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The metric space of geodesic laminations on a surface II: small surfaces

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Abstract We continue our investigation of the space of geodesic laminations on a surface, endowed with the Hausdorff topology. We determine the topology of this space for the once-punctured torus and the 4–times-punctured sphere. For these two surfaces, we also compute the Hausdorff dimension of the space of geodesic laminations, when it is endowed with the natural metric which, for small distances, is -1 over the logarithm of the Hausdorff metric. The key ingredient is an estimate of the Hausdorff metric between two simple closed geodesics in terms of their respective slopes.

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This article is a continuation of the study of the Hausdorff metric d_{H} on the space $\mathcal{L}(S)$ of all geodesic laminations on a surface S, which we began in the article [10]. The impetus for these two papers originated in the monograph [3] by Andrew Casson and Steve Bleiler, which was the first to systematically exploit the Hausdorff topology on the space of geodesic laminations.

In this paper, we restrict attention to the case where the surface S is the once-punctured torus or the 4–times-punctured sphere. To some extent, these are the first non-trivial examples, since $\mathcal{L}(S)$ is defined only when the Euler characteristic of S is negative, is finite when S is the 3–times-punctured sphere or the twice-punctured projective plane, and is countable infinite when S is the once-punctured Klein bottle (see for instance Section 9).

We will also restrict attention to the open and closed subset $\mathcal{L}_0(S)$ of $\mathcal{L}(S)$ consisting of those geodesic laminations which are disjoint from the boundary. This second restriction is only an expository choice. The results and techniques of the paper can be relatively easily extended to the full space $\mathcal{L}(S)$, but at the expense of many more cases to consider; the corresponding strengthening of the results did not seem to be worth the increase in size of the article.

The first two results deal with the topology of $\mathcal{L}_0(S)$ for these two surfaces.

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Theorem 1 *When S is the once-punctured torus, the space $\mathcal{L}_0(S)$ naturally splits as the disjoint union of two compact subsets, the closure $\mathcal{L}_0^{\mathrm{cr}}(S)$ of the set of simple closed curves and its complement $\mathcal{L}_0(S) - \mathcal{L}_0^{\mathrm{cr}}(S)$. The first subspace $\mathcal{L}_0^{\mathrm{cr}}(S)$ is homeomorphic to a subspace $K \cup L_1$ of the circle $\mathbb{S}^1$, where K is the standard Cantor set and where L_1 is a countable set consisting of one isolated point in each component of $\mathbb{S}^1 - K$. The complement $\mathcal{L}_0(S) - \mathcal{L}_0^{\mathrm{cr}}(S)$ is homeomorphic to a subspace $K \cup L_3$ of $\mathbb{S}^1$, union of the Cantor set $K \subset \mathbb{S}^1$ and of a countable set L_3 consisting of exactly 3 isolated points in each component of $\mathbb{S}^1 - K$.*

Theorem 2 *When S is the 4–times-punctured sphere, the space $\mathcal{L}_0(S)$ is homeomorphic to a subspace $K \cup L_7$ of $\mathbb{S}^1$, union of the Cantor set K and of a countable set L_7 consisting of exactly 7 isolated points in each component of $\mathbb{S}^1 - K$. In this case, the closure $\mathcal{L}_0^{\mathrm{cr}}(S)$ of the set of simple closed curves is the union $K \cup L_1$ of K and of a discrete set $L_1 \subset L_7$ consisting of exactly one point in each component of $\mathbb{S}^1 - K$; in particular, its complement $\mathcal{L}_0(S) - \mathcal{L}_0^{\mathrm{cr}}(S)$ is countable infinite.*

The above subspaces $K \cup L_1$, $K \cup L_3$ and $K \cup L_7$ are all homeomorphic. However, it is convenient to keep a distinction between these spaces, because the proofs of Theorems 1 and 2 make the corresponding embeddings of $\mathcal{L}_0(S)$ and $\mathcal{L}_0^{\mathrm{cr}}(S)$ in $\mathbb{S}^1$ relatively natural. In particular, these establish a one-to-one correspondence between the components of $\mathbb{S}^1 - K$ and the simple closed curves of S. These embeddings are also well behaved with respect to the action of the homeomorphism group of S on $\mathcal{L}_0(S)$.

We now consider metric properties of the Hausdorff metric d_{H} on $\mathcal{L}_0(S)$. In [10], we showed that the metric space $(\mathcal{L}(S), d_{\mathrm{H}})$ has Hausdorff dimension 0. In particular, it is totally disconnected, which is consistent with Theorems 1 and 2. However, we also observed that, to some extent, the Hausdorff metric d_{H} of $\mathcal{L}(S)$ is not very canonical because it is only defined up to Hölder equivalence. This lead us to consider on $\mathcal{L}(S)$ another metric $d_{\log}$ which, for small distances, is just equal to $-1/\log d_{\mathrm{H}}$. This new metric $d_{\log}$ has better invariance properties because it is well-defined up to Lipschitz equivalence; in particular, its Hausdorff dimension is well-defined. We refer to [10] and Section 1 for precise definitions.

Theorem 3 *When S is the once-punctured torus or the 4–times-punctured sphere, the Hausdorff dimension of the metric space $(\mathcal{L}_0(S), d_{\log})$ is equal to 2. Its 2–dimensional Hausdorff measure is equal to 0.*

Theorem 3 was used in [10] to show that, for a general surface S of negative Euler characteristic which is not the 3–times-punctured sphere, the twice-punctured projective plane or the once-punctured Klein bottle, the Hausdorff dimension of $(\mathcal{L}_0(S), d_{\log})$ is positive and finite.

These results should be contrasted with the more familiar Thurston completion of the set of simple closed curves on S, by the space $\mathcal{PML}(S)$ of projective measured laminations [5, 8]. For the once–punctured torus and the 4–times-punctured sphere, $\mathcal{PML}(S)$ is homeomorphic to the circle and has Hausdorff dimension 1.

What is special about the once-punctured torus and the 4–times-punctured sphere is that there is a relatively simple classification of their simple closed curves, or more generally of their recurrent geodesic laminations, in terms of their slope. The key technical result of this article is an estimate, proved in Section 2, which relates the Hausdorff distance of two simple closed curves to their slopes.

Proposition 4 *Let λ and λ' be two simple closed geodesics on the once-punctured torus or on the 4–times-punctured sphere, with respective slopes $\frac{p}{q} < \frac{p'}{q'} \in \mathbb{Q} \cup \{\infty\}$. Their Hausdorff distance $d_{\mathrm{H}}(\lambda, \lambda')$ is such that*

$$\mathrm{e}^{-c_1/d\left(\frac{p}{q},\frac{p'}{q'}\right)} \leqslant d_{\mathrm{H}}(\lambda, \lambda') \leqslant \mathrm{e}^{-c_2/d\left(\frac{p}{q},\frac{p'}{q'}\right)}$$

where the constants c_1, $c_2 > 0$ depend only on the metric on the surface, and where

$$d\left(\tfrac{p}{q}, \tfrac{p'}{q'}\right) = \max\left\{\tfrac{1}{|p''|+|q''|}; \tfrac{p}{q} \leqslant \tfrac{p''}{q''} \leqslant \tfrac{p'}{q'}\right\}.$$

The other key ingredient is an analysis of the above metric d on $\mathbb{Q} \cup \{\infty\}$, which is provided in the Appendix.

A large number of the results of this paper were part of the dissertation [9].

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1 Train tracks

We will not repeat the basic definitions on geodesic laminations, referring instead to the standard literature [3, 2, 8, 1], or to [10]. However, it is probably worth reminding the reader of our definition of the *Hausdorff distance* $d_{\mathrm{H}}(\lambda,\lambda')$ between two geodesic laminations λ, λ' on the surface S, namely

$$d_{\mathrm{H}}(\lambda,\lambda') = \min\left\{\varepsilon;\ \begin{array}{l}\forall x\in\lambda, \exists x'\in\lambda', d\left((x,T_x\lambda),(x',T_{x'}\lambda')\right) < \varepsilon\\ \forall x'\in\lambda', \exists x\in\lambda, d\left((x,T_x\lambda),(x',T_{x'}\lambda')\right) < \varepsilon\end{array}\right\}$$

where the distance d is measured in the projective tangent bundle $PT(S)$ consisting of all pairs (x,l) with $x\in S$ and with l a line through the origin in the tangent space T_xS, and where $T_x\lambda$ denotes the tangent line at x of the leaf of λ passing through x. In particular, $d_{\mathrm{H}}(\lambda,\lambda')$ is not the Hausdorff distance between λ and λ' considered as closed subsets of S, but the Hausdorff distance between their canonical lifts to $PT(S)$. As indicated in [10], this definition guarantees that $d_{\mathrm{H}}(\lambda,\lambda')$ is independent of the metric of S up to Hölder equivalence, whereas it is unclear whether the same property holds for the Hausdorff metric as closed subsets of S. This subtlety is relevant only when we consider metric properties since, as proved in [3, Lemma 3.5], the two metrics define the same topology on $\mathcal{L}(S)$.

A classical tool in 2–dimensional topology/geometry is the notion of train track. A *train track* on the surface S is a graph Θ contained in the interior of S which consists of finitely many vertices, also called *switches*, and of finitely many edges joining them such that:

(1) The edges of Θ are differentiable arcs whose interiors are embedded and pairwise disjoint (the two end points of an edge may coincide).

(2) At each switch s of Θ, the edges of Θ that contain s are all tangent to the same line L_s in the tangent space T_sS and, for each of the two directions of L_s, there is at least one edge which is tangent to that direction.

(3) Observe that the complement $S-\Theta$ has a certain number of spikes, each leading to a switch s and locally delimited by two edges that are tangent to the same direction at s; we require that no component of $S-\Theta$ is a disc with 0, 1 or 2 spikes or an open annulus with no spike.

A curve c *carried* by the train track Θ is a differentiable immersed curve $c\colon I\to S$ whose image is contained in Θ, where I is an interval in $\mathbb{R}$. The geodesic lamination λ is *weakly carried* by the train track Θ if, for every leaf g of λ, there is a curve c carried by Θ which is homotopic to g by a homotopy moving points by a bounded amount. In this case, the bi-infinite sequence $\langle \dots, e_{-1}, e_0, e_1, \dots, e_n, \dots \rangle$ of the edges of Θ that are crossed in this order by the curve c is the *edge path realized by* the leaf g; it can be shown that the curve c is uniquely determined by the leaf g, up to reparametrization, so that the edge path realized by g is well-defined up to order reversal.

Let $\mathcal{L}(\Theta)$ be the set of geodesic laminations that are weakly carried by Θ. (This was denoted by $\mathcal{L}^{\mathrm{w}}(\Theta)$ in [10] where, unlike in the current paper, we had to distinguish between "strongly carried" and "weakly carried".)

We introduced two different metrics on $\mathcal{L}(\Theta)$ in [10]. The first one is defined over all of $\mathcal{L}(S)$, and is just a variation of the Hausdorff metric d_{H}. The distance function $d_{\log}$ on $\mathcal{L}(S)$ is defined by the formula

$$d_{\log}(\lambda, \lambda') = \frac{1}{\left|\log\left(\min\left\{d_{\mathrm{H}}(\lambda, \lambda'), \frac{1}{4}\right\}\right)\right|}.$$

In particular, $d_{\log}(\lambda, \lambda') = |1/\log d_{\mathrm{H}}(\lambda, \lambda')|$ when λ and λ' are close enough from each other. The min in the formula was only introduced to make $d_{\log}$ satisfy the triangle inequality, and is essentially cosmetic.

The other metric is the *combinatorial distance* between λ and $\lambda' \in \mathcal{L}(\Theta)$, defined by

$$d_\Theta(\lambda, \lambda') = \min\left\{\tfrac{1}{r+1}; \lambda \text{ and } \lambda' \text{ realize the same edge paths of length } r\right\},$$

where we say that an edge path is realized by a geodesic lamination when it is realized by one of its leaves. This metric is actually an ultrametric, in the sense that it satisfies the stronger triangle inequality

$$d_\Theta(\lambda, \lambda'') \leqslant \max\left\{d_\Theta(\lambda, \lambda'), d_\Theta(\lambda', \lambda'')\right\}.$$

The main interest of this combinatorial distance is the following fact, proved in [10].

Proposition 5 *For every train track Θ on the surface S, the combinatorial metric d_Θ is Lipschitz equivalent to the restriction of the metric $d_{\log}$ to $\mathcal{L}(\Theta)$.*

The statement that d_Θ and $d_{\log}$ are Lipschitz equivalent means that there exists constants c_1 and $c_2 > 0$ such that

$$c_1\, d_\Theta(\lambda, \lambda') \leqslant d_{\log}(\lambda, \lambda') \leqslant c_2\, d_\Theta(\lambda, \lambda')$$

for every λ, $\lambda' \in \mathcal{L}(\Theta)$. In particular, the two metrics define the same topology, and have the same Hausdorff dimension.

For future reference, we note the following property, whose proof can be found in [1, Chapter 1] (and also easily follows from Proposition 5).

Proposition 6 *The space $\mathcal{L}(\Theta)$ is compact.*

2 Distance estimates on the once-punctured torus

In this section, we focus attention on the case where the surface S is the once-punctured torus, which we will here denote by T. As indicated above, there is a very convenient classification of simple closed geodesics or, equivalently, isotopy classes of simple closed curves, on T; see for instance [8].

Let $\mathcal{S}(T) \subset \mathcal{L}(T)$ denote the set of all simple closed geodesics which are contained in the interior of T. In the plane $\mathbb{R}^2$, consider the lattice $\mathbb{Z}^2$. The quotient of $\mathbb{R}^2 - \mathbb{Z}^2$ under the group $\mathbb{Z}^2$ acting by translations is diffeomorphic to the interior of T. Fix such an identification $\mathrm{int}(T) \cong \left(\mathbb{R}^2 - \mathbb{Z}^2\right)/\mathbb{Z}^2$. Then every straight line in $\mathbb{R}^2$ which has rational slope and avoids $\mathbb{Z}^2$ projects to a simple closed curve in $\mathrm{int}(T) = \left(\mathbb{R}^2 - \mathbb{Z}^2\right)/\mathbb{Z}^2$, which itself is isotopic to a unique simple closed geodesic of $\mathcal{S}(T)$. This element of $\mathcal{S}(T)$ depends only on the slope of the line, and this construction induces a bijection $\mathcal{S}(T) \cong \mathbb{Q} \cup \{\infty\}$.

By definition, the element of $\mathbb{Q} \cup \{\infty\}$ thus associated to $\lambda \in \mathcal{S}(T)$ is the *slope* of λ.

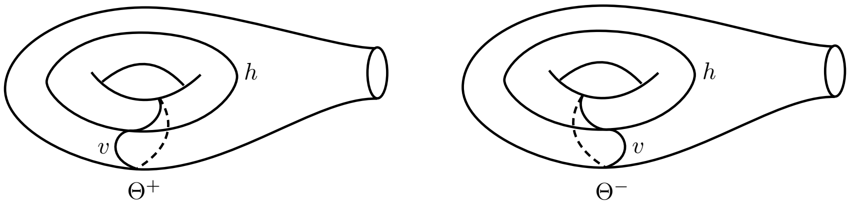

Figure 1: The train tracks Θ^+ and Θ^- on the once-punctured torus T

From this description, one concludes that every simple closed geodesic $\lambda \in \mathcal{S}(T)$ is weakly carried by one of the two train tracks Θ^+ and Θ^- represented on Figure 1. These two train tracks each consist of two edges h and v meeting at one single switch. The identification $\mathrm{int}(T) \cong \left(\mathbb{R}^2 - \mathbb{Z}^2\right)/\mathbb{Z}^2$ can be chosen so that the preimage of Θ^+ in $\mathbb{R}^2 - \mathbb{Z}^2$ is the one described in Figure 2. In particular, the preimage of the edge h is a family of 'horizontal' curves, each properly isotopic to a horizontal line in $\mathbb{R}^2 - \mathbb{Z}^2$, and the preimage of the edge v is a family of 'vertical' curves. Similarly, the preimage of Θ^- is obtained from that of Θ^+ by reflection across the x–axis.

The simple closed geodesic $\lambda \in \mathcal{S}(T)$ is weakly carried by Θ^+ (respectively Θ^-) exactly when its slope $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$ is non-negative (respectively non-positive),

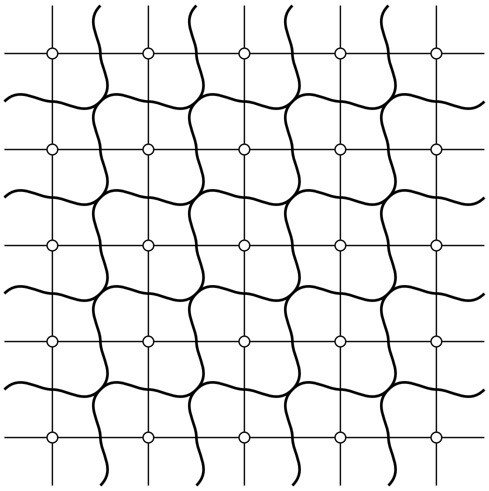

Figure 2: The preimage of Θ^+ in $\mathbb{R}^2 - \mathbb{Z}^2$

by consideration of a line of slope $\frac{p}{q}$ in $\mathbb{R}^2 - \mathbb{Z}^2$ and of its translates under the action of $\mathbb{Z}^2$. In this case, it is tracked by a simple closed curve c carried by Θ^+ (respectively Θ^-) which crosses $|p|$ times the edge v and q times the edge h. We are here requiring the integers p and q to be coprime with $q \geqslant 0$, and the slope $\infty = \frac{1}{0} = \frac{-1}{0}$ is considered to be both non-negative and non-positive. We will use the same convention for slopes throughout the paper.

The following result, which computes the combinatorial distance between two simple closed geodesics in terms of their slopes, is the key to our analysis of $\mathcal{L}_0(T)$.

Proposition 7 *Let the simple closed geodesics λ, $\lambda' \in \mathcal{S}(T)$ have slopes $\frac{p}{q}$, $\frac{p'}{q'} \in \mathbb{Q} \cup \{\infty\}$ with $0 \leqslant \frac{p}{q} < \frac{p'}{q'} \leqslant \infty$. Then*

$$d_{\Theta^+}(\lambda, \lambda') = \max\left\{\tfrac{1}{p''+q''}; \tfrac{p}{q} \leqslant \tfrac{p''}{q''} \leqslant \tfrac{p'}{q'}\right\}.$$

Proof For this, we first have to understand the edge paths realized by a simple closed geodesic $\lambda \in \mathcal{S}(T)$ in terms of its slope $\frac{p}{q}$.

Let L be a line in $\mathbb{R}^2$ of slope $\frac{p}{q}$ which avoids the lattice $\mathbb{Z}^2$. Look at its intersection points with the grid $\mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z}$, and label them as

$$\ldots, x_{-1}, x_0, x_1, \ldots, x_i, \ldots$$

in this order along L. This defines a periodic bi-infinite edge path

$$\langle \dots, e_{-1}, e_0, e_1, \dots, e_i, \dots \rangle$$

in Θ^+, where e_i is equal to the edge h if the point x_i is in a vertical line $\{n\} \times \mathbb{R}$ of the grid, and $e_i = v$ if x_i is in a horizontal line $\mathbb{R} \times \{n\}$. By consideration of Figures 1 and 2, it is then immediate that a (finite) edge path is realized by λ if and only if it is contained in this bi-infinite edge path $\langle \dots, e_{-1}, e_0, e_1, \dots, e_i, \dots \rangle$.

The main step in the proof of Proposition 7 is the following special case.

Lemma 8 *If λ, $\lambda' \in \mathcal{S}(T)$ have finite positive slopes $\frac{p}{q}$, $\frac{p'}{q'} \in \mathbb{Q} \cap \,]0, \infty[$ such that $pq' - p'q = \pm 1$, then $d_{\Theta^+}(\lambda, \lambda') = \max \left\{ \frac{1}{p+q}, \frac{1}{p'+q'} \right\}$.*

Proof of Lemma 8 Let L and L' be lines of respective slopes $\frac{p}{q}$ and $\frac{p'}{q'}$ in $\mathbb{R}^2$, avoiding the lattice $\mathbb{Z}^2$. Let c and c' be the projections of L and L' to $\mathrm{int}(T) \cong \left(\mathbb{R}^2 - \mathbb{Z}^2 \right) / \mathbb{Z}^2$. For suitable orientations, the algebraic intersection number of c and c' is equal to $pq' - p'q = \pm 1$. Since all intersection points have the same sign (depending on slopes and orientations), we conclude that c and c' meet in exactly one point.

Let A be the surface obtained by splitting $\mathrm{int}(T)$ along the curve c. Topologically, A is a closed annulus minus one point. Since c and c' transversely meet in one point, c' gives in A an arc c'_1 going from one component of ∂A to the other.

Similarly, the grid $\mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z}$ projects to a family of arcs in A. Most of these arcs go from one boundary component of A to the other. However, exactly four of these arcs go from ∂A to the puncture. We will call the union of these four arcs the *cross* of A. As one goes around the puncture, the arcs of the cross are alternately horizontal and vertical. Also, the cross divides A into one hexagon and two triangles Δ and Δ'. See Figure 3 for the case where $\frac{p}{q} = \frac{3}{5}$ and $\frac{p'}{q'} = \frac{2}{3}$.

Set $r = \min \{p + q - 1, p' + q' - 1\} \geqslant 0$. We want to show that every edge path $\gamma = \langle e_1, e_2, \dots, e_r \rangle$ of length r in Θ^+ which is realized by c' is also realized by c.

Given such an edge path γ, there exists an arc a immersed in c' which cuts the image of the grid $\mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z}$ at the points x_1, x_2, $\dots$, x_r in this order, and such that x_i is in the image of a vertical line $\{n\} \times \mathbb{R}$ if $e_i = h$ and in the

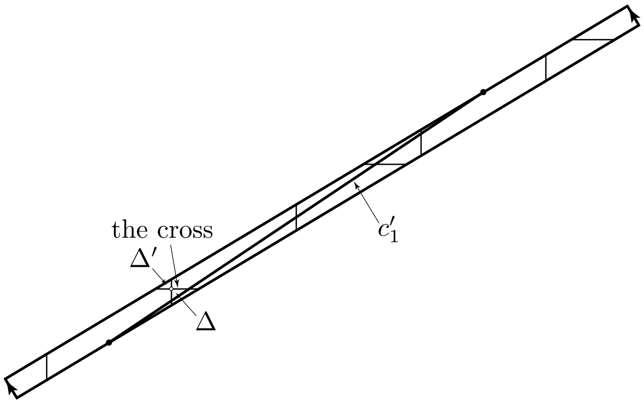

Figure 3: Splitting the once-punctured torus T along the image of a straight line (glue the two short sides of the rectangle)

image of a horizontal line $\mathbb{R} \times \{n\}$ if $e_i = v$. Since c' crosses the image of the grid in $p' + q' > r$ points, the arc a' is actually embedded in c'.

We had a degree of freedom in choosing the closed curve c, since it only needs to be the projection of a line L with slope $\frac{p}{q}$. We can choose this line L so that c contains the starting point of the arc a' (we may need to slightly shorten a' for this, in order to make sure that $L \subset \mathbb{R}^2$ avoids the lattice $\mathbb{Z}^2$).

The arc a' now projects to an arc a'_1 embedded in the arc $c'_1 \subset A$ traced by c', such that the starting point of a'_1 is on the boundary of A.

Note that each boundary component of A crosses the image of the grid $\mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z}$ in $p+q > r$ points. Since a'_1 cuts the image of this grid in r points, we conclude that a'_1 "turns less than once" around A, in the sense that it cuts each arc of the image of the grid in at most one point. Similarly, if Δ and Δ' are the two triangles delimited in A by the cross of the grid, a'_1 can meet the union $\Delta \cup \Delta'$ in at most one single arc. It follows that there exists an arc in ∂A which cuts exactly the same components of the image of the grid as a'_1. This arc a_1 shows that the edge path γ is also realized by c, and therefore by λ.

We conclude that every edge path of length r which is realized by λ' is realized by λ. Exchanging the rôles of λ and λ', every edge path of length r which is realized by λ is also realized by λ'. Consequently, $d_{\Theta^+}(\lambda, \lambda') \leqslant \frac{1}{r+1}$.

To show that $d_{\Theta^+}(\lambda, \lambda') = \frac{1}{r+1}$, we need to find an edge path of length $r+1$ which is realized by one of λ, λ' and not by the other one. Without loss of generality, we can assume that $p + q \leqslant p' + q'$, so that $r + 1 = p + q$.

Consider c, c', A and c'_1 as above. The curves c and c' cross the image of the grid in $p+q$ and $p'+q'$ points, respectively. By our hypothesis that $p+q \leqslant p'+q'$, it follows that c'_1 turns at least once around the annulus A, and therefore meets at least one of the two triangles Δ delimited by the cross. By moving the line $L \subset \mathbb{R}^2$ projecting to c (while fixing c' and the corresponding line L'), we can arrange that $\Delta \cap c'_1$ consists of a single arc, and contains the initial point of c'_1. Let a'_2 be an arc in c'_1 which starts at this initial point and crosses exactly $p+q$ points of the grid. Note that this is possible because $p+q \leqslant p'+q'$. Because each of the two components of ∂A meets the grid in $p+q$ points, the ending point of a'_2 is contained in the other triangle $\Delta' \neq \Delta$ delimited by the cross. Consider the edge path γ' of length $p+q$ described by a'_2.

By construction, the edge path γ' is realized by λ'. We claim that it is not realized by λ. Indeed, let γ be the edge path described by an arc a_2 in ∂A which goes once around the component of ∂A that contains the starting point of c'_2, and starts and ends at this point. By construction, the edge path γ' is obtained from γ by switching the last edge, either from v to h, or from h to v. In particular, the edge paths γ and γ' contain different numbers of edges v. However, because c cuts the grid in exactly $p+q$ points, every edge path of length $p+q$ which is realized by λ must contain the edge v exactly p times. It follows that γ' is not realized by λ.

We consequently found an edge path γ' of length $p+q = r+1$ which is realized by λ' but not by λ. This proves that $d_{\Theta^+}(\lambda, \lambda') \geqslant \frac{1}{r+1}$, and therefore that $d_{\Theta^+}(\lambda, \lambda') = \frac{1}{r+1}$. Since $r = \min\{p+q-1, p'+q'-1\}$, this concludes the proof of Lemma 8. $\square$

Remark 9 For future reference, note that we actually proved the following property: Under the hypotheses of Lemma 8 and if $r = p+q-1 \leqslant p'+q'-1$, then λ and λ' realize exactly the same edge paths of length r, and there exists an edge path of length $r+1$ which is realized by λ' and not by λ.

Lemma 10 *If λ, λ', $\lambda'' \in \mathcal{S}(T)$ have slopes $\frac{p}{q}$, $\frac{p'}{q'}$, $\frac{p''}{q''} \in \mathbb{Q}$ with $0 < \frac{p}{q} \leqslant \frac{p''}{q''} \leqslant \frac{p'}{q'} < \infty$, then every edge path γ in Θ^+ which is realized by both λ and λ' is also realized by λ''.*

Proof of Lemma 10 Let L and L' be lines of respective slopes $\frac{p}{q}$ and $\frac{p'}{q'}$ in $\mathbb{R}^2 - \mathbb{Z}^2$. Since λ realizes $\gamma = \langle e_1, e_2, \ldots, e_n \rangle$, there is an arc $a \subset L$ which meets the grid $\mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z}$ at the points x_1, x_2, …, x_n in this order, and so that the point x_i is in a vertical line of the grid when $e_i = h$ and in a horizontal line when $e_i = v$. Since λ' also realizes γ', there is a similar arc $a' \subset L'$ which meets the grid at points x'_1, x'_2, $\dots$, x'_n.

Applying to L and L' elements of the translation group $\mathbb{Z}^2$ if necessary, we can assume without loss of generality that the starting points of a and a' are both in the square $]0,1[\times]0,1[$. Then, because the slopes are both positive, the fact that the arcs a and a' cut the grid according to the same vertical/horizontal pattern implies that each x_i is in the same line segment component I_i of $(\mathbb{Z}\times\mathbb{R}\cup\mathbb{R}\times\mathbb{Z})-\mathbb{Z}^2$ as x'_i.

The set of lines which cut these line segments I_i is connected. Therefore, one of them must have slope $\frac{p''}{q''}$. By a small perturbation, we can arrange that this line L'' of slope $\frac{p''}{q''}$ is also disjoint from the lattice $\mathbb{Z}^2$. The fact that L'' cuts I_1, I_2, $\dots$, I_n in this order then shows that the corresponding simple closed geodesic λ'' realizes the edge path γ. □

We can now conclude the proof of Proposition 7. Temporarily setting aside the slopes 0 and ∞, let the simple closed geodesics λ, $\lambda' \in \mathcal{S}(T)$ have slopes $\frac{p}{q}$, $\frac{p'}{q'} \in \mathbb{Q}$ with $0 < \frac{p}{q} < \frac{p'}{q'} < \infty$. Then, by elementary number theory (see for instance [6, Section 3.1]), there is a finite sequence of slopes

$$\frac{p}{q} = \frac{p_0}{q_0} < \frac{p_1}{q_1} < \cdots < \frac{p_n}{q_n} = \frac{p'}{q'}$$

such that $p_i q_{i-1} - p_{i-1} q_i = 1$ for every i. Let $\lambda_i \in \mathcal{S}(T)$ be the simple closed geodesic with slope $\frac{p_i}{q_i}$

By the ultrametric property and by Lemma 8,

$$d_{\Theta^+}(\lambda,\lambda') \leqslant \max\{d_{\Theta^+}(\lambda_{i-1},\lambda_i);\ i=1,\dots,n\} = \tfrac{1}{r+1}$$

if $r = \inf\{p_i + q_i - 1;\ i = 0,\dots,n\}$. We want to prove that this inequality is actually an equality, namely that there is an edge path of length $r+1$ which is realized by one of λ, λ' and not by the other.

First consider the case where $p+q-1 > r$, and examine the first i such that $p_i+q_i-1 = r$. By Lemma 8 and Remark 9, there is an edge path γ of length $r+1$ which is realized by λ and λ_{i-1}, and not by λ_i. Since $\frac{p}{q} < \frac{p_i}{q_i} < \frac{p'}{q'}$, Lemma 10 shows that γ cannot be realized by λ', which proves that $d_{\Theta^+}(\lambda,\lambda') = \frac{1}{r+1}$.

When $p'+q'-1 > r$, the same argument provides an edge path which is realized by λ' and not by λ, again showing that $d_{\Theta^+}(\lambda,\lambda') = \frac{1}{r+1}$ in this case.

Finally, consider the case where $p+q-1 = p'+q'-1 = r$. Let γ be any edge path of length $r+1$ which is realized by λ. Note that γ goes exactly once around λ. We conclude that γ contains exactly p times the edge v, and q times the edge h. Similarly, any edge path γ' of length $r+1$ which is realized by λ' must contain p' times the edge v, and q' times the edge h. Since $\frac{p}{q} \neq \frac{p'}{q'}$, we conclude that such a γ' cannot be realized by λ. Therefore, $d_{\Theta^+}(\lambda, \lambda') = \frac{1}{r+1}$ again in this case.

This proves that

$$\begin{aligned} d_{\Theta^+}(\lambda,\lambda') &= \tfrac{1}{r+1} \\ &= \max\left\{ \tfrac{1}{p_i+q_i};\ i = 0, \dots, n \right\} \\ &= \max\left\{ \tfrac{1}{p''+q''};\ \tfrac{p}{q} \leqslant \tfrac{p''}{q''} \leqslant \tfrac{p'}{q'} \right\} \end{aligned}$$

where the last equality comes from the elementary property that $p'' \geqslant p_{i-1}+p_i$ and $q'' \geqslant q_{i-1}+q_i$ whenever $\frac{p_{i-1}}{q_{i-1}} < \frac{p''}{q''} < \frac{p_i}{q_i}$ (hint: $p_i q_{i-1} - p_{i-1} q_i = 1$).

This concludes the proof of Proposition 7 in the case where $0 < \frac{p}{q} < \frac{p'}{q'} < \infty$.

When, $\frac{p}{q} = \frac{0}{1}$, note that λ never crosses the edge v, but that λ' does. This provides an edge path of length 1 which is realized by λ' and not by λ. Therefore, $d_{\Theta^+}(\lambda, \lambda') = 1 = \max\left\{ \frac{1}{p''+q''};\ \frac{0}{1} \leqslant \frac{p''}{q''} \leqslant \frac{p'}{q'} \right\}$ in this case as well. The case where $\frac{p'}{q'} = \infty = \frac{1}{0}$ is similar. □

Corollary 11 *The slope map $\mathcal{S}(T) \to \mathbb{Q} \cup \{\infty\}$ sends the metric $d_{\log}$ to a metric which is Lipschitz equivalent to the metric d on $\mathbb{Q} \cup \{\infty\}$ defined by*

$$d\left(\tfrac{p}{q}, \tfrac{p'}{q'}\right) = \max\left\{ \tfrac{1}{|p''|+|q''|}; \tfrac{p}{q} \leqslant \tfrac{p''}{q''} \leqslant \tfrac{p'}{q'} \right\}$$

for $\frac{p}{q} < \frac{p'}{q'}$.

Proof Propositions 5 and 7 prove this property for the restrictions of $d_{\log}$ to $\mathcal{L}(\Theta^+) \cap \mathcal{S}(T)$ and $\mathcal{L}(\Theta^-) \cap \mathcal{S}(T)$. It therefore suffices to show that there is a positive lower bound for the distances $d_{\log}(\lambda, \lambda')$ as λ, λ' range over all simple closed geodesics such that λ has finite negative slope and λ' has finite positive slope; indeed the d–distance between the slopes of such λ and λ' is equal to 1.

We could prove this geometrically, but we will instead use Proposition 7 and the fact that the Lipschitz equivalence class of $d_{\log}$ is invariant under diffeomorphisms of T. Recall that every diffeomorphism of T acts on the slope set $\mathbb{Q} \cup \{\infty\}$ by linear fractional maps $x \mapsto \frac{ax+b}{cx+d}$, with a, b, c, $d \in \mathbb{Z}$ and

$ad - bc = \pm 1$, and that every such linear fractional map is realized by a diffeomorphism of T.

First consider the case where the slope $\frac{p}{q}$ of λ is in the interval $[-1, 0[$. Let φ_1 be a diffeomorphism of T whose action on the slopes is given by $x \mapsto x+1 = \frac{x+1}{0x+1}$. Now $\varphi_1(\lambda)$ and $\varphi_1(\lambda')$ both have non-negative slopes, which are on different sides of the number 1. It follows from Propositions 5 and 7 that $d_{\log}(\varphi_1(\lambda), \varphi_1(\lambda')) \geqslant c_0$ for some constant $c_0 > 0$. Since φ_1 does not change the Lipschitz class of $d_{\log}$, it follows that there exists a constant $c_1 > 0$ such that $d_{\log}(\lambda, \lambda') \geqslant c_1 d_{\log}(\varphi_1(\lambda), \varphi_1(\lambda'))$. Therefore, $d_{\log}(\lambda, \lambda') \geqslant c_1 c_0$.

Similarly, when $\frac{p}{q}$ is in the interval $]\infty, -1]$, consider the diffeomorphism φ_2 of T whose action on the slopes is given by $x \mapsto \frac{x}{x+1}$. The same argument as above gives $d_{\log}(\lambda, \lambda') \geqslant c_2 c_0$. □

3 Chain-recurrent geodesic laminations on the once-punctured torus

A geodesic lamination $\lambda \in \mathcal{L}(S)$ is *chain-recurrent* if it is in the closure of the set of all multicurves (consisting of finitely many simple closed geodesics) in S. See for instance [1, Chapter 1] for an equivalent definition of chain-recurrent geodesic laminations which better explains the terminology.

When the surface S is the once-punctured torus T, a multicurve is, either a simple closed geodesic in the interior fo T (namely an element of $\mathcal{S}(T)$), or the union of ∂T and of an element of $\mathcal{S}(T)$, or just ∂T. As a consequence, a chain-recurrent geodesic lamination in the interior of T is a limit of simple closed geodesics.

Let $\mathcal{L}_0^{\mathrm{cr}}(T)$ denote the set of chain-recurrent geodesic laminations that are contained in the interior of T. By the above remarks, $\mathcal{L}_0^{\mathrm{cr}}(T)$ is also the closure in $\mathcal{L}_0(T)$ of the set $\mathcal{S}(T)$ of all simple closed geodesics.

The space $\mathcal{L}(S)$ is compact; see for instance [3, Section 3], [2, Section 4.1] or [1, Section 1.2]. Also, there is a neighborhood U of ∂T such that every complete geodesic meeting U must, either cross itself, or be asymptotic to ∂T, or be ∂T; in particular, every geodesic lamination which meets U must contain ∂T. It follows that $\mathcal{L}_0(T)$ is both open and closed in $\mathcal{L}(T)$. As a consequence, $\mathcal{L}_0^{\mathrm{cr}}(T)$ is compact.

We conclude that $(\mathcal{L}_0^{\mathrm{cr}}(T), d_{\log})$ is the completion of $(\mathcal{S}(T), d_{\log})$. By Corollary 11, $(\mathcal{L}_0^{\mathrm{cr}}(T), d_{\log})$ is therefore Lipschitz equivalent to the completion $\left(\widehat{\mathbb{Q}}, d\right)$

of $(\mathbb{Q} \cup \{\infty\}, d)$, where the metric d is defined by

$$d\left(\tfrac{p}{q}, \tfrac{p'}{q'}\right) = \max\left\{\tfrac{1}{|p''|+|q''|}; \tfrac{p}{q} \leqslant \tfrac{p''}{q''} \leqslant \tfrac{p'}{q'}\right\}$$

for $\frac{p}{q} < \frac{p'}{q'}$.

This completion $(\widehat{\mathbb{Q}}, d)$ is studied in detail in the Appendix. In particular, Proposition 35 determines its topology, and Proposition 36 computes its Hausdorff dimension and its Hausdorff measure in this dimension. These two results prove:

Theorem 12 *The space $\mathcal{L}_0^{\mathrm{cr}}(T)$ is homeomorphic to the subspace $K \cup L_1$ of the circle $\mathbb{R} \cup \{\infty\}$ obtained by adding to the standard middle third Cantor set $K \subset [0,1] \subset \mathbb{R}$ a family L_1 of isolated points consisting of exactly one point in each component of $\mathbb{R} \cup \{\infty\} - K$.*

Theorem 13 *The metric space $(\mathcal{L}_0^{\mathrm{cr}}(T), d_{\log})$ has Hausdorff dimension 2, and its 2–dimensional Hausdorff measure is equal to 0.*

4 Dynamical properties of geodesic laminations

We collect in this section a few general facts on geodesic laminations which will be useful to extend our analysis from chain-recurrent geodesic laminations to all geodesic laminations.

A geodesic lamination λ is *recurrent* is every half-leaf of λ comes back arbitrarily close to its starting point, and in the same direction. For instance, a multicurve (consisting of finitely many disjoint simple closed geodesics) is recurrent.

A geodesic lamination λ cannot be recurrent if it contains an *infinite isolated leaf*, namely a leaf g which is not closed and for which there exists a small arc k transverse to g such that $k \cap \lambda = k \cap g$ consists of a single point.

Proposition 14 *A geodesic lamination λ has finitely many connected component. It can be uniquely decomposed as the union of a recurrent geodesic lamination λ^{r} and of finitely many infinite isolated leaves which spiral along λ^{r}.*

Proof See for instance [2, Theorem 4.2.8] or [1, Chapter 1]. □

Here the statement that an infinite leaf g spirals along λ^{r} means that each half of g is asymptotic to a half-leaf contained in λ^{r}.

Let a *sink* in the geodesic lamination λ be an oriented sublamination $\lambda_1 \subset \lambda$ such that every half-leaf of $\lambda - \lambda_1$ which spirals along λ_1 does so in the direction of the orientation, and such that there is at least one such half-leaf spiralling along λ_1.

Proposition 15 [1, Chapter 1] *A geodesic lamination is chain-recurrent if and only if it contains no sink.*

As a special case of Proposition 15, every recurrent geodesic lamination is also chain-recurrent.

In our analysis of the once-punctured torus and the 4–times-punctured sphere, the following lemma will be convenient to push our arguments from chain-recurrent geodesic laminations to all geodesic laminations. We prove it in full generality since it may be of independent interest.

Lemma 16 *There exists constants c_0, $r_0 > 0$, depending only on the (negative) curvature of the metric m on S, with the following property. Let λ_1 be a geodesic lamination contained in the geodesic lamination λ and containing the recurrent part λ^{r} of λ. Then any geodesic lamination λ_1' with $d_{\mathrm{H}}(\lambda_1, \lambda_1') < r_0$ is contained in a geodesic lamination λ' with $d_{\mathrm{H}}(\lambda, \lambda') \leqslant c_0\, d_{\mathrm{H}}(\lambda_1, \lambda_1')$.*

Proof We will explain how to choose c_0 and r_0 in the course of the proof. Right now, assume that r_0 is given, and pick r with $r/2 < d_{\mathrm{H}}(\lambda_1, \lambda_1') < r \leqslant r_0$.

We claim that there is a constant $c_1 > 1$ such that, at each $x \in \lambda \cap \lambda_1'$, the angle between the lines $T_x\lambda$ and $T_x\lambda_1'$ is bounded by $c_1 r$. Indeed, since $d_{\mathrm{H}}(\lambda_1, \lambda_1') < r$, there is a point $y \in \lambda_1$ such that $\big(y, T_y\lambda_1\big)$ is at distance less than r from $\big(x, T_x\lambda_1'\big)$ in the projective tangent bundle $PT(S)$. In particular, the distance between the two points x, $y \in \lambda$ is less than r. In this situation, a classical lemma (see [4, Corollary 2.5.2] or [1, Appendix B]) asserts that, because the two leaves of λ passing through x and y are disjoint or equal, there is a constant c_2, depending only on the curvature of the metric m, such that the distance from $\big(x, T_x\lambda\big)$ to $\big(y, T_y\lambda\big) = \big(y, T_y\lambda_1\big)$ in $PT(S)$ is bounded by $c_2 d(x,y)$, and therefore by $c_2 r$. Consequently, the angle from $T_x\lambda$ to $T_x\lambda_1'$ at x, namely the distance from $\big(x, T_x\lambda\big)$ to $\big(x, T_x\lambda_1'\big)$ in $PT(S)$, is bounded by $(1 + c_2)r = c_1 r$.

Since λ_1 contains the recurrent part of λ, Proposition 14 shows that λ is the union of λ_1 and of finitely many infinite isolated leaves. Let $\widehat{\lambda}_1'$ denote the canonical lift of λ_1' to the projective tangent bundle $PT(S)$, consisting of those $\big(x, T_x\lambda_1\big) \in PT(S)$ where $x \in \lambda_1$. Let A consist of those points x in $\lambda - \lambda_1$ such that $\big(x, T_x\lambda\big)$ is at distance greater than $c_1 r$ from $\widehat{\lambda}_1'$ in $PT(S)$, where c_1 is the constant defined above. The set A is disjoint from λ_1' by choice of c_1. Because $d_{\mathrm{H}}(\lambda_1, \lambda_1') < r < c_1 r$ and because the leaves of $\lambda - \lambda_1$ spiral along the recurrent part $\lambda^{\mathrm{r}} \subset \lambda_1$, the set A stays away from the ends of $\lambda - \lambda_1$. As a consequence, A has only finitely many components a_1, a_2, …, a_n whose length is at least r.

Let us focus attention on one of these a_i, contained in an infinite isolated leaf g_i of $\lambda - \lambda_1$. Let b_i be the component of $g_i - \lambda_1'$ that contains a_i. The open interval b_i can have 0, 1 or 2 end points in g_i (corresponding to points where λ_1' transversely cuts g_i).

Let x_i be an end point of b_i. Then x_i is contained in a leaf g_i' of λ_1'. We observed that the angle between g_i and g_i' at x_i is bounded by $c_1 r$. Let k_i be the half-leaf of λ_1' delimited by x_i in g_i' which makes an angle of at least $\pi - c_1 r$ with b_i at x_i; note that k_i is uniquely determined if we choose r_0 small enough that $c_1 r \leqslant c_1 r_0 < \pi/2$.

We now construct a family h of bi-infinite or closed piecewise geodesics such that:

(1) h is the union of all the arcs b_i and of pieces of the half-leaves k_j considered above.

(2) The external angle of h_i at each corner is at most $c_1 r$.

(3) h can be perturbed to a family of disjoint simple curves contained in the complement of λ_1'.

(4) One of the two geodesic pieces meeting at each corner of h has length at least 1 (say).

As a first approximation and if we do not worry about the third condition, we can just take h to be the union of the arcs b_i and of the half-leaves k_j (of infinite length). However, with respect to this third condition, a problem arises when one half-leaf k_i collides with another k_j; more precisely when, as one follows the half-leaf k_i away from the end point x_i of b_i, one meets an end point x_j of another arc b_j (with possibly $b_j = b_i$) such that b_j is on the same side of k_i as b_i and such that the half-leaf k_j associated to x_j goes in the direction opposite to k_i. In this situation, remove from h the two half-leaves k_i and k_j and add the arc c_{ij} connecting x_i to x_j in k_i. Because the leaves of λ containing b_i and b_j do not cross each other, the length of c_{ij} will be at least 1 if we choose r_0 so that $c_1 r \leqslant c_1 r_0$ is small enough, depending on the curvature of the metric.

Iterating this process, one eventually reaches an h satisfying the required conditions.

Consider a component h_i of h. By construction, h_i is piecewise geodesic, the external angles at its corners are at most $c_1 r$, and every other straight piece of h_i has length at least 1. A Jacobi field argument then provides a constant c_3 such that h_i can be deformed to a geodesic h'_i by a homotopy which moves points by a distance bounded by $c_3 r$. Actually, a little more holds: if the homotopy sends $x \in h_i$ to $x \in h'_i$, then the distance from $\big(x, T_x h_i\big)$ to $\big(x', T_{x'} h'_i\big)$ in $PT(S)$ is bounded by $c_3 r$.

Consider the geodesics h'_i thus associated to the components h_i of h. By the Condition (3) imposed on h, the h_i are simple, two h_i and h_j are either disjoint or equal, and each h_i is either disjoint from λ'_1 or contained in it. Also, by construction of h, each end of a geodesic h'_i which is not closed is asymptotic either to a leaf of λ'_1 (containing a half-leaf k_j) or to a leaf of λ^{r} which is disjoint from λ'_1 (and containing an infinite arc b_j).

Let λ' be the union of λ'_1 and of the closure of the geodesics h'_i thus associated to the components a_i of length $\geqslant r$ of A. By the above observations, λ' is a geodesic lamination.

We want to prove that $d_{\mathrm{H}}(\lambda, \lambda') \leqslant c_0 r$ for some constant c_0. Let $\widehat{\lambda}$, $\widehat{\lambda}'$, $\widehat{\lambda}_1$ and $\widehat{\lambda}'_1$ denote the respective lifts of λ, λ', λ_1 and λ'_1 to the projective tangent bundle $PT(S)$.

If x' is a point of λ', either it belongs to λ'_1, or it belongs to one of the geodesics h'_i, or it belongs to one of the components of λ^{r} (in the closure of some h'_i). In the first and last case, it is immediate that the corresponding point $\big(x', T_{x'}\lambda'\big) \in \widehat{\lambda}'$ is at distance less than r from $\widehat{\lambda}$ in $PT(S)$. If x' is in the geodesic h'_i, then we saw that there is a point $x \in h_i$ such that the distance from $\big(x', T_{x'}\lambda'\big) = \big(x', T_{x'}h'_i\big)$ to $\big(x, T_x h_i\big)$ is at most $c_3 r$. Then $\big(x, T_x h_i\big)$ belongs to $\widehat{\lambda}$ if x is some arc b_j, and belongs to $\widehat{\lambda}'_1$ otherwise. Since $d_{\mathrm{H}}(\widehat{\lambda}_1, \widehat{\lambda}'_1) < r$, we conclude that $\big(x', T_{x'}\lambda'\big)$ is at distance at most $(c_3+1)r$ from $\widehat{\lambda}$ in this case. This proves that $\widehat{\lambda}'$ is contained in the $(c_3+1)r$–neighborhood of $\widehat{\lambda}$.

Conversely, if x is a point of λ, either $\big(x, T_x\lambda\big)$ is at distance at most $c_1 r$ from $\widehat{\lambda}'_1 \subset \widehat{\lambda}'$, or x belongs to the subset A of $\lambda - \lambda_1$ introduced at the beginning of this proof. If x belongs to one of the components a_i of A used to construct the leaves h'_i of λ', then $\big(x, T_x\lambda\big)$ is at distance less than $c_3 r$ from $\big(x, T_x h'_i\big) \in \widehat{\lambda}'$ for some $x \in h'_i \subset \lambda'$. If x belongs to a component a (of length $< r$) of A which is not one of the a_i, then x is at distance less than $\frac{1}{2}r$ from an end point y of a; in this case $\big(x, T_x\lambda\big)$ is at distance less than $\frac{1}{2}r$ from $\big(y, T_y\lambda\big)$

by definition of the metric of $PT(S)$, and $\bigl(y, T_y\lambda\bigr)$ is at distance $c_1 r$ from $\widehat{\lambda}'_1 \subset \widehat{\lambda}'$ by definition of A. We conclude that $\bigl(x, T_x\lambda\bigr)$ is at distance at most $\max\{(c_1+\frac{1}{2})r, c_3 r\}$ from $\widehat{\lambda}'$ in all cases. Consequently, $\widehat{\lambda}$ is contained in the $\max\{(c_1+\frac{1}{2})r, c_3 r\}$–neighborhood of $\widehat{\lambda}'$.

This proves that, if we set $c_0 = 2\max\bigl\{c_1+\frac{1}{2}, c_3+1\bigr\}$, then $d_{\mathrm{H}}(\lambda,\lambda') = d_{\mathrm{H}}(\widehat{\lambda},\widehat{\lambda}') \leqslant c_0 r/2 < c_0 d_{\mathrm{H}}(\lambda_1,\lambda'_1)$ by choice of r. □

5 The topology of geodesic laminations of the once-punctured torus

Every recurrent geodesic lamination admits a full support transverse measure. The following is a consequence of the fact that there is a relatively simple classification of measured geodesic laminations on the once-punctured torus. See for instance [8], or [5] using the closely related notion of measured foliations.

Proposition 17 *Every recurrent geodesic lamination in the interior of the once-punctured torus T is orientable, and admits a unique transverse measure up to multiplication by a positive real number. This establishes a correspondence between the set of recurrent geodesic laminations in the interior of T and the set of lines passing through the origin in the homology space $H_1(T;\mathbb{R})$.*

When λ corresponds to a rational line, namely to a line passing through non-zero points of $H_1(T;\mathbb{Z}) \subset H_1(T;\mathbb{R})$, the geodesic lamination λ is a simple closed geodesic, and the completion of its complement is a once-punctured open annulus. Otherwise, λ has uncountably many leaves and the completion of its complement is a once-punctured bigon, with two infinite spikes.

In this statement, the completion of the complement $S-\lambda$ of a geodesic lamination λ in a surface S means its completion for the path metric induced by the metric of S. It is always a surface with geodesic boundary and with finite area, possibly with a finite number of infinite spikes. See for instance [2, Section 4.2] or [1, Chapter 1]. For instance, when λ corresponds to an irrational line in $H_1(T;\mathbb{Z})$ in Proposition 17, the completion of $T-\lambda$ topologically is a closed annulus minus two points on one of its boundary components; the boundary of this completion consists of ∂T and of two geodesics corresponding to infinite leaves of λ and whose ends are separated by two infinite spikes.

Fix an identification of the interior of the once-punctured torus with $\bigl(\mathbb{R}^2 - \mathbb{Z}^2\bigr)/\mathbb{Z}^2$. This determines an identification $H_1(T;\mathbb{R}) \cong \mathbb{R}^2$, and a line in

$H_1(T;\mathbb{R})$ is now determined by its slope $s \in \mathbb{R} \cup \{\infty\}$. Let λ_s be the recurrent geodesic lamination associated to the line of slope s.

The identification $\mathrm{int}(T) \cong (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ also determines an orientation for T.

An immediate corollary of Proposition 17 is that, if λ is a geodesic lamination in the interior of T with recurrent part λ^{r}, the completion of $T - \lambda^{\mathrm{r}}$ contains only finitely many simple geodesics (finite or infinite). As a consequence, if we are given the recurrent part λ^{r}, there are only finitely many possibilities for λ and it is a simple exercise to list all of them. Using Proposition 15, we begin by enumerating the possibilities for chain-recurrent geodesic laminations.

Proposition 18 *The chain-recurrent geodesic laminations in the interior of the once-punctured torus T fall into the following categories:*

(1) *The recurrent geodesic lamination λ_s with irrational slope $s \in \mathbb{R} - \mathbb{Q}$.*

(2) *The simple closed geodesic λ_s with rational slope $s \in \mathbb{Q} \cup \{\infty\}$.*

(3) *The union λ_s^+ of the simple closed geodesic λ_s with slope $s \in \mathbb{Q} \cup \{\infty\}$ and of one infinite geodesic g such that, for an arbitrary orientation of λ_s, one end of g spirals on the right side of g in the direction of the orientation and the other end spirals on the left side of g in the opposite direction.*

(4) *The union λ_s^- of the simple closed geodesic λ_s with slope $s \in \mathbb{Q} \cup \{\infty\}$ and of one infinite geodesic g such that, for an arbitrary orientation of λ_s, one end of g spirals on the left side of g in the direction of the orientation and the other end spirals on the right side of g in the opposite direction.*

Note that, in Cases 3 and 4, reversing the orientation of λ_s exchanges left and right, so that λ_s^+ and λ_s^- do not depend on the choice of orientation for λ_s. These two geodesic laminations are illustrated in Figure 4.

A corollary of Proposition 18 is that every chain-recurrent geodesic lamination in the interior of the punctured torus is connected and orientable.

In Theorem 12 (based on Proposition 35 in the Appendix), we constructed a homeomorphism φ from the space $\mathcal{L}_0^{\mathrm{cr}}(T)$ of the chain-recurrent geodesic laminations to the subspace $K \cup L_1$ of $\mathbb{R} \cup \{\infty\}$ union of the standard middle third Cantor set $K \subset [0,1]$ and of a family L_1 of isolated points consisting of the point ∞ and of exactly one point in each component of $[0,1] - K$. We can revisit this construction within the framework of Proposition 18.

Proposition 19 *The homeomorphism $\varphi \colon \mathcal{L}_0^{\mathrm{cr}}(T) \to K \cup L_1$ constructed in the proof of Theorem 12 is such that:*

(i) *The image under* φ *of the subset* $\mathcal{S}(T) \subset \mathcal{L}_0^{\mathrm{cr}}(T)$ *of simple closed curves is exactly the set* L_1 *of isolated points.*

(ii) *If* s, $t \in \mathbb{Q}$ *are such that* $s < t$, *then* $\varphi(\lambda_s) < \varphi(\lambda_t)$ *in* $\mathbb{R}$.

(iii) *If* I_s *is the component of* $[0,1] - K$ *containing the image* $\varphi(\lambda_s)$ *of the simple closed geodesic* λ_s *of finite slope* $s \in \mathbb{Q}$, *then, in the notation of Proposition 18, the left end point of* I_s *is* $\varphi(\lambda_s^-)$ *and its right end point is* $\varphi(\lambda_s^+)$.

(iv) $\varphi(\lambda_\infty) = \infty$, $\varphi(\lambda_\infty^-) = 1$ *and* $\varphi(\lambda_\infty^+) = 0$.

Proof Properties (i) and (ii) are immediate from the construction of φ in Theorem 12 and Proposition 35. Note that Property (i) is satisfied by an arbitrary homeomorphism (since a homeomorphism sends isolated point to isolated point), but that this is false for Property (ii).

A consequence of the order-preserving condition of Property (ii) is that the left end point of the interval I_s is the limit of $\varphi(\lambda_t)$ as t tends to s on the left. Since λ_t tends to λ_s^- as t tends to s on the left, it follows by continuity of φ that the left end point of I_s is equal to $\varphi(\lambda_s^-)$. Similarly, the right end point of I_s is equal to the limit of $\varphi(\lambda_t)$ as t tends to s on the right, namely $\varphi(\lambda_s^+)$.

At $s = \infty$, $\varphi(\lambda_\infty) = \infty$ by construction. By Property (ii), the point 1 is equal to the limit of $\varphi(\lambda_t)$ as t tends to ∞ on the left (namely as t tends to $+\infty$). As t tends to ∞ on the left, λ_t tends to λ_∞^-, and it follows that $\varphi(\lambda_\infty^-) = 1$. Similarly, 0 is equal to the limit of $\varphi(\lambda_t)$ as t tends to ∞ on the right (namely as t tends to $-\infty$), and is therefore equal to $\varphi(\lambda_\infty^+)$. □

We saw that a recurrent geodesic lamination is connected and orientable, and that it is classified by its slope $s \in \mathbb{R} \cup \{\infty\}$. We can interpret the slope as an element of the space of unoriented lines passing through the origin in $\mathbb{R}^2$, namely as an element of the projective plane $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$.

We will need to consider the space of *oriented* recurrent geodesic laminations. This clearly is a 2–fold cover of the space of unoriented recurrent geodesic laminations, and an oriented recurrent geodesic lamination is therefore classified by its *oriented slope* $\vec{s}$, defined as an element of the space of oriented lines passing through the origin in $\mathbb{R}^2$, namely as an element of the unit circle $\mathbb{S}^1$ in $\mathbb{R}^2$. Let $\lambda_{\vec{s}}$ be the oriented recurrent geodesic lamination associated to the oriented slope $\vec{s}$. We similarly define the oriented chain-recurrent geodesic lamination $\lambda_{\vec{s}}^+$ and $\lambda_{\vec{s}}^-$ associated to the irrational oriented slope $\vec{s}$, union of the oriented geodesic lamination $\lambda_{\vec{s}}$ and of one additional geodesic as in Cases 3 and 4 of Proposition 18.

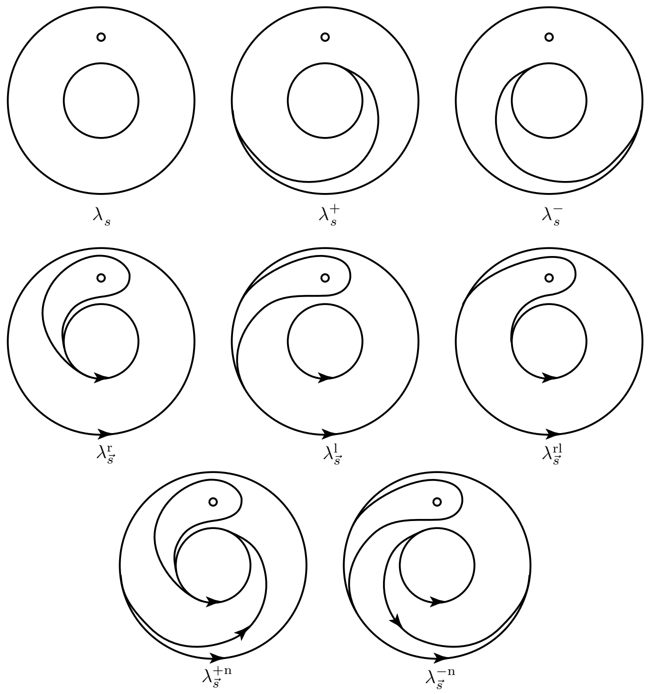

Figure 4: The geodesic laminations containing the simple closed geodesic λ_s, as seen in the completion of $T - \lambda_s$

We will say that the oriented slope $\vec{s} \in \mathbb{S}^1$ is *rational* when its associated unoriented slope $s \in \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$ is rational.

Again, a case-by-case analysis combining Propositions 15 and 17 provides:

Proposition 20 *The geodesic laminations in the interior of the punctured torus T which are not chain-recurrent fall into the following categories:*

(1) *The union $\lambda_{\vec{s}}^{\mathrm{n}}$ of the oriented geodesic lamination $\lambda_{\vec{s}}$, with irrational oriented slope $\vec{s}$, and of one additional geodesic g whose two ends converge to the same spike of $T - \lambda_{\vec{s}}$, in the direction given by the orientation.*

(2) *The union $\lambda_{\vec{s}}^{\mathrm{r}}$ of the oriented simple closed geodesic $\lambda_{\vec{s}}$, with rational oriented slope $\vec{s}$, and of one additional geodesic g whose two ends spiral on the right side of $\lambda_{\vec{s}}$ in the direction given by the orientation.*

(3) *The union $\lambda_{\vec{s}}^{\mathrm{l}}$ of the oriented simple closed geodesic $\lambda_{\vec{s}}$, with rational oriented slope $\vec{s}$, and of one additional geodesic g whose two ends spiral on the left side of $\lambda_{\vec{s}}$ in the direction given by the orientation.*

(4) *The union* $\lambda_{\vec{s}}^{\mathrm{rl}}$ *of the oriented simple closed geodesic* $\lambda_{\vec{s}}$*, with rational oriented slope* $\vec{s}$*, and of one additional geodesic* g *whose two ends spiral around* $\lambda_{\vec{s}}$ *in the direction given by the orientation, one on the right side and one on the left side.*

(5) *The union* $\lambda_{\vec{s}}^{+\mathrm{n}}$ *of the oriented chain-recurrent geodesic lamination* $\lambda_{\vec{s}}^{+}$*, with rational oriented slope* $\vec{s}$*, and of one additional geodesic* g *whose two ends converge to the same spike of* $T - \lambda_{\vec{s}}^{+}$*, in the direction of the orientation.*

(6) *The union* $\lambda_{\vec{s}}^{-\mathrm{n}}$ *of the oriented chain-recurrent geodesic lamination* $\lambda_{\vec{s}}^{-}$*, with rational oriented slope* $\vec{s}$*, and of one additional geodesic* g *whose two ends converge to the same spike of* $T - \lambda_{\vec{s}}^{-}$*, in the direction given by the orientation.*

Here the letters n, r and l respectively stand for "non-chain-recurrent", "right" and "left". Figure 5 shows the geodesic lamination $\lambda_{\vec{s}}^{\mathrm{n}}$, for an irrational oriented slope $\vec{s}$. Figure 4 illustrates the geodesic laminations $\lambda_{\vec{s}}^{\mathrm{r}}$, $\lambda_{\vec{s}}^{\mathrm{l}}$, $\lambda_{\vec{s}}^{\mathrm{rl}}$, $\lambda_{\vec{s}}^{+\mathrm{n}}$ and $\lambda_{\vec{s}}^{-\mathrm{n}}$ when the oriented slope $\vec{s}$ corresponds to one orientation of the rational slope s.

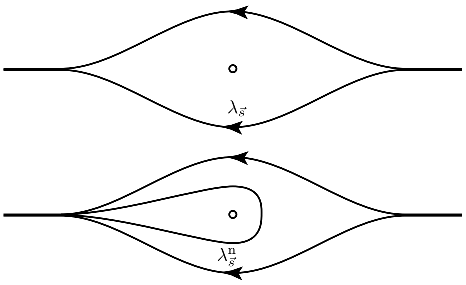

Figure 5: The geodesic laminations $\lambda_{\vec{s}}$ and $\lambda_{\vec{s}}^{\mathrm{n}}$ with $\vec{s}$ irrational, as seen in the completion of $T - \lambda_{\vec{s}}$

6 The topology of $\mathcal{L}_0(T)$ for the once-punctured torus

We already determined the topology of the space $\mathcal{L}_0^{\mathrm{cr}}(T)$ of all chain-recurrent geodesic laminations in Theorem 12. Let $\mathcal{L}_0^{\mathrm{n}}(T)$ denote its complement $\mathcal{L}_0(T) - \mathcal{L}_0^{\mathrm{cr}}(T)$, namely the space of all non-chain-recurrent geodesic laminations in the interior of the once-punctured torus T.

The following property is very specific to the once-punctured torus. For instance, we will see in Section 8 that it is false for the 4–times-punctured sphere.

Lemma 21 *The space $\mathcal{L}_0^{\mathrm{n}}(T)$ is compact.*

Proof By Proposition 15 and by inspection in Proposition 20, a geodesic lamination λ on the once-punctured torus which is not chain-recurrent admits a unique decomposition as the union of an oriented chain-recurrent geodesic lamination $\sigma(\lambda)$ (its unique sink) and of exactly one infinite isolated leaf whose two ends spiral along $\sigma(\lambda)$ in the direction of the orientation. The chain-recurrent geodesic lamination $\sigma(\lambda)$ is weakly carried by one of the two train tracks Θ^+ and Θ^- of Figure 1. By inspection, it follows that λ is weakly carried by one of the four train tracks $\Theta^\pm_\pm$ of Figure 6, unless λ is of the form $\lambda^{\mathrm{rl}}_{\vec{s}}$ where the oriented slope $\vec{s}$ corresponds to the unoriented slopes 0 or ∞. The train track $\Theta^\pm_\pm$ is made up of $\Theta^\pm$ and of one additional edge e going from one "armpit" of $\Theta^\pm$ to itself; in addition the edges of $\Theta^\pm$ are oriented in such a way that the orientations match at the switch, and that the two ends of the additional edge e merge with $\Theta^\pm$ in the direction of the orientation. Note that the infinite isolated leaf of λ is tracked by a curve carried by $\Theta^\pm_\pm$ which crosses the edge e exactly once.

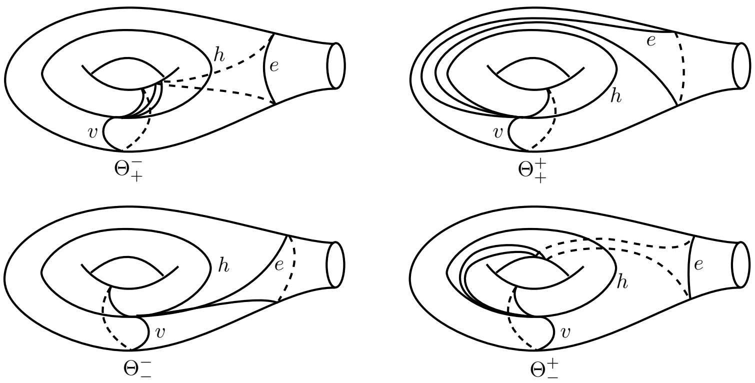

Figure 6: The train tracks $\Theta^\pm_\pm$

Let $\mathcal{L}^{\mathrm{n}}(\Theta^\pm_\pm)$ denote the space of non-chain-recurrent geodesic laminations that are weakly carried by the train track $\Theta^\pm_\pm$. We claim that $\mathcal{L}^{\mathrm{n}}(\Theta^\pm_\pm)$ is equal to the set of those $\lambda \in \mathcal{L}(\Theta^\pm_\pm)$ (namely those λ which are weakly carried by $\Theta^\pm_\pm$) which realize the edge path $\langle e \rangle$ consisting of the single edge e. Indeed, a curve which is carried by $\Theta^\pm_\pm$ can cross e at most once. If $\lambda \in \mathcal{L}(\Theta^\pm_\pm)$ realizes $\langle e \rangle$, it follows that every leaf of λ realizing $\langle e \rangle$ is isolated, and consequently that those leaves which do not realize $\langle e \rangle$ form a (closed) non-empty sublamination $\lambda^1 \subset \lambda$. Since the train track $\Theta^\pm \subset \Theta^\pm_\pm$ is oriented, we conclude that λ_1 is a sink for λ, and therefore that λ is not chain-recurrent by Proposition 15.

A corollary of this observation is that $\mathcal{L}^{\mathrm{n}}(\Theta_{\pm}^{\pm})$ is closed in $\mathcal{L}(\Theta_{\pm}^{\pm})$ (since the topology can be defined with the metric $d_{\Theta_{\pm}^{\pm}}$). Because $\mathcal{L}(\Theta_{\pm}^{\pm})$ is compact by Proposition 6, it follows that $\mathcal{L}^{\mathrm{n}}(\Theta_{\pm}^{\pm})$ is compact.

As a consequence, the space

$$\mathcal{L}^{\mathrm{n}}(T) = \mathcal{L}^{\mathrm{n}}(\Theta_{+}^{+}) \cup \mathcal{L}^{\mathrm{n}}(\Theta_{-}^{+}) \cup \mathcal{L}^{\mathrm{n}}(\Theta_{+}^{-}) \cup \mathcal{L}^{\mathrm{n}}(\Theta_{-}^{-}) \cup \{\lambda^{\mathrm{rl}}_{\vec{0}_1}, \lambda^{\mathrm{rl}}_{\vec{0}_2} \lambda^{\mathrm{rl}}_{\vec{\infty}_1} \lambda^{\mathrm{rl}}_{\vec{\infty}_2}\},$$

where the oriented slopes $\vec{0}_1$, $\vec{0}_2$, $\vec{\infty}_1$, $\vec{\infty}_2$ correspond to the unoriented slopes 0 and ∞, is compact. □

Let $\mathcal{L}_0^{\mathrm{ocr}}(T)$ denote the space of all oriented chain-recurrent geodesic laminations in the interior of T. There is a natural map $\pi\colon \mathcal{L}_0^{\mathrm{ocr}}(T) \to \mathcal{L}_0^{\mathrm{cr}}(T)$ defined by forgetting the orientation. This is a 2–fold covering map, since Proposition 18 shows that every chain-recurrent geodesic lamination is orientable and connected.

By Proposition 15 and by inspection in Proposition 20, each $\lambda \in \mathcal{L}_0^{\mathrm{n}}(T)$ uniquely decomposes as the union of a sink $\sigma(\lambda) \in \mathcal{L}_0^{\mathrm{ocr}}(T)$ and of one infinite isolated leaf whose two ends spiral around $\sigma(\lambda)$ in the direction of its orientation. This defines a map $\sigma\colon \mathcal{L}^{\mathrm{n}}(T) \to \mathcal{L}_0^{\mathrm{ocr}}(T)$.

Lemma 22 *The map $\sigma\colon \mathcal{L}^{\mathrm{n}}(T) \to \mathcal{L}_0^{\mathrm{ocr}}(T)$ is continuous.*

Proof It clearly suffices to show that the restriction of $\pi \circ \sigma$ to each subset $\mathcal{L}^{\mathrm{n}}(\Theta_{\pm}^{\pm})$ is continuous. For $\lambda \in \mathcal{L}^{\mathrm{n}}(\Theta_{\pm}^{\pm})$, we observed in the proof of Lemma 21 that $\pi \circ \sigma(\lambda) \in \mathcal{L}_0^{\mathrm{cr}}(T)$ is obtained by removing from λ the infinite isolated leaf that realizes the edge path $\langle e \rangle$. It follows that the restriction of $\pi\circ\sigma$ to $\mathcal{L}^{\mathrm{n}}(\Theta_{\pm}^{\pm})$ is distance non-increasing for the metric $d_{\Theta_{\pm}^{\pm}}$, and is therefore continuous. □

Theorem 23 *The space $\mathcal{L}_0^{\mathrm{n}}(T) = \mathcal{L}_0(T) - \mathcal{L}_0^{\mathrm{cr}}(T)$ is homeomorphic to the subspace $K \cup L_3$ of $\mathbb{R}\cup\{\infty\}$ union of the standard Cantor set $K \subset [0,1] \subset \mathbb{R}$ and of a countable set L_3 consisting of exactly 3 isolated points in each component of $\mathbb{R}\cup\{\infty\} - K$.*

Proof In Theorem 12, we constructed a homeomorphism φ from the space $\mathcal{L}_0^{\mathrm{cr}}(T)$ of the chain-recurrent geodesic laminations to the union $K \cup L_1$ of K and of a family L_1 of isolated points consisting of the point ∞ and of exactly one point in each component of $[0,1] - K$. Select the homeomorphism φ so that it satisfies the conditions of Proposition 19. In particular, φ establishes an order-preserving one-to-one correspondence between $\mathbb{Q}$ and the set of components of

$[0,1]-K$, by associating to $s\in\mathbb{Q}$ the component of $[0,1]-K$ that contains the isolated point $\varphi(\lambda_s)$.

Consider the 2–fold covering map $\mathcal{L}_0^{\mathrm{ocr}}(T)\to\mathcal{L}_0^{\mathrm{cr}}(T)$ defined by forgetting the orientation. For every component I of $\mathbb{R}\cup\{\infty\}-\varphi\left(\mathcal{L}_0^{\mathrm{cr}}(T)\right)$, it follows from Proposition 19(iii) that an orientation of the geodesic orientation corresponding under φ to one end point of I uniquely determines an orientation of the geodesic lamination corresponding to the other end point. We can consequently lift $\varphi\colon\ \mathcal{L}_0^{\mathrm{cr}}(T)\to\mathbb{R}\cup\{\infty\}$ to a continuous map $\widetilde{\varphi}\colon\ \mathcal{L}_0^{\mathrm{ocr}}(T)\to\mathbb{S}^1$, where we denote by $\mathbb{S}^1$ the circle that is the 2–fold covering of $\mathbb{R}\cup\{\infty\}$. (Of course, $\mathbb{S}^1$ is homeomorphic to $\mathbb{R}\cup\{\infty\}$, but we prefer to use a different letter to emphasize the distinction).

Let $\widetilde{K}$ denote the preimage of K in $\mathbb{S}^1$. Pick any family L_3 of isolated points in $\mathbb{S}^1-\widetilde{K}$ such that, for every component I of $\mathbb{S}^1-\widetilde{K}$, the intersection $I\cap L_3$ consists of exactly 3 points x_I^{r}, x_I^{l} and x_I^{rl}. Note that each component I of $\mathbb{S}^1-\widetilde{K}$ is now indexed by an oriented slope $\vec{s}$, namely I is the component $I_{\vec{s}}$ containing the image under $\widetilde{\varphi}$ of the oriented simple closed geodesic $\lambda_{\vec{s}}$ of oriented slope $\vec{s}$, and whose boundary points are consequently $\widetilde{\varphi}(\lambda_{\vec{s}}^-)$ and $\widetilde{\varphi}(\lambda_{\vec{s}}^+)$ by Proposition 19.

We now define a map $\psi\colon\ \mathcal{L}_0^{\mathrm{n}}(T)\to\widetilde{K}\cup L_3$ as follows. If the sink $\sigma(\lambda)\in\mathcal{L}_0^{\mathrm{ocr}}(T)$ of $\lambda\in\mathcal{L}_0^{\mathrm{n}}(T)$ is not a simple closed geodesic, define $\psi(\lambda)$ as $\widetilde{\varphi}(\sigma(\lambda))\in\widetilde{K}$. Otherwise, $\sigma(\lambda)$ is the oriented simple closed geodesic $\lambda_{\vec{s}}$ for some oriented slope $\vec{s}$, and λ is the geodesic lamination $\lambda_{\vec{s}}^{\mathrm{r}}$, $\lambda_{\vec{s}}^{\mathrm{l}}$ or $\lambda_{\vec{s}}^{\mathrm{rl}}$ with the notation of Proposition 20; in this case, define $\psi(\lambda)$ as the isolated point $x_{I_{\vec{s}}}^{\mathrm{r}}$, $x_{I_{\vec{s}}}^{\mathrm{l}}$ or $x_{I_{\vec{s}}}^{\mathrm{rl}}\in L_3$, respectively.

We will show that $\psi\colon\ \mathcal{L}_0^{\mathrm{n}}(T)\to\widetilde{K}\cup L_3$ is a homeomorphism.

Because φ is a homeomorphism, $\widetilde{\varphi}$ is injective and it immediately follows from the construction that ψ is a bijection.

To prove that $\psi\colon\ \mathcal{L}_0^{\mathrm{n}}(T)\to\widetilde{K}\cup L_3$ is continuous, we will show that for every sequence $\alpha_n\in\mathcal{L}_0^{\mathrm{n}}(T)$, $n\in\mathbb{N}$, converging to $\lambda\in\mathcal{L}_0^{\mathrm{n}}(T)$, the sequence $\psi(\alpha_n)$ admits a subsequence converging to $\psi(\lambda)$. Passing to a subsequence if necessary, we can assume that, either the sink $\sigma(\alpha_n)\in\mathcal{L}_0^{\mathrm{ocr}}(T)$ is a closed geodesic for every n, or it is a closed geodesic for no n. If the $\sigma(\alpha_n)$ are not closed geodesics, then $\psi(\alpha_n)=\widetilde{\varphi}(\sigma(\alpha_n))$ converges to $\widetilde{\varphi}(\sigma(\lambda)=\psi(\lambda)$ by continuity of $\widetilde{\varphi}$ and σ, and we are done. We can consequently assume that each $\sigma(\alpha_n)$ is an oriented closed geodesic $\lambda_{\vec{s}_n}$ of oriented rational slope $\vec{s}_n$.

If the limit $\sigma(\lambda)$ of $\sigma(\alpha_n)$ is a closed geodesic $\lambda_{\vec{s}}$, then it is isolated in $\mathcal{L}_0^{\mathrm{ocr}}(T)$ and $\sigma(\alpha_n)=\lambda_{\vec{s}}$ for n large enough. It follows that $\alpha_n=\lambda_{\vec{s}}^{\mathrm{r}}$, $\lambda_{\vec{s}}^{\mathrm{l}}$ or $\lambda_{\vec{s}}^{\mathrm{rl}}$, and therefore that the converging sequence α_n is eventually constant, equal to its limit λ. In particular, $\psi(\alpha_n)$ converges to $\psi(\lambda)$.

If $\sigma(\lambda)$ is not a closed geodesic, the sequence $\vec{s}_n$ has no constant subsequence. It follows that the length of the component of $\mathbb{S}^1 - \widetilde{K}$ containing $\psi(\alpha_n)$ tends to 0 as n tends to ∞. Since this component also contains the point $\widetilde{\varphi} \circ \sigma(\alpha_n)$ by construction of ψ, we conclude that the sequence $\psi(\alpha_n)$ converges to the limit of $\widetilde{\varphi} \circ \sigma(\alpha_n)$, namely to $\widetilde{\varphi} \circ \sigma(\lambda) = \psi(\lambda)$ by continuity of $\widetilde{\varphi}$ and σ.

This concludes the proof that the bijection $\psi \colon \mathcal{L}_0^{\mathrm{n}}(T) \to \widetilde{K} \cup L_3$ is continuous. Because $\mathcal{L}_0^{\mathrm{n}}(T)$ is compact by Lemma 21, it follows that ψ is a homeomorphism. Since there is a homeomorphism from $\mathbb{S}^1$ to $\mathbb{R} \cup \{\infty\}$ sending $\widetilde{K}$ to K, this concludes the proof of Theorem 23. □

7 The Hausdorff dimension of $\mathcal{L}_0(T)$ for the once-punctured torus

Theorem 24 *The space $(\mathcal{L}_0(T), d_{\log})$ has Hausdorff dimension 2, and its 2–dimensional Hausdorff measure is equal to 0.*

Proof Since $\mathcal{L}_0(T)$ contains $\mathcal{L}_0^{\mathrm{cr}}(T)$, which has Hausdorff dimension 2 by Theorem 13, the Hausdorff dimension of $(\mathcal{L}_0(T), d_{\log})$ is at least 2. Therefore, it suffices to show that its 2–dimensional Hausdorff measure is equal to 0.

Fix $\varepsilon > 0$ and $r > 0$. By Theorem 13, $(\mathcal{L}_0^{\mathrm{cr}}(T), d_{\log})$ has 2–dimensional measure 0. In particular, the subset $\mathcal{L}_0^{\mathrm{r}}(T)$ consisting of all recurrent geodesic laminations also has 2–dimensional Hausdorff measure 0. Therefore, we can cover $\mathcal{L}_0^{\mathrm{r}}(T)$ by a family of $d_{\log}$–balls $B(\lambda_i, r_i)$, $i \in I$, with $\lambda_i \in \mathcal{L}_0^{\mathrm{r}}(T)$, $r_i < r$ and $\sum_{i \in I} r_i^2 < \varepsilon$.

Let r_0 and c_0 be the constants of Lemma 16, and assume $r < r_0$ without loss of generality. For each of the above balls $B(\lambda_i, r_i)$, consider the balls $B(\lambda', r')$ where λ' contains λ_i and where $r' = c_0 r_i$. By Propositions 18 and 20, there are at most 11 such λ'. By Lemma 16, we can therefore cover the whole space $\mathcal{L}_0(T)$ by a family of balls $B(\lambda'_j, r'_j)$, $j \in J$, such that $r'_j < c_0 r$ and $\sum_{j \in J} (r'_j)^2 < 11 c_0^2 \varepsilon$. Since this holds for every $r < r_0$ and every ε, this proves that $(\mathcal{L}_0(T), d_{\log})$ has 2–dimensional Hausdorff measure 0. □

8 The 4–times-punctured sphere

We now consider the case where the surface S is the 4–times-punctured sphere. The analysis is very similar to that of the once-punctured torus, and we will only sketch the arguments.

Consider the group Γ of diffeomorphisms of $\mathbb{R}^2 - \mathbb{Z}^2$ consisting of all rotations of π around the points of the lattice $\mathbb{Z}^2$, and of all translations by the elements of the sublattice $(2\mathbb{Z})^2$. The quotient space $\left(\mathbb{R}^2 - \mathbb{Z}^2\right)/\Gamma$ is diffeomorphic to the interior of the 4–times-punctured sphere S.

What makes the 4–times-punctured sphere so similar to the once-punctured torus is that, in both cases, simple closed geodesics in the interior of the surface are characterized by their slope. As in the case of the once-punctured torus, every straight line with rational slope $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$ projects to a simple closed curve in S. Conversely, every simple closed curve that is not isotopic to a boundary component is obtained in this way. This establishes a one-to-one correspondence between the set $\mathcal{S}(S)$ of simple closed geodesics in the interior of the 4–times-punctured sphere S and the set of rational slopes $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$.

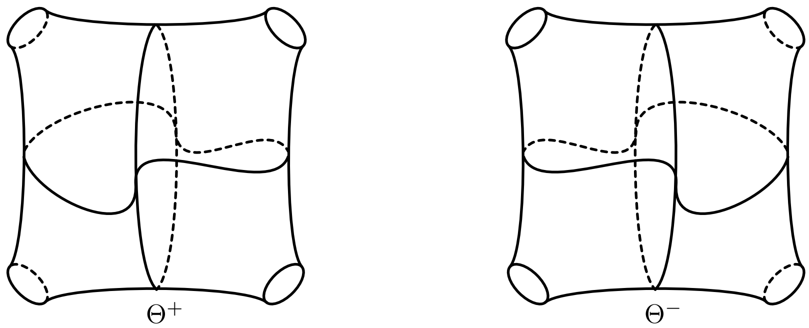

Figure 7: The train tracks Θ^+ and Θ^- on the 4–times-punctured sphere S

Consider the train tracks Θ^+ and Θ^- shown on Figure 7. For the appropriate identification between $\mathbb{R}^2/\Gamma$ and the interior of S, the preimage of Θ^+ is exactly the train track which already appeared in Figure 2 for the once-punctured torus. In particular, every simple closed geodesic $\lambda \in \mathcal{S}(S)$ with non-negative slope $\frac{p}{q} \in [0, \infty] \cap \mathbb{Q}$ is weakly carried by Θ^+, and every simple closed geodesic with non-positive slope is weakly carried by Θ^-.

The key estimate is the following analog of Proposition 7, whose proof very closely follows that of that first result.

Proposition 25 *On the 4–times-punctured sphere S, let the simple closed geodesics λ, $\lambda' \in \mathcal{S}(S)$ have slopes $\frac{p}{q}$, $\frac{p'}{q'} \in \mathbb{Q} \cup \{\infty\}$ with $0 \leqslant \frac{p}{q} < \frac{p'}{q'} \leqslant \infty$. Then*

$$d_{\Theta^+}(\lambda, \lambda') = \max\left\{ \tfrac{1}{|p''|+|q''|}; \tfrac{p}{q} \leqslant \tfrac{p''}{q''} \leqslant \tfrac{p'}{q'} \right\}.$$

As in the case of the once-punctured torus, the combination of Proposition 25 and of Proposition 35 in the Appendix provide a homeomorphism between the space $\mathcal{L}_0^{\mathrm{cr}}(S)$ of chain-recurrent geodesic laminations and the subspace $K \cup L_1$ of $\mathbb{R} \cup \{\infty\}$ obtained by adding to the standard middle third Cantor set $K \subset [0,1]$ a family L_1 of isolated points, consisting of the point ∞ and of exactly one isolated point in each component of $[0,1] - K$.

The space $\mathcal{L}_0^{\mathrm{ncr}}(S)$ of non-chain-recurrent geodesic laminations is much simpler for the 4–times-punctured sphere S than for the once-punctured torus T. To see this, we need to analyze the topology of geodesic laminations in S.

We begin by borrowing from [8] or [5] the classification of measured geodesic laminations. For the identifications $T \cong (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$ and $S \cong (\mathbb{R}^2 - \mathbb{Z}^2)/\Gamma$, it can be shown that, for every slope $s \in \mathbb{R} \cup \{\infty\}$, there is a geodesic lamination μ_s in the interior of the 4–times-punctured sphere S whose preimage to $\mathbb{R}^2 - \mathbb{Z}^2$ coincides with the preimage of the geodesic lamination $\lambda_s \in \mathcal{L}(T)$ of Section 5.

Proposition 26 *Every recurrent geodesic lamination in the interior of the 4–times-punctured sphere is of the form μ_s for some $s \in \mathbb{R} \cup \{\infty\}$. When s is rational, μ_s is a simple closed geodesic, and the completion of its complement consists of two twice-punctured disks. Otherwise, μ_s has uncountably many leaves and the completion of its complement consists of four once-punctured monogons, each with one spike.*

In particular, the completion of the complement of each μ_s again contains only finitely many simple geodesics. A case-by-case analysis then provides the following two statements.

Proposition 27 *The chain-recurrent geodesic laminations in the interior of the 4–times-punctured sphere S fall into the following categories:*

1. *The recurrent geodesic lamination μ_s, with $s \in \mathbb{R} \cup \{\infty\}$.*
2. *The union μ_s^+ of the simple closed geodesic μ_s, with rational slope $s \in \mathbb{Q} \cup \{\infty\}$, and of two infinite isolated leaves, one in each component of $S - \mu_s$; for an arbitrary orientation of μ_s, the two ends of the leaf in the left component of $S - \mu_s$ spiral along μ_s in the direction of the orientation, and the ends of the leaf in the right component of $S - \mu_s$ spiral along μ_s in the opposite direction.*

(3) *The union μ_s^- of the simple closed geodesic μ_s, with rational slope $s \in \mathbb{Q} \cup \{\infty\}$, and of two infinite isolated leaves, one in each component of $S - \mu_s$; for an arbitrary orientation of μ_s, the two ends of the leaf in the right component of $S - \mu_s$ spirals along μ_s in the direction of the orientation, and the ends of the leaf in the left component of $S - \mu_s$ spiral along μ_s in the opposite direction.*

Note that the μ_s with irrational slopes s, as well as the μ_s^+ and μ_s^- (with rational slopes) are non-orientable. In particular, by Proposition 15, every non-chain-recurrent geodesic lamination is obtained by adding to a simple closed geodesic μ_s, with rational slope $s \in \mathbb{Q} \cup \{\infty\}$, a certain number of infinite isolated leaves whose ends all spiral along μ_s in the same direction. Looking at possibilities, one concludes:

Proposition 28 *Any non-chain-recurrent geodesic lamination in the interior of the 4–times-punctured sphere is the union of a closed geodesic μ_s, with rational slope $s \in \mathbb{Q} \cup \{\infty\}$, and of 1 or 2 infinite isolated leaves whose ends all spiral along μ_s in the same direction. A given simple closed geodesic μ_s is contained in exactly 6 such non-chain-recurrent geodesic laminations.*

As in the proof of Theorem 23, we can then use Propositions 27 and 28 to push our analysis of the topology of $\mathcal{L}_0^{\mathrm{cr}}(S)$ to $\mathcal{L}_0(S)$.

Theorem 29 *For the 4–times-punctured sphere S, the space $\mathcal{L}_0(S)$ is homeomorphic to the subspace $K \cup L_7$ of $\mathbb{R} \cup \{\infty\}$ union of the standard middle third Cantor set $K \subset [0,1]$ and of a set L_7 of isolated points consisting of exactly 7 points in each component of $\mathbb{R} \cup \{\infty\} - K$. The homeomorphism can be chosen so that the set $\mathcal{S}(S) \subset \mathcal{L}_0(S)$ of all simple closed geodesics corresponds to a subset $L_1 \subset L_7$ consisting of exactly 1 point in each component of $\mathbb{R} \cup \{\infty\} - K$. The closure of $\mathcal{S}(S)$, namely the space $\mathcal{L}_0^{\mathrm{cr}}(S)$ of chain-recurrent geodesic laminations, then corresponds to the union $K \cup L_1$. Its complement, the space $\mathcal{L}_0^{\mathrm{n}}(S)$ of non-chain-recurrent geodesic laminations, is countable.*

More precisely, each component I of $\mathbb{R} \cup \{\infty\} - K$ is indexed by the rational slope $s \in \mathbb{Q} \cup \{\infty\}$ of the simple closed geodesic μ_s whose image is contained in I. The end points of the interval I then correspond to the chain-recurrent geodesic laminations $\mu_s^{\pm}$, and the intersection of the interior of I with $\widehat{X}$ corresponds to μ_s and to the 6 non-chain-recurrent geodesic laminations containing it.

The Hausdorff dimension and measure of $(\mathcal{L}_0(S), d_{\log})$ are obtained by combining Propositions 5, 25, 36 and the fact that $\mathcal{L}_0^{\mathrm{ncr}}(S)$ is countable.

Theorem 30 *For the 4–times-punctured sphere S, the Hausdorff dimension of the metric space $(\mathcal{L}_0(S), d_{\log})$ is equal to 2. Its 2–dimensional Hausdorff measure is equal to 0.*

9 Very small surfaces

Having considered the once-punctured torus or the 4–times-punctured sphere, we may wonder about surfaces of lower complexity. Geodesic laminations on a surface S make sense only when S admits a metric of negative curvature for which the boundary is totally geodesic, namely when the Euler characteristic $\chi(S)$ is negative. This leaves the 3–times-punctured sphere, the twice-punctured projective plane and the once-punctured Klein bottle.

We will see that the geodesic lamination spaces of these surfaces are relatively trivial, thereby justifying the emphasis of this paper on the once-punctured torus and on the 4–times-punctured sphere.

Proposition 31 *If the surface S is the 3–times-punctured sphere or the twice-punctured projective plane, then the space $\mathcal{L}(S)$ of geodesic laminations on S is finite.*

Proof Each of these surfaces has only finitely many homotopy classes of simple closed curves. It follows that they contain only finitely many multicurves and therefore that every chain-recurrent geodesic lamination is a multicurve. In particular, every recurrent geodesic lamination is a multicurve, since it is chain-recurrent by Proposition 15.

If S is the 3–times-punctured sphere, each multicurve is in addition contained in the boundary ∂S. There are only finitely many simple arcs $a \subset S$ with $\partial a \subset \partial S$, modulo homotopy keeping ∂a in ∂S. It easily follows that there are only finitely many infinite simple geodesics in S, spiralling along boundary components. This implies that the 3–times-punctured sphere S contains only finitely many geodesic lamination.

When S is the twice-punctured projective plane, splitting S open along a multicurve λ_1 produces a twice-punctured projective plane (when $\lambda_1 \subset \partial S$) or a 3–times-punctured sphere. Again, the twice-punctured projective plane contains only finitely many homotopy classes of simple arcs, relative to the boundary. In both cases, it follows that λ_1 can be extended to a finite number of geodesic laminations. Since S contains a finite number of multicurves, $\mathcal{L}(S)$ is finite for the twice-punctured projective plane S. □

For the once-punctured Klein bottle, we restrict attention to the closed subspace $\mathcal{L}_0(S) \subset \mathcal{L}(S)$ consisting of those geodesic laminations that are contained in the interior of S. As indicated in the introduction, this is essentially an exposition choice as the results easily extend to the whose space $\mathcal{L}(S)$, but at the expense of more cases to consider.

Proposition 32 *If S is the once-punctured Klein Bottle, then the space $\mathcal{L}_0(S)$ of geodesic laminations in the interior of S is countable infinite*[1]*. All of its points are isolated, with the exception of six limit points. The closure $\mathcal{L}_0^{\mathrm{cr}}(S)$ of the set of multicurves consists of infinitely many isolated points and of two limit points; these two limit points of $\mathcal{L}_0^{\mathrm{cr}}(S)$ are also limit points of $\mathcal{L}_0(S) - \mathcal{L}_0^{\mathrm{cr}}(S)$.*

Proof Up to homotopy, the interior of the once-punctured Klein bottle S contains only one orientation-preserving simple closed curve that is essential, in the sense that it is not parallel to the boundary and that it bounds neither a disk nor a Möbius strip. It follows that the interior of S contains only one orientation-preserving simple closed geodesic λ_∞ in the interior of S.

There are infinitely many homotopy classes of simple, orientation-reversing, closed curves, but these are easily classified. The corresponding orientation-reversing simple closed geodesics can be listed as λ_n, $n \in \mathbb{Z}$, is such a way that each λ_n is homotopic to $T^n(\lambda_0)$ where T denotes the Dehn twist around λ_∞ (well-defined once we fix an orientation on a neighborhood of the orientation-preserving simple closed geodesic λ_∞, and once we arbitrarily decide which simple closed geodesic will be called λ_0). In particular, each λ_n meets λ_∞ in exactly one point.

In addition, λ_m is disjoint from λ_n exactly when $m = n \pm 1$.

As n tends to $+\infty$, the simple closed geodesic λ_n converges to a geodesic lamination λ_∞^-, union of λ_∞ and of an infinite simple geodesic whose ends spiral on each side of λ_∞, in opposite direction. As n tends to $-\infty$, λ_n converges to a geodesic lamination λ_∞^+ of the same type, but with opposite spiralling directions on each side of λ_∞.

It follows that chain-recurrent geodesic laminations are of the following four possible types: λ_∞, λ_n, $\lambda_n \cup \lambda_{n+1}$, λ_∞^+ or λ_∞^-. In particular, the subspace $\mathcal{L}_0^{\mathrm{cr}}(S)$ consisting of all chain-recurrent geodesic lamination is countable infinite. All of its points are isolated, with the exception of two limit points corresponding to $\lambda_\infty^\pm$.

[1] And not finite, as erroneously stated in [10] and in an earlier version of the current paper!

From the above list, we see that splitting S along a recurrent geodesic lamination λ produces a 3–times-punctured sphere, or a once-punctured bigon, or a twice-punctured projective plane. By Proposition 31, it follows that such a recurrent geodesic lamination can be enlarged to finitely many geodesic laminations. By Proposition 14, this implies that the space $\mathcal{L}_0(S)$ is countable.

A limit point of $\mathcal{L}_0(S)$ must contain a limit point of $\mathcal{L}_0^{\mathrm{cr}}(S)$, by consideration of the recurrent parts of geodesic laminations converging to that limit. Therefore, a limit point of $\mathcal{L}_0(S)$ must contain λ_∞^+ or λ_∞^-. The complement of the geodesic lamination $\lambda_\infty^\pm$ is a punctured bigon. As such, this complement contains exactly two simple geodesics, each of which has its two ends converging towards one of the spikes of the bigon. It follows that $\lambda_\infty^\pm$ can be enlarged to exactly two geodesic laminations. By considering suitable enlargements of λ_n and letting n tend to $\pm\infty$, one easily sees that these enlarged geodesic laminations are indeed limit points of $\mathcal{L}_0(S)$. Therefore, $\mathcal{L}_0(S)$ has exactly 6 limit points.

Similarly, the two elements $\lambda_\infty^\pm$ are limit points of $\mathcal{L}_0(S) - \mathcal{L}_0^{\mathrm{cr}}(S)$. □

Since the geodesic lamination spaces $\mathcal{L}_0(S)$ of the 3–times-punctured sphere, the twice-punctured projective plane and the once-punctured Klein bottle are countable, their Hausdorff dimension is of course 0 for any metric, and in particular for the metric $d_{\log}$.

Appendix

In this appendix, we study the space $\mathbb{Q}\cup\{\infty\}$ with the (ultra)metric d defined by

$$d\left(\tfrac{p}{q}, \tfrac{p'}{q'}\right) = \max\left\{ \tfrac{1}{|p''|+q''} ; \tfrac{p}{q} \leqslant \tfrac{p''}{q''} \leqslant \tfrac{p'}{q'} \right\}$$

when $\frac{p}{q} < \frac{p'}{q'}$. More precisely, we will study the completion $\widehat{\mathbb{Q}}$ of $\mathbb{Q}\cup\{\infty\}$ for this metric.

By convention, whenever we consider a rational number $\frac{p}{q} \in \mathbb{Q}\cup\{\infty\}$, we always implicitly assume that p and q are integer and coprime, and that $q \geqslant 0$. In particular, $0 = \frac{0}{1}$ and $\infty = \frac{1}{0} = \frac{-1}{0}$.

Recall that the completion $\widehat{\mathbb{Q}}$ can be defined as the set of equivalence classes of Cauchy sequences in $\mathbb{Q}\cup\{\infty\}$, where two Cauchy sequences are equivalent when their union is a Cauchy sequence. We consequently need to analyze Cauchy sequences in $(\mathbb{Q}\cup\{\infty\}, d)$.

Lemma 33 *A sequence $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ in $(\mathbb{Q}\cup\{\infty\}, d)$ is Cauchy if and only if one of the following holds:*

(1) *The sequence converges to an irrational number for the usual topology of $\mathbb{R}\cup\{\infty\}$.*

(2) *The sequence converges to a rational number $\frac{p}{q}\in\mathbb{Q}\cup\{\infty\}$ for the usual topology of $\mathbb{R}\cup\{\infty\}$ and, for n sufficiently large, $\frac{p_n}{q_n}$ stays on one side of $\frac{p}{q}$.*

(3) *For n sufficiently large, $\frac{p_n}{q_n}$ is equal to a fixed rational number $\frac{p}{q}\in\mathbb{Q}\cup\{\infty\}$.*

Proof First assume that the sequence $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ is Cauchy for the metric d.

By compactness, $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ admits a subsequence which converges for the usual topology of $\mathbb{R}\cup\{\infty\}$. If two subsequences of $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ had different limits $l\neq l'$ then, for an arbitrary rational number $\frac{p}{q}$ between l and l', we would have $d\left(\frac{p_m}{q_m},\frac{p_n}{q_n}\right)\geqslant\frac{1}{|p|+q}$ whenever $\frac{p_m}{q_m}$ is close to l and $\frac{p_n}{q_n}$ is close to l', contradicting the fact that the sequence is Cauchy. We conclude that $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ must have a limit x for the usual topology of $\mathbb{R}\cup\{\infty\}$.

If the limit x is a rational number $\frac{p}{q}$, then $d\left(\frac{p_m}{q_m},\frac{p_n}{q_n}\right)\geqslant\frac{1}{|p|+q}$ if $\frac{p_m}{q_m}\leqslant\frac{p}{q}\leqslant\frac{p_n}{q_n}$, unless $\frac{p_m}{q_m}=\frac{p}{q}=\frac{p_n}{q_n}$. This proves that, either $\frac{p_m}{q_m}$ stays on one side of $\frac{p}{q}$ for n sufficiently large, or $\frac{p_m}{q_m}=\frac{p}{q}$ for n sufficiently large.

Consequently, if the sequence $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ is Cauchy for d, then it satisfies Conditions 1–3 of the lemma.

Conversely, if $\left(\frac{p_n}{q_n}\right)_{n\in\mathbb{N}}$ satisfies Conditions (1)–(3), one easily sees that it is Cauchy for d. □

Corollary 34 *The completion $\widehat{\mathbb{Q}}$ contains a subset A which is isometric to $\mathbb{R}-\mathbb{Q}$ endowed with the metric d defined by*

$$d(x,y)=\max\left\{\frac{1}{|p|+q}; x<\frac{p}{q}<y\right\}$$

for every $x<y$ in $\mathbb{R}-\mathbb{Q}$. In addition, the complement $\widehat{\mathbb{Q}}-A$ is countable.

Proof The fact that every Cauchy sequence in $(\mathbb{Q}\cup\{\infty\}, d)$ admits a limit for the usual topology of $\mathbb{R}\cup\{\infty\}$ defines a continuous map $\pi\colon\widehat{\mathbb{Q}}\to\mathbb{R}\cup\{\infty\}$. By Lemma 33, the preimage of an irrational point under π consists of a single point of $\widehat{\mathbb{Q}}$. It follows that the restriction of π induces a bijection between

$A = \pi^{-1}(\mathbb{R} - \mathbb{Q})$ and $\mathbb{R} - \mathbb{Q}$. In addition, this restriction of π to A clearly sends the metric of $\widehat{\mathbb{Q}}$ to the metric on $\mathbb{R} - \mathbb{Q}$ indicated.

By Lemma 33, the preimage of a rational point $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$ under π consists of 3 points in $\widehat{\mathbb{Q}}$: one corresponding to the constant sequence $\frac{p}{q}$, and the other two corresponding to each side of $\frac{p}{q}$. It follows that the complement $\pi^{-1}(\mathbb{Q} \cup \{\infty\})$ of A is countable. □

We now describe a topological model for $\widehat{\mathbb{Q}}$. Let $K \subset [0,1] \subset \mathbb{R}$ be the standard middle third Cantor set, and let X be a family of isolated points consisting of the point ∞ and of exactly one point in each component of $[0,1] - K$. Note that $\widehat{X} = K \cup X$ is equal to the closure of X in $\mathbb{R} \cup \{\infty\}$.

Proposition 35 *The completion $\widehat{\mathbb{Q}}$ of $\mathbb{Q} \cup \{\infty\}$ for the metric d is homeomorphic to the subspace $\widehat{X} \subset \mathbb{R} \cup \{\infty\}$ described above.*

Proof We first construct an order-preserving map $\varphi\colon X \to \mathbb{Q} \cup \{\infty\}$, using the Farey combinatorics of rational numbers; see for instance [6, Section 3.1]. (There are other possibilities to construct φ, but this one seems prettier).

In the standard construction of the Cantor set K, the set $\mathcal{I}$ of components of $[0,1] - K$ is written as an increasing union $\mathcal{I}_n$ of $2^n - 1$ disjoint intervals where: $\mathcal{I}_1$ consists of the interval $\left]\frac{1}{3}, \frac{2}{3}\right[$; the set $\mathcal{I}_{n+1}$ is obtained from $\mathcal{I}_n$ by inserting one interval of length $\frac{1}{3^n}$ between any pair of consecutive intervals of $\mathcal{I}_n$, as well as before the first interval and after the last one. Since the set $X - \{\infty\}$ was defined by picking one point in each interval of $\mathcal{I}$, this provides a description of X as an increasing union of finite sets X_n, each with 2^n elements, such that $X_0 = \{\infty\}$ and such that, for $n \geqslant 1$, X_n consists of ∞ and of exactly one point in each interval of $\mathcal{I}_n$. In particular, X_{n+1} is obtained from X_n by adding one point between each pair of consecutive elements of X_n.

The set $Y = \mathbb{Q} \cup \{\infty\}$ can similarly be written as an increasing union of finite sets with 2^n elements such that: $Y_0 = \{\infty\}$; $Y_1 = \{0, \infty\} = \{\frac{0}{1}, \frac{1}{0}\}$, $Y_2 = \{\frac{-1}{2}, \frac{0}{1}, \frac{1}{2}, \frac{1}{0}\}$; more generally, the set Y_{n+1} is obtained from Y_n by adding the point $\frac{p+p'}{q+q'}$ between any two consecutive elements $\frac{p}{q}$, $\frac{p'}{q'}$ of Y_n (counting ∞ as both $\frac{1}{0}$ and $\frac{-1}{0}$). See [6, Section 3] for a proof that Y_n contains all the $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$ with $|p| + q \leqslant n+1$, which implies that the union of the Y_n is really equal to $Y = \mathbb{Q} \cup \{\infty\}$.

Define $\varphi\colon X \to \mathbb{Q} \cup \{\infty\}$ as the unique order-preserving map which sends each 2^n–element set X_n to the 2^n–element set Y_n.

From Lemma 33, one easily sees that φ establishes a one-to-one correspondence between Cauchy sequences in $(\mathbb{Q}\cup\{\infty\},d)$ and Cauchy sequences in X (namely sequences in X which have a limit in $\widehat{X}$). Therefore, φ induces a homeomorphism between the completion $\widehat{\mathbb{Q}}$ of $(\mathbb{Q}\cup\{\infty\},d)$ and the completion $\widehat{X}$ of X. $\square$

Proposition 36 *The completion $\widehat{\mathbb{Q}}$ of $\mathbb{Q}\cup\{\infty\}$ for the metric d has Hausdorff dimension 2, and its 2–dimensional Hausdorff measure is equal to 0.*

Proof By Corollary 34, it suffices to show that $\mathbb{R}-\mathbb{Q}$, endowed with the metric d, has Hausdorff dimension 2 and 2–dimensional Hausdorff measure 0. By symmetry, it suffices to show this for $]0,\infty[-\mathbb{Q}$ endowed with the metric d.

We will make a further reduction. The map $x\mapsto\frac{x}{x+1}$ induces an isometry between the metric spaces $(]0,\infty[-\mathbb{Q},d)$ and $([0,1]-\mathbb{Q},\delta)$, where

$$\delta(x,y)=\max\{\tfrac{1}{q};x\leqslant\tfrac{p}{q}\leqslant y\}.$$

It therefore suffices to show that $([0,1]-\mathbb{Q},\delta)$ has Hausdorff dimension 2 and 2–dimensional Hausdorff measure 0.

We begin with a few elementary observations on δ–balls in $[0,1]-\mathbb{Q}$. Recall that a *Farey interval* is an interval of the form $\left[\frac{p}{q},\frac{p'}{q'}\right]$ with $p'q-pq'=1$.

Lemma 37 *Let $B\subset[0,1]-\mathbb{Q}$ be an open δ–ball of radius r. Then there is a Farey interval $I=\left[\frac{p}{q},\frac{p'}{q'}\right]$ such that $B=I-\mathbb{Q}$. In addition, the diameter of (B,δ) is equal to $\Delta(B)=\Delta(I)=\frac{1}{q+q'}$, and its Lebesgue measure is $l(I)=\frac{1}{qq'}$.*

Proof Let N_r be the finite set of all rational numbers $\frac{p}{q}\in[0,1]$ with $q\leqslant\frac{1}{r}$. It immediately follows from the definition of the metric δ that the ball B is equal to $I-\mathbb{Q}$ for the closure I of some component of $[0,1]-N_r$. It is well-known that such an I must be a Farey interval; see for instance [6, Section 3]. The formula for the diameter $\Delta(B)$ is an immediate consequence of the fact that $\min\{q'';\frac{p}{q}<\frac{p''}{q''}<\frac{p'}{q'}\}=q+q'$ for every Farey interval $I=\left[\frac{p}{q},\frac{p'}{q'}\right]$, which is elementary. The formula for the length $l(I)$ of the interval I is a straightforward computation. $\square$

The reader should beware of unexpected properties of δ–balls. If $I=\left[\frac{p}{q},\frac{p'}{q'}\right]$ is a Farey interval, then $B=I-\mathbb{Q}$ is an open δ–ball whose center is any point of B and whose radius is any number r with $\frac{1}{q+q'}<r\leqslant\min\{\frac{1}{q},\frac{1}{q'}\}$.

Lemma 38 *The Hausdorff dimension of* $([0,1]-\mathbb{Q},\delta)$ *is at least* 2.

Proof We will show that, for every s with $1<s<2$, the s–dimensional Hausdorff measure of $[0,1]-\mathbb{Q}$ is strictly positive.

For this, we will use the classical fact that, for every $\varepsilon>0$ and for almost every $x\in[0,1]$, the set $\left\{\frac{p}{q}\in\mathbb{Q};\left|x-\frac{p}{q}\right|<\frac{1}{q^{2+\varepsilon}}\right\}$ is finite; see for instance [6, 7]. As a consequence, there exists a subset $A_\varepsilon\subset[0,1]-\mathbb{Q}$ with non-zero Lebesgue measure and a number $N_\varepsilon>0$ such that $\left|x-\frac{p}{q}\right|\geqslant\frac{1}{q^{2+\varepsilon}}$ for every $x\in A_\varepsilon$ and every $\frac{p}{q}\in\mathbb{Q}$ with $q\geqslant N_\varepsilon$.

We apply this to $\varepsilon=\frac{2-s}{s-1}>0$. If $x\in A_\varepsilon$ is contained in a Farey interval $I=\left[\frac{p}{q},\frac{p'}{q'}\right]$ where both q and q' are greater then N_ε, suppose $q'>q$ without loss of generality. Then,

$$\frac{1}{qq'}=l(I)\geqslant\left|x-\frac{p}{q}\right|\geqslant\frac{1}{q^{2+\varepsilon}}$$

so that $q\geqslant(q')^{\frac{1}{1+\varepsilon}}=(q')^{s-1}$. It follows that

$$\Delta(I)^s=\frac{1}{(q+q')^s}\geqslant\frac{1}{2^s(q')^s}>\frac{1}{4qq'}=\frac{1}{4}l(I)$$

in this case.

Let $r>0$ be small. Cover $[0,1]-\mathbb{Q}$ by a family of δ–balls B_i, $i\in\mathcal{I}$, with respective radii $r_i\leqslant r$. By the ultrametric property of δ, we can assume that the B_i are pairwise disjoint. (If two δ–balls meet, one is contained in the other one). We saw that each ball B_i is equal to $I_i-\mathbb{Q}$ for some Farey interval $I_i=\left[\frac{p_i}{q_i},\frac{p_i'}{q_i'}\right]$. By the ultrametric property, $r_i\geqslant\Delta(B_i)=\frac{1}{q_i+q_i'}$. In particular, at least one of q_i, q_i' is greater than $\frac{1}{2r_i}\geqslant\frac{1}{2r}$, and therefore is greater than N_ε if we choose r small enough.

Decompose the index set $\mathcal{I}$ as the disjoint union of $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ where:

(1) $\mathcal{I}_1$ consists of those i such that I_i contains a point of A_ε and such that both q_i and q_i' are greater than N_ε.

(2) $\mathcal{I}_2$ consists of those i such that I_i meets A_ε and such that only one of q_i and q_i' is greater than N_ε.

(3) $\mathcal{I}_3$ consists of those i such that I_i does not meet A_ε.

If $i\in\mathcal{I}_1$, namely if I_i contains a point of A_ε and if both q_i and q_i' are greater than N_ε, we saw that $r_i^s\geqslant\Delta(B_i)^s=\Delta(I_i)^s\geqslant\frac{1}{4}l(I_i)$.

If $i\in\mathcal{I}_2$, namely if I_i contains a point of A_ε and if only one of q_i and q_i' is greater than N_ε, we observed that $\max\{q_i,q_i'\}\geqslant\frac{1}{2r_i}\geqslant\frac{1}{2r}$. Therefore,

$l(I_i)=\frac{1}{q_iq_i'}<2r_i<2r$. Also, note that there are at most N_ε^2 such elements of $\mathcal{I}_2$.

We conclude that

$$l(A_\varepsilon)\leqslant\sum_{i\in\mathcal{I}_1}l(I_i)+\sum_{i\in\mathcal{I}_2}l(I_i)\leqslant 4\sum_{i\in\mathcal{I}_1}r_i^s+2rN_\varepsilon^2\leqslant 4\sum_{i\in\mathcal{I}}r_i^s+2rN_\varepsilon^2$$

Taking the infimum over all coverings of $[0,1]-\mathbb{Q}$ by δ–balls of radius at most r, and letting r tend to 0, we conclude that the s–dimensional Hausdorff measure $\mathcal{H}^s\left([0,1]-\mathbb{Q},\delta\right)$ of $([0,1]-\mathbb{Q},\delta)$ is bounded from below by $\frac{1}{4}l(A_\varepsilon)>0$.

This proves that $([0,1]-\mathbb{Q},\delta)$ has non-zero s–dimensional Hausdorff measure for every $1<s<2$. It follows that the Hausdorff dimension of $([0,1]-\mathbb{Q},\delta)$ is at least 2. □

Lemma 39 *The 2–dimensional Hausdorff measure of* $([0,1]-\mathbb{Q},\delta)$ *is equal to* 0*. In particular, its Hausdorff dimension is at most* 2*.*

Proof We can use the well-known connection between Farey intervals and continued fractions. If $x\in[0,1]-\mathbb{Q}$ has continued fraction expansion

$$x=\cfrac{1}{a_1(x)+\cfrac{1}{a_2(x)+\cfrac{1}{\cdots+\cfrac{1}{a_n(x)+\cfrac{1}{\cdots}}}}}=[a_1(x),a_2(x),\dots,a_n(x),\dots],$$

then successive finite continued fractions $\frac{p_n}{q_n}=[a_1(x),a_2(x),\dots,a_n(x)]$ form Farey intervals $\left[\frac{p_n}{q_n},\frac{p_{n-1}}{q_{n-1}}\right]$ (with $\frac{p_n}{q_n}>\frac{p_{n-1}}{q_{n-1}}$ for n odd) containing x. In addition, $a_n(x)\leqslant\frac{q_n}{q_{n-1}}<a_n(x)+1$ because $q_n=a_n(x)q_{n-1}+q_{n-2}$ and $0<q_{n-2}<q_{n-1}$.

The set A of those $x=[a_1(x),a_2(x),\dots,a_n(x),\dots]$ for which the sequence $(a_n(x))_{n\in\mathbb{N}}$ is unbounded has full Lebesgue measure in $[0,1]$; see for instance [6, 7]. By our comparison between $\frac{q_n}{q_{n-1}}$ and $a_n(x)$, for every small $r>0$ and for every large $M>1$, we can cover A by a family of Farey intervals $I_i=\left[\frac{p_i'}{q_i'},\frac{p_i''}{q_i''}\right]$, $i\in\mathcal{I}$, such that $\frac{q_i'}{q_i''}>M$ and $q_i'+q_i''>\frac{1}{r}$. (We do not necessarily assume that $\frac{p_i'}{q_i'}<\frac{p_i''}{q_i''}$). For such an interval,

$$\frac{\Delta(I_i)^2}{l(I_i)}=\frac{q_i'q_i''}{(q_i'+q_i'')^2}=f\left(\frac{q_i'}{q_i''}\right)<f(M)$$

where $f(x) = \frac{x}{(x+1)^2}$ and where the inequality comes from the fact that $f(x)$ is decreasing for $x \geqslant 1$. In particular, the intersection B_i of I_i with $[0,1] - \mathbb{Q}$ is a δ–ball of radius $r_i < r$ with $r_i^2 = \Delta(I_i)^2 < f(M) l(I_i)$.

Since the complement B of A has Lebesgue measure 0, we can cover B by a family of Farey intervals I_j, $j \in \mathcal{J}$, whose total length is arbitrarily small, say $\sum_{j\in\mathcal{J}} l(I_j) < \frac{1}{M}$. In this case, the intersection B_j of I_j with $[0,1] - \mathbb{Q}$ is a d–ball of radius $r_j < r$ with $r_j^2 = \Delta(I_j)^2 \leqslant l(I_j)\,(\max f(x)) = \frac{1}{4} l(I_j)$.

In this way, we have covered $[0,1] - \mathbb{Q}$ by a family of δ–balls B_i, $i \in \mathcal{I} \cup \mathcal{J}$, with respective radii $r_i < r$ such that

$$\sum_{i\in\mathcal{I}\cup\mathcal{J}} r_i^2 \leqslant f(M) \sum_{i\in\mathcal{I}} l(I_i) + \tfrac{1}{4} \sum_{j\in\mathcal{J}} l(I_j) \leqslant f(M) + \frac{1}{4M}.$$

The estimate is valid for every $r > 0$ and every $M \geqslant 1$. Since $f(M)$ tends to 0 as M tends to ∞, it follows that the 2–dimensional Hausdorff measure of $([0,1] - \mathbb{Q}, \delta)$ is equal to 0. □

This concludes the proof of Proposition 36. □

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