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Tori in symplectic 4–manifolds

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Abstract We study the question of how many embedded symplectic or Lagrangian tori can represent the same homology class in a simply connected symplectic 4–manifold.

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1 Introduction

The basic question addressed in this paper is:

> Let X be a simply connected symplectic 4–manifold and let $x \in H_2(X,\mathbb{Z})$. How unique is an embedded symplectic or Lagrangian representative of x?

It is only in the last few years that an answer to this question has begun to emerge. The answer is 'not very' for symplectic tori of self-intersection 0 and remains elusive for higher genus surfaces. As we show below:

Theorem *If X is a simply connected symplectic 4–manifold containing an embedded symplectic torus T of self-intersection 0, then for each fixed integer $m \geq 2$, there are infinitely many embedded symplectic tori, each representing the homology class $m[T]$, no two of which are equivalent under a smooth isotopy of X.*

The first such examples were produced by the present authors in [6], and the technique therein was enhanced to produce further examples in [19, 3]. In section 4 we give a proof of the above theorem. In section 3 we give a proof of the theorem for even $m \geq 6$ which is straightfoward, and which depends on some nice theorems of Montesinos and Morton [12] and of Kanenobu [9] rather than on explicit constructions.

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Some intriguing questions remain. Siebert and Tian have conjectured that for symplectic 4–manifolds with $b^+ = 1$ and $c_1^2 > 0$ any embedded symplectic surface must be symplectically isotopic to a holomorphic curve. (Of course, no such manifold contains an embedded symplectic torus of square 0.) They have shown that in $\mathbb{CP}^2$ this is true for each curve of degree ≤ 17, and they have also some results to this effect in $S^2 \times S^2$. However the general problem is still wide open, as is the case in general for surfaces of higher genus or for other self-intersections. However, in the case where $\pi_1(X) \neq 0$ Ivan Smith has constructed examples of nonisotopic but homologous surfaces of square 0 distinguished by π_1 of their complements.

Much less is known in the case of Lagrangian tori. Until this year, it was unknown if there existed Lagrangian tori which were homologous but inequivalent (under either isotopy or orientation-preserving diffeomorphism). The first examples are due to Stefano Vidussi:

Theorem [20] *Let K denote the trefoil knot. Then in the symplectic manifold $E(2)_K$ there is a primitive homology class α so that for each positive integer m, there are infinitely many embedded Lagrangian tori representing $m\alpha$, no two of which are equivalent under orientation-preserving diffeomorphisms.*

Utilizing an invariant coming from Seiberg–Witten theory and the geometry of fibered knots, the current authors improved this theorem as follows:

Theorem [7] (a) *Let X be any symplectic manifold with $b_2^+(X) > 1$ which contains an embedded symplectic torus with a vanishing cycle. Then for each fibered knot K in S^3, the result of knot surgery X_K contains infinitely many nullhomologous Lagrangian tori, pairwise inequivalent under orientation-preserving diffeomorphisms.*

(b) *Let X_i, $i = 1, 2$, be symplectic 4–manifolds containing embedded symplectic tori F_i and assume that F_1 contains a vanishing cycle. Let X be the fiber sum, $X = X_1 \#_{F_1=F_2} X_2$. Then for each fibered knot K in S^3, the manifold X_K contains an infinite family of homologically primitive and homologous Lagrangian tori which are pairwise inequivalent.*

In sections 5–8 we show how this theorem works in a specific example constructed via double branched covers. The discussion here differs somewhat from the more general arguments of [7], however we feel that it is helpful to understand specific examples from different points of view.

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2 Seiberg–Witten invariants

The Seiberg–Witten invariant of a smooth closed oriented 4–manifold X with $b_2^+(X) > 1$ is an integer-valued function which is defined on the set of Spin^c structures over X (cf [21]). In case $H_1(X;\mathbb{Z})$ has no 2–torsion, there is a natural identification of the Spin^c structures of X with the characteristic elements of $H_2(X;\mathbb{Z})$ (ie those elements k whose Poincaré duals $\hat{k}$ reduce mod 2 to $w_2(X)$). In this case we view the Seiberg–Witten invariant as

$$\mathrm{SW}_X\colon \{k \in H_2(X;\mathbb{Z}) | \hat{k} \equiv w_2(TX) \pmod 2)\} \to \mathbb{Z}.$$

The sign of SW_X depends on an orientation of $H^0(X;\mathbb{R}) \otimes \det H^2_+(X;\mathbb{R}) \otimes \det H^1(X;\mathbb{R})$; however, when X has a symplectic structure, there is a preferred sign for SW_X (see [16]).

If $\mathrm{SW}_X(\beta) \neq 0$, then β is called a *basic class* of X. It is a fundamental fact that the set of basic classes is finite. Furthermore, if β is a basic class, then so is $-\beta$ with $\mathrm{SW}_X(-\beta) = (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\,\mathrm{SW}_X(\beta)$ where $\mathrm{e}(X)$ is the Euler number and $\mathrm{sign}(X)$ is the signature of X.

It is convenient to view the Seiberg–Witten invariant as an element of the integral group ring $\mathbb{Z}H_2(X)$. For $\alpha \in H_2(X)$ we let t_α denote the corresponding element in $\mathbb{Z}H_2(X)$. More specifically, suppose that $\{\pm\beta_1, \dots, \pm\beta_n\}$ is the set of nonzero basic classes for X. Then the Seiberg–Witten invariant of X is the Laurent polynomial

$$\mathcal{SW}_X = \mathrm{SW}_X(0) + \sum_{j=1}^n \mathrm{SW}_X(\beta_j) \cdot (t_{\beta_j} + (-1)^{(\mathrm{e}+\mathrm{sign})(X)/4}\, t_{\beta_j}^{-1}) \in \mathbb{Z}H_2(X).$$

A key vanishing theorem for the Seiberg–Witten invariants is:

Theorem [21] *Let X be a smooth closed 4–manifold which admits a decomposition $X = A \cup B$ into 4–manifolds with $\partial A = \partial B = Y$. Suppose that $b_2^+(A) > 0$, $b_2^+(B) > 0$, and that Y admits a metric of positive scalar curvature, then $\mathcal{SW}_X = 0$.*

Another important and extremely useful fact about Seiberg–Witten invariants is the adjunction inequality: If X is a smooth closed 4–manifold with $b_2^+ > 1$ and Σ is an embedded surface of positive genus g in X representing a nontrivial element of $H_2(X;\mathbb{R})$ then for any basic class β of X

$$2g - 2 \geq \Sigma^2 + \beta \cdot \Sigma \tag{1}$$

We next recall the link surgery construction of [5]. This construction starts with an oriented n–component link $L = \{K_1, \dots, K_n\}$ in S^3 and n pairs (X_i, T_i) of smoothly embedded self-intersection 0 tori in simply connected 4–manifolds. (In the original article [5], an extra condition ('c-embedded') was placed on these tori; however, recent work of Cliff Taubes [17] has shown this condition to be unnecessary.)

Let $\alpha_L\colon \pi_1(S^3 \setminus L) \to \mathbb{Z}$ denote the homomorphism characterized by the property that it sends the meridian m_i of each component K_i to 1, and let ℓ_i denote the longitude of K_i. The curves $\gamma_i = \ell_i - \alpha_L(\ell_i) m_i$ on $\partial N(K_i)$ form the boundary of a Seifert surface for the link, and in case L is a fibered link, the γ_i are given by the boundary components of a fiber.

In $S^1 \times (S^3 \setminus N(L))$ let $T_{m_i} = S^1 \times m_i$, and define the link-surgery manifold $X(X_1, \dots X_n; L)$ by

$$X(X_1, \dots X_n; L) = (S^1 \times (S^3 \setminus N(L)) \cup \bigcup_{i=1}^{n} (X_i \setminus (T_i \times D^2))$$

where $S^1 \times \partial N(K_i)$ is identified with $\partial N(T_i)$ so that for each i

$$[T_{m_i}] = [T_i], \quad \text{and} \quad [\gamma_i] = [\mathrm{pt} \times \partial D^2].$$

It is not clear whether or not this determines $X(X_1, \dots X_n; L)$ up to diffeomorphism, however any such manifold will have the same Seiberg–Witten invariant:

Theorem [5] *If each $\pi_1(X \backslash T_i) = 1$, then $X(X_1, \dots X_n; L)$ is simply-connected and its Seiberg–Witten invariant is*

$$\mathcal{SW}_{X(X_1, \dots X_n; L)} = \Delta_L^{sym}(t_1^2, \dots, t_n^2) \cdot \prod_{j=1}^{n} \mathcal{SW}_{X_j} \cdot (t_j - t_j^{-1})$$

where $t_j = t_{[T_j]}$ and $\Delta_L^{sym}(t_1, \dots, t_n)$ is the symmetric multivariable Alexander polynomial of the link L.

In case each $(X_i, T_i) \cong (X, T)$, a fixed pair, we write

$$X(X_1, \dots X_n; L) = X_L$$

(We implicitly remember T, but it is removed from the notation.) As an example, consider the case where each $X_i = E(1)$, the rational elliptic surface ($E(1) \cong \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$) and each $T_i = F$ is a smooth elliptic fiber. Since $SW_{E(1)} = (t - t^{-1})^{-1}$, we have that

$$\mathcal{SW}_{E(1)_L} = \Delta_L^{sym}(t_1^2, \dots, t_n^2).$$

In case the link L is actually a knot K, we call the procedure 'knot surgery' and the resulting manifold X_K. The formula for the Seiberg–Witten invariant looks slightly different in this case due to the difference in the relationship of the Seiberg–Witten invariant of a 3–manifold and its Alexander polynomial when $b_1 > 1$ and $b_1 = 1$.

Theorem [5] *If $\pi_1(X \setminus T) = 1$, then X_K is simply connected and its Seiberg–Witten invariant is*

$$\mathcal{SW}_{X_K} = \Delta_K^{sym}(t^2) \cdot \mathcal{SW}_X$$

where $t = t_{[T]}$.

3 Tori and simple covers

Our first construction utilizes an extremely interesting theorem of José Montesinos and Hugh Morton which characterizes fibered links in the 3–sphere. To begin, let X be a simply connected symplectic 4–manifold containing an embedded symplectic torus T of self-intersection 0, and identify a tubular neighborhood of T with $S^1 \times (S^1 \times D^2)$. A closed braid may be viewed as contained in $S^1 \times D^2 \subset S^3 = (S^1 \times D^2) \cup (D^2 \times S^1)$ and then its *axis* is $\{0\} \times S^1$. The theorem of Montesinos and Morton is:

Theorem (Montesinos and Morton [12]) *Every fibered link in S^3 with k components can be obtained as the preimage of the braid axis for a d–sheeted simple branched cover of S^3 branched along a suitable closed braid, where $d = \max\{k, 3\}$.*

(Recall that a *simple* branched cover of degree d is one whose branch points have exactly $d-1$ points in their preimages.)

A second important ingredient in this construction is a theorem of Kanenobu concerning the Hosokawa polynomial of fibered links. The Alexander polynomial of a link L of k components is a polynomial $\Delta_L(t_1, \dots, t_k)$ in k variables (corresponding to the meridians of the components of the link). The polynomial $\Delta_L(t, \dots, t)$ obtained by setting all the variables equal is always divisible by $(t-1)^{k-2}$, and the Hosokawa polynomial of L is defined to be $\nabla_L(t) = \Delta_L(t, \dots, t)/(t-1)^{k-2}$.

Theorem (Kanenobu [9]) *Let $f(t)$ be any symmetric polynomial of even degree with integral coefficients satisfying $f(0) = \pm 1$, then for any $k \geq 2$ there is a fibered link L of k components in S^3 with $\nabla_L(t) = f(t)$.*

We now use these two theorems to build symplectic tori homologous to multiples of T. We have described a tubular neighborhood N of T as $N = S^1 \times (S^1 \times D^2)$. Fix a three-component fibered link L in S^3 and let B_L be the braid corresponding to L by the Montesinos–Morton Theorem. As above, we view B_L as contained in $S^1\times D^2 \subset S^3 = (S^1\times D^2)\cup(D^2\times S^1)$ with axis $A = \{0\}\times S^1$. (See Figure 1 for an example.) Then $T_L = S^1 \times B_L \subset N$ is a symplectic torus [6], and if B_L has m strands, then T_L is homologous to mT.

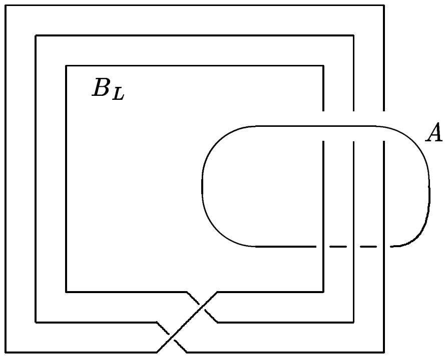

Figure 1

Let $\pi\colon (S^3, L) \to (S^3, A)$ be the threefold branched cover with branch set B_L given by the Montesinos–Morton Theorem. Because $L = \pi^{-1}(A)$ is a three-component link, the covering restricted to A is trivial. This means that the restriction of π over $\partial(S^1 \times D^2)$ is a trivial covering, and the induced branched cover over $N = S^1\times(S^1\times D^2)$ extends trivially over X. We thus get a threefold simple branched cover $p = p_L\colon \widetilde{X} \to X$ with branch set T_L. We have

$$\widetilde{X} = \widetilde{N} \cup \bigcup_{i=1}^{3} (X \setminus N)_i$$

where $\widetilde{N} = p^{-1}(N) = S^1 \times (S^3 \setminus L)$, and $(X \setminus N)_i$ denotes a copy of $X \setminus N$.

This means that $\widetilde{X}$ is obtained via link surgery on the link L using (X,T). The Seiberg–Witten invariant of $\widetilde{X}$ (viewed as an element of $\mathbb{Z}H_2(\widetilde{X})$) may be calculated via the techniques of [17, 14, 5]:

$$\mathcal{SW}_{\widetilde{X}} = \Delta_L^{sym}(t_1^2, t_2^2, t_3^2) \cdot \prod_{i=1}^{3} \mathcal{SW}_{X_i} \cdot (t_i - t_i^{-1})$$

The induced map $p_*\colon \mathbb{Z}H_2(\widetilde{X}) \to \mathbb{Z}H_2(X)$ satisfies $p_*(\mathcal{SW}_{X_i}) = \mathcal{SW}_X$. Also, since t_i is the element of $\mathbb{Z}H_2(\widetilde{X})$ corresponding to the homology class of $S^1\times\mu_i$

where μ_i are the meridians of the components of L, $p_*(t_i)$ is the element of $\mathbb{Z}H_2(X)$ corresponding to $S^1 \times \mu_A$, where μ_A is a meridian to A. Since μ_A is the core circle $S^1 \times \{0\} \subset S^1 \times D^2$, we have $[S^1 \times \mu_A] = [T]$ in $H_2(X)$. Thus $p_*(t_i) = t$, and

$$p_*(\mathcal{SW}_{\widetilde{X}}) = \Delta_L^{sym}(t^2, t^2, t^2) \cdot \mathcal{SW}_X^3 \cdot (t - t^{-1})^3 \tag{2}$$

Now suppose that we are given another three-component link L' which is a threefold simple cover of S^3 with branch set $B_{L'}$ and symplectic torus $T_{L'} = S^1 \times B_{L'}$. The covering projections p_L, $p_{L'}$ are determined by homomorphisms φ_L (or $\varphi_{L'}$) from $\pi_1(X \setminus T_L)$ (or $\pi_1(X \setminus T_{L'})$) to the symmetric group S_3 such that each meridian of T_L (or $T_{L'}$) is sent to a transposition.

Any isotopy of X taking T_L to $T_{L'}$ and which carries the covering data for p_L to that of $p_{L'}$ gives rise to

$$\begin{tikzcd} \widetilde{X}_L \arrow[r, "\tilde{f}\ \cong"] \arrow[d, "p_L"'] & \widetilde{X}_{L'} \arrow[d, "p_{L'}"] \\ X \arrow[r, "f\ \cong"'] & X \end{tikzcd} \tag{3}$$

where $f(T_L) = T_{L'}$ and f_* is the identity on homology.

Since $\tilde{f}_*(\mathcal{SW}_{\widetilde{X}_L}) = \mathcal{SW}_{\widetilde{X}_{L'}}$ it follows from (2) and (3) that

$$\Delta_L^{sym}(t^2, t^2, t^2) = \Delta_{L'}^{sym}(t^2, t^2, t^2)$$

In other words, $\nabla_L^{sym}(t^2) = \nabla_{L'}^{sym}(t^2)$. Using Kanenobu's theorem, one sees that there are infinite families of fibered links $\{L_i\}$ whose $\nabla_{L_i}^{sym}(t)$ are distinct and have arbitrary fixed even degree (> 0). The genus g_L of the fibered link L is half the degree of its Hosokawa polynomial. (See, for instance, [2].) Furthermore, the fiber of L is the thrice-punctured surface which is a simple threefold branched cover of D^2 (a normal fiber to $S^1 \times \{0\}$) with m branch points. Thus the number of strands m of B_L is determined by $m = 2g_L + 4$.

This means that for any even $m \geq 6$ we get an infinite family $\{T_i\}$ of symplectic tori homologous to mT with distinct threefold simple branched covers. Note that each braided torus T_L admits at most finitely many simple threefold branched covers of X with T_L as branch set, since there are finitely many distinct homomorphisms $\pi_1(X \setminus T_L) \to S_3$. Thus we have:

Theorem 3.1 *Let X be a simply connected symplectic 4–manifold containing an embedded symplectic torus T of self-intersection 0. Then for each even $m \geq 6$ there are infinitely many pairwise nonsmoothly isotopic embedded symplectic tori homologous to mT.* □

4 Fiber sums

We begin this section with the same hypotheses as the last: We are given a simply connected symplectic 4–manifold X containing an embedded symplectic torus T of self-intersection 0. The construction of new symplectic tori is similar to that of the last section (and of [6]). For each $m \geq 2$ consider closed braids B with m strands. Then the braided torus $T_B = S^1 \times B$ is embedded in the tubular neighborhood $S^1 \times (S^1 \times D^2)$ of $T = S^1 \times S^1 \times \{0\}$. Furthermore, T_B is symplectic and homologous to mT.

Suppose that B and B' are m–strand closed braids and that $T_{B'}$ is smoothly isotopic to T_B in X. Then there is a diffeomorphism $f\colon X \to X$ satisfying: $f(T_B) = T_{B'}$, $f(\mu_B) = \mu_{B'}$, and $f_* = \mathrm{id}$ on $H_*(X)$. (Here μ_B and $\mu_{B'}$ are meridians to the braids; so they also may be viewed as meridians to the tori T_B and $T_{B'}$.)

Our goal is to use relative Seiberg–Witten invariants $\mathcal{SW}_{(X,T_B)}$ to distinguish the tori T_B up to isotopy. Let $E(1)$ denote the rational elliptic surface. Because of the gluing theorems of [17, 14] and the fact that the relative Seiberg–Witten invariant of $E(1)$ minus a a smooth elliptic fiber is $\mathcal{SW}_{E(1)\setminus F} = 1$ (see eg [10]), the relative Seiberg–Witten invariant of (X, T_B) may be expressed as the absolute Seiberg–Witten invariant of the fiber sum of X and $E(1)$ along T_B and F:

$$\mathcal{SW}_{(X,T_B)} = \mathcal{SW}_{X\#_{T_B=F}E(1)}$$

Now write $N(T_B)$ for a tubular neighborhood of T_B in X and also write $N(T) = S^1 \times (S^1 \times D^2)$, the original tubular neighborhood of T. We have

$$X \setminus N(T_B) = \big(X \setminus N(T)\big) \cup \big(S^1 \times \big((S^1 \times D^2) \setminus N(B)\big)\big) \tag{4}$$

Let L_B be the link in S^3 consisting of the closed braid B together with its axis A. If μ_A denotes a meridian to A, then T is homologous to $S^1 \times \mu_A$. Let $t = t_T$ denote the corresponding element in $\mathbb{Z}H_2(X)$.

We may now rewrite (4) as

$$X \setminus N(T_B) = (X \setminus N(T)) \cup \big(S^1 \times (S^3 \setminus N(L_B))\big)$$

The manifold $X\#_{T_B=F}E(1)$ is obtained from the same components as link surgery using the link L_B and the manifolds $(E(1), F)$ and (X, T); however the gluings are not necessarily those specified in section 2. Since $E(1)$ has big diffeomorphism group with respect to F (see eg [8]), each diffeomorphism $\partial N(F) \to \partial N(F)$ extends to a self-diffeomorphism of $E(1) \setminus N(F)$; so the diffeomorphism used to glue in $E(1) \setminus N(F)$ is inconsequential. However, it is useful to demand that the fiber F of $E(1)$ should be identified with $S^1 \times \lambda_B$ where λ_B is the longitude of B in S^3.

According to [17, 14], $\mathcal{SW}_X \cdot (t - t^{-1})$ is the relative Seiberg–Witten invariant of (X, T), and by [5], as described in section 2, the relative invariant of the manifold $S^1 \times (S^3 \setminus N(L_B))$ is $\Delta^{sym}_{L_B}(t^2, \tau^2)$. Applying [5] and [17] we obtain:

$$\mathcal{SW}_{(X,T_B)} = \mathcal{SW}_{X\#_{T_B=F}E(1)} = \Delta^{sym}_{L_B}(t^2, \tau^2) \cdot \mathcal{SW}_X \cdot (t - t^{-1})$$

where τ is the element of $\mathbb{Z}H_2(X)$ corresponding to $[S^1 \times \mu_B]$. Since $[F] = [S^1 \times \lambda_B] = m[S^1 \times \mu_A] = m[T]$. When applying this formula, we need to remember that $t_T = t$ and $t_F = t^m$.

Theorem 4.1 *Let X be a simply connected symplectic 4–manifold with $b_2^+ > 1$ containing an embedded symplectic torus T of self-intersection 0. For a fixed integer $m \geq 2$, let B and B' be closed m-strand braids in S^3. Then T_B and $T_{B'}$ are embedded symplectic tori in X which are homologous to mT. If there is an isotopy of X taking T_B to $T_{B'}$, then $\Delta^{sym}_{L_{B'}}(t^2, \tau'^2) = \Delta^{sym}_{L_B}(t^2, \tau^{\pm 2})$.*

Proof We first describe $H_2(X\#_{T_B=F}E(1))$. Let R_B denote the group of rim tori of the torus T_B; ; ie $R_B = \ker(H_2(X \setminus T_B) \to H_2(X)) \cong \mathbb{Z} \oplus \mathbb{Z}$. A basis for R_B is given by $\tau = [S^1 \times \mu_B]$ and $v = [\lambda_B \times \mu_B]$ where λ_B is the longitude of the knot B in S^3. The classes τ and v are primitive (because of the definition of R_B), thus there is a group $D_B \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by the dual classes to τ and v in $H_2(X\#_{T_B=F}E(1))$.

Let $A = T_B^{\perp} = T^{\perp} \subset H_2(X)$. Note that the adjunction inequality (1) implies that no basic class of X has nontrivial intersection with $[T]$. Thus $\mathcal{SW}_X \in \mathbb{Z}A$. We have $H_2(X \setminus T_B) = A \oplus R_B$. Finally, suppose that $[T]$ is n times a primitive homology class, and let S denote the class in $H_2(X\#_{T_B=F}E(1))$ which has a representative built from mn punctured sections in $E(1) \setminus F$ and a surface in $X \setminus T_B$ which has boundary mn copies of the meridian μ_B to T_B.

A Mayer–Vietoris argument shows that the homology of $X\#_{T_B=F}E(1)$ splits as

$$H_2(X\#_{T_B=F}E(1)) = A \oplus (R_B \oplus D_B) \oplus \mathbb{Z}(S) \oplus E_8$$

where the E_8 comes from $H_2(E(1) \setminus F)$. There is a similar splitting of the homology of $H_2(X\#_{T_{B'}=F}E(1))$.

If there is an isotopy of T_B to $T_{B'}$, there is a diffeomorphism

$$\bar{f}\colon\ X\#_{T_B=F}E(1) \to X\#_{T_{B'}=F}E(1)$$

satisfying $\bar{f}_*|_A = \mathrm{id}$ and $\bar{f}_*(R_B) = R_{B'}$ (since $f(\mu_B) = \mu_{B'}$). Thus the induced homomorphism of group rings satisfies $\bar{f}_*(\mathcal{SW}_X) = \mathcal{SW}_X$ and $\bar{f}_*(t_F) = t_F$; ie $\bar{f}_*(t)^m = t^m$, and so $\bar{f}_*(t) = t$ because $H_2(X\#_{T_B=F}E(1))$ is torsion-free. It follows that the fact that $\bar{f}_*(\mathcal{SW}_{(X,T_B)}) = \mathcal{SW}_{(X,T_{B'})}$ implies that

$$\Delta^{sym}_{L_B}(t^2, \bar{f}_*(\tau)^2) = \Delta^{sym}_{L_{B'}}(t^2, \tau'^2) \tag{5}$$

Write $\bar{f}_*(\tau) = a\tau' + bv'$ (where $\tau' = [S^1 \times \mu_{B'}]$ and $v' = [\lambda_{B'} \times \mu_{B'}]$). Each term $nt^{2r'}\tau'^{2s'}$ of $\Delta^{sym}_{L_{B'}}(t^2, \tau'^2)$ corresponds to basic classes of $X\#_{T_{B'}=F}E(1)$ of the form $\alpha + (2r' \pm 1)[T] + 2s'\tau'$ where $\alpha \in H_2(X)$ is a basic class, and so $\alpha \cdot \tau' = 0$, and $\alpha \cdot [T] = 0$. Furthermore each class in $R_{B'}$ is orthogonal to T.

Terms of the form $nt^{2r}\bar{f}_*(\tau)^{2s}$ of $\Delta^{sym}_{L_{B'}}(t^2, \bar{f}_*(\tau)^2)$ correspond to basic classes of $X\#_{T_{B'}=F}E(1)$ of the form $\beta + (2r \pm 1)[T] + 2s(a\tau' + bv')$, and each basic class can be written like this. Since τ' and v' are independent, it is clear that $b = 0$. This means that $\bar{f}_*(\tau) = a\tau'$, and $a = \pm 1$ since τ is primitive. Thus $\Delta^{sym}_{L_{B'}}(t^2, \tau'^2) = \Delta^{sym}_{L_B}(t^2, \tau^{\pm 2})$. □

We have as a corollary:

Theorem 4.2 *Let X be a simply-connected symplectic 4–manifold satisfying $b_2^+(X) > 1$ and containing an embedded symplectic torus T of self-intersection 0. For each $m \geq 2$ there are infinitely many pairwise nonisotopic embedded symplectic tori in X which are homologous to mT.*

Proof This follows from the above theorem provided for each $m \geq 2$ there are infinitely many closed m-strand braids B whose 2-component links $L_B = A \cup B$ have distinct 2-variable Alexander polynomials. Such examples are given, for example, in the work of Etgu and Park [3]. □

5 Lagrangian tori

In this section we use branched covers as a means for constructing examples of Lagrangian tori in symplectic 4–manifolds whose homology classes are equal but which are not equivalent under symplectic diffeomorphisms. There are already two papers [20, 7] dealing with this phenomenon, and the invariants of [7] can be used to distinguish the examples given in this section. However we believe that the constructions below are interesting in their own right and are certainly different from those cited.

To begin, let K be the trefoil knot, and M_K the 3–manifold obtained from 0–framed surgery on S^3 along K. Since K is a genus–1 fibered knot, M_K fibers over the circle with fiber a torus, $M_K = T^2 \times_\varphi S^1$. Let $E(1)_K$ be the result of knot surgery on $E(1)$, $E(1)_K = E(1)\#_{F=S^1\times m_0} S^1 \times M_K$, where m_0 is a meridian to K. This manifold has a symplectic structure induced from that on $E(1)$ and the structure on $S^1 \times M_K$ in which the fiber and section are symplectic submanifolds. (See [5].)

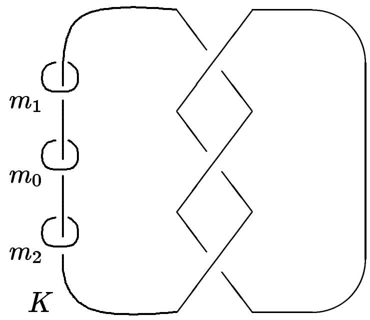

Figure 2

Let m_1 and m_2 be meridians of K as in Figure 2, and let X denote the double branched cover of $E(1)_K$ with branch set $S^1 \times (m_1 \cup m_2)$. Since $S^1 \times m_i$ is a section to the fibration $S^1 \times M_K \to S^1 \times S^1$, the branch set of this cover is symplectic, hence X inherits a symplectic structure. We have

$$X = E(1)' \#_{F'=S^1\times m_0'} S^1 \times \widetilde{M}_K \#_{S^1\times m_0''=F''} E(1)''$$

where $E(1)'$ and $E(1)''$ are copies of $E(1)$ and $\widetilde{M}_K$ is the double cover of M_K branched over $m_1 \cup m_2$. It follows that $\widetilde{M}_K$ also fibers over the circle, and its fiber is the double branched cover of the fiber of M_K, branched over two points. Thus the fiber of $\widetilde{M}_K \to S^1$ has genus 2. We can say more:

Lemma 5.1 *Let K be any knot in S^3 and M_K the result of 0–surgery along K. The double cover of M_K branched over two meridians to K is $M_{K\#K}$, the result of 0–surgery on S^3 along the connected sum $K\#K$.*

Proof This proof is an exercise in Kirby calculus. The double branched cover of S^3 branched over the two-component unlink is $S^2 \times S^1$. This means that the double branched cover of M_K branched along two meridians to K is the result of surgery on the lift of K in $S^2 \times S^1$. (See Figure 3.) Note that K lifts to two components. Referring to Figure 3, slide one copy of K over the other copy of K to obtain Figure 4. In this figure, 0–surgery on K together with 0–surgery on a meridian form a cancelling pair. We are left with 0–surgery on $K\#K$. □

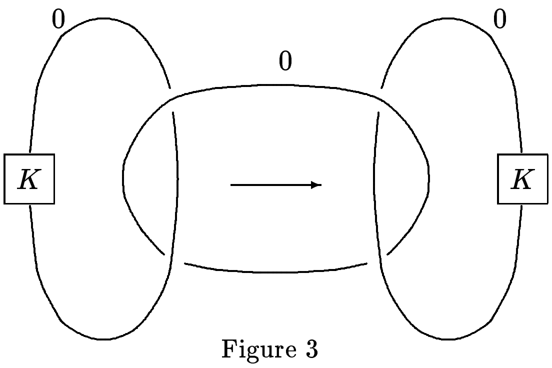

Figure 3

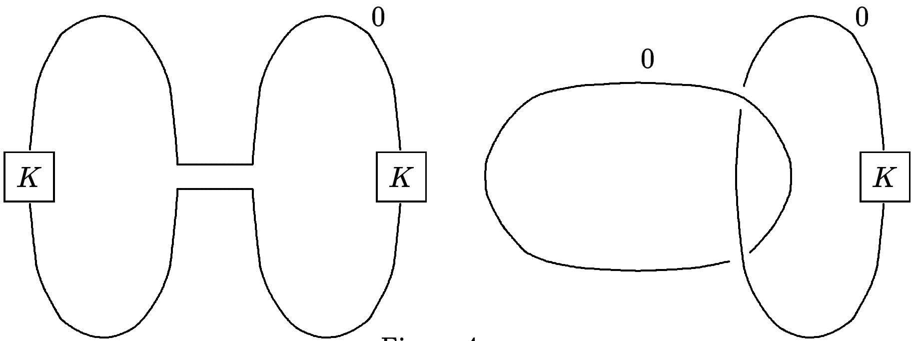

Figure 4

Since $X = E(1)'\#_{F'=S^1\times m_0'} S^1\times M_{K\#K}\#_{S^1\times m_0''=F''} E(1)''$, and the complement of a fiber in $E(1)$ is simply-connected, we have

$$\pi_1(X) = \pi_1(S^1\times m_0'\times\mu')\backslash\pi_1(S^1\times(M_{K\#K}\setminus(m_0'\cup m_0''))/\pi_1(S^1\times m_0''\times\mu'')$$

where μ' and μ'' are the meridians to m_0' and m_0''. The group in the middle, $\pi_1(S^1\times M_{K\#K}\setminus(m_0'\cup m_0''))$, is normally generated by the classes of $S^1\times\mathrm{pt}$, any meridian to $K\#K$, and by μ' and μ''. These loops all lie on $S^1\times m_0'\times\mu'$ or $S^1\times m_0''\times\mu''$; so we see that X is simply-connected.

Consider the paths P and P' shown in Figure 5, each running from a point $y_1\in m_1$ to $y_2\in m_2$. These paths lie in a fiber of the fibration of $S^1\times M_K$ over $S^1\times S^1$; thus the construction of the symplectic structure (essentially 'area form on base' plus 'area form on fiber') implies that the surfaces $S^1\times P$ and $S^1\times P'$ are Lagrangian.

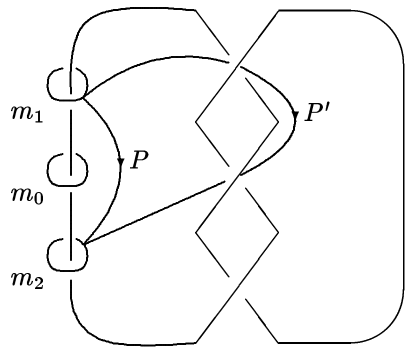

Figure 5

Since the endpoints of P and P' lie in the branch set of the cover, their lifts γ and γ' in $\widetilde{M}_K = M_{K\#K}$ are circles in the fibers (which are genus–2 surfaces). We thus obtain Lagrangian tori $T = T_\gamma = S^1 \times \gamma$ and $T' = T'_\gamma = S^1 \times \gamma'$ in $S^1 \times M_{K\#K}$. These tori are disjoint from the lifts of m_0, where the gluing in the construction of X takes place, so T and T' are Lagrangian tori in X.

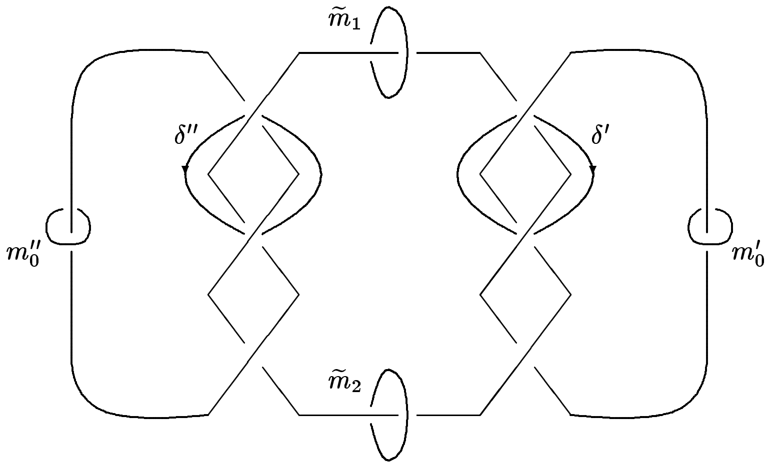

Figure 6

The meridian m_0 in M_K lifts to a pair of meridians, m'_0, m''_0 in $M_{K\#K}$ as in Figure 6. $H_1(M_{K\#K} \setminus (m'_0 \cup m''_0)) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by $[m'_0] = [m''_0]$ and by the classes of the meridians $[\mu'] = [\mu'']$ to m'_0 and m''_0. (Note that $M_{K\#K} \setminus (m'_0 \cup m''_0)$ is fibered over the circle and its fibers are genus 2 surfaces with two boundary components. The meridians μ' and μ'' form the boundary of one fiber.)

Referring to Figure 6, in $H_1(M_{K\#K} \setminus (m_0' \cup m_0''))$ we have $[\gamma'] - [\gamma] = [\delta'] + [\delta'']$. Because δ' and δ'' link neither m_0' (or m_0'') nor μ' (or μ''), the loops δ' and δ'' are nullhomologous in $M_{K\#K} \setminus (m_0' \cup m_0'')$. This means that the corresponding Lagrangian tori, $\Sigma' = S^1 \times \delta'$ and $\Sigma'' = S^1 \times \delta''$ are nullhomologous in X. Since $T' - T$ is homologous to $\Sigma' + \Sigma''$, we see that T and T' are homologous in X.

The loop γ is a separating curve in the fiber of $M_{K\#K} \to S^1$. See Figure 7. We see that γ is homologous to μ' in the fiber of $M_{K\#K} \setminus (m_0' \cup m_0'') \to S^1$. Thus in $S^1 \times M_{K\#K} \setminus (m_0' \cup m_0'')$, the Lagrangian torus $T = S^1 \times \gamma$ is homologous to $R_{\mu'} = S^1 \times \mu'$. But $R_{\mu'}$ is a rim torus to $S^1 \times m_0'$, one of the tori along which the fiber sum

$$X = E(1)' \#_{F' = S^1 \times m_0'} S^1 \times M_{K\#K} \#_{S^1 \times m_0' = F''} E(1)''$$

is made. In such a fiber sum, the rim tori give essential homology classes – thus we see that the Lagrangian tori T and T' are essential in X.

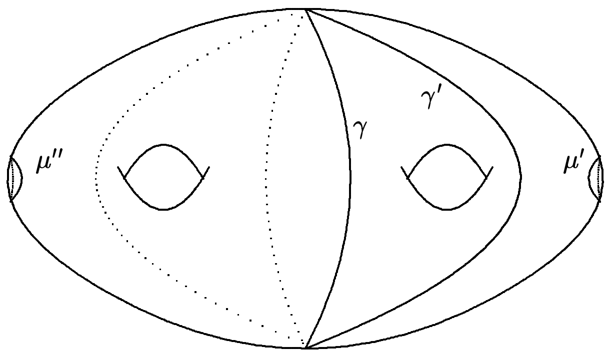

Figure 7

We claim that there is no diffeomorphism of X which takes T to T'. To see this we shall use an invariant obtained from Seiberg–Witten theory. To do this we need the notion of 'surgery on T'. As usual, this means the result of removing a tubular neighborhood $N(T) \cong T^2 \times D^2$ and regluing it.

$$X(T, \psi) = (X \setminus N(T)) \cup_\psi (T^2 \times D^2)$$

The key quantity in this operation is the class $\omega \in H_1(\partial N(T))$ which is killed by the composition of $\psi\colon\ \partial N(T) \to T^2 \times \partial D^2$ with the inclusion $T^2 \times \partial D^2 \to T^2 \times D^2$. This class determines $X(T, \psi)$ up to diffeomorphism; so we write $X_T(\omega)$ instead of $X(T, \psi)$.

Note that if there is a diffeomorphism f of X taking T to T', then each manifold $X_T(\omega)$ corresponds to a unique $X_{T'}(f_*(\omega))$. Thus the collection of all manifolds

$$\{X_T(\omega) \mid \omega \in H_2(\partial N(T))\}$$

is a diffeomorphism invariant of (X,T). Our invariant $I(X,T)$, defined and computed below, will be the set of Seiberg–Witten invariants of these manifolds.

6 Product formulas for the Seiberg–Witten invariant

Before formally defining $I(X,T)$ we need to discuss techniques for calculating the Seiberg–Witten invariants of the manifolds $X_T(\omega)$. Fix simple loops α, β, δ on $\partial N(T)$ whose homology classes generate $H_1(\partial N(T))$. If $\omega = p\alpha + q\beta + r\delta$ write $X_T(p,q,r)$ instead of $X_T(\omega)$. An important formula for calculating the Seiberg–Witten invariants of surgeries on tori is due to Morgan, Mrowka, and Szabo [13] (see also [11], [17]). Given a class $k \in H_2(X)$:

$$\begin{aligned}\sum_i \mathrm{SW}_{X_T(p,q,r)}(k_{(p,q,r)} + 2i[T]) &= p \sum_i \mathrm{SW}_{X_T(1,0,0)}(k_{(1,0,0)} + 2i[T]) \\ &+ q \sum_i \mathrm{SW}_{X_T(0,1,0)}(k_{(0,1,0)} + 2i[T]) + r \sum_i \mathrm{SW}_{X_T(0,0,1)}(k_{(0,0,1)} + 2i[T]) \qquad (6)\end{aligned}$$

In this formula, T denotes the torus which is the core $T^2 \times 0 \subset T^2 \times D^2$ in each specific manifold $X(a,b,c)$ in the formula, and $k_{(a,b,c)} \in H_2(X_T(a,b,c))$ is any class which agrees with the restriction of k in $H_2(X \setminus T \times D^2, \partial)$ in the diagram:

$$\begin{array}{ccc} H_2(X_T(a,b,c)) & \longrightarrow & H_2(X_T(a,b,c), T \times D^2) \\ & & \downarrow \cong \\ & & H_2(X \setminus T \times D^2, \partial) \\ & & \uparrow \cong \\ H_2(X) & \longrightarrow & H_2(X, T \times D^2) \end{array}$$

Let $\pi(a,b,c)\colon H_2(X_T(a,b,c)) \to H_2(X \setminus T \times D^2, \partial)$ be the composition of maps in the above diagram, and $\pi(a,b,c)_*$ the induced map of integral group rings. Since we are interested in invariants of the pair (X,T), we shall work with

$$\overline{\mathcal{SW}}_{(X_T(a,b,c),T)} = \pi(a,b,c)_*(\mathcal{SW}_{X_T(a,b,c)}) \in \mathbb{Z}H_2(X \setminus T \times D^2, \partial).$$

The indeterminacy due to the sum in (6) is caused by multiples of $[T]$; so passing to $\overline{\mathcal{SW}}$ removes this indeterminacy, and the Morgan–Mrowka–Szabo formula becomes

$$\overline{\mathcal{SW}}_{(X_T(p,q,r),T)} = p\overline{\mathcal{SW}}_{(X_T(1,0,0),T)} + q\overline{\mathcal{SW}}_{(X_T(0,1,0),T)} + r\overline{\mathcal{SW}}_{(X_T(0,0,1),T)}. \quad (7)$$

Proposition 6.1 *The collection of Seiberg–Witten invariants*

$$I(X,T) = \{\overline{\mathcal{SW}}_{X_T(a,b,c)} | \, a,b,c \in \mathbb{Z}\}$$

is an orientation-preserving diffeomorphism invariant of the pair (X,T). □

7 Calculation of $I(X,T)$: $X(0,1,0)$

We first specify a basis for $H_1(\partial N(T))$ as follows: Recall that $T = S^1 \times \gamma$ where γ lies in a fiber of the fibration $M_{K\#K} \setminus (m_0' \cup m_0'') \to S^1$ (Figure 7). Then $N(T)$ may be identified with $S^1 \times \gamma \times D^2$, and we take the basis $\alpha = [S^1 \times \mathrm{pt} \times \mathrm{pt}]$, $\beta = [\mathrm{pt} \times \gamma_L]$, where γ_L is a pushoff of γ in the fiber of the fibration $M_{K\#K} \setminus (m_0' \cup m_0'') \to S^1$ (this is called the 'Lagrangian framing' in [7]), and $\delta = [m_T]$, where $m_T = \mathrm{pt} \times \mathrm{pt} \times \partial D^2$, the meridian to T. It is then clear from (7) that in order to calculate $I(X,T)$, one needs to calculate $\overline{\mathcal{SW}}_Y$ for $Y = X_T(1,0,0)$, $X_T(0,1,0)$, and $X_T(0,0,1)$; however, from our choice of basis, we have $X_T(0,0,1) \cong X$. This leaves us with two invariants to calculate below. For a different approach to these calculations see [7].

The calculation of $\mathcal{SW}_{X_T(0,1,0)}$ depends on some basic facts about double covers of 3–manifolds branched over closed braids. Suppose that B is a braid in a solid torus with $2m$ strands, ie $B \subset S^1 \times D^2 \subset M^3$ is a link such that each disk $\mathrm{pt} \times D^2$ intersects B in exactly $2m$ points. There is then a double cover $Y_B \to M^3$ branched over B for which each meridian to B is covered nontrivially, and this cover is trivial outside the solid torus.

The pertinent question is: 'What is the effect on Y_B of putting half-twists into the braid B?' In other words, suppose that ζ is an arc in $(\mathrm{pt} \times D^2) \subset S^1 \times D^2$ whose endpoints lie on B, but so that no other point of ζ is on B. Then we can put half-twists in B by twisting in a small neighborhood of ζ. Figure 8 shows a local picture.

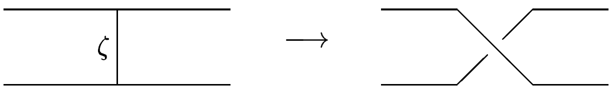

Figure 8

In the double cover, Y_B, the solid torus $S^1 \times D^2$ lifts to a bundle V over the circle with fiber the double cover of D^2 branched over $2m$ points, a twice-punctured surface S of genus $m-1$. The path ζ lifts to a simple closed loop $\widetilde{\zeta} \subset S \subset Y_B$, and changing B by a half-twist of along ζ as described corresponds to changing the monodromy of the lifted bundle by a single Dehn twist along $\widetilde{\zeta}$. (This is true essentially because each half-twist along ζ lifts to a full twist in the double cover.)

Thus if B' is the braid with the new positive half-twist, then its corresponding double branched cover, $Y_{B'}$, is obtained from Y_B by cutting out V and replacing it with the bundle over S^1 with fiber S but whose monodromy is the monodromy of V composed with a Dehn twist about $\widetilde{\zeta}$. This means that $Y_{B'}$ is obtained by $(+1)$-Dehn surgery on $\widetilde{\zeta}$ with respect to the 0–framing given by the pushoff of $\widetilde{\zeta}$ in the fiber S of V. (For example, see [1].)

Proposition 7.1 *The result of 0–surgery on $\widetilde{\zeta}$ in Y_B is the double cover of M^3 branched along the link obtained from B by the operation of Figure 9.*

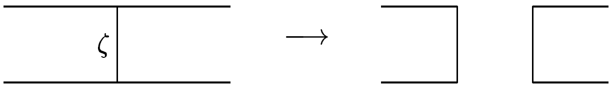

Figure 9

Proof If we restrict the deck transformation $\tau\colon Y_B \to Y_B$ of the branched cover to an annular neighborhood of $\widetilde{\zeta}$ in a fiber S of V, then we see an annulus double covering a disk with two branch points. Identify a neighborhood of $\widetilde{\zeta}$ in Y_B with $\widetilde{\zeta} \times I \times I$ where $I=[-1,1]$. The restriction of τ to this neighborhood is equivalent to $\tau(z,s,t)=(\bar{z},-s,t)$, and its fixed point set consists of two arcs $\{(\pm 1,0)\} \times I$ (identifying $\widetilde{\zeta}$ with S^1). If we now change coordinates so that $I \times I$ becomes $D^2 \subset \mathbf{C}$, then we get $\tau(z,w)=(\bar{z},\rho w)$, where ρ is reflection in the imaginary axis, and the fixed set is $\{\pm 1\} \times \{\text{the imaginary axis} \cap D^2\}$.

According to our framing convention, 0–surgery on $\widetilde{\zeta}$ is the one that kills the homology class of a pushoff of $\widetilde{\zeta}$ in S, ie the class of $\widetilde{\zeta} \times \mathrm{pt}$ in $\widetilde{\zeta} \times D^2$. Thus the result of 0–surgery is $Z = Y_B \setminus (\widetilde{\zeta} \times D^2) \cup_\varphi (S^1 \times D^2)$ where $\varphi_*[\mathrm{pt} \times \partial D^2] = [\widetilde{\zeta} \times \mathrm{pt}]$. Such a map φ is given by $\varphi(z,w)=(w,z)$. Define the involution σ

on $S^1\times D^2$ by $\sigma(z,w)=(\rho(z),\bar w)$. Then we see that the diagram

$$\begin{array}{ccc} S^1\times\partial D^2 & \xrightarrow{\ \sigma\ } & S^1\times\partial D^2 \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi} \\ \widetilde{\zeta}\times\partial D^2 & \xrightarrow[\ \tau\ ]{} & \widetilde{\zeta}\times\partial D^2 \end{array} \tag{8}$$

commutes. Thus, the restriction of τ to $Y_B\setminus(\widetilde{\zeta}\times D^2)$ extends to an involution τ' over all of Z via σ.

On the solid torus $S^1\times D^2$ the fixed set of $\tau'=\sigma$ is $\{\pm i\}\times\{\text{the real axis}\cap D^2\}$. Thus the picture in the quotient is exactly that of Figure 9. □

We now apply this proposition to the case at hand, where the 3–manifold is M_K, the braid is the trivial braid with components m_1 and m_2, and the arc ζ is the path P of Figure 5. It follows that $X_T(0,1,0)$ is the double branched cover of $E(1)_K$ with branch set $S^1\times C$ where C is the loop shown in Figure 10. Notice that C is an unknotted circle which is unlinked from K.

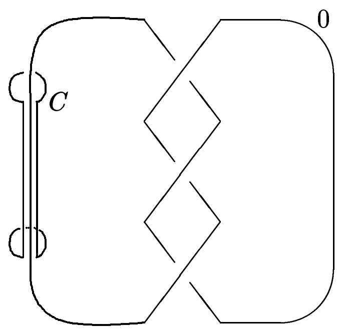

Figure 10

The double cover of a 3–ball, branched over an unknot, is $S^2\times I$ so it follows that the double cover of M_K branched over C is $M_K\#M_K$. Thus

$$X_T(0,1,0)=E(1)\#_F\left(S^1\times(M_K\#M_K)\right)\#_F E(1)$$

This means that $X_T(0,1,0)$ is split by $S^1\times S^2$ with b_2^+ positive on each side. It follows that $\mathcal{SW}_{X_T(0,1,0)}=0$.

Next we need to make a similar calculation for $X_{T'}(0,1,0)$. This time the arc ζ is the path P' of Figure 5, and $X_{T'}(0,1,0)$ is the double branched cover of $E(1)_K$ with branch set $S^1\times C'$ where C' is the loop shown in Figure 11.

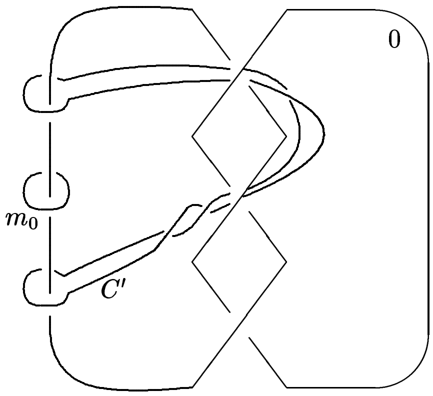

Figure 11

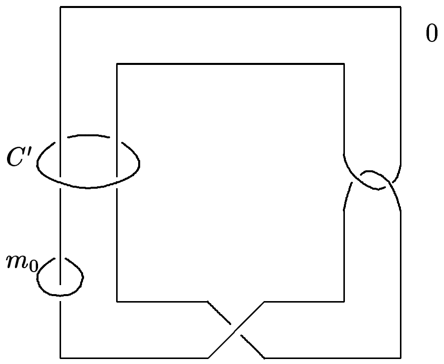

Figure 12

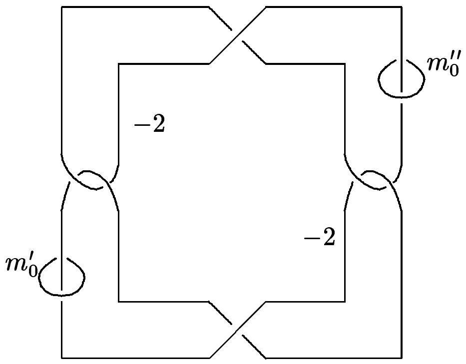

Figure 13

An isotopy of Figure 11 gives Figure 12, and the corresponding double branched cover is shown in Figure 13. (For techniques for determining this double cover, see [15].) Using Fox calculus, one calculates the torsion of the link $m_0' \cup m_0''$ in the double cover $(M_K)_{C'}$ (the 3–manifold shown in Figure 13). This torsion is 1. According to [11] this means that the Seiberg–Witten invariant of $S^1 \times ((M_K)_{C'} \setminus (m_0' \cup m_0''))$ is 1. Since the Seiberg–Witten invariant of $E(1) \setminus F$ is also equal to 1, the gluing theorem of Taubes [17] tells us that the Seiberg–Witten invariant of $E(1)\#_{F=S^1\times m_0'}(S^1 \times (M_K)_{C'})\#_{S^1\times m_0'=F}E(1)$ is equal to 1. However, we have just shown that this manifold is $X_{T'}(0,1,0)$.

Proposition 7.2 *For the nullhomologous Lagrangian tori T, T' in X, we have*

$$\mathcal{SW}_{X_T(0,1,0)} = 0, \quad \mathcal{SW}_{X_{T'}(0,1,0)} = 1$$

8 Calculation of $I(X,T)$: $X(1,0,0)$

The key calculation of this section will show that the Seiberg–Witten invariants of the manifolds $X_T(1,0,0)$ and $X_{T'}(1,0,0)$ vanish. Our approach here is to describe the surgered manifolds in terms of a branched covering. (It would be useful to compare with [7], where a more general approach is utilized.)

Proposition 8.1 *Let $\bar{\gamma}$ denote either γ or γ', and let Z be the result of the surgery on $T = S^1 \times \bar{\gamma} \subset X$ which kills $S^1 \times pt \times pt \subset S^1 \times \bar{\gamma} \times \partial D^2 \subset S^1 \times (M_{K\#K} \setminus (m_0' \cup m_0''))$. Then Z is the double branched cover of the manifold W obtained from $E(1)_K$ by a surgery on a circle $S^1 \times \{\text{point on } \bar{\gamma}\}$ (trading a neighborhood $S^1 \times D^3$ for $D^2 \times S^2$). The branch set in W of this cover consists of a pair of disjoint 2–spheres of self-intersection 0.*

Proof As we have seen in the proof of Proposition 7.1, the deck transformation of $X \to E(1)_K$ in a neighborhood $S^1 \times \bar{\gamma} \times D^2$ of T is given by $\tau(t,z,w) = (t,\bar{z},\rho(w))$ where ρ is reflection through the imaginary axis. The manifold Z is:

$$Z = \big(X \setminus (S^1 \times \bar{\gamma} \times D^2)\big) \cup_\vartheta (S^1 \times S^1 \times D^2)$$

where $\vartheta(t,z,w) = (w,z,t)$. Then the diagram

$$\begin{array}{ccc} S^1 \times S^1 \times \partial D^2 & \xrightarrow{\ v\ } & S^1 \times S^1 \times \partial D^2 \\ {\scriptstyle\vartheta}\Big\downarrow & & \Big\downarrow{\scriptstyle\vartheta} \\ S^1 \times \bar{\gamma} \times \partial D^2 & \xrightarrow[\tau]{} & S^1 \times \bar{\gamma} \times \partial D^2 \end{array} \tag{9}$$

commutes, where $v(t,z,w) = (\rho(t),\bar{z},w)$. Thus v extends the deck transformation τ over the surgered manifold Z.

The quotient $(S^1 \times S^1 \times D^2)/\tau \cong S^1 \times (D^2 \times I) \cong S^1 \times D^3$, but $(S^1 \times S^1 \times D^2)/v \cong S^2 \times D^2$, since the action of v restricted to $S^1 \times S^1 \times \{\mathrm{pt}\}$ is equivalent to the action of the deck transformation of the double covering $T^2 \to S^2$ with four branch points. Thus the effect of the surgery on the base is to perform surgery on the circle $S^1 \times \{\mathrm{pt}\} \subset S^1 \times D^3$. Before performing the surgery, the branch set consists of two tori. Since the fixed point set of v on $S^1 \times S^1 \times D^2$ is $\{\pm i\} \times \{\pm 1\} \times D^2$, the surgery trades a pair of annuli for four disks. Removing the annuli leaves us with a pair of complementary annuli in the branch set, and the addition of the four disks caps them off, giving a pair of 2–spheres.

To see that the components of the branch set of W have self-intersection 0, first consider the branch torus $S^1 \times m_1$ of X. Write $m_1 = J_1 \cup J_2$, the union of two intervals meeting only at their endpoints. We do this so that the intersection of $S^1 \times m_1$ with $(S^1 \times S^1 \times D^2)/\tau \cong S^1 \times D^3$ is $S^1 \times J_2$ where $J_2 \cap \partial D^3 = \partial J_2$. Then the corresponding component of the branch set in W is $(S^1 \times J_1) \cup (D^2 \times \partial J_2)$. In X we can isotop $S^1 \times m_1$ slightly by moving m_1 to $\bar{m}_1 = I_1 \cup I_2$ in M_K; so that $\bar{m}_1 \cap m_1 = \emptyset$, $\bar{m}_1 \cap D^3 = I_2$, and $I_2 \cap \partial D^3 = \partial I_2$. Then in W, $(S^1 \times I_1) \cup (D^2 \times \partial I_2)$ is disjoint from $(S^1 \times J_1) \cup (D^2 \times \partial J_2)$. □

Let Γ_i denote the components of the branch set in Z. The Γ_i are also 2–spheres of self-intersection 0.

In X there is a 'section class' C which arises from the sections of the elliptic fibrations on the copies of $E(1)$. To build a representative for C, start with a fixed Seifert surface B_0 of $K\#K$ whch is a fiber of the fibration of $S^3 \setminus K\#K \to S^1$. The boundary of B_0 is capped off by the 2–disk introduced when we do 0–surgery on $K\#K$ to form $M_{K\#K}$. The tori $S^1 \times m_0'$ and $S^1 \times m_0''$ which are identified with fibers of $E(1)'$ and $E(1)''$ each intersect $\{\mathrm{pt}\} \times \mathrm{B}_0$ in a single point. Remove disks in $\{\mathrm{pt}\} \times \mathrm{B}_0$ about each of these points. The boundaries then bound disks of self-intersection -1, sections of $E(1)$ minus the neighborhood of a fiber. The union of these surfaces gives a genus–2 surface of self-intersection -2 representing C.

The loop $\bar{\gamma} \subset X$ is contained in a Seifert surface for $K\#K$, and we may assume that it is disjoint from B_0. Thus the surgery torus, $S^1 \times \bar{\gamma}$ is disjoint from C. Since $C \cdot (S^1 \times m_1) = C \cdot F' = 1$, after surgery in Z, we still have $C \cdot \Gamma_1 = 1$; so Γ_1 is an essential 2–sphere in Z (and similarly for Γ_2). Thus Z contains an essential 2–sphere of self-intersection 0, and that means that $\mathcal{SW}_Z = 0$ [4].

Theorem 8.2 *T and T' are essential and homologous Lagrangian tori of X; however, there is no orientation-preserving diffeomorphism f of X with $f(T) = T'$.*

Proof We have $\mathcal{SW}_{X_T(1,0,0)} = 0$, $\mathcal{SW}_{X_T(0,1,0)} = 0$, and, since $X_T(0,0,1) = X$, $\mathcal{SW}_{X_T(0,0,1)} = (t_F^2 - 1 + t_F^{-2})^2$. Hence $I(X,T) = \{r(t_F^2 - 1 + t_F^{-2})^2 | \, r \in \mathbb{Z}\}$. On the other hand, $\mathcal{SW}_{X_{T'}(1,0,0)} = 0$, $\mathcal{SW}_{X_{T'}(0,1,0)} = 1$, and $X_{T'}(0,0,1) = X$; so $I(X,T') = \{q + r(t_F^2 - 1 + t_F^{-2})^2 | \, q, r \in \mathbb{Z}\}$. This concludes the proof since $I(X,T)$ is an orientation-preserving diffeomorphism invariant of (X,T). □

Auroux, Donaldson, and Katzarkov have shown in [1] that the surgery manifolds $X_T(0,k,1)$ and $X_{T'}(0,k,1)$ are symplectic for all $k \in \mathbb{Z}$. The corresponding Seiberg–Witten invariants are $\mathcal{SW}_{X_T(0,k,1)} = (t_F^2 - 1 + t_F^{-2})^2$ and $\mathcal{SW}_{X_{T'}(0,k,1)} = k + (t_F^2 - 1 + t_F^{-2})^2$. Note that the leading coefficient of these polynomials is ± 1, as required by Taubes' theorem.

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