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Homological representations of the Iwahori–Hecke algebra

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Abstract Representations of the Iwahori–Hecke algebra of type A_{n-1} are equivalent to representations of the braid group B_n for which the generators satisfy a certain quadratic relation. We show how to construct such representations from the natural action of B_n on the homology of configuration spaces of the punctured disk. We conjecture that all irreducible representations of $\mathcal{H}_n$ can be obtained in this way, even for non-generic values of q.

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Dedicated to Andrew Casson, whose commitment to accuracy and elegance has greatly influenced my work

1 Introduction

This paper is concerned with the representation theory of the braid group B_n and the Iwahori–Hecke algebra $\mathcal{H}_n$. The simplest and least motivated definition of these objects is by generators and relations. The braid group B_n is the group generated by $\sigma_1, \dots, \sigma_{n-1}$ with defining relations

- $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$,
- $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ if $|i-j| = 1$.

Let q be a unit in an integral domain K. The Iwahori–Hecke algebra $\mathcal{H}_n(K, q)$, or simply $\mathcal{H}_n$, is the K–algebra generated by $T_1, \dots, T_{n-1}$ with defining relations

- $T_iT_j = T_jT_i$ if $|i-j| > 1$,
- $T_iT_jT_i = T_jT_iT_j$ if $|i-j| = 1$,

- $(T_i + 1)(T_i - q) = 0$.

The Iwahori–Hecke algebra plays a central role in representation theory. If $q = 1$ then $\mathcal{H}_n$ is the group algebra KS_n of the symmetric group S_n. If K has characteristic zero and q is *generic* then $\mathcal{H}_n$ is isomorphic to KS_n. Here, generic means $q^i \neq 1$ for $i = 2, \dots, n$. Most research on $\mathcal{H}_n$ is now concerned with understanding the non-generic case.

There is an obvious map from B_n to $\mathcal{H}_n$ given by $\sigma_i \mapsto T_i$. Thus the representations of $\mathcal{H}_n$ are equivalent to those representations of B_n for which the generators satisfy $(\sigma_i + 1)(\sigma_i - q) = 0$. The aim of this paper is to describe a topological method of constructing such representations.

The main construction is given in Section 2. Briefly, the idea is as follows. A braid can be represented by a homeomorphism from the punctured disk D_n to itself. This induces a homeomorphism from a configuration space $C_m(D_n)$ to itself. This in turn induces an automorphism of a certain module obtained from $C_m(D_n)$ using homology theory.

The homology module is defined using a local coefficient system that depends on a representation of B_m. The main result of this paper, Theorem 5.1, states that if the representation of B_m satisfies the required quadratic relation then so does the resulting representation of B_n, up to some rescaling.

In preparation for this result, Section 3 studies a slightly simpler homological representation, and Section 4 uses Poincaré duality to define a Hermitian form that is preserved by the action of B_n. Section 6 gives a conjectural method to obtain all irreducible representations of Iwahori–Hecke algebras by starting from the trivial representation and inductively applying the construction from Section 2. We conclude with some examples in support of this conjecture, and some more open questions motivated by it.

Lawrence [6] has some results very similar to those in this paper. The construction and the outcome are almost identical to mine, although the method of proof is quite different. There are two main advantages to the approach in this paper. First, it explicitly identifies the representation as an image of a natural map from homology to relative homology, whereas [6] uses a less well motivated quotient. Second, it works even when q is a root of unity, whereas [6] needs to assume q is generic. On the other hand, [6] has the advantage of identifying precisely what representations are obtained, whereas this paper only does so conjecturally in Conjecture 6.1.

We use the following notation throughout this paper.

- D is the unit disk centered at 0 in the complex plane.
- $-1 < p_1 < \cdots < p_n < 1$ are distinct points on the real line.
- D_n is the n–times punctured disk $D \setminus \{p_1, \ldots, p_n\}$.

Then B_n is the group of homeomorphisms $f\colon D_n \to D_n$ such that $f|\partial D$ is the identity, taken up to isotopy relative to ∂D. The generator σ_i corresponds to a homeomorphism that exchanges p_i and p_{i+1} by a counterclockwise half twist.

We also use the definition of B_n as the fundamental group of a configuration space. If X is a surface, let $C_n(X)$ denote the configuration space of unordered n–tuples of distinct points in X. Then $B_n = \pi_1(C_n(D), \{p_1, \ldots, p_n\})$. The generator σ_i corresponds to a loop in which p_i and p_{i+1} switch places by a counterclockwise half twist, while the other points remain fixed.

The main results of this paper require K to be a field with a conjugation operation. However the construction in Section 2 and the results in Section 3 apply when K is any integral domain.

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2 Constructing a representation

In this section we describe a construction that produces a representation W of B_n from a representation V of B_m, where n and m are any non-negative integers. We leave open the possibility that W is the zero vector space, although this should not technically be called a representation of B_n. In Section 6 we give some examples and a conjecture suggesting that in interesting cases W will be in some sense more sophisticated than V.

Let B_m act by automorphisms on a non-zero K–module V. Let C be the configuration space $C_m(D_n)$. Let $c_0 \in C$ be a set of m points on ∂D. We will use the action of B_m on V to define an action of $\pi_1(C, c_0)$ on V. The construction does not require a deep understanding of $\pi_1(C, c_0)$. However we mention that $\pi_1(C, c_0)$ can be described succinctly as the subgroup of B_{n+m} consisting of geometric braids for which the first n strands are straight lines.

Let $f\colon C \to \mathbb{C} \setminus \{0\}$ be the map

$$f(\{z_1, \ldots, z_m\}) = \prod_{i=1}^{m} \prod_{j=1}^{n} (z_i - p_j).$$

Let $w\colon \pi_1(C,c_0) \to \mathbb{Z}$ be the map that takes a loop γ in C to the winding number of $f \circ \gamma$ around 0.

Let $i\colon C \to C_m(D)$ be the map induced by the inclusion $D_n \subset D$. Then i_* is a surjective homomorphism from $\pi_1(C,c_0)$ to B_m. Let $\pi_1(C)$ act on V by

$$g(v) = q^{w(g)} i_*(g)(v)$$

for all $g \in \pi_1(C)$ and $v \in V$. Let $\mathcal{L}$ be the V–bundle over C whose monodromy is given by this action.

Let $H_i(C;\mathcal{L})$ be the homology module with local coefficients given by $\mathcal{L}$. We recall briefly how this is defined. A singular chain is a formal sum of terms of the form $n\sigma$, where $\sigma\colon \Delta^i \to C$ is a singular i–simplex in C and $n\colon \Delta^i \to \mathcal{L}$ is a lifting of σ. The obvious boundary map gives rise to a chain complex, whose homology modules are $H_i(C;\mathcal{L})$. We will be particularly interested in the middle homology $H_m(C;\mathcal{L})$.

Relative homology with local coefficients is defined in the usual way. We also use *Borel–Moore* homology, defined by

$$H_m^{\mathrm{BM}}(C;\mathcal{L}) = \lim_{\leftarrow} H_m(C, C \setminus A; \mathcal{L}),$$

where the inverse limit is taken over all compact subsets A of C.

There is an action of the braid group B_n on $H_m(C;\mathcal{L})$ defined as follows. Let $f\colon D_n \to D_n$ be a homeomorphism that acts as the identity on ∂D. Then f induces a homeomorphism $f'\colon C \to C$ that fixes c_0. The induced automorphism f'_* of $\pi_1(C,c_0)$ is such that $f'_*(g)(v) = g(v)$ for all $g \in \pi_1(C,c_0)$ and $v \in V$. Thus there is a unique lift $\tilde{f}'\colon \mathcal{L} \to \mathcal{L}$ that acts as the identity on the fiber over c_0. This map induces an automorphism of $H_m(C;\mathcal{L})$. This automorphism depends only on the isotopy class of f relative to ∂D, so gives a well defined action of B_n on $H_m(C;\mathcal{L})$.

Let W be the image of the map from $H_m(C;\mathcal{L})$ to $H_m^{\mathrm{BM}}(C,\partial C;\mathcal{L})$ induced by inclusion. Then B_n acts on W. This completes our construction of a representation of B_n from a representation of B_m.

For improved readability, we suppress mention of $\mathcal{L}$ from now on, writing for example $H_m(C)$ instead of $H_m(C;\mathcal{L})$.

3 The structure of $H_m^{\mathrm{BM}}(C)$

Let q be a unit in an integral domain K, let V be a K–module with an action of B_m, and let C and $\mathcal{L}$ be as in Section 2. In this section we compute $H_m^{\mathrm{BM}}(C)$.

Our main interest will ultimately be in the representation W, but $H_m^{\mathrm{BM}}(C)$ is easier to understand.

Definition Let Π be the set of sequences $\pi = (\pi_1, \dots, \pi_{n-1})$ such that π_i is a non-negative integer and $\sum \pi_i = m$.

For $\pi \in \Pi$, let U_π be the set of all $\{x_1, \dots, x_m\} \in C$ such that $x_1, \dots, x_m \in (p_1, p_n)$ and

$$\sharp(\{x_1, \dots, x_m\} \cap (p_i, p_{i+1})) = \pi_i$$

for $i = 1, \dots, n-1$. This is an open m–ball,

Lemma 3.1 *$H_m^{\mathrm{BM}}(C)$ is the direct sum of $\binom{n+m-1}{m}$ copies of V, namely the images of $H_m^{\mathrm{BM}}(U_\pi)$ for $\pi \in \Pi$.*

Proof Let C_0 be the set of all $\{x_1, \dots, x_m\} \in C$ such that $x_1, \dots, x_m \in (p_1, p_n)$. Then C_0 is the disjoint union of U_π for $\pi \in \Pi$. Thus $H_m^{\mathrm{BM}}(C_0)$ is the direct sum of V_π over all $\pi \in \Pi$. It remains to prove that the map $H_m^{\mathrm{BM}}(C_0) \to H_m^{\mathrm{BM}}(C)$ induced by inclusion is an isomorphism.

Definition For $\epsilon > 0$, let A_ϵ be the set of all $\{z_1, \dots, z_m\} \in C$ such that $|z_i - z_j| \geq \epsilon$ for all distinct $i, j = 1, \dots, m$, and $|z_i - p_j| \geq \epsilon$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$.

Note that each A_ϵ is compact, and every compact subset of C is contained in some A_ϵ. Thus it suffices to show that for all sufficiently small $\epsilon > 0$, the map

$$H_m(C_0, C_0 \setminus A_\epsilon) \to H_m(C, C \setminus A_\epsilon)$$

induced by inclusion is an isomorphism.

Let $D_n' \subset D_n$ be a closed $\epsilon/2$ neighborhood of the interval $[p_1, p_n]$. Let $C' = C_m(D_n')$. The map

$$H_m(C', C' \setminus A_\epsilon) \to H_m(C, C \setminus A_\epsilon)$$

induced by inclusion is an isomorphism. To see this, note that the obvious homotopy shrinking C to C' is a homotopy through maps from $(C, C \setminus A_\epsilon)$ to itself.

Let U be the set of $\{z_1, \dots, z_m\} \in C'$ such that the real parts of $z_1, \dots, z_m$ are all distinct and lie in $[p_1, p_n] \setminus \{p_1, \dots, p_n\}$. Then U is open and contains $A_\epsilon \cap C'$. By excision, the map

$$H_m(U, U \setminus A_\epsilon) \to H_m(C', C' \setminus A_\epsilon)$$

induced by inclusion is an isomorphism.

There is an obvious deformation retraction from U to C_0 taking $\{z_1,\dots,z_m\}$ to $\{x_1,\dots,x_m\}$ where x_i is the real part of z_i. This is a homotopy through maps of pairs of spaces from $(U, U\setminus A_\epsilon)$ to itself. Thus the map

$$H_m(C_0, C_0\setminus A_\epsilon) \to H_m(U, U\setminus A_\epsilon))$$

induced by inclusion is an isomorphism. □

3.1 The Krammer representation

As an example, we now describe a homological definition of the representation that Krammer defined in [4] and proved to be faithful in [5].

Let $V = K$, let t be a unit of K, and let the generator of B_2 act on V by multiplication by $-t$. Then C is the space of unordered pairs of distinct points in D_n and $\mathcal{L}$ is a K–bundle with monodromy as follows. A loop in C in which the two points switch places by a small counterclockwise twist has monodromy $-t$. A loop in which one point makes a small counterclockwise loop around one puncture while the other point remains fixed has monodromy q.

By Lemma 3.1, $H_2^{\mathrm{BM}}(C)$ is a free K–module of rank $\binom{n}{2}$. For $1 \le i \le j \le n-1$, let $U_{i,j}$ be the set of $\{x,y\} \in C$ such that $x \in (p_i, p_{i+1})$ and $y \in (p_j, p_{j+1})$. Let $u_{i,j}$ be the element of $H_2^{\mathrm{BM}}(C)$ represented by some non-zero lift of $U_{i,j}$ to $\mathcal{L}$. These form the basis of $H_2^{\mathrm{BM}}(C)$ given by the proof of Lemma 3.1.

An alternative basis is as follows. For $1 \le i < j \le n$, let $T_{i,j}$ be an edge from p_i to p_j whose interior lies in the upper half plane. Let $V_{i,j}$ be the set of pairs of distinct points in $T_{i,j}$. Let $v_{i,j}$ be the element of $H_2^{\mathrm{BM}}(C)$ represented by some non-zero lift of $V_{i,j}$ to $\mathcal{L}$. With appropriate choices of local coordinates,

$$v_{i,j} = \sum_{i \le k \le l < j} u_{k,l}.$$

These form a basis for $H_2^{\mathrm{BM}}(C)$.

It is not hard to compute the matrices for the action of B_n with respect to this basis. They are the same as the matrices used by Krammer in [4] and [5] to produce a faithful representation of B_n. Thus we have a homological construction of the "Krammer representation" over any ring. This proves [1, Conjecture 5.7].

4 A Hermitian form

For the rest of this paper, we assume the following.

- K is a field,
- $x \mapsto \bar{x}$ is an automorphism of K such that $\bar{\bar{x}} = x$,
- $q \in K$ satisfies $q\bar{q} = 1$ and $q \neq 1$,
- $\langle \cdot, \cdot \rangle_V \colon V \times V \to K$ is a non-singular Hermitian form that is invariant under the action of B_m.

An interesting example is $K = \mathbb{C}$, q is a root of unity, and V is a unitary representation of B_m. We will not explicitly use the assumption $q \neq 1$, but our constructions produce trivial results if $q = 1$.

Under these assumptions, we will define a Hermitian form on W that is invariant under the action of B_n. This form is something of a relic from my previous less efficient proofs of the results in this paper. It now is only used to prove Lemma 4.1 below. With more work it could probably be completely eliminated, thus allowing us to work over a more general integral domain K. However I still expect this form to play an important role in a proof of Conjecture 6.1.

Suppose $\pi = (\pi_1, \ldots, \pi_{n-1}) \in \Pi$, that is, π_i is a non-negative integer and $\sum \pi_i = m$. Let

$$f_1, \ldots, f_m \colon (I, \partial I) \to (D_n, \partial D)$$

be disjoint vertical edges, oriented upwards, numbered from left to right, so that for all $i = 1, \ldots, n-1$ there are π_i edges passing between p_i and p_{i+1}. The map

$$F = f_1 \times \cdots \times f_m$$

induces an embedding of the closed m–ball into C. The set of lifts of F to $\mathcal{L}$ defines a subspace V_π of $H_m(C, \partial C)$.

Lemma 4.1 *$H_m(C, \partial C)$ is the direct sum of V_π for $\pi \in \Pi$.*

This follows immediately from Lemma 3.1 and Poincaré duality. We therefore devote the rest of this section to reviewing standard duality results from homology theory, since these may be less familiar in the context of homology with local coefficients.

Let C and $\mathcal{L}$ be as in Section 2. Recall that singular cohomology $H^m(C)$ is the homology of the space of singular cochains. A singular cochain is a function ϕ that assigns to each singular cochain $\sigma \colon \Delta^i \to C$ a lift $\phi(\sigma) \colon \Delta^i \to \mathcal{L}$.

There are versions of Poincaré duality and the universal coefficient theorem using local coefficients. Poincaré duality is proved using a cap product with a fundamental class of the usual singular homology $H_{2m}(C;K)$. The universal coefficient theorem gives an anti-isomorphism from cohomology to the dual of homology. It is proved using the anti-isomorphism from V to its dual given by the effective Hermitian form on V.

These results give rise to forms

$$\langle\cdot,\cdot\rangle_1\colon\ H_m^{\mathrm{BM}}(C)\times H_m(C,\partial C)\to K,$$
$$\langle\cdot,\cdot\rangle_2\colon\ H_m(C)\times H_m^{\mathrm{BM}}(C,\partial C)\to K,$$

that are non-singular, sesquilinear, and invariant under the action of B_n.

Define a form

$$\langle\cdot,\cdot\rangle_0\colon\ H_m(C)\times H_m(C)\to K$$

as follows. Suppose $u,v\in H_m(C)$. Let $u_1\in H_m^{\mathrm{BM}}(C)$, $v_1\in H_m(C,\partial C)$, and $v_2\in H_m^{\mathrm{BM}}(C,\partial C)$ be the images of u and v under maps induced by inclusion. Let

$$\langle u,v\rangle_0=\langle u_1,v_1\rangle_1.$$

This is equivalent to

$$\langle u,v\rangle_0=\langle u,v_2\rangle_2.$$

This is Hermitian, but not necessarily non-singular.

In practice, $u,v\in H_m(C)$ are often represented by immersed closed orientable m–manifolds in C, together with lifts $\tilde{u}$, $\tilde{v}$ to $\mathcal{L}$. In this case $\langle u,v\rangle$ can be computed as follows. Let the m–manifolds be intersect transversely in C. For each point x of intersection, let u_x and v_x be the corresponding points of $\tilde{u}$ and $\tilde{v}$ in the fiber over x. Then $\langle u,v\rangle_0$ is the sum of $\langle u_x,v_x\rangle_{\mathcal{L}_x}$ over all points x of intersection.

Recall that W is the image of the map from $H_m(C)$ to $H_m^{\mathrm{BM}}(C,\partial C)$. Define

$$\langle\cdot,\cdot\rangle_W\colon\ W\times W\to K$$

as follows. Suppose $w_1,w_2\in W$ are the images of $u_1,u_2\in H_m(C)$ under the map induced by inclusion. Then let

$$\langle w_1,w_2\rangle_W=\langle u_1,u_2\rangle_0.$$

Using the properties of $\langle\cdot,\cdot\rangle_0$ that we have mentioned, it is not hard to show that $\langle\cdot,\cdot\rangle_W$ is a well defined non-singular Hermitian form that is invariant under the action of B_n.

For example, suppose $V = K = \mathbb{C}$ and the generator of B_2 acts as multiplication by $-t$, a complex number with unit norm. Then $H_2^{\mathrm{BM}}(C)$ is the Krammer representation described in Section 2. If q and t are algebraically independent unit complex numbers then it can be shown to map isomorphically onto W. Thus we obtain a non-singular Hermitian form preserved by the action of B_n. In [2], Budney proves that this form is negative definite, and hence the representation is unitary.

5 A representation of the Iwahori–Hecke algebra

Let V be a finite dimensional K–vector space with an action of B_m that preserves an effective Hermitian form. Let W be the representation of B_n constructed as in Section 2. To avoid confusion, denote the generators of B_m by $\tau_1, \dots, \tau_{m-1}$, and the generators of B_n by $\sigma_1, \dots, \sigma_{n-1}$.

Theorem 5.1 *If the action of B_m on V satisfies*

$$(\tau_i - 1)(q\tau_i + 1) = 0$$

for $i = 1, \dots, m-1$, then the resulting action of B_n on W satisfies

$$(\sigma_i - 1)(\sigma_i + q) = 0$$

for $i = 1, \dots, n-1$.

Proof It suffices to show that the action of B_n on W satisfies $(\sigma_1 - 1)(\sigma_1 + q) = 0$, since the generators σ_i are all conjugate to σ_1. In fact, we will show that this relation holds for the action of B_n on the image of the map

$$H_m(C, \partial C) \to H_m^{\mathrm{BM}}(C, \partial C)$$

induced by inclusion.

Suppose $\pi = (\pi_1, \dots, \pi_{n-1}) \in \Pi$, that is, π_i is a non-negative integer and $\sum \pi_i = m$. Let

$$f_1, \dots, f_m \colon (I, \partial I) \to (D_n, \partial D)$$

be disjoint vertical edges, oriented upwards, numbered from left to right, so that for all $i = 1, \dots, n-1$ there are π_i edges passing between p_i and p_{i+1}. The map

$$F = f_1 \times \cdots \times f_m$$

induces an embedding of the closed m–ball into C. Let v be represented by a lift $\tilde{F}$ of F to $\mathcal{L}$. By Lemma 4.1, such elements v generate $H_m(C, \partial C)$. We show that either $(\sigma_1 - 1)(\sigma_1 + q)v = 0$ or the image of v in $H_m^{\mathrm{BM}}(C, \partial C)$ is zero.

First consider the case $\pi_1 = 0$. Then σ_1 is the identity on the image of F, so σ_1 acts as the identity on V_π. In particular, $(\sigma_1 - 1)v = 0$.

Next consider the case $\pi_1 = 1$. Then σ_1 is the identity on the images of $f_2, \dots, f_n$. Thus

$$\sigma_1 F = \sigma_1 f_1 \times f_2 \times \cdots \times f_m.$$

We can homotope the edge $\sigma_1 f_1$, keeping it to the left of f_2 and keeping the endpoints on ∂D, to obtain a composition of paths $g\bar{f}_1$ where g goes from $f_1(0)$ to $f_1(1)$, passing upwards between p_2 and p_3, and $\bar{f}_1$ is f_1 with the reverse orientation. Let

$$\begin{aligned} G &= g \times f_2 \times \cdots \times f_m, \\ F' &= \bar{f}_1 \times f_2 \times \cdots \times f_m, \end{aligned}$$

Then $\sigma_1 v = u + v'$ where u and v' are represented by appropriate lifts $\tilde{G}$ and $\tilde{F}'$ of G and F'.

Now F' is the same closed ball as F but with the opposite orientation. It remains to compare the lifts $\tilde{F}'$ and $\tilde{F}$. Consider the loop in α in C given by

$$\alpha(t) = \{(g\bar{f}_1)(t), f_2(0), \dots, f_m(0)\}.$$

Let $\tilde{\alpha}$ be the lift of α such that $\tilde{\alpha}(0) \in \tilde{F}$. Then $\tilde{\alpha}$ goes through $\tilde{G}$ to a point in the intersection of $\tilde{G}$ and $\tilde{F}'$, then through $\tilde{F}'$ to $\tilde{\alpha}(1)$. Thus $\tilde{\alpha}(1)$ is the point in $\tilde{F}'$ corresponding to the point $\tilde{\alpha}(0)$ in $\tilde{F}$. Now α has monodromy q, so we we conclude that $v' = -qv$. Thus $(\sigma_1 + q)v = u$. But $(\sigma_1 - 1)u = 0$ by the previous case. Thus $(\sigma_1 - 1)(\sigma_1 + q)v = 0$.

Finally, consider the case $\pi_1 \geq 2$. Let $\epsilon > 0$ and let A_ϵ be as defined in Section 3. Let $U = C \setminus A_\epsilon$. This is the set of $\{z_1, \dots, z_m\} \in C$ such that some distinct z_i and z_j are within distance ϵ of each other, or some z_i is within ϵ of a puncture. We show that the image of u in $H_m(C, U \cup \partial C)$ is zero, and hence that the image of u in $H_m^{\mathrm{BM}}(C, \partial C)$ is zero.

We can homotope f_1, keeping it to the left of f_2 and keeping the endpoints on ∂D, to obtain a composition $f f_\epsilon \bar{f}$, where f is the straight line from -1 to $p_1 - \epsilon/2$ and f_ϵ is a circular loop of radius $\epsilon/2$ around p_1. The image of $f_\epsilon \times f_2 \times \cdots \times f_m$ lies in U. Thus the image of v in $H_m(C, U \cup \partial C)$ is $(1-q)v'$, where v' is represented by some lift of $f \times f_2 \times \cdots \times f_m$.

Next homotope f_2, keeping it disjoint from f and f_3, and keeping the endpoints on ∂D, to obtain a composition $g_1 g_\epsilon g_2$, where g_1 is a straight line going right from ∂D to $p_1 - (\epsilon/2)i$, g_ϵ is a semicircle of radius $\epsilon/2$ centered at p_1, and g_2 is a straight line going left from $p_1 + (\epsilon/2)i$ to ∂D. Let

$$G_1 = f \times g_1 \times f_3 \times \cdots \times f_m,$$
$$G_2 = f \times g_2 \times f_3 \times \cdots \times f_m.$$

Then $v' = v_1 + v_2$ where v_i is represented by an appropriate lift of G_i.

We can simultaneously homotope g_2 to $\bar{f}$, and f to g_1, by "pushing downwards", keeping one endpoint of each edge on ∂D and the other in an $\epsilon-$neighborhood of p_1, and keeping the two edges disjoint. Thus v_2 is represented by an appropriate lift of

$$G_2' = g_1 \times \bar{f} \times f_3 \times \cdots \times f_m.$$

This has the same image as G_1. It also has the same orientation, since the first two coordinates have switched and the second coordinate has reversed orientation. It remains to compare the lifts of G_1 and G_2' that represent v_1 and v_2.

Let α be the path in C given by

$$\alpha(t) = \{f(1), g_\epsilon(t), f_3(0), \ldots, f_m(0)\}.$$

Let $\tilde{\alpha}$ be the lift of α to $\mathcal{L}$ such that $\tilde{\alpha}(0) \in \tilde{G}_1$. Then $\tilde{\alpha}(1) \in \tilde{G}_2$. Let β be the path in C given by

$$\beta(t) = \{\beta_1(t), \beta_2(t), f_3(0), \ldots, f_m(0)\}$$

where β_1 is a path from $h_\epsilon(1)$ down to $g(1)$ and β_2 is a path from $g(1)$ down to $h_\epsilon(0)$. Let $\tilde{\beta}$ be the lift of β such that $\tilde{\beta}(0) \in \tilde{G}_2$. Then $\tilde{\beta}$ follows our homotopy from $\tilde{G}_2$ to $\tilde{G}_2'$, so $\tilde{\beta}(1) \in \tilde{G}_2'$. Now $\tilde{\alpha}\tilde{\beta}$ is a lift of $\alpha\beta$ that starts in $\tilde{G}_1$ and ends at the corresponding point in $\tilde{G}_2'$. But $\alpha\beta$ is a loop with monodromy $q\tau$. Thus $v_2 = q\tau_1 v_1$, so

$$v' = (1 + q\tau_1)v_1.$$

We can assume $|f(t) - g_1(t)| < \epsilon$ for all $t \in I$. Let

$$T_1 = \{(t_1, t_2, \ldots, t_m) \in I^m : t_1 > t_2\},$$
$$T_2 = \{(t_1, t_2, \ldots, t_m) \in I^m : t_1 < t_2\}.$$

Then $\tilde{G_1}|T_1$ and $\tilde{G_1}|T_2$ represent elements w_1 and w_2 of $H_m(C, U \cup \partial C)$. We have $v_1 = w_1 + w_2$.

Let

$$H_1 = f \times f \times f_3 \times \cdots \times f_m|T_1 \to C,$$

$$H_2 = f \times f \times f_3 \times \cdots \times f_m | T_2 \to C.$$

Then w_i is represented by an appropriate lift $\tilde{H}_i$ of H_i. The image of H_2 is the same as that of H_1, but the orientation is reversed since the first two coordinates have switched places. A path from a point in $\tilde{H}_1$ to the corresponding point in $\tilde{H}_2$ is given by a lift of a loop in C in which the leftmost two points switch places by a half twist. The monodromy of this loop is τ_1. Thus $w_2 = -\tau_1 w_1$.

We conclude that the image of v in $H_m^{\mathrm{BM}}(C, \partial C)$ is $(1-q)(1+q\tau_1)(1-\tau_1)w_1$. But by assumption, the action of B_m on V satisfies $(1+q\tau_1)(1-\tau_1) = 0$. Thus the image of v in $H_m^{\mathrm{BM}}(C, \partial C)$ is zero. □

6 Conclusion

We can use Theorem 5.1 to construct representations of the Iwahori–Hecke algebra as follows. Suppose V is a representation of $\mathcal{H}_m$. Let B_m act on V by

$$\tau_i \mapsto -T_i^{-1}.$$

Let W be the representation of B_n constructed in Section 2. By Theorem 5.1, the action of B_n on W factors through the map

$$\sigma_i \mapsto qT_i^{-1},$$

and so gives a representation of $\mathcal{H}_n$.

In their groundbreaking paper [3], Dipper and James classify the irreducible representations of $\mathcal{H}_n$. They define a representation D_λ of $\mathcal{H}_n$ for every partition λ of n. They show that the non-zero D_λ enumerate the irreducible representations of $\mathcal{H}_n$. Furthermore they give a simple classification of which partitions λ give rise to non-zero D_λ.

Let λ be a partition of n. That is, $\lambda = (\lambda_1, \dots, \lambda_k)$ where $\lambda_1 \geq \cdots \geq \lambda_k > 0$ and $\lambda_1 + \cdots + \lambda_k = n$. Let μ be the partition $(\lambda_2, \dots, \lambda_k)$ of $m = n - \lambda_1$.

Conjecture 6.1 *Suppose $q \neq 1$. If V is the representation D_μ of $\mathcal{H}_m$ and W is the representation of $\mathcal{H}_n$ constructed by Theorem 5.1 then $W = D_\lambda$.*

This would make it possible to inductively construct all irreducible representations of Iwahori–Hecke algebras from $D_\emptyset$, the trivial representation of the algebra $\mathcal{H}_0 = K$. The conjecture is true when $K = \mathbb{C}$ and q is a generic unit complex number, because in this case my construction is the same as that of Lawrence [6]. We now give some examples that can be computed directly. These confirm the conjecture in some simple non-generic cases, and give an idea of the sort of behavior to expect in general.

6.1 The trivial representation

Let λ be the partition (n). Then μ is the empty partition and $D_\mu = K$. The configuration space C consists of a single point (namely the empty set). Thus $W = H_0(C) = K$ with the trivial action of B_n. We obtain the representation $T_i \mapsto q$ of $\mathcal{H}_n$, which is indeed D_λ.

6.2 The Burau representation

Let λ be the partition $(n-1,1)$. Then $\mu = (1)$, $m = 1$, and $V = K$. The configuration space C is simply D_n. The local coefficients $\mathcal{L}$ are such that a counterclockwise loop around a puncture has monodromy q. The homology $H_1(D_n)$ is an $(n-1)$–dimensional vector space. The induced action of B_n is the well-known homological construction of the Burau representation.

The map from $H_1(D_n)$ to $H_1^{\mathrm{BM}}(D_n, \partial D)$ is an isomorphism except if q is an nth root of unity. In this case, the map has a one-dimensional kernel. The kernel is generated by a non-zero lift of ∂D to $\mathcal{L}$. Such a lift exists because the monodromy around ∂D is trivial when $q^n = 1$.

Thus W is either the Burau representation or an $(n-2)$–dimensional quotient of the Burau representation. This is indeed isomorphic to D_λ.

6.3 The $(n-2,2)$ representation

Suppose $n \geq 4$ and λ is the partition $(n-2,2)$. Then $\mu = (2)$, and D_μ is the one-dimensional representation of $\mathcal{H}_2$ given by $T_1 \mapsto q$. Thus V is the one-dimensional representation of B_2 given by $\tau_1 \mapsto -q^{-1}$.

We obtain a K–bundle $\mathcal{L}$ over the space C of unordered pairs of distinct points in D_n. A loop in C in which one point goes counterclockwise around a puncture has monodromy q. A loop in which the points switch places by a counterclockwise twist has monodromy $-q^{-1}$.

Let W_0 be the image of the map from $H_2(C)$ to $H_2^{\mathrm{BM}}(C)$. The main result of [1] is that W_0 has dimension $n(n-3)/2$ and is the "Specht module" corresponding to λ. However I used the assumption that q is not a square or cube root of 1, which I now think I can do without.

If $q^{n-2} = 1$ then the kernel of the map from W_0 to W has dimension $n-1$. A basis is given by lifts of annuli of the form $\partial D \times (p_i, p_{i+1})$. Such lifts exist because the loop in which one point goes around the other point and all punctures has monodromy $q^{n-2} = 1$. The action of B_n on this kernel is the Burau representation.

If $q^{n-1} = 1$ then the kernel of the map from W_0 to W has dimension 1. It is the space of lifts of the surface of pairs of distinct points in ∂D.

These results agree with what is known about D_λ.

6.4 Open questions

Assuming Conjecture 6.1 is true, it is natural to ask the following questions.

- Does Conjecture 6.1 have any applications to the representation theory of Iwahori–Hecke algebras?
- What is the dimension of D_λ?
- Does Conjecture 6.1 hold when K is a local ring?
- Is there a similar construction for representations of the Birman–Murakami–Wenzl algebra?

The first question is deliberately vague. The representation theory of Iwahori–Hecke algebras is a well trammeled area of study, and it might be too optimistic to think the topological approach in this paper will help resolve any of the outstanding open problems. At least I hope it will give an amusing new perspective.

The second question is a specific example of a major open problem in representation theory. The answer is known in the sense that there is an unfeasibly slow algorithm to compute the dimension of D_λ. A great deal of more practical information is also known - enough to make it clear that it is a very deep problem.

Viewing this problem in the light of Conjecture 6.1, we are led to investigate the nature of the kernel of the map from $H_m(C)$ to $H_m^{\mathrm{BM}}(C, \partial C)$. This kernel comes from the previous term in a long exact sequence of relative homology. Thus we are led to ask what is the dimension of $H_{m+1}(\partial \bar{C})$, where $\bar{C}$ is the set A_ϵ defined in Section 3, or perhaps some more sophisticated compactification of C.

The third question has applications to the representation theory of the symmetric group, which is still not well understood over non-zero characteristic. The idea is to first understand the representations of $\mathcal{H}_n$ when K is a ring localized at $(q-1)$, and then understand the effect of taking the quotient by $(q-1)$. The main difficulty if K is a ring seems to be in defining the non-singular Hermitian form. The problem is that the universal coefficient theorem is more complicated in the presence of torsion.

The last question is inspired by the result due to Zinno [7], which states that the Lawrence–Krammer representation is a representation of the Birman–Murakami–Wenzl (BMW) algebra. The BMW algebra, like the Iwahori–Hecke algebra, is a quotient of the braid group algebra by some relations in the generators. By analogy with the proof of Theorem 5.1 it should be possible to show that, up to some mild rescaling, if V satisfies the relations of the BMW algebra then so does W.

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