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On the characteristic and deformation varieties of a knot

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Abstract The colored Jones function of a knot is a sequence of Laurent polynomials in one variable, whose nth term is the Jones polynomial of the knot colored with the n–dimensional irreducible representation of $\mathfrak{sl}_2$. It was recently shown by TTQ Le and the author that the colored Jones function of a knot is q–holonomic, ie, that it satisfies a nontrivial linear recursion relation with appropriate coefficients. Using holonomicity, we introduce a geometric invariant of a knot: the characteristic variety, an affine 1–dimensional variety in $\mathbb{C}^2$. We then compare it with the character variety of $\mathrm{SL}_2(\mathbb{C})$ representations, viewed from the boundary. The comparison is stated as a conjecture which we verify (by a direct computation) in the case of the trefoil and figure eight knots.

We also propose a geometric relation between the peripheral subgroup of the knot group, and basic operators that act on the colored Jones function. We also define a noncommutative version (the so-called noncommutative A–polynomial) of the characteristic variety of a knot.

Holonomicity works well for higher rank groups and goes beyond hyperbolic geometry, as we explain in the last chapter.

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Dedicated to Andrew Casson on the occasion of his 60th birthday

1 Introduction

1.1 The colored Jones function of a knot

The *colored Jones function*

$$J_K\colon \mathbb{N} \longrightarrow \mathbb{Z}[q^{\pm}]$$

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of a knot K in 3–space is a sequence of Laurent polynomials, whose nth term $J_K(n)$ is the Jones polynomial of a knot colored with the n–dimensional irreducible representation of $\mathfrak{sl}_2$; see [22]. We will normalize it by $J_{\mathrm{unknot}}(n) = 1$ for all n, and (for those who worry about framings), we will assume that K is zero-framed.

The first two terms of the colored Jones function of a knot K are better known. Indeed, $J_K(1) = 1$, and $J_K(2)$ coincides with the *Jones polynomial* of a knot K, defined by Jones in [10]. Although we will not use it, note that the colored Jones function of a knot essentially encodes the Jones polynomial of a knot and its connected parallels.

The starting point for our paper is the key property that the colored Jones function is q–holonomic, as was shown in joint work with TTQ Le; see [6]. Informally, a q–holonomic function is one that satisfies a nontrivial linear recursion relation, with appropriate coefficients. A convenient way to describe recursion relations is the *operator point of view* which we now describe.

1.2 The characteristic variety of a knot

Consider the ring $\mathcal{F}$ of *discrete functions* $f\colon \mathbb{N} \longrightarrow \mathbb{Q}(q)$, and define the linear operators E and Q on $\mathcal{F}$ which act on a discrete function f by:

$$(Qf)(n) = q^n f(n) \qquad (Ef)(n) = f(n+1).$$

It is easy to see that $EQ = qQE$, and that E, Q generate a noncommutative *Weyl algebra* (often called a q–Weyl algebra) with presentation

$$\mathcal{A} = \mathbb{Z}[q^{\pm}]\langle Q, E\rangle/(EQ = qQE).$$

Given a discrete function f, consider the set

$$\mathcal{I}_f = \{P \in \mathcal{A} \,|\, Pf = 0\}.$$

It is easy to see that $\mathcal{I}_f$ is a left ideal of the Weyl algebra, the so-called *recursion ideal* of f.

If $P \in \mathcal{I}_f$, we may think of the equation $Pf = 0$ as a *linear recursion relation* on f. Thus, the set of linear recursion relations that f satisfies may be identified with the recursion ideal $\mathcal{I}_f$.

Definition 1.1 We say that f is *q–holonomic* iff $\mathcal{I}_f \neq 0$. In other words, a discrete function is q–holonomic iff it satisfies a nontrivial linear recursion relation.

Consider the quotient $\mathcal{B} = \mathbb{Z}[E, Q]$ of the Weyl algebra and let

$$\epsilon\colon \mathcal{A} \longrightarrow \mathcal{B} \tag{1}$$

be the *evaluation map* at $q = 1$.

Definition 1.2 If I is a left ideal in $\mathcal{A}$, we define its *characteristic variety* $\mathrm{ch}(I) \subset (\mathbb{C}^\star)^2$ by

$$\mathrm{ch}(I) = \{(x, y) \in (\mathbb{C}^\star)^2 | P(x, y) = 0 \text{ for all } P \in \epsilon(I)\}.$$

If f is a q–holonomic function, then we define its *characteristic variety* to be $\mathrm{ch}(\mathcal{I}_f)$. Finally, if K is a knot in 3–space, we define its *characteristic variety* $\mathrm{ch}(K)$ to be $\mathrm{ch}(J_K)$.

We will make little distinction between a variety $V \subset (\mathbb{C}^\star)^2$ and its closure $\overline{V} \subset \mathbb{C}^2$. For those proficient in holonomic functions, please note that our definition of characteristic variety does *not* agree with the one commonly used in holonomic functions. The latter uses only the symbol (ie the leading E–term) of recursion relations.

1.3 The deformation variety of a knot

The deformation variety of a knot is the character variety of $\mathrm{SL}_2(\mathbb{C})$ representations of the knot complement, viewed from their restriction to the boundary torus. The deformation variety of a knot is of fundamental importance to hyperbolic geometry, and to geometrization, and was studied extensively by Cooper et al and Thurston; see [2] and [21].

Given a knot K in S^3, consider the complement $M = S^3 - \mathrm{nbd}(K)$ (a 3–manifold with torus boundary $\partial M \cong T^2$), and the set

$$R(M) = \mathrm{Hom}(\pi_1(M), \mathrm{SL}_2(\mathbb{C}))$$

of representations of $\pi_1(M)$ into $\mathrm{SL}_2(\mathbb{C})$. This has the structure of an affine algebraic variety defined over $\mathbb{Q}$, on which $\mathrm{SL}_2(\mathbb{C})$ acts by conjugation on representations. Let $X(M)$ denote the algebrogeometric quotient. There is a natural restriction map $X(M) \longrightarrow X(\partial M)$, induced by the inclusion $\partial M \subset M$. Notice that $\pi_1(\partial M) \cong \mathbb{Z}^2$, generated by a meridian and longitude of K. Restricting attention to representations of $\pi_1(\partial M)$ which are upper diagonal, we may identify the character variety of ∂M with $(\mathbb{C}^2)^\star$, parametrized by L and M, the upper left entry of meridian and longitude. Cooper, Culler, Gillet, Long and Shalen [2] define the *deformation variety* $D(K)$ to be the image of $X(\partial M)$ in $(\mathbb{C}^\star)^2$.

1.4 The conjecture

Recall that every affine subvariety V in $\mathbb{C}^2$ is the disjoint union $V_0 \sqcup V_1 \sqcup V_2$ where V_i is a subvariety of V of *pure dimension* i.

We say that two algebraic subvarieties V and V' of $\mathbb{C}^2$ are *essentially equal* iff V_1 is equal to V_1' union some y–lines, where a y–line in $\mathbb{C}^2 = \{(x,y)\,|\,x,y \in \mathbb{C}\}$ is a line $y = a$ for some a.

Conjecture 1 (The Characteristic equals Deformation Variety Conjecture) *For every knot in S^3, the characteristic and deformation varieties are essentially equal.*

Questions similar to the above conjecture and its polynomial version (Conjecture 2 below) were also raised by Frohman and Gelca who studied the colored Jones function of a knot via *Kauffman bracket skein theory*, [8]. Our approach to recursion relations in [6] and here is via statistical mechanics sums and holonomic functions.

A modest corollary of the above conjecture is the following:

Corollary 1.3 *If a knot has nontrivial deformation variety (eg. the knot is hyperbolic), then it has nontrivial colored Jones function.*

Remark 1.4 Despite our improved understanding of the geometry of 3–manifolds, it is unknown at present whether the deformation variety of a knot complement is positive dimensional. If a knot is hyperbolic or torus, then it is, by above mentioned work of Thurston and Cooper et al. If a knot is a satellite, then it is not known, due to the presence of *forbidden representations*, explained by Cooper–Long in [3, section 9].

As evidence for the conjecture, we will show by a direct calculation that:

Proposition 1.5 *Conjectures 1 and 2 are true for the trefoil and figure–8 knots.*

Let us end this section with three comments:

Remark 1.6 Conjecture 1 may be translated as an equality of two polynomials with two commuting variables and integer coefficients; see Conjecture 2 below. Since these polynomials are computable by elimination, it follows that Conjecture 1 is in principle a decidable question. This is in contrast to the *Hyperbolic Volume Conjecture* (due to Kashaev–Murakami–Murakami; see [11, 13]) which involves the existence and identification of a limit of complex numbers.

Remark 1.7 Both Conjecture 1 and the Hyperbolic Volume Conjecture state a relationship between the colored Jones function of a knot and hyperbolic geometry. Combining both conjectures, it follows that the colored Jones function of a hyperbolic knot determines the volume of the hyperbolic 3–manifolds obtained by Dehn surgery on the knot. Indeed, the variation of the volume function depends on the restriction of a path of $\mathrm{SL}_2(\mathbb{C})$ representations to the boundary of the knot complement. Furthermore, the polynomial that defines the deformation variety can compute the variation of the volume function; see Cooper et al [2, section 4.5] and also Yoshida [25] and Neumann–Zagier [14, equation (47)].

Remark 1.8 Conjecture 1 reveals a close relation between the colored Jones function of a knot and its deformation variety. It does not explain though why we ought to look at characters of $\mathrm{SL}_2(\mathbb{C})$ representations. There is a generalization to higher rank groups, which we present in Section 4. We warn the reader that there is no evidence for this generalization.

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2 A polynomial version of Conjecture 1

2.1 The A–polynomial of a knot

Recall the definition of the deformation variery of a knot from Section 1.3. Since projection of affine algebraic varieties corresponds to elimination in their corresponding ideals (see [4]), it is clear that the deformation variety of a knot can in principle be computed via elimination.

In fact, according to [2], the deformation variety $D(K)$ of a knot K is essentially equal to a complex curve in $\mathbb{C}^2$ which is defined by the zero-locus of the so-called *A–polynomial* $A(K)$ of K, where the latter lies in $\mathbb{Z}[L, M^2]$. Here A stands for *affine* and not for Alexander.

2.2 A noncommutative version of the A–polynomial

In this section we define a noncommutative version of the A–polynomial of a knot.

If the Weyl algebra $\mathcal{A}$ were a principal ideal domain, every left ideal (such as the recursion ideal of a discrete function) would be generated by a polynomial in noncommuting variables E and Q. This polynomial would be the noncommutative A–polynomial of an ideal. Applying this to the recursion ideal of J_K would allow us to define the noncommutative A–polynomial of a knot.

Unfortunately, the algebra $\mathcal{A}$ is not a principal ideal domain. One way to get around this problem is to invert polynomials in Q, as we now explain. Consider the *Ore algebra* $\mathcal{A}_{\mathrm{loc}} = \mathbb{K}[E, \sigma]$ over the field $\mathbb{K} = \mathbb{Q}(q, Q)$, where σ is the automorphism of $\mathbb{K}$ given by

$$\sigma(f)(q, Q) = f(q, qQ). \tag{2}$$

Additively, we have

$$\mathcal{A}_{\mathrm{loc}} = \left\{ \sum_{k=0}^{\infty} a_k E^k \quad | \quad a_k \in \mathbb{K}, \quad a_k = 0 \text{ for } k \gg 0 \right\},$$

where the multiplication of monomials given by $aE^k \cdot bE^l = a\sigma^k(b)E^{k+l}$.

Recall the ring $\mathcal{F}$ of discrete functions $f\colon \mathbb{N} \longrightarrow \mathbb{Q}(q)$, and its quotient ring $\tilde{\mathcal{F}}$ under the equivalence relation $f \sim g$ iff $f(n) = g(n)$ for all but finitely many n. Then, $\mathcal{A}_{\mathrm{loc}}$ acts on $\tilde{\mathcal{F}}$. In particular, if f is a discrete function, we may define its recursion ideal, with respect to $\mathcal{A}_{\mathrm{loc}}$. We will call f q–holonomic with respect to $\mathcal{A}_{\mathrm{loc}}$ iff its recursion ideal with respect to $\mathcal{A}_{\mathrm{loc}}$ does not vanish.

By clearing out denominators, it is easy to see that if f is a discrete function, then it is q–holonomic with respect to $\mathcal{A}$ iff it is q–holonomic with respect to $\mathcal{A}_{\mathrm{loc}}$.

It turns out that every left ideal in $\mathcal{A}_{\mathrm{loc}}$ is *principal*; see [5, chapter 2, exercise 4.5]. Given a left ideal I of $\mathcal{A}_{\mathrm{loc}}$, let $A_q(I)$ denote a generator of I, with the following properties:

- $A_q(I)$ has smallest E–degree and lies in $\mathcal{A}$.
- We can write $A_q(I) = \sum_k a_k E^k$ where $a_k \in \mathbb{Z}[q, Q]$ are coprime (this makes sense since $\mathbb{Z}[q, Q]$ is a unique factorization domain).

These properties uniquely determine $A_q(I)$ up to left multiplication by $\pm q^a Q^b$ for integers a, b.

Definition 2.1 Given a left ideal I in $\mathcal{A}$, we define its *A_q–polynomial* $A_q(I) \in \mathcal{A}$ to be $A_q(I)$. Given a knot K in S^3, we define its A_q–polynomial $A_q(K)$ to be the A_q–polynomial of the $\mathcal{A}_{\mathrm{loc}}$–recursion ideal of J_K.

Recall from Section 2.1 that the A polynomial of a knot lies in the ring $\mathbb{Z}[L, M^2]$ which we will identify with $\mathbb{Z}[E, Q]$ by $L = E$ and $M = Q^{1/2}$. In other words,

Definition 2.2 We identify the geometric pair (L, M^2) of (*meridian, longitude*) of a knot K with the pair (E, Q) of *basic operators* which act on the colored Jones function of K.

Let us comment on this definition. It is not too surprising that the meridian variable M is identified with Q, the multiplication by q^n. This is foreshadowed by the *Euler expansion* of the colored Jones function in terms of powers of q^n and $q-1$, [7]. The physical meaning of this expansion is, according to Rozansky, a Feynman diagram expansion around a $U(1)$–connection in the knot complement with holonomy q^n, [18]. Thus, it is not surprising that $M^2 = Q$.

It is more surprising that the longitude variable L corresponds to the shift operator E. This can be explained in the following way. According to Witten (see [24]), the Jones polynomial $J_K(n)$ of a knot K is the average over an infinite dimensional space of connections, of the *trace of the holonomy around K*, where the trace is computed in the n–dimensional representation of $\mathfrak{sl}_2$. To a leading order term, computing traces in the n–dimensional representation is equivalent to computing traces of an $(n-1, 1)$–connected parallel of the knot in the 2–dimensional representation. Thus, increasing n by 1 corresponds to going once more around the knot. Since holonomy and longitude are synonymous notions, this explains in some sense the relation $E = L$.

Conjecture 2 (The AJ Conjecture)[1] *For every knot in S^3, $A(K)(L, M) = \epsilon A_q(K)(L, M^2)$.*

Lemma 2.3 *Conjecture 2 implies Conjecture 1.*

Proof Consider $f, g \in \mathbb{Z}[E, Q]$. Let us say that f is *essentially equal* to q if their images in $\mathbb{Q}(Q)[E]$ are equal. In other words, f is essentially equal to g iff f/g is a rational function of Q.

[1] AJ are the initials of the A–polynomial and the colored Jones polynomial

If $V(f) = \{(x,y) \in \mathbb{C}^2 \,|\, f(x,y) = 0\}$ denotes the variety of zeros of f, then it is easy to see that if f is essentially equal to g, then $V(f)$ is essentially equal to $V(g)$.

It is easy to see that the characteristic (resp. deformation) variety is essentially equal to $V(\epsilon A_q)$ (resp. $V(A)$). The result follows. □

Remark 2.4 Conjecture 2 is consistent with the behavior of the colored Jones function and the A–polynomial under mirror image, changing the orientation of the knot, and $\mathbb{Z}_2$–symmetry. For the behavior of the A–polynomial under these operations, see Cooper–Long: [3, proposition 4.2]. On the other hand, the colored Jones function satisfies the symmetry $J(n) = J(-n)$. Moreover, J is invariant under the change of orientation of a knot and changes under $q \to q^{-1}$ under mirror image.

2.3 Computing the A_q polynomial of a knot

Section 2 defines the A_q polynomial of a knot K. This section explains how to compute the A_q polynomial of a knot. For more details, we refer the reader to [6].

Starting from a generic planar projection of a knot K, it was shown in [6, section 3.2] that the colored Jones function of a knot K can be written as a *multisum*

$$J_K(n) = \sum_{k_1,\dots,k_r=0}^{\infty} F(n,k_1,\dots,k_r) \tag{3}$$

of a *proper q–hypergeometric function* $F(n,k_1,\dots,k_r)$. For a fixed positive n, only finitely many terms are nonzero. Of course, F depends on a planar projection of K. The key property is that F is q–holonomic in all $r+1$ variables, and that it follows from first principles that multisums of q–holonomic functions are q–holonomic in all remaining free variables.

Working with the Weyl algebra $\mathcal{A}_r$ of $r+1$ variables, and using the fact that F is q–proper hypergeometric, we may write $EF/F = A/B$ and $E_iF/F = A_i/B_i$ for polynomials $A, B, A_i, B_i \in \mathbb{Q}(q)[q^n, q^{k_1}, \dots, q^{k_r}]$. Replacing q^n by Q and q^{k_i} by Q_i, it follows that the recursion ideal of F in the Weyl algebra $\mathcal{A}_{r+1}$ is generated by $BE - A, B_1E_1 - A_1, \dots, B_rE_r - A_r$.

The creative telescoping method of Wilf–Zeilberger (the so-called *WZ algorithm*) produces from these generators of F, via noncommutative elimination, operators that annihilate J_K. For a discussion of Wilf–Zeilberger's algorithm, see [26, 23, 17] and also [6, section 5]. For an implementation of the algorithm, see [15, 16].

Applying the WZ algorithm to equation (3), we are guaranteed to get an operator $P \in \mathcal{A}_{\text{loc}}$ such that $PJ_K = 0$. It follows that $A_q(K)$ is a right divisor of P. In other words, there exist an operator $P_1 \in \mathcal{A}_{\text{loc}}$ such that $P_1 P = A_q(K)$. We caution however that the WZ algorithm does not give in general a minimal order difference operator. For a thorough discussion of this matter, see [17, p164]. In other words, P need not equal to $A_q(K)$.

The problem of computing right factors of an operator has been solved in theory by Petkovšek in [1]. A computer implementation of this solution is not available at present.

In case we are looking for right factors of degree 1 (this is equivalent to deciding whether a discrete function has closed form), there is an algorithm `qHyper` of Petkovšek which decides about this problem in real time; see [17].

In the special examples that we will consider, namely the colored Jones function of 3_1 and 4_1 knots, we can bypass the thorny issue of right factorization of an operator.

3 Proof of the conjecture for the trefoil and figure–8 knots

3.1 The colored Jones function and the A–polynomial of the 3_1 and 4_1 knots

Habiro [9] and Le give the following formula for the colored Jones function of the left-handed trefoil (3_1) and figure–8 (4_1) knots:

$$J_{3_1}(n) = \sum_{k=0}^{\infty} (-1)^k q^{k(k+3)/2} q^{nk} (q^{-n-1}; q^{-1})_k (q^{-n+1}; q)_k \tag{4}$$

$$J_{4_1}(n) = \sum_{k=0}^{\infty} q^{nk} (q^{-n-1}; q^{-1})_k (q^{-n+1}; q)_k. \tag{5}$$

where we define the *rising* and *falling factorials* for $k > 0$ by:

$$\begin{aligned} (a;q)_k &= (1-a)(1-aq)\dots(1-aq^{k-1}) \\ (a;q^{-1})_k &= (1-a)(1-aq^{-1})\dots(1-aq^{-k+1}) \end{aligned}$$

and $(a;q)_0 = (a;q)_0 = 1$. Notice that the sums in in equations (4) and (5) have compact support, namely for each positive n, only the terms with $k \leq n$ contribute.

These formulas are discussed in detail in Masbaum [12, theorem 5.1], in relation to the cyclotomic expansion of the colored Jones function of twist knots. To compare Masbaum's formula with the one given above, keep in mind that:

$$\begin{aligned} S(n,k) &:= q^{nk}(q^{-n-1};q^{-1})_k(q^{-n+1};q)_k \\ &= \frac{\{n-k\}\{n-k+1\}\dots\{n+k\}}{\{n\}} \\ &= \prod_{j=1}^{k}((q^{n/2}-q^{-n/2})^2-(q^{j/2}-q^{-j/2})^2) \end{aligned}$$

where $\{m\} = q^{m/2} - q^{-m/2}$.

On the other hand, Cooper et al [2] compute the A–polynomial of the 3_1 and 4_1 knots, as follows:

$$A(3_1) = (L-1)(L+M^6) \tag{6}$$

$$A(4_1) = (L-1)(-L+LM^2+M^4+2LM^4+L^2M^4+LM^6-LM^8) \tag{7}$$

where we include the factor $L-1$ in the A–polynomial which corresponds to the abelian representations of the knot complement.

3.2 Computer calculations

The colored Jones function of the 3_1 and 4_1 knots given in Equations (4) and (5) has no *closed form*. However, it is *guaranteed* to obey nontrivial recursion relations. Moreover, these relations can be found by computer. There are various programs that can compute the recursion relations for multisums. In maple, one may use `qEKHAD` developed by Zeilberger [17]. In Mathematica, one may use `qZeil.m` developed by Paule and Riese [15, 16]. We will give explicit examples in Mathematica, using Paule and Riese's `qZeil.m` package.

We start in computer talk by loading the packages:

```
Mathematica 5.0 for Sun Solaris
Copyright 1988-2000 Wolfram Research, Inc.
 -- Motif graphics initialized --
In[1]:=<< qZeil.m
q-Zeilberger Package by Axel Riese -- ©RISC Linz -- V 2.35 (04/29/03)
In[2]:= << qMultiSum.m
```

```
qMultiSum Package by Axel Riese -- ©RISC Linz -- V 2.45 (04/02/03)
```

Let us type the colored Jones function J_{3_1} from Equation (4):

```
In[3]:= summandtrefoil = (-1)^k q^(k(k + 3)/2) q^(n  k) qfac[q^(-n - 1),
      q^(-1), k] qfac[q^(-n + 1), q, k]
               k  (k (3 + k))/2 + k n                -1 - n  1
Out[3]= (-1)  q                         qPochhammer[q      , -, k]
                                                             q
             1 - n
>    qPochhammer[q     , q, k]
```

We now ask for a recursion relation for J_{3_1}:

```
In[4]:= qZeil[summandtrefoil, {k, 0, Infinity}, n, 1]
qZeil::natbounds: Assuming appropriate convergence.
                        -2 + n          2 n      -1 + 3 n        -1 + n
                       q       (-q + q   )   q          (1 - q      ) SUM[-1 + n]
Out[4]= SUM[n] == ------------------- - ---------------------------------
                                 n                                n
                           -1 + q                            1 - q
```

In other words, for $J(n) = J_{3_1}(n)$ we have:

$$J(n) = q^{-2+n}\frac{-q+q^{2n}}{-1+q^n} - q^{-1+3n}\frac{1-q^{-1+n}}{1-q^n}J(n-1),$$

The above relation is a first order inhomogeneous recursion relation. We may convert it into a second order homogeneous recursion relation as follows:

```
In[5]:= rec31 = MakeHomRec[%, SUM[n]]
         -1 + 2 n   2    n
        q          (q  - q ) SUM[-2 + n]
Out[5]= ------------------------------- +
                       3    2 n
                      q  - q
              n         n    4    4 n    3 + n    2 + 2 n    3 + 2 n
>     ((q - q ) (q + q ) (q  + q    - q      + q        - q        -
           1 + 3 n                   n        2 n   3    2 n
>         q        ) SUM[-1 + n]) / (q  (q - q   ) (q  - q   )) +
        2 - n        n
       q      (-1 + q ) SUM[n]
>     ---------------------- == 0
                 2 n
             q - q
```

Perhaps the reader is displeased to see the above recursion relation written in *backwards shifts*, ie, `SUM[-k+n]` where $k \geq 0$. This can be converted into a recursion relation using *forward shifts* by:

```
In[6]:= ForwardShifts[% ]
         3 + 2 n   2    2 + n
        q         (q  - q     ) SUM[n]
Out[6]= ----------------------------- +
                     3    4 + 2 n
                    q  - q
        -2 - n        2 + n        2 + n
>     (q       (q - q      ) (q + q      )
           4    5 + n    6 + 2 n    7 + 2 n    7 + 3 n    8 + 4 n
```

```
>         (q  - q       + q         - q         - q         + q       ) SUM[1 + n])
                                                          2 + n
                  4 + 2 n    3    4 + 2 n     (-1 + q      ) SUM[2 + n]
>         / ((q - q       ) (q  - q       )) + ------------------------ == 0
                                                    n      4 + 2 n
                                                   q  (q - q       )
```

The next command converts the recursion relation `rec31` into an operator, where (due to `Mathematica` annoyance), we use the symbol X to denote the shift E:

```
In[7]:= ToqHyper[rec31[[1]] - rec31[[2]]] /. {SUM[N] -> 1, SUM[N q^c_.] :> X^c}
  /. N -> Q
          2               2       2
         q  (-1 + Q)    (q  - Q) Q
Out[7]= ----------- + -------------- +
                  2        3    2  2
         Q (q - Q )    q (q  - Q ) X
                              4    3       2  2    3  2      3    4
     (q - Q) (q + Q) (q  - q  Q + q  Q  - q  Q  - q Q  + Q )
>    --------------------------------------------------------
                              2    3    2
                     Q (q - Q ) (q  - Q ) X
```

This operator right-divides the A_q polynomial of the 3_1 knot. Let us assume for now that it equals to the A_q polynomial, after clearing denominators. Setting $q = 1$, and replacing X by L and Q by M^2, and obtain:

```
In[8]:= Factor[ToqHyper[rec31[[1]] - rec31[[2]]] /. {SUM[N] -> 1,
          SUM[N q^c_.] :> X^c} /. {N -> Q, q -> 1}] /. {Q -> M^2, X -> L}
                          6
            (-1 + L) (L + M )
Out[8]= -(-----------------)
             2  2       2
            L  M  (1 + M )
```

The result agrees, up to multiplication by a rational function of M and a power of E, with the A–polynomial of 3_1 from (6).

It remains to prove that rec31:=`Out[7]` coincides with $A_q(3_1)$, after clearing denominators. Notice that $\text{rec31} = PA_q(3_1)$ for some operator P and $\text{ord}_E(\text{rec31}) = 2$, where $\text{ord}_E(P)$ denotes the E–order of an operator E. Thus $\text{ord}_E(A_q(3_1))$ is 1 or 2. If $\text{ord}_E(A_q(3_1)) = 1$, then J_{3_1} would have a closed form. This problem can be decided by computer using `qHyper` (see [17]), which indeed confirms that J_{3_1} does not have closed form. Thus $\text{ord}_E(A_q(3_1)) = 2 = \text{ord}_E(\text{rec31})$. It follows that (up to left multiplication by units), $A_q(3_1)$ equals to rec31. This completes the proof in the case of the trefoil.

Now, let us repeat the process for the colored Jones function of the figure 8 knot, given in Equation (5).

```
In[9]:= summandfigure8 = q^(n  k) qfac[q^(-n - 1), q^(-1), k] qfac[q^(-n + 1),
  q, k]
```

```
          k n                  -1 - n  1                         1 - n
Out[9]= q     qPochhammer[q      , -, k] qPochhammer[q      , q, k]
                                       q
In[10]:= qZeil[summandfigure8, {k, 0, Infinity}, n, 2]
qZeil::natbounds: Assuming appropriate convergence.
                   -1 - n        n        2 n
                  q       (q + q ) (-q + q   )
Out[10]= SUM[n] == --------------------------- -
                                   n
                             -1 + q
          -2 + n          -1 + 2 n
      (1 - q      ) (1 - q        ) SUM[-2 + n]
>     --------------------------------------- +
               n         -3 + 2 n
         (1 - q ) (1 - q        )
        -2 - 2 n         -1 + n 2        -1 + n
>     (q         (1 - q       )  (1 + q       )
          4    4 n    3 + n    1 + 2 n    3 + 2 n    1 + 3 n
>        (q  + q    - q      - q        - q        - q       ) SUM[-1 + n]) /
            n        -3 + 2 n
>      ((1 - q ) (1 - q        ))
```

gives a second-order inhomogeneous recursion relation, which we convert into a third-order homogeneous recursion relation:

```
In[11]:= rec41 = MakeHomRec[%, SUM[n]]
           2 + n    3    n
          q      (-q  + q ) SUM[-3 + n]
Out[11]= ---------------------------- -
              2    n     5    2 n
            (q  + q ) (-q  + q   )
        -2 - n  2    n    8    4 n      6 + n    7 + n    3 + 2 n
>     (q       (q  - q ) (q  + q    - 2 q      + q      - q        +
          4 + 2 n    5 + 2 n    1 + 3 n      2 + 3 n
>        q        - q        + q        - 2 q       ) SUM[-2 + n]) /
           n    5    2 n
>      ((q + q ) (q  - q   )) +
        -1 - n        n    4    4 n    2 + n      3 + n    1 + 2 n
>     (q       (-q + q ) (q  + q    + q      - 2 q      - q        +
          2 + 2 n    3 + 2 n      1 + 3 n    2 + 3 n
>        q        - q        - 2 q        + q       ) SUM[-1 + n]) /
                                   1 + n        n
          2    n          2 n     q      (-1 + q ) SUM[n]
>      ((q  + q ) (-q + q   )) + ---------------------- == 0
                                         n       2 n
                                   (q + q ) (q - q   )
```

In forward shifts, we have:

```
In[12]:= ForwardShifts[%]
           5 + n    3    3 + n
          q      (-q  + q     ) SUM[n]
Out[12]= ------------------------------ -
           2    3 + n     5    6 + 2 n
         (q  + q     ) (-q  + q       )
        -5 - n  2    3 + n    8      9 + n    10 + n    9 + 2 n
>     (q       (q  - q     ) (q  - 2 q      + q       - q        +
          10 + 2 n    11 + 2 n    10 + 3 n      11 + 3 n    12 + 4 n
>        q         - q         + q         - 2 q         + q        )
                                 3 + n     5    6 + 2 n
>        SUM[1 + n]) / ((q + q     ) (q  - q       )) +
        -4 - n          3 + n    4    5 + n      6 + n    7 + 2 n    8 + 2 n
>     (q       (-q + q     ) (q  + q      - 2 q      - q        + q        -
```

```
             9 + 2 n       10 + 3 n      11 + 3 n      12 + 4 n
>           q        - 2 q         + q          + q         ) SUM[2 + n]) /
                                                4 + n       3 + n
        2    3 + n          6 + 2 n        q         (-1 + q     ) SUM[3 + n]
>     ((q  + q     ) (-q + q       )) + ------------------------------- == 0
                                                   3 + n          6 + 2 n
                                            (q + q     ) (q - q        )
```

In operator form, `rec41` becomes:

```
In[13]:= ToqHyper[rec41[[1]] - rec41[[2]]] /. {SUM[N] -> 1, SUM[N q^c_.] :>
  X^c} /. N -> Q
                                    2      3
          q (-1 + Q) Q           q  Q (-q  + Q)
Out[13]= ---------------- + ---------------------- -
                        2      2          5    2  3
          (q + Q) (q - Q )    (q  + Q) (-q  + Q ) X
        2          8      6      7      3  2    4  2    5  2        3     2  3
>    ((q  - Q) (q  - 2 q  Q + q  Q - q  Q  + q  Q  - q  Q  + q Q  - 2 q  Q  +
         4          2               5    2   2
>       Q )) / (q  Q (q + Q) (q  - Q ) X ) +
                   4    2      3        2    2  2    3  2        3    2  3
>    ((-q + Q) (q  + q  Q - 2 q  Q - q Q  + q  Q  - q  Q  - 2 q Q  + q  Q  +
        4               2            2
>       Q )) / (q Q (q  + Q) (-q + Q ) X)
```

where $X=E$. Let us assume that this coincides with $A_q(4_1)$, after we clear denominators. Setting $q=1$, and replacing X by L and Q by M^2, and obtain:

```
In[14]:= Factor[ToqHyper[rec41[[1]] - rec41[[2]]] /. {SUM[N] -> 1,
             SUM[N q^c_.] :> X^c} /. {N -> Q , q -> 1}] /. {Q -> M^2, X -> L}
                         2    4        4    2  4       6      8
          (-1 + L) (L - L M  - M  - 2 L M  - L  M  - L M  + L M )
Out[14]= ---------------------------------------------------------
                                3  2       2 2
                               L  M  (1 + M )
```

The result agrees, up to multiplication by a rational function of M and a power of E, with the A–polynomial of 4_1 from (7).

It remains to prove that rec41:=`Out[13]` is equal, up to units, to $A_q(4_1)$. Notice that $\mathrm{rec41}=PA_q(4_1)$ for some operator P and $\mathrm{ord}_E(\mathrm{rec41})=3$. Thus $\mathrm{ord}_E(A_q(4_1))$ is 1 or 2 or 3.

If $\mathrm{ord}_E(A_q(4_1))=1$, then J_{4_1} would have a closed form. This problem can be decided by computer using `qHyper` (see [17]), which indeed confirms that J_{3_1} does not have closed form.

If $\mathrm{ord}_E(A_q(4_1))=2$, recall the map ϵ which evaluates at $q=1$. We have: $\epsilon\mathrm{rec41}=\epsilon P\,\epsilon A_q(4_1)$. Since $\mathrm{ord}_E(\epsilon\mathrm{rec41})=3$, it follows that we must have $\mathrm{ord}_E(\epsilon A_q(4_1))=2$.

Furthermore, the computer calculation above shows that $\epsilon A_q(4_1)$ divides $A(4_1)$. The latter, given by Equation (7) can be factored as a product of two irreducible polynomials of E–degree 1 and 2.

On the other hand, Lemma 3.1 below implies that $E-1$ divides $(\epsilon A_q(4_1))|_{Q=1}$. Combining these facts, it follows that $\epsilon A_q(4_1) = A(4_1)$ (and therefore, also $A_q(4_1)$) is of E–degree 3, a contradiction to our hypothesis.

Thus, it follows that $\mathrm{ord}_E(A_q(4_1)) = 3 = \mathrm{ord}_E(\mathrm{rec41})$. This implies that, up to left multiplication by units, $A_q(4_1)$ coincides with rec41. This concludes the proof in the case of the figure–8 knot.

Lemma 3.1 *For every knot K, $\epsilon\mathcal{A}_q(K)(1,1) = 0$.*

Proof Recall that the colored Jones function of a knot K is given by a multisum formula of a q–proper hypergeometric function. Consider the evaluation of the colored Jones function ϵJ_K at $q = 1$. This is a discrete function which is given by a multisum of a proper hypergeometric function. Applying the WZ algorithm, it follows that $\epsilon_Q\epsilon\mathcal{A}_q(K)$ annihilates ϵJ_K, where ϵ_Q is the evaluation at $Q = 1$. However, $\epsilon J_K(n) = 1$ for all n; see [6]. Thus $E-1$ divides $\epsilon_Q\epsilon\mathcal{A}_q(K)$. The result follows. □

4 Higher rank groups

The purpose of this section is to formulate a generalization of the characteristic and deformation varieties of a knot to higher rank groups.

Consider a *simple* simply connected compact Lie group G with Lie algebra $\mathfrak{g}$ and complexified group $G_{\mathbb{C}}$. Let $\Lambda \cong \mathbb{Z}^r$ denote its weight lattice, which is a free abelian group of rank r, the rank of G, and let $\Lambda_+ \cong \mathbb{N}^r$ denote the cone of positive dominant weights.

One can define the $\mathfrak{g}$–*colored Jones function*

$$J_{\mathfrak{g}}\colon \mathbb{N}^r \longrightarrow \mathbb{Z}[q^{\pm}].$$

In [6], we showed that $J_{\mathfrak{g}}$ is q–holonomic, with respect to the Weyl algebra of r variables:

$$\mathcal{A}_r = \frac{\mathbb{Z}[q^{\pm}]\langle Q_1, \dots, Q_r, E_1, \dots, E_r\rangle}{(\mathrm{Rel}_q)}$$

where the relations are given by:

$$\begin{aligned} Q_iQ_j &= Q_jQ_i & E_iE_j &= E_jE_i \\ Q_iE_j &= E_jQ_i \text{ for } i \neq j & E_iQ_i &= qQ_iE_i \end{aligned} \qquad (\mathrm{Rel}_q)$$

Loosely speaking, holonomicity of a discrete function of r variables means that it satisfies r independent linear recursion relations.

A precise definition in several equivalent forms was given in [6, section 2]. For the benefit of the reader, we recall here the definition in its form most useful for our purposes.

Given a discrete function $f\colon \mathbb{N}^r \longrightarrow \mathbb{Q}(q)$, we define the *recursion ideal* $\mathcal{I}_f$ and the *q–Weyl module* M_f by:

$$\mathcal{I}_f = \{P \in \mathcal{A}_r \,|Pf = 0\} \qquad M_f := \mathcal{A}_r f \cong \mathcal{A}_r/\mathcal{I}_f.$$

M_f is a cyclic left $\mathcal{A}_r$ module. Every finitely generated left $\mathcal{A}_r$ module has a Hilbert dimension. In case $M = \mathcal{A}_r/I$ is cyclic, its *Hilbert dimension* $d(M)$ is defined as follows. Let F_m be the sub-space of $\mathcal{A}_r$ spanned by polynomials in Q_i, E_i of total degree $\leq m$. Then the module $\mathcal{A}_r/I$ can be approximated by the sequence $F_m/(F_m \cap I), m = 1, 2, \ldots$. It turns out that, for $m \gg 1$, the dimension of the vector space $F_m/(F_m \cap I) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Q}(q)$ (over the field $\mathbb{Q}(q)$) is a polynomial in m of degree equal (by definition) to $d(M)$.

Bernstein's *famous inequality* (proved by Sabbah in the q–case, [19]) states that $d(M) \geq r$, if $M \neq 0$ and M has *no monomial torsions*, ie, any non-trivial element of M cannot be annihilated by a monomial in Q_i, E_i. Note that the left $\mathcal{A}_r$ module $M_f := \mathcal{A}_r \cdot f \cong \mathcal{A}_r/\mathcal{I}_f$ does not have monomial torsion.

Definition 4.1 We say that a discrete function f is q–holonomic if $d(M_f) \leq r$.

Note that if $d(M_f) \leq r$, then by Bernstein's inequality, either $M_f = 0$ or $d(M_f) = r$. The former can happen only if $f = 0$. Of course, for $r = 1$, definitions 1.1 and 4.1 agree.

Let us now define the characteristic variety of a cyclic $\mathcal{A}_r$ module $M = \mathcal{A}_r/I$. Let

$$\mathcal{B}_r = \mathbb{Z}[Q_1, \ldots, Q_r, E_1, \ldots, E_r]$$

and $\epsilon\colon \mathcal{A}_r \longrightarrow \mathcal{B}_r$ denote the evaluation map at $q = 1$.

Definition 4.2 The *characteristic variery* $\mathrm{ch}(M)$ of M is defined by

$$\mathrm{ch}(M) = \{(x, y) \in (\mathbb{C}^\star)^{2r} \mid P(x, y) = 0 \text{ for all } P \in \epsilon(I \cap \mathcal{A}_r)\}$$

This definition may be extended to define the characteristic variety of finitely generated left $\mathcal{A}_r$ modules. As before, we will make little distinction between the characteristic variety and its closure in $\mathbb{C}^{2r}$.

Lemma 4.3 *If M is a q–holonomic $\mathcal{A}_r$ module, then* $\dim_{\mathbb{C}} \mathrm{ch}(M) \geq r$.

Proof Since M is q–holonomic, it follows that the Hilbert dimension of $(\mathcal{A}_r \otimes \mathbb{Q}(q))/I$ is r, and from this it follows that the Hilbert dimension of $(\mathcal{A}_r \otimes \mathbb{Q}(q))/I$ for generic $q \in \mathbb{C}$ is r. Since dimension is upper semicontinuous and it coincides with the Hilbert dimension at the generic point [20], the result follows. □

Definition 4.4 If K is a knot in S^3, and G as above, we define its *G–characteristic variety* $V_G(K) \subset \mathbb{C}^{2r}$ to be the characteristic variery of its $\mathfrak{g}$–colored Jones function.

Similarly to the case of $\mathrm{SL}_2(\mathbb{C})$, given a knot K in S^3, consider the complement $M = S^3 - \mathrm{nbd}(K)$ and the set $R_{G_{\mathbb{C}}}(M)$ of representations of $\pi_1(M)$ into $G_{\mathbb{C}}$. This has the structure of an affine algebraic variety, on which $G_{\mathbb{C}}$ acts by conjugation on representations. Let $X_{G_{\mathbb{C}}}(M)$ denote the algebrogeometric quotient. There is a natural restriction map $X_{G_{\mathbb{C}}}(M) \longrightarrow X_{G_{\mathbb{C}}}(\partial M)$. Notice that $\pi_1(\partial M) \cong \mathbb{Z}^2$, generated by the meridian and longitude of K. Restricting attention to representations of $\pi_1(\partial M)$ which are upper diagonal with respect to a Borel decomposition, we may identify the character variety $X_{G_{\mathbb{C}}}(\partial M)$ with T^2 where T is a maximal torus in $G_{\mathbb{C}}$.

Definition 4.5 The *$G_{\mathbb{C}}$–deformation variety* $D_{G_{\mathbb{C}}}(K)$ of K is the image of $X_{G_{\mathbb{C}}}(\partial M)$ in T^2.

Notice that the maximal torus T of $G_{\mathbb{C}}$ can be identified with $(\mathbb{C}^\star)^r$, once we choose fundamental weights λ_i. This allows us to identify the values of meridian and longitude with T^2. Notice further that the deformation variety of a knot contains an r–dimensional component which corresponds to abelian representations.

Let us say that two varieties V and V' in $\mathbb{C}^{2r} = \{(x,y) | \, x, y \in \mathbb{C}^r\}$ are *essentially equal* if the pure r–dimensional part of V equals to that of V' union some r–dimensional varieties of the form $f(y) = 0$.

Question 1 Is it true that for every G as above and for every knot K, the characteristic and deformation varieties $V_G(K)$ and $D_{G_{\mathbb{C}}}(K)$ are essentially equal?

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