When this is the case, the result of the construction is essentially unique.

From now on we consider a (connected) orientable PD^3 pair (Y, X) with incompressible boundary. We would like to show that any splitting of G is induced by a connected sum splitting of Y. Following the arguments of Turaev in the preceding section, we first seek a characterisation of the possible homotopy types of (Y, X). Here we follow the preprint [1] of Bleile, but we omit most of the details.

In view of Theorem 1.5 we may suppose given an oriented 2–manifold X, with no 2–sphere components, a group G, and a homotopy class of maps $X \to K$. We will later assume (X is incompressible) that each map $H_r \to G$ is injective, thus in this case each component of $\widetilde{X}$ is contractible. Suppose also given a class $\mu \in H_3(K, X;\mathbb{Z})$ such that $\partial_*(\mu) = [X]$ is the fundamental class of X.

We may suppose the map $X \to K$ an inclusion, so have an exact sequence $C_*(X) \to C_*(K) \to C_*(K, X)$ of R–free chain complexes. As before, a class $\mu \in H_3(K, X;\mathbb{Z})$ induces cap products $\mu \frown\colon H^{3-i}(K, X; -) \to H_i(K; -)$ via a chain mapping $C^{3-*}(K, X) \to C_*(K)$, which is determined up to chain homotopy. The induced map of the cokernel $FC^2(K, X)$ of $\delta\colon C^1(K, X) \to$

$C^2(K,X)$ to the cokernel of $d\colon C_2(K) \to C_1(K)$ is unique up to stable equivalence. As before, since $C_*(K)$ is acyclic in low dimensions, the cokernel of $d\colon C_2(K) \to C_1(K)$ is stably isomorphic to the augmentation ideal $\mathfrak{G}$. Thus we have a map in $[FC^2(K,X), \mathfrak{G}]$, which we denote $\nu(\mu)$. Extending Turaev's argument, it is shown that this coincides with $\nu_{C(K,X),3}(\partial_*(\mu))$. We are now ready for Bleile's main result.

Theorem 3.6 [1] *Given a triple (X, G, μ) as above; in particular with G an (f.p.) group, X incompressible, and a class $\mu \in H_3(K, X; \mathbb{Z})$ with $\partial_*(\mu) = [X]$, then if there is an oriented PD^3 pair (Y, X) with $\pi_1(Y) = G$ and $i_*[Y] = \mu$, $\nu(\mu)$ is a stable equivalence.*

If also X is incompressible, then conversely if $\nu(\mu)$ is a stable equivalence, such a PD^3 pair exists. Moreover, (Y, X) is unique up to oriented homotopy equivalence.

First suppose (Y, X) an oriented PD^3 pair with the desired properties. Then cap product with $[Y]$ gives a chain equivalence $C^{3-*}(Y,X) \to C_*(Y)$, and thus a stable equivalence $\nu[Y]\colon F^2C \to \mathfrak{G}$. Since $i\colon Y \to K$ is 2–connected, we may suppose K formed from Y by attaching cells of dimension ≥ 3, and then inclusion induces an isomorphism $F^2C(Y,X) \to F^2C(K,X)$. Thus by naturality, $\nu(i_*[Y])$ also is a stable isomorphism.

Conversely, suppose $\nu(\mu) \in [F^2C(K,X), \mathfrak{G}]$ is a stable equivalence and X is incompressible. Form Z by attaching 1– and 2–cells to X and extending the map to $K(G,1)$ until $Z \to K(G,1)$ is 2–connected. Then $F^2C(K,X) \cong F^2(Z,X)$. The stable equivalence $\nu(\mu)$ thus arises from an isomorphism $F^2C(Z,X) \oplus P_1 \to \mathfrak{G} \oplus P_2$ for some projective modules P_1, P_2, where we may suppose P_1 free. Replacing Z (if necessary) by its bouquet with a number of 2–spheres, we may suppose $P_1 = 0$. As before, using an infinite process if necessary, we may suppose P_2 free, of rank t, say.

We have an isomorphism of $F^2C(Z,X)$ to $\mathfrak{G} \oplus R^t \subset R^{t+1}$. Composing with the projection gives a map $C^2(Z,X) \to R^{t+1}$, which dualises to a map $R^{t+1} \to C_2(Z,X)$ whose image lies in the kernel of $d_2\colon C_2(Z,X) \to C_1(Z,X)$. Thus the images of the generators of R^{t+1} define elements of $H_2(Z,X)$.

Since, by the incompressibility hypothesis, $\widetilde{X}$ has each component contractible, the induced map $H_2(Z) \to H_2(Z,X)$ is an isomorphism. According to our convention, since $\pi_1(Z) = G$, $H_2(Z)$ denotes the homology group of the universal cover of Z; thus by Hurewicz' theorem is isomorphic to $\pi_2(Z)$. Thus each generator of R^{t+1} defines an element of $\pi_2(Z)$, and we use corresponding maps $S^2 \to Z$ to attach 3–discs to Z to form Y. We can then identify $R^{t+1} = C_3(Y) = C_3(Y, X)$.

It follows as before that $H_3(Y, X; \mathbb{Z}) \cong \mathbb{Z}$. Write $[Y]$ for the generator corresponding to $+1 \in \mathbb{Z}$. Since our construction was based on the given stable equivalence $\nu(\mu) \in [F^2C(Z), \mathfrak{G}]$, this equivalence is induced by cap product with $[Y]$, so is equal to $\nu[Y]$. By naturality, it is also equal to $\nu(i_*[Y])$. Since $\widetilde{K}$ is contractible and $\widetilde{X}$ is homotopy equivalent to a discrete set, the hypothesis $H_3(K, X) = H_2(K, X) = 0$ is satisfied. It follows by Turaev's Lemma 2.6 that we have $i_*[Y] = \mu$.

In particular, $\partial_*[Y] = \partial_*\mu = [X]$. Hence cap products with $[X]$ and $[Y]$ induce (up to homotopy and sign) a map of exact triangles of chain complexes

$$(C^{2-*}(X) \to C^{3-*}(Y, X) \to C^{3-*}(Y)) \to (C_*(X) \to C_*(Y) \to C_*(Y, X)).$$

Since X is a PD^2 complex, the map $C^{2-*}(X) \to C_*(X)$ is a chain equivalence. In the induced map of homology sequences

$$\begin{array}{ccccccccccccc}
H^1(X) & \to & H^2(Y,X) & \to & H^2(Y) & \to & H^2(X) & \to & H^3(Y,X) & \to & H^3(Y) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_1(X) & \to & H_1(Y) & \to & H_1(Y,X) & \to & H_0(X) & \to & H_0(Y) & \to & H_0(Y,X) & \to & 0
\end{array}$$

we have $H_0(Y) \cong \mathbb{Z}$ and $H_1(Y) = 0$ since this is the homology of the universal cover of Y. Since by construction the map $F^2C(Y, X) \to C^3(Y, X)$ can be identified with the inclusion $\mathfrak{G} \oplus R^t \subset R^{t+1}$, we have $H^2(Y, X) = 0$ and $H^3(Y, X) \cong \mathbb{Z}$, so by the definition of $[Y]$, the induced map $H^3(Y, X) \to H_0(Y)$ is an isomorphism. Thus all the vertical maps in the diagram are isomorphisms. As the map of triangles of chain complexes is self-dual, it follows that we also have isomorphisms for the other half of the sequence, so that we have chain homotopy equivalences, and so duality holds, as required.

Uniqueness can also be established much as in the former case.

This result enables us to prove the desired splitting theorem.

Theorem 3.7 *If (Y, X) is a PD^3 pair such that X is incompressible and G is a free product $G' * G''$, then (Y, X) is homotopy equivalent to the connected sum of PD^3 pairs (Y', X') and (Y'', X''), with $X = X' \dot\cup X''$, $\pi_1(Y') \cong G'$ and $\pi_1(Y'') \cong G''$. Moreover, this splitting is unique up to homotopy.*

As before, set $R' := \mathbb{Z}G'$ etc. For each boundary component X_r, since $H_r = \pi_1(X_r)$ maps injectively to G and is neither free nor a free product, H_r must be conjugate to a subgroup of G' or of G''. We thus have a natural partition of the components of X, and denote the corresponding partition by $X = X' \dot{\cup} X''$.

To apply Theorem 3.6 we need a triple (X', G', μ') with G' an (f.p.) group, X' incompressible, and $\mu' \in H_3(K', X'; \mathbb{Z})$ with $\partial_*(\mu') = [X']$ such that $\nu(\mu')$ is a stable equivalence. We are given G' and have just constructed X'; since $K = K(G, 1)$ can be taken as a bouquet $K' \vee K''$ there is a natural splitting $H_3(K, X; \mathbb{Z}) \cong H_3(K', X'; \mathbb{Z}) \oplus H_3(K'', X''; \mathbb{Z})$ and we write $\mu = \mu' \oplus \mu''$. It follows that $\partial_*(\mu') = [X']$, so it remains to discuss $\nu(\mu')$.

The situation is now essentially the same as before. The class $\nu(\mu')$ is represented by a map, $\phi' \colon A' \to B'$ say, of R'–modules; similarly for μ''. The ring R is free as an R'–module, and since everything splits naturally, $\nu(\mu)$ (which, by hypothesis, is a stable isomorphism) is represented by $(\phi' \otimes_{R'} R) \oplus (\phi'' \otimes_{R''} R)$. We apply the functor $\otimes_R R'$ and deduce that $\phi' \oplus ((\phi'' \otimes_{R'} \mathbb{Z}) \otimes_{\mathbb{Z}} R')$ is a stable isomorphism. We thus need to consider $\phi'' \otimes_{R'} \mathbb{Z}$.

But now, as before, since ϕ'' is a stable isomorphism of f.g. R–modules, $\phi'' \otimes_{R'} \mathbb{Z}$ is a stable isomorphism of f.g. abelian groups. For stable equivalence we can ignore the free part, and have an isomorphism of the torsion subgroups. Thus, up to stable equivalence $(\phi'' \otimes_{R'} \mathbb{Z}) \otimes_{\mathbb{Z}} R'$ is an isomorphism of torsion groups, and the stable isomorphism $\phi \otimes_R R'$ induces this isomorphism on torsion subgroups, so the induced map on the torsion-free quotient also is a stable isomorphism. Hence so is ϕ', as required.

4 PD^3 group pairs

We now assume that Y is a $K(G, 1)$; each component X_r of X is a $K(H_r, 1)$, say, and H_r injects in G. Thus $(G, \{H_r\})$ is an orientable PD^3 pair in the group theoretic sense. An equivalent notation is to consider a pair $(G, \mathbf{S})$, where G acts on the set $\mathbf{S}$: given a collection of subgroups H_r we can define $\mathbf{S}$ to be the union of cosets $H \backslash G$, while given a G–set $\mathbf{S}$ we can choose a set $\{s_i\}$ of orbit representatives in $\mathbf{S}$ and define H_i to be the stabiliser of s_i. The group G is now torsion free. It is our belief that in this case there is always a compact 3–manifold M such that $(M, \partial M)$ is homotopy equivalent to $(K(G, 1), \bigcup_r K(H_r, 1))$. We break this into a series of lesser problems.

First we need some terminology. A *group pair* consists of a group G and a finite list $\{H_r\}$, unordered but perhaps with repetitions, of subgroups H_r. $(G, \{H_r\})$ is a *2–orbifold pair* if there is a 2–orbifold V with infinite fundamental group G and the H_r are the fundamental groups of the boundary components; if

$\chi_{orb}(V) < 0$ it is a *Fuchsian pair*. If there is a normal subgroup $K \lhd G$ with $K \subseteq \bigcap\{H_r\}$ we have a quotient pair $(G/K, \{H_r/K\})$; if this is a 2–orbifold pair, we speak of a K–by–2–orbifold pair. If G acts on a tree, with v a vertex, the *vertex pair* consists of the stabiliser G_v of v and the set of stabilisers of the edges incident to v. A pair $(G; G, G)$ is said to be *inessential*, a pair (G, H) with G a torus group and $|G : H| = 2$ is *weakly inessential* (this corresponds to the product of S^1 with a Möbius strip); otherwise we call it essential.

For any field k and group H, write $\overline{kH} := \mathrm{Hom}_k(kH, k)$ for the set of infinite formal k–linear combinations of elements of H. If H is a subgroup of G, G acts on $\overline{kG}/(kG \otimes_{kH} \overline{kH})$ and the number of coends $\widetilde{e}(G, H)$ is the dimension of the submodule of invariants:

$$\widetilde{e}(G, H) := \dim_k(\overline{kG}/(kG \otimes_{kH} \overline{kH}))^G.$$

Denote by $\mathcal{C}_n$ the class of virtually polycyclic groups of Hirsch rank n: thus $\mathcal{C}_1$ is the class of groups with a subgroup $\mathbb{Z}$ of finite index, ie 2–ended groups; $\mathcal{C}_2$ is the class of groups with a torus subgroup $\mathbb{Z} \times \mathbb{Z}$ of finite index.

We will apply the following major result of Dunwoody and Swenson (there are similar theorems due to Fujiwara and Papasoglu and to Scott and Swarup):

Theorem 4.1 [9] *Let $n \geq 1$, let G be f.p., and suppose G has no subgroup $H \in \mathcal{C}_{n-1}$ with $\widetilde{e}(G, H) \geq 2$. Then either*

(i) *$G \in \mathcal{C}_{n+1}$,*

(ii) *G is $\mathcal{C}_{n-1}$–by–Fuchsian, or*

(iii) *G has a finite bipartite graph of groups decomposition such that each black vertex pair is either inessential with $G_v \in \mathcal{C}_n$, or a $\mathcal{C}_{n-1}$–by–Fuchsian pair. (Thus every edge group is in $\mathcal{C}_n$.)*

Moreover,

(a) *For every splitting $G = A *_C B$ or $G = A*_C$ of G over a $\mathcal{C}_n$ group C, each white vertex group is conjugate into A or B.*

(b) *For every $\mathcal{C}_{n-1}$–by–Fuchsian pair which is a vertex pair in some splitting of G, its vertex group is conjugate into a black vertex group.*

(c) *Every subgroup $J \in \mathcal{C}_n$ of G with $\widetilde{e}(G, J) \geq 2$ is conjugate into a vertex group and has a subgroup of finite index conjugate into a black vertex group.*

Theorem 4.2 *If G is an f.p. orientable PD^3 group, one of the following holds.*

(i) $G \in \mathcal{C}_3$.

(ii) G *is a* $\mathbb{Z}$*–by–Fuchsian group.*

(iii) G *has a finite bipartite graph of groups decomposition such that each edge group is a torus group; for each black vertex, either the vertex group is a torus group and the vertex pair inessential, or the vertex pair is a* $\mathbb{Z}$*–by–Fuchsian pair.*

Moreover,

(a) *For every splitting* $G = A *_C B$ *or* $G = A*_C$ *of* G *over a torus group* C*, each white vertex group is conjugate into* A *or* B*.*

(b) *For every* $\mathbb{Z}$*–by–Fuchsian pair which is a vertex pair in some splitting of* G*, its vertex group is conjugate into a black vertex group.*

(c) *Every torus subgroup* J *of* G *is conjugate into a vertex group and has a subgroup of finite index conjugate into a black vertex group.*

Proof Since G is a PD^3 group, we have $\operatorname{cd} G = 3$; in particular, G is torsion free (so any $\mathcal{C}_1$ subgroup is isomorphic to $\mathbb{Z}$). It also follows that for a subgroup $H \cong \mathbb{Z}$, $\widetilde{e}(G,H) = 1$. Thus the hypotheses of Theorem 4.1 are satisfied, taking $n = 2$. Since a torsion-free $\mathcal{C}_2$ group is either a torus group or a Klein bottle group, and as G is orientable it cannot split over a Klein bottle group, and since for any torus subgroup $J \subset G$, $\widetilde{e}(G,J) = 2$, the result follows. □

Addendum 4.3 *In case (iii), we also have*

(d) *every torus subgroup* J *of* G *is conjugate into a black vertex group,*

(e) *each essential white vertex pair is atoroidal.*

Proof To prove (d), we may suppose J contained in a white vertex group. Thus (d) will follow from (e).

Denote the white vertex pair in question by $(G', \mathbf{S})$. By (c), any torus subgroup $J \subset G'$ has a subgroup J' of finite index conjugate into a black vertex group. Thus J' is conjugate into two distinct vertex groups of the decomposition. It follows from properties of subgroups of split groups that J' is contained in one of the edge groups $H' \in \mathbf{S}$.

We may suppose without loss of generality that J is a maximal torus subgroup of G', and that J' is a maximal subgroup which is conjugate into H'. Visualise $(G', \mathbf{S})$ as a manifold pair $(M, \partial M)$; consider the covering space $\widetilde{M}$ corresponding to the subgroup J'. The boundary components over H' correspond to double cosets of H' and J' in G'; to the double coset $J'gH'$ corresponds something with fundamental group $g^{-1}J'g \cap H'$. If $g \in J$, this is just the torus group J'. If the cosets $J'g, J'g'$ (with $g, g' \in J$) are distinct, so are $J'gH'$ and $J'g'H'$, for otherwise there exists $h \in H'$ with $J'gh = J'g'$ and so $h \in J$; by maximality of J', $h \in J'$, so $J'g = J'g'$. Thus in the cover $\widetilde{M}$ — itself homotopy equivalent to a torus — there are at least $|J : J'|$ boundary components which are tori. Hence $H_3(\widetilde{M}, \partial\widetilde{M})$ has rank at least $|J : J'| - 1$. By duality, so has $H_c^0(\widetilde{M})$. But this group vanishes if the covering is infinite and $\widetilde{M}$ noncompact; if the cover is finite, the group has rank 1. Hence if $|J : J'| > 1$, the index is 2; the covering is finite, so J' has finite index in G'; as G' is torsion free, it is a torus group, so $G' = J$. Now $H' = J'$ is a torus group of index 2 in G', thus (G', H') is weakly inessential. Otherwise $J = J'$ is contained in a boundary group. Since this holds for any torus subgroup of G', the vertex pair is atoroidal. □

In the case when $Y = K(G,1)$ is a 3–manifold, one can show that the essential white vertex pairs are not only atoroidal but acylindrical. We expect this to be the case here also.

Hope 4.1 *Let $(G; \{H_r\})$ be an essential white vertex pair in the above decomposition, and suppose that $g \in G$, $s_i \in H_r$ and $s_j \in H_j$ satisfy $g^{-1}s_i g = s_j$. Then either $s_i = 1$ (and hence $s_j = 1$) or $i = j$ and $g^{-1}H_r g = H_r$.*

Peter Scott indicated to me a proof in the geometric case. Trying to push it through in the PD case leads me to the following.

Suppose s belongs to two different edge groups. Since each edge group is a torus group, we can choose elements t_1, t_2 in them such that $T_i = \langle s, t_i \rangle$ is a subgroup of finite index in the edge group. Define $t_1t_2t_3 = 1$. Then s also commutes with t_3, so the group T_3 they generate is either cyclic or is contained in a further edge group. Suppose $H := \langle s, t_1, t_2, t_3 \rangle$ is the direct product of $\langle s \rangle$ and a free group on the t_i. Then $(H; T_1, T_2, T_3)$ is a PD^3 pair (the 'product of a circle and a pair of pants'). The covering map from $K(H,1)$ to $K(G,1)$ gives a map of Poincaré pairs, of finite degree on each boundary component, hence of finite degree. Thus H has finite index in G. But H is not atoroidal (there are too many torus subgroups), hence nor is G.

We now wish to extend the above results to the case $X \neq \emptyset$. If G satisfies the maximal condition on centraliser subgroups (denoted Max–c), a theorem of Kropholler in the case $n = 3$ gives:

Theorem 4.4 [17] *Let $(G, \mathbf{S})$ be a PD^3–pair such that G satisfies Max–c but is not in $\mathcal{C}_3$. Then there is a unique reduced G–tree Y, adapted to $\mathbf{S}$, such that $G\backslash Y$ is finite, each edge group is in $\mathcal{C}_2$ (hence is a torus group), each vertex pair is either of Seifert type or atoroidal; and every torus subgroup of G fixes a vertex of Y.*

We expect that the hypothesis Max–c is unnecessary, more precisely:

Hope 4.2 *There is an analogue of Theorem 4.1 for group pairs.*

It would suffice for us to have an extension of Theorem 4.2. One line of attack is to apply Theorem 4.2 to the double $D(G, \mathbf{S})$. Since the resulting splitting is unique (this follows from a theorem of [12], as apart from inessential components, the tree is 'strongly slide-free'), it is compatible with the involution σ interchanging the two copies of G. It would be necessary to investigate how σ restricts to the vertex pairs of the splitting.

I abstain from further discussion, as it seems likely that this question can be resolved by the methods used for the proof of Theorem 4.1.

5 Geometrisation

The hope is to find some sort of decomposition of any PD^3 pair into pieces which are geometric in some homotopy-theoretic sense. We refer to [20] for an account of the Thurston geometries.

In sections 2 and 3 we discussed decomposition of an arbitrary pair into pieces with G not a free product: while this does not hold in all cases, it does if G is torsion free. Suppose now G not a free product. If $e(G) = 2$ we have $S^2 \times S^1$ and $D^2 \times S^1$. If $e(G) = 0$, we have complexes Y finitely covered (up to homotopy) by S^3. Each of these may be regarded as geometric. The group G has periodic cohomology with period 4. Such groups have been classified, and each choice of a generator $g \in H^4(G; \mathbb{Z})$ determines a PD^3 complex, unique up to homotopy equivalence.

In the case $e(G) = 1$, we gave a further decomposition in Theorem 4.2. In cases (i) and (ii), $K(G, 1)$ is homotopy equivalent to a 3–manifold with geometric structure; of type $\mathbb{R}^3$, *Nil* or *Sol* for (i) and $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{SL_2(\mathbb{R})}$ for (ii). In case (iii) the black vertex pairs correspond to Seifert manifolds with boundary, which have geometric structures of Seifert type. This identification is essentially due to the theorem that convergence groups are Fuchsian. If Hope 4.2 is true, corresponding results also hold in the case of pairs.

Much harder is the case of the essential white vertex pairs. It was conjectured by Kropholler that if $(G, \mathbf{S})$ is atoroidal, G is isomorphic to a discrete subgroup of $PSL_2(\mathbb{C})$ and $\mathbf{S}$ to the collection of peripheral subgroups. We attempt a more precise formulation as two separate questions.

Hope 5.1 *If the PD^3 pair $(G, \mathbf{S})$ is atoroidal and acylindrical, then G is hyperbolic relative to $\mathbf{S}$.*

Misha Kapovich and Bruce Kleiner have recently shown that if G is a PD^3 group which acts discretely, isometrically and cocompactly on a CAT(0) space, then G is either a hyperbolic group, a Seifert manifold group, or splits over a virtually abelian subgroup.

Hope 5.2 *If the PD^3 pair $(G, \mathbf{S})$ is atoroidal and acylindrical, and G is hyperbolic relative to $\mathbf{S}$, then G is isomorphic to a discrete subgroup of $PSL_2(\mathbb{C})$ and $\mathbf{S}$ to the collection of peripheral subgroups.*

Cannon and co-workers have written several papers developing an approach to this problem in the case $\mathbf{S} = \emptyset$; I confine myself to citing [5] and [6], and the outline in [15]. It is known that a hyperbolic group is a PD^3 group if and only if ∂G has the Čech homology of a 2–sphere; and that then ∂G is homeomorphic to S^2. If there is a homeomorphism preserving the quasi-conformal structure, or equivalently if G is itself quasi-isometric to hyperbolic space $\mathbb{H}^3$, a theorem of Sullivan applies to give the desired result.

If $\mathbf{S} \neq \emptyset$, then $\operatorname{cd} G = 2$. Hence $\dim \partial G = 1$. It follows from duality that G cannot split over a finite group, so ∂G is connected; it follows from the acylindricity hypothesis that ∂G has no local cut points. It now follows from a theorem of Kapovich and Kleiner [16] that ∂G is homeomorphic to either S^1, a Sierpinski gasket or a Menger curve.

Consider the double $D(G, \mathbf{S})$. Either G is a surface group or $D(G, \mathbf{S})$ is hyperbolic, with G a quasiconvex subgroup. Hence ∂G embeds in $\partial D(G, \mathbf{S})$, which is homeomorphic to S^2, so is planar. Thus ∂G cannot be a Menger curve, so is either S^1 or the Sierpinski gasket, and if $\partial G = S^1$, G is Fuchsian. The Sierpinski gasket is obtained from a 2–disc by removing the interiors of a sequence of disjoint discs, so has a natural boundary which is a union of circles S_r. In this case Kapovich and Kleiner further show:

Theorem 5.1 *The circles S_r fall into finitely many orbits under G. The stabiliser of S_r is a virtually Fuchsian group H_r, quasi-convex in G, which acts on S_r as a uniform convergence group.*

Let $\mathbf{H}$ consist of one representative from each conjugacy class of subgroups H_r. Then the double $D(G,\mathbf{H})$ is hyperbolic, G is a quasiconvex subgroup of it, $\partial D(G,\mathbf{H})$ is homeomorphic to S^2, and $D(G,\mathbf{H})$ is a PD^3 group (over $\mathbb{Q}$; if torsion-free, over $\mathbb{Z}$).

Thus if Hope 5.2 can be established when $X = \emptyset$, the general case seems to follow.

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