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Minimal surfaces in germs of hyperbolic 3–manifolds

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Abstract This article introduces a universal moduli space for the set whose archetypal element is a pair that consists of a metric and second fundamental form from a compact, oriented, positive genus minimal surface in some hyperbolic 3–manifold. This moduli space is a smooth, finite dimensional manifold with canonical maps to both the cotangent bundle of the Teichmüller space and the space of $SO_3(\mathbb{C})$ representations for the given genus surface. These two maps embed the universal moduli space as a Lagrangian submanifold in the product of the latter two spaces.

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1 Introduction

Immersed, compact, minimal surfaces are now known to appear in every compact, hyperbolic 3–manifold. Indeed, Schoen and Yau [14] proved that a Haken hyperbolic manifold has at least one stable minimal surface. More recently, Pitts and Rubinstein [11, 10, 12], (see also [3]) proved that all compact, hyperbolic 3–manifolds have at least one unstable, immersed, minimal surface. Other authors, for example Freedman, Hass and Scott [8] and Hass and Scott [9] also have foundational papers on this subject. The ubiquity of minimal surfaces in hyperbolic 3–manifolds motivates the introduction and study of a universal moduli space for the set whose archetypal element is a pair that consists of a metric and second fundamental form from a compact, oriented, positive genus minimal surface in some hyperbolic 3–manifold. This article introduces such a moduli space and takes some (very) small steps to elucidate its properties.

In this regard, the moduli space introduced below has components that are labeled in part by the Euler class, $-\chi$, of the surface. As explained below, the component with label χ is a smooth, orientable manifold of dimension 6χ. Numerologists might notice that this number is the dimension of the cotangent bundle to the Euler class $-\chi$ Teichmüller space, and that it is also the dimension of the adjoint action quotient of the space of homomorphisms from the surface fundamental group to $\mathrm{PSL}_2(\mathbb{C})$ in its guise as the group $\mathrm{SO}_3(\mathbb{C})$. In fact, the moduli space introduced here admits a canonical map to each of the latter two spaces, and these maps play a central role in what follows.

The symbol $\mathcal{H}$ is used below to denote the union of the fixed χ components of the moduli space.

Before starting, please note that there is a tremendous body of published research on the subject of minimal surfaces in 3–manifolds. The fact is that such surfaces are an old and well used tool for studying hyperbolic 3–manifolds and 3–manifolds in general. Meanwhile, the author pleads the case of a noviate to the subjects of minimal surfaces and hyperbolic 3–manifolds. On the basis of this meager excuse, the author hereby asks to be forgiven for his rhyolitic ignorance of the fundamental work of others, and also for belaboring what may appear obvious to the experts.

In any event, Rubinstein [12] has an excellent review of various aspects of the minimal surface story as applied to 3–manifolds. Meanwhile, Colding and Minnicozzi [4] have a recent monograph that reviews many of the more analytic aspects of the subject. On the bigger subject of 3–manifolds, Scott's exposition [15] is still very much worth reading.

Finally, take note that moduli spaces of minimal surfaces have been introduced by others (Brian White [17] and also Colding and Minnicozzi [5], for example); however, the results in these studies do not appear to speak directly to the moduli space defined here.

The remainder of this article is organized as follows: The subsequent parts of this section provide the precise definition of $\mathcal{H}$ and describes its canonical maps to the cotangent bundle of Teichmüller space and to

$$\mathrm{Hom}(\pi_1(\Sigma); \mathrm{SO}_3(\mathbb{C}))/\,\mathrm{SO}_3(\mathbb{C}).$$

Section 2 explains why $\mathcal{H}$ is a smooth manifold, while Section 3 describes the critical loci of its two canonical maps. Section 4 next describes the pull-backs via these maps of certain natural symplectic structures on the Teichmüller space cotangent bundle and on $\mathrm{Hom}(\pi_1(\Sigma); \mathrm{SO}_3(\mathbb{C}))/\,\mathrm{SO}_3(\mathbb{C})$. Section 5 explains why every element in $\mathcal{H}$ arises as a minimal surface in some (usually incomplete) hyperbolic 3–manifold. Finally, Section 6 describes an open subset of $\mathcal{H}$ whose elements arise as minimal surfaces in certain complete, quasi-Fuchsian hyperbolic metrics on $\Sigma \times \mathbb{R}$. There are also a number of appendices for novices that provide derivations of more or less classical formulas.

1.1 The definition of $\mathcal{H}$

In what follows, Σ denotes a compact, oriented, 2–dimensional manifold with negative Euler characteristic. The absolute value of the Euler characteristic is denoted as χ. A pair (g, m) of Riemannian metric and symmetric section of $T^*\Sigma \otimes T^*\Sigma$ will be called a "hyperbolic germ" on Σ if the following conditions are met:

$$\begin{aligned} d_C m_{AB} - d_B m_{AC} &= 0 \\ r + (|m|^2 + \frac{1}{3}) - k^2 &= 0 \end{aligned} \tag{1.1a}$$

Here, and below, the notation is as follows: First, the subscripts indicate components with respect to some local frame for $T^*\Sigma$. Second, d_C denotes the covariant derivative defined by the metric g, the norms are defined by the metric g, and the respective functions r and k are the scalar curvature for the metric g and the trace of m as defined using the metric g. Finally, repeated indices are to be summed.

A "minimal hyperbolic germ" on Σ is a pair, (g, m), of metric and symmetric tensor that obeys both (1.1a) together with the auxiliary condition

$$k = g^{AB} m_{AB} \equiv 0. \tag{1.1b}$$

With regards to the terminology, an argument is given below to prove that there is an honest hyperbolic metric on a neighborhood of $\Sigma \times \{0\}$ in $\Sigma \times \mathbb{R}$ whose respective first and second fundamental forms on $\Sigma \times \{0\}$ are g and m when (g, m) is a hyperbolic germ. The latter metric and that defined by the line element

$$ds^2 = (g_{AB} + 2t m_{AB} + \frac{1}{2} t^2 (|m|^2 + \frac{1}{3}) g_{AB}) dz^A dz^B + dt^2 \tag{1.2}$$

agree to order t^3 near $t = 0$. In this regard, a metric given by (1.2) has $R_{ij} = -\frac{1}{3} g_{ij}$ at $t = 0$ if and only if (1.1a) holds. Thus, (1.1a) insures that the metric in (1.2) is hyperbolic to first order at $t = 0$. Granted (1.1a), the surface $\Sigma \times \{0\}$ has zero mean, extrinsic curvature with respect to this same metric if and only if (1.1b) holds. Thus, it is a minimal surface with respect to the metric in (1.2) and to the associated hyperbolic metric on a neighborhood of $\Sigma \times \{0\}$ in $\Sigma \times \mathbb{R}$. In this regard, keep in mind that the normalization used here is such that the scalar curvature of the 3–dimensional hyperbolic metric is -1; thus its sectional curvatures are $-\frac{1}{3}$.

Define an equivalence relation on the space of hyperbolic germs whereby any given pair of such germs are identified when one is obtained from the other by a diffeomorphism of Σ that lies in the component of the identity in $\mathrm{Diff}(\Sigma)$. In this regard, the infinitesimal form of the diffeomorphism group's action on the space of hyperbolic germs has a vector field, $v^A\partial_A$, sending g_{AB} and m_{AB} to

$$\begin{aligned} &\delta g_{AB} = d_A v_B + d_B v_A \\ \text{and} \quad &\delta m_{AB} = v^C d_C m_{AB} + m_{BC} d_A v_C + m_{AC} d_B v_C. \end{aligned} \tag{1.3}$$

Note that this equivalence relation preserves the subset of minimal hyperbolic germs.

Let $\mathcal{H}$ denote the quotient. Thus, $\mathcal{H} = \{\text{minimal hyperbolic germs}\}/\,\mathrm{Diff}_0(\Sigma)$, where $\mathrm{Diff}_0(\Sigma)$ is the component of the identity in the diffeomorphism group of Σ. The set $\mathcal{H}$ inherits the quotient topology with the understanding that the hyperbolic germs have the induced topology as a subset of the space of smooth, symmetric, 2×2 tensor fields on Σ. Please tolerate the notation used below whereby a pair of metric and traceless, symmetric tensor is said to "be in" $\mathcal{H}$. Of course, this means that the $\mathrm{Diff}_0(\Sigma)$ orbit of the given pair is in $\mathcal{H}$.

By the way, $\mathcal{H}$ is assuredly non-empty; indeed, if g is a hyperbolic metric on Σ with scalar curvature $-\frac{1}{3}$, then the pair $(g, 0)$ defines a point in $\mathcal{H}$. According to the upcoming Theorem 2.1, this is but a small slice of $\mathcal{H}$. In any event, the space $\mathcal{H}$ is the subject of this article. What follows is a brief summary of the story.

The space $\mathcal{H}$ is a smooth, orientable manifold whose dimension is 6χ where χ denotes the absolute value of the Euler characteristic of Σ. Moreover, $\mathcal{H}$ admits smooth maps to the cotangent bundle of Σ's Teichmüller space and to the moduli space of flat, $\mathrm{SO}_3(\mathbb{C})$ connections on Σ, both spaces with dimension 6χ. Neither map is proper and both admit critical points. This said, here is a surprise: The critical loci of these maps are identical, this being the loci of pairs in $\mathcal{H}$ where $\Sigma \times \{0\}$ has isotopies in $\Sigma \times \mathbb{R}$ that preserve its minimality to first order. However, even as the critical loci agree, the kernels of the respective differentials are linearly independent. The reasons for this coincidence are mysterious, though almost surely related to the following added surprise: The canonical symplectic forms on the cotangent bundle to Teichmüller space and on the space of flat $\mathrm{SO}_3(\mathbb{C})$ connections agree upon pull-back to $\mathcal{H}$. In particular, with the signs of these symplectic forms suitably chosen, these maps immerse $\mathcal{H}$ as a Lagrangian subvariety in the product of the cotangent bundle to Teichmüller space and the smooth portion of $\mathrm{Hom}(\pi_1; \mathrm{SO}_3(\mathbb{C}))/\,\mathrm{SO}_3(\mathbb{C})$.

By the way, an analysis of the critical loci of these maps from $\mathcal{H}$ leads to the following observation: The nullity of a compact, oriented and immersed minimal surface in a hyperbolic 3–manifold is no larger than 3 times the absolute value of its Euler class.

1.2 $\mathcal{H}$ and Teichmüller space

The additive group of smooth functions acts on the space of metrics to change the conformal factor. This is to say that a function u sends a metric g to $e^{-u}g$. This action extends to one on the space of pairs (g, m) with m left unchanged. Defined in this way, the first equation in (1.1a) is preserved by this action, as is the equation in (1.1b). The second equation is not invariant under such a change. As demonstrated in the appendix, the second equation changes as follows:

$$r_{e^{-u}g} + |m|^2_{e^{-u}g} + \frac{1}{3} = e^u(r_g + \Delta_g u + e^u|m|^2_g + \frac{1}{3}e^{-u}). \tag{1.4}$$

In any event, the quotient of the space of smooth metrics on Σ by the action of the semi-direct product of Diff_0 and $C^\infty(\Sigma)$ is called Teichmüller space. Here it is denoted by $\mathcal{T}$; it is a smooth manifold of dimension 3χ. The projection from the space of metrics to $\mathcal{T}$ induces a smooth map from $\mathcal{H}$ to $T\mathcal{T}$. For certain purposes, it is often convenient to view this projection as a map to $T^*\mathcal{T}$; this is done by using the second component of any pair (g, m) to define a measure on Σ with values in $\mathrm{Sym}^2(T\Sigma)$, this denoted by $\hat{m}$. In particular $\hat{m}$ has components

$$\hat{m}^{AB} = \det(g)^{\frac{1}{2}} g^{AC} g^{BD} m_{CD} \tag{1.5}$$

in a local coordinate frame. The tensor-valued measure $\hat{m}$ then defines a linear functional on the tangent space to the space of metrics, this the functional whose value on a symmetric tensor h is given by

$$\int_\Sigma \hat{m}^{AB} h_{AB} = \int_\Sigma g^{AC} g^{BD} m_{CD} h_{AB} d\,\mathrm{vol}_g\,. \tag{1.6}$$

Because m is traceless and obeys the top equation in (1.1a), the linear function defined by (1.6) annihilates tangent vectors at g to the orbit of $\mathrm{Diff}_0(\Sigma) \times C^\infty(\Sigma)$. Thus, it descends with g to define an element in $T^*\mathcal{T}$.

Granted the preceding, any pair $(g, m) \in \mathcal{H}$ gives a point in $T^*\mathcal{T}$, and these assignments thus define a canonical map from $\mathcal{H}$ to $T^*\mathcal{T}$.

1.3 $\mathcal{H}$ and $\mathrm{SO}_3(\mathbb{C})$

Change gears now to consider the assertion that any given pair $(g, m) \in \mathcal{H}$ can also be used to define a flat $\mathrm{SO}_3(\mathbb{C})$ connection over Σ. To elaborate, these will be connections on the complexification, E, of $T^*\Sigma \oplus \mathbb{R}$. For the purpose of defining such a connection, use the metric g with the Euclidean metric to define an inner product, $\langle , \rangle_g$, on $T^*\Sigma \oplus \mathbb{R}$. The latter induces a $\mathbb{C}$–bilinear, quadratic form on E which will be denoted in the same way. Since E is the complexification of a real 3–plane bundle, it inherits a tautological complex conjugation involution; this denoted by an overbar. Of course, this complex conjugation and the bilinear form define a hermitian inner product on E in the usual way; this the polarization of the norm whose square sends η to $\langle \overline{\eta}, \eta \rangle_g$.

An $\mathrm{SO}_3(\mathbb{C})$ connection on E is defined by its covariant derivative, ∇. In this regard, the latter must have the property that

$$d\left\langle \eta, \eta' \right\rangle_g = \left\langle \nabla\eta, \eta' \right\rangle_g + \left\langle \eta, \nabla\eta' \right\rangle_g \tag{1.7}$$

whenever η and η' are sections of E.

The flat $\mathrm{SO}_3(\mathbb{C})$ connection defined by (g, m) is best expressed using a local, oriented orthonormal frame $\{e^A\}_{A=1,2}$ for $T^*\Sigma$ and the unit vector, e^3, on $\mathbb{R}$ that points in the positive direction. With respect to such a frame, any given section η of E that is defined where the local frame is defined can be written as a column vector,

$$\eta = \begin{pmatrix} \eta_B \\ \eta_3 \end{pmatrix} \tag{1.8}$$

The covariant derivative defined by the pair (g, m) sends such a section to $\nabla\eta = \nabla_A \eta e^A$ with

$$\nabla_A \eta \equiv \begin{pmatrix} d_A \eta_B + \theta_{AB}\eta_3 \\ d_A \eta_3 - \theta_{AC}\eta_C \end{pmatrix} \tag{1.9}$$

and

$$\theta_{AB} = m_{AB} + \frac{i}{\sqrt{6}} \varepsilon_{AB}. \tag{1.10}$$

Here, d_A now denotes the metric's covariant derivative when acting on a section of $T^*\Sigma_{\mathbb{C}}$, and it denotes the exterior derivative when acting on a complex-valued function.

For use below, note parenthetically that three related, flat connections can also be defined on E. To define the first, introduce the isometry $\mathbb{T}\colon E \to E$ that acts trivially on the $\mathbb{R}$ factor and as multiplication by -1 on the $T^*\Sigma$ factor. This done, the new covariant derivative, ∇', is given by the formula $\nabla' \equiv \mathbb{T}\nabla\mathbb{T}$. In particular, ∇' is given with respect to the orthonormal frame $\{e^A\}$ by replacing every θ_{AB} by $-\theta_{AB}$ in (1.9). Note that $\mathbb{T}$ defines an automorphism of E with values in $\mathrm{SO}(3)$, so ∇ and ∇' are gauge equivalent.

The remaining two flat connections are also defined by their covariant derivatives, these denoted by $\overline{\nabla}$ and $\overline{\nabla}'$. In this regard, $\overline{\nabla}' \equiv \mathbb{T}\overline{\nabla}\mathbb{T}$ while $\overline{\nabla}$ is defined by the formula

$$\overline{\nabla}\eta \equiv \overline{\nabla\overline{\eta}}. \tag{1.11}$$

In particular, the formula for $\overline{\nabla}$ with respect to a local frame is obtained from (1.9) by replacing every θ_{AB} by its complex conjugate. Note that the formal L^2 adjoint of ∇ as defined using the Hermitian inner product on $C^\infty(E)$ is $\nabla^\dagger = -\overline{\nabla}$. Likewise, $\nabla'^\dagger = -\overline{\nabla}'$.

With an inner product fixed on $T*\Sigma \oplus \mathbb{R}$, let $\mathcal{M}$ denote the moduli space of flat $\mathrm{SO}_3(\mathbb{C})$ connections on the complexification, E. This is the quotient of the space of flat $\mathrm{SO}_3(\mathbb{C})$ connections by the action of the group of automorphisms of E. In this regard, note that parallel transport around a fixed basis for $\pi_1(\Sigma)$ identifies $\mathcal{M}$ with the quotient in $(\mathrm{SO}_3(\mathbb{C}))^{\chi+2}$ of a codimension 6 subvariety by the adjoint action of $\mathrm{SO}_3(\mathbb{C})$. Here, the subvariety in question is the inverse image of the identity in $\mathrm{SO}_3(\mathbb{C})$ under the map that sends a $(\chi+2)$–tuple of matrices $(U_1, \ldots, U_{\chi+2})$ to

$$(U_1U_2U_1^{-1}U_2^{-1})\ldots(U_{\chi+1}U_{\chi+2}U_{\chi+1}^{-1}U_{\chi+2}^{-1}). \tag{1.12}$$

It is useful at times to fix a base point, $z_0 \in \Sigma$, and take the quotient by the group of automorphisms that act as the identity on the fiber of E over z_0. The choice of a generating basis for the fundamental group $\pi_1(\Sigma; z_0)$ identifies the latter space, $\mathcal{M}^0$, with the aforementioned codimension 6 subvariety in $(\mathrm{SO}_3(\mathbb{C}))^{\chi+2}$. With regards to the structure of $\mathcal{M}^0$, note that the differential of the map f has a cokernel only at those $(\chi+2)$–tuples that consist of matrices that all fix a non-zero element in the lie algebra under the adjoint representation. A connection that corresponds to the latter sort of $(\chi+2)$–tuple is said to be "reducible". In particular, note that a connection is reducible if and only if there exists a non-zero section of E that is annihilated by the corresponding covariant derivative. The complement in $\mathcal{M}$ of the set reducible connections is a smooth manifold of dimension 6χ.

The association of the covariant derivative ∇ to a pair $(g, m) \in \mathcal{H}$ defines a continuous map from $\mathcal{H}$ to $\mathcal{M}$. As argued below, the image of $\mathcal{H}$ avoids the reducible connections.

2 The structure of $\mathcal{H}$

The following theorem is the principle result of this subsection:

Theorem 2.1 *The space $\mathcal{H}$ has the structure of a smooth, orientable manifold of dimension 6χ. Moreover, $\mathcal{H}$ comes equipped with a smooth action of S^1,* *this provided by the map that sends* $\tau \in S^1 = \mathbb{R}/(2\pi\mathbb{Z})$ *and* $(g,m) \in \mathcal{H}$ *to* $(g, \cos\tau\, m + \sin\tau\, \varepsilon \cdot m)$ *where* $\varepsilon \cdot m$ *is the symmetric, traceless tensor with components* $\varepsilon_{AC}m_{CB}$ *in a local,* g*–orthonormal frame.*

Note that this circle action is free away from the locus in $\mathcal{H}$ whose elements are pairs of the form $(g,0)$ where g is a metric on Σ with constant scalar curvature $-\frac{1}{3}$.

The remainder of this subsection is occupied with the proof of this proposition.

To start the proof, remark first that the linearization of the equations in (1.1a) and (1.1b) about any given pair (g,m) defines a differential operator; the latter denoted by $L_{(g,m)}$ in what follows. In local coordinates, this operator sends a pair consisting of a symmetric tensor and a symmetric, traceless tensor to a pair consisting of a 1–form and a functions. In particular, the map sends a pair (h_{AB}, n_{AB}) to the vector/function pair

$$\begin{aligned}
\gamma_B &\equiv \varepsilon_{CA}d_Cn_{AB} + \frac{1}{2}\varepsilon_{AC}(d_Ch_{DB})m_{AD} \\
&\quad + \frac{1}{2}(\varepsilon_{EF}d_Eh_{FC})m_{BC} + \frac{1}{2}\varepsilon_{CB}d_C(h_{EF}m_{EF}), \\
\gamma_3 &\equiv \frac{1}{2}(\frac{1}{3} - |m|^2)h_{AA} + d_Ad_Bh_{AB} - d_Bd_Bh_{AA} + 2n_{AB}m_{AB}.
\end{aligned} \tag{2.1}$$

In this regard, the first order variation of m_{AB} is not n_{AB}, but rather $n_{AB} + \frac{1}{2}g_{AB}(h_{EF}m_{EF})$; this an imposition from (1.1b).

This $L_{(g,m)}$ extends as a bounded, linear map from various Sobolev space completions of its domain to corresponding completions of the range space as a semi-Fredholm map, a map with infinite dimensional kernel, closed range and finite dimensional cokernel. For example, it has such an extension from $L^2_2(\mathrm{Sym}^2\, T^*\Sigma) \oplus L^2_1(\mathrm{Sym}^2\, T^*\Sigma)$ to $L^2(T^*\Sigma) \oplus L^2(\Sigma)$.

By the way, the kernel is infinite dimensional due to the fact that all pairs (h,n) that induce (1.3) are in its kernel. This said, introduce the operator $l_{(g,m)}$ that maps 1–forms to the domain of $L_{(g,m)}$ by the rule

$$v_B \to (d_Av_B + d_Bv_A, v^C d_Cm_{AB} + m_{BC}d_Av_C + m_{AC}d_Bv_C - g_{AB}(m_{EF}d_Ev_F)). \tag{2.2}$$

The restriction of $L_{(g,m)}$ to the L^2–orthogonal complement of the image of $l_{(g,m)}$ is then Fredholm. Let $L_{*(g,m)}$ denote this restricted operator. The implicit function theorem in conjunction with "off the shelf" differential equation technology can be employed to prove the following:

Lemma 2.2 *Let $(g, m) \in \mathcal{H}$. There exists a ball B about the origin in the kernel of $L_{*(g,m)}$, a smooth map,*

$$f\colon\ B \to \operatorname{cokernel}(L_{*(g,m)}) \tag{2.3}$$

that maps 0 to 0, and a homeomorphism from $f^{-1}(0)$ to a neighborhood of (g, m) in $\mathcal{H}$ that sends the origin to (g, m). Moreover, if $L_{*(g,m)}$ has trivial cokernel at a given (g, m), then the corresponding cokernel is trivial at all points in some neighborhood of (g, m) in $\mathcal{H}$; and this neighborhood has the structure of a smooth manifold of dimension 6χ.

Those points in $\mathcal{H}$ where the cokernel of the operator $L_{(\cdot)}$ is trivial will be called "regular points". This understood, the fact that the space $\mathcal{H}$ is a manifold follows from

Proposition 2.3 *All points in $\mathcal{H}$ are regular points.*

Proof of Proposition 2.3 A pair (σ_B, σ_3) is in the cokernel of $L_{(g,m)}$ if and only if it is L^2–orthogonal to all pairs (γ_B, γ_3) that can be written as in (2.1). It proves useful for this and other purposes to replace (σ_B, σ_3) with $v_b \equiv -\varepsilon_{BC}\sigma_C$ and $v_3 \equiv -2\sigma_3$. As is explained in Appendix D, the fact that (σ_B, σ_3) is in the cokernel of $L_{(g,m)}$ implies that

$$\begin{aligned} &\eta_B \equiv v_B + i\sqrt{6}(-\varepsilon_{CB} d_C v_3 + v_E \varepsilon_{CB} m_{EC}),\\ &\eta_3 \equiv v_3 + i\sqrt{\frac{3}{2}}\varepsilon_{EF} d_E v_F,\\ &u_B \equiv 0, \quad \text{and}\\ &u_3 = -i\sqrt{\frac{3}{2}} d_C v_C \end{aligned} \tag{2.4}$$

gives respective complex-valued sections η and u of $T^*\Sigma \oplus \mathbb{R}$ that obey the equation

$$\nabla_A \eta = \varepsilon_{AB} \nabla'_B u. \tag{2.5}$$

Granted that such is the case, it then follows that both

$$\nabla'_A \nabla_A \eta = 0 \quad \text{and} \quad \nabla_A \nabla'_A u = 0. \tag{2.6}$$

With (2.7) noted, the proof of Proposition 2.3 is obtained by invoking

Lemma 2.4 *The operators $\nabla'_A \nabla_A$ and $\nabla_A \nabla'_A$ are equal. Moreover, if η is annihilated by either, then $\eta_B = 0$ and η_3 obeys the equation*

$$d_A d_A \eta_3 + (|m|^2 - \frac{1}{3})\eta_3 = 0. \tag{2.7}$$

The proof of the Lemma 2.4 is given momentarily. To obtain Proposition 2.3, note that in the case at hand, with η given by (2.4), this lemma implies that the cokernel element to $L_{(g,m)}$ defined by (v_B, v_3) has $v_B \equiv 0$ since the latter is the real part of η_B. This then implies via the third and fourth lines in (2.4) that $u \equiv 0$ and then (2.5) requires that $v_3 = 0$ as well.

Proof of Lemma 2.4 Written in their components, the equation $\nabla'_A \nabla_A \eta = 0$ and the equation $\nabla_A \nabla'_A u = 0$ read:

$$\begin{aligned} d_A d_A \eta_B + \frac{1}{2}(|m|^2 - \frac{1}{3})\eta_B + \frac{i}{\sqrt{6}}(m_{AB}\varepsilon_{AC} + \varepsilon_{AB} m_{AC})\eta_C = 0 \\ d_A d_A \eta_3 + (|m|^2 - \frac{1}{3})\eta_3 = 0. \end{aligned} \tag{2.8}$$

To see that only $\eta_B = 0$ solves the top equation, take its $\mathbb{C}$–linear inner product with $\overline{\eta_B}$ and then integrate over Σ. After an integration by parts, one obtains the equality

$$\int_\Sigma (d_A \overline{\eta_B} d_A \eta_B) = \frac{1}{2}\int_\Sigma (|m|^2 - \frac{1}{3})\overline{\eta_B}\eta_B + \frac{i}{\sqrt{6}}\int_\Sigma (m_{AB}\varepsilon_{AC} + \varepsilon_{AB} m_{AC})\overline{\eta_B}\eta_C. \tag{2.9}$$

Now, the second term on the right side in (2.9) is purely imaginary by virtue of the fact that the tensor $m_{AB}\varepsilon_{AC}$ is symmetric when (1.1b) holds. As the other terms in (2.9) are real, it follows that

$$\int_\Sigma (d_A \overline{\eta_B} d_A \eta_B) = \frac{1}{2}\int_\Sigma (|m|^2 - \frac{1}{3})\overline{\eta_B}\eta_B. \tag{2.10}$$

To see that no such equality can hold if η_B is nonzero, note that

$$\int_\Sigma (d_A \overline{\eta_B} d_A \eta_B) = \int_\Sigma (|\varepsilon_{AC} d_A \eta_C|^2 + |d_A \eta_A|^2) - \frac{1}{2}\int_\Sigma r \overline{\eta_B}\eta_B. \tag{2.11}$$

In particular, with the value for r from (1.1a), these last two equations imply that

$$\frac{1}{2}\int_\Sigma (|m|^2 + \frac{1}{3})\overline{\eta_B}\eta_B \le \frac{1}{2}\int_\Sigma (|m|^2 - \frac{1}{3})\overline{\eta_B}\eta_B \tag{2.12}$$

which requires that $\eta_B \equiv 0$. □

As is explained next, the fact that $\mathcal{H}$ is orientable follows using the Atiyah–Singer index theorem for families of operators. To elaborate, remark first that the family in question is that defined by the pair of operators $D_{(g,m)} \equiv (L_{(g,m)}, l^*_{(g,m)})$, where $l^*_{(g,m)}$ is the formal L^2–adjoint of the operator $l_{(g,m)}$ that appears in (2.2). Thus, $D_{(g,m)}$ maps a pair consisting of a symmetric tensor and a symmetric, traceless tensor to a triple consisting of a 1–form, a function, and a vector field on Σ. This operator is defined for any pair (g, m) of metric on Σ and traceless, symmetric tensor. It is equivariant with respect to the action of $\mathrm{Diff}_0(\Sigma)$ on the space of such pairs, and it is Fredholm for any given pair (g, m). This noted, the Atiyah–Singer index theorem for families of operators [2] asserts that the kernels and cokernels of this operator as the pair (g, m) vary defines a class in the real, $\mathrm{Diff}_0(\Sigma)$–invariant real K–theory of the space of pairs of metric and symmetric, traceless tensor on Σ.

The relevance of this last point to the orientation question stems from the fact that the kernel of $D_{(g,m)}$ along the inverse image of $\mathcal{H}$ projects to $\mathcal{H}$ as its tangent space. Thus, the first Stiefel–Whitney class of the corresponding K–theory class along $\mathcal{H}$ is that of the tangent bundle to $\mathcal{H}$. In particular, $\mathcal{H}$ is orientable if the first Stiefel–Whitney class of this K–theory class is zero.

To prove that such is the case, note that the space of pairs (g, m) deformation retracts in the obvious manner onto its $m = 0$ subspace. On the latter subspace, a given version of this operator sends a pair (h, n) in its domain to

$$(\varepsilon_{AC} d_C n_{AB}, \frac{1}{6} h_{AA} + d_A d_B h_{AB} - d_B d_B h_{AA}, -2 d_A h_{AB}) \\ \in C^\infty(T^*\Sigma) \oplus C^\infty(\Sigma) \oplus C^\infty(T\Sigma). \quad (2.13)$$

The kernel of such a $(g, 0)$ version of $D_{(g,m)}$ consists of the pairs (h, n) where both are traceless and symmetric and both the $w = h$ and $w = n$ versions of the equation $\varepsilon_{AC} d_C w_{CB} = 0$ are satisfied. Meanwhile, the cokernel is trivial in all cases because no metric on a surface with genus greater then 1 has a non-trivial, conformal Killing vector field.

As just identified, the kernel space of $D_{(g,0)}$ admits the evident almost complex structure that sends a pair (h, n) to $(\varepsilon \cdot h, \varepsilon \cdot n)$; thus it can be viewed as a vector space over $\mathbb{C}$. This complex structure is compatible with the $\mathrm{Diff}_0(\Sigma)$ action, and so the $\mathrm{Diff}_0(\Sigma)$–equivariant K–theory class in question comes from a complex K–theory class. As such, its first Stiefel–Whitney class is automatically zero.

3 The maps from $\mathcal{H}$ to $\mathcal{M}$ and $T^*\mathcal{T}$

The purpose of this section is to describe certain aspects of the maps from $\mathcal{H}$ to $\mathcal{M}$ and from $\mathcal{H}$ to $T^*\mathcal{T}$. The map to $\mathcal{M}$ is considered first. In this regard, note that Lemma 2.4 has the following immediate corollary:

Proposition 3.1 *The image of $\mathcal{H}$ in $\mathcal{M}$ has no reducible connections.*

Proof of Proposition 3.1 Indeed, if A is reducible, then there is a non-trivial section of E that is annihilated by ∇_A. Such a section is then annihilated by $\nabla'_A\nabla_A$ and so it has only its third component non-trivial. However, since θ_{AB} is invertible, this is impossible if it is annihilated by ∇_A. □

To study the differentiable aspects of the map to $\mathcal{M}$, a preliminary digression is needed to introduce a "Zariski tangent space" to the $\mathrm{SO}_3(\mathbb{C})$–orbit of a flat connection: If ∇ is the corresponding covariant derivative on E, then this Zariski tangent space is the vector space of E–valued 1–forms that obey

$$\varepsilon_{AB}\nabla_A w_B = 0 \quad \text{and} \quad -\overline{\nabla}_A w_A = 0. \tag{3.1}$$

Use $T\mathcal{M}|_\nabla$ to denote the vector space of such forms. In the case that the connection is irreducible, then $T\mathcal{M}|_\nabla$ has dimension 6χ. Note that when a is any given E–valued 1–form, then Hodge theory provides its decomposition as

$$a_A = \nabla_A\alpha + \varepsilon_{AB}\overline{\nabla}_B\beta + w_A \tag{3.2}$$

where $w \in T\mathcal{M}|_\nabla$ and both α and β are sections of E. Note that this decomposition is orthogonal with respect to the L^2–hermitian inner product on $T^*\Sigma \otimes E$. Furthermore, w is uniquely defined by v; and so are α and β when the connection is irreducible.

Now, the tangent space to a pair $(g, m) \in \mathcal{H}$ can be identified with the vector space of pairs (h, n) that make (γ_B, γ_3) from (2.1) vanish identically and are such that

$$-d_A h_{AC} + \frac{1}{2} n_{AB} d_C m_{AB} - d_A(n_{AB}m_{BC} + \frac{1}{2} h_{EF} m_{EF} m_{AC}) = 0 \tag{3.3}$$

In this regard, remember that both h and n are symmetric tensors with n traceless, and that they define respective first order deformations of g and m that are given by $\delta g_{AB} = h_{AB}$ and $\delta m_{AB} = n_{AB} + h_{EF}m_{EF}g_{AB}$. The condition in (3.3) asserts only that this version of δg and δm is L^2–orthogonal to all versions of δg and δm that are given by (1.3).

To proceed, define an E–valued 1–form v by writing its components (v_{AB}, v_{A3}) as

$$\begin{aligned} v_{AB} &= -\varepsilon_{AE}n_{EB} + \frac{1}{2}\varepsilon_{AB}h_{CD}m_{CD} - \frac{1}{2}\varepsilon_{CB}m_{FA}h_{CF} - \frac{i}{2\sqrt{6}}h_{AB} \\ v_{A3} &= \varepsilon_{AE}(\frac{1}{2}d_C h_{CE} - \frac{1}{2}d_E h_{CC}). \end{aligned} \tag{3.4}$$

The vanishing of the right hand side of the equations in (2.1) is then the assertion that

$$\varepsilon_{AC}\nabla_A v_C = 0. \tag{3.5}$$

This section v of $E\otimes T^*\Sigma$ is introduced for the following reason: The "Zariski" tangent space to $\mathcal{M}$ at the connection that defines ∇ is the vector space of sections of $E\otimes T^*\Sigma$ that obey the equations $\nabla^\dagger_A q_A = 0 = \varepsilon_{AB}\nabla_A q_B$. Moreover, the L^2–orthogonal projection of v onto this space provides the image of the tangent vector (h, n) to $\mathcal{H}$ in this Zariski tangent space. This understood, the kernel of the differential at (g, m) of the map from $\mathcal{H}$ to $\mathcal{M}$ is isomorphic to the subspace of pairs (h, n) for which the corresponding v in (3.4) can be written as $v_A = \nabla_A u$ with u a smooth section of E.

Proposition 3.2 *The kernel of the differential at (g, m) of the map from $\mathcal{H}$ to $\mathcal{M}$ is canonically isomorphic to the kernel of the operator $\Delta_{(g,m)} \equiv -d_C d_C - |m|^2 + \frac{1}{3}$. Moreover, the isomorphism in question sends a function $\sigma \in \mathrm{kernel}(\Delta_{(g,m)})$ to the pair $(h, n) \in T\mathcal{H}_{(g,m)}$ that are given in a local frame by*

$$\begin{aligned} h_{AB} &= -\,(d_A\sigma_B + d_B\sigma_A) - 2m_{AB}\sigma \\ n_{CB} &= -\,(\sigma_F d_F m_{BC} + m_{CF}d_B\sigma_F + m_{BF}d_C\sigma_F - g_{CB}m_{AF}d_A\sigma_F) \\ &\quad + (d_B d_C\sigma - \frac{1}{2}g_{BC}d_A d_A\sigma); \end{aligned} \tag{3.6}$$

here $\{\sigma_A\}$ are determined by requiring (h, n) to be orthogonal to the tangent space to the orbit of $\mathrm{Diff}_0(\Sigma)$ *through (g, m).*

Proof of Proposition 3.2 There are two parts to the proof. The first argues that any pair (h, n) as given by the proposition defines a non-trivial tangent vector to $\mathcal{H}$ at (g, m). The second part proves that the kernel of the differential at (g, m) of the map to $\mathcal{M}$ has the asserted form.

Part 1: Suppose that $d_C d_C\sigma + (|m|^2 - \frac{1}{3})\sigma = 0$ and σ is not identically zero. To prove that the pair (h, n) in (3.6) defines a non-zero tangent vector to $\mathcal{H}$ it is necessary to explain why there is no choice for $\{\sigma_A\}$ that makes (h, n) identically zero, and why the pair (h, n) makes (γ_B, γ_3) in (2.1) equal to zero. For the first task, note that if $h_{AB} = 0$, then $d_C\sigma_C = 0$ and also $-|m|^2 u = m_{AB}d_A\sigma_B$. Thus, the vanishing of (3.6) is equivalent to the equations

$$\begin{aligned} d_C\sigma_C &= 0, \\ d_A\sigma_B + d_B\sigma_A - g_{AB}d_C\sigma_C - 2m_{AB}\sigma &= 0, \\ \frac{1}{2}(\frac{1}{3} + |m|^2)g_{AB} - d_A d_B\sigma + \sigma_F d_F m_{AB} + m_{CF}d_B\sigma_F + m_{BF}d_C\sigma_F &= 0. \end{aligned} \tag{3.7}$$

With the substitute notation $\sigma_A \to v_A$ and $\sigma \to v_3$, this is a version of (D.2) in Appendix D. As established in Appendix D, the latter equation is equivalent to the assertion that complex-valued section, η, of $T^*\Sigma \oplus \mathbb{R}$ that is given by the top line of (2.4) obeys the $u \equiv 0$ version of (2.5). However, according to Proposition 3.1, this means that $\eta \equiv 0$, and such can not be the case unless $\sigma \equiv 0$ as well.

The proof that (h, n) makes (γ_A, γ_3) vanish is left as a chore for the reader.

Part 2: If (h, n) is mapped to zero by the differential of the map to $\mathcal{M}$, then v given by (3.4) as the form $v = \nabla u$. This assumed, write $u \equiv \alpha + i\beta$ where α and β have purely real components. The equation $v = \nabla u$ implies that

$$h_{AB} = -\sqrt{6}(d_A\beta_B + d_B\beta_A + \varepsilon_{AB}(\varepsilon_{EF}d_E\beta_F + \sqrt{\frac{2}{3}}\alpha_3) + 2m_{AB}\beta_3). \tag{3.8}$$

Note that this last equation implies straight off that

$$\varepsilon_{EF}d_E\beta_F + \frac{\sqrt{2}}{\sqrt{3}}\alpha_3 = 0 \tag{3.9}$$

because h_{AB} is symmetric. Thus, the first line in (3.6) is seen to hold using $\sigma_B = -\sqrt{6}\beta_B$ and $\sigma \equiv -\sqrt{6}\beta_3$.

To obtain the required equation of $\sigma = -\sqrt{6}\beta_3$, note first that as v_{A3} is purely real, the equation $v = \nabla u$ also implies that

$$d_A\beta_3 - m_{AB}\beta_B - \frac{1}{\sqrt{6}}\varepsilon_{AB}\alpha_B = 0. \tag{3.10}$$

To continue, act by d_A on both sides of this last equation, sum over the index A, and invoke the requirement from (3.8) that $m_{AB}h_{AB} = -2\sqrt{6}m_{AB}d_A\beta_B - 2\sqrt{6}|m|^2\beta_3$. The resulting equation reads

$$d_Ad_A\beta_3 + |m|^2\beta_3 + \frac{1}{2\sqrt{6}}m_{AB}h_{AB} - \frac{1}{\sqrt{6}}\varepsilon_{AB}d_A\alpha_B = 0. \tag{3.11}$$

Meanwhile, a return to (3.4) for v_{AB} finds that

$$m_{AB}h_{AB} = 2\varepsilon_{AB}v_{AB} = 2\varepsilon_{AB}d_A\alpha_B - \frac{4}{\sqrt{6}}\beta_3 \tag{3.12}$$

and so (3.11) asserts that

$$d_Ad_A\beta_3 + (|m|^2 - \frac{1}{3})\beta_3 = 0. \tag{3.13}$$

To obtain the second line of (3.6), note that under the given assumptions, (3.4) implies that

$$n_{CB} - \frac{1}{2}m_{FB}h_{CF} = \varepsilon_{CA}d_A\alpha_B + \varepsilon_{CA}m_{AB}\alpha_3 + \frac{1}{\sqrt{6}}g_{CB}\beta_3. \tag{3.14}$$

Using (3.10) to write $\alpha_B = \sqrt{6}(\varepsilon_{DB} d_D \beta_3 - \varepsilon_{DB} m_{DC} \beta_C)$ then finds

$$n_{CB} = \frac{1}{2} m_{FB} h_{CF} + \sqrt{6}\varepsilon_{CA} d_A (\varepsilon_{DB} d_D \beta_3 - \varepsilon_{DB} m_{DF} \beta_F) \\ - \sqrt{\frac{3}{2}} \varepsilon_{CA} m_{AB} \varepsilon_{EF} d_E \beta_F + \frac{1}{\sqrt{6}} g_{CB} \beta_3. \quad (3.15)$$

This then rearranges as

$$n_{CB} = \frac{1}{2} m_{FB} h_{CF} + \sqrt{6} d_B d_C \beta_3 - \sqrt{6} g_{CB} d_A d_A \beta_3 \\ + \frac{1}{\sqrt{6}} g_{CB} \beta_3 - \sqrt{6} d_B (m_{CF} \beta_F) + \sqrt{6} g_{CB} m_{AF} d_A \beta_F \\ - \sqrt{\frac{3}{2}} m_{AB} d_C \beta_A + \sqrt{\frac{3}{2}} m_{AB} d_A \beta_C. \quad (3.16)$$

Meanwhile, (3.8) also has

$$-\frac{1}{2} m_{FB} h_{CF} = \frac{1}{2}\sqrt{6}(m_{AB} d_A \beta_C + m_{AB} d_C \beta_A) + \frac{1}{2}\sqrt{6} g_{CB} |m|^2 \beta_3. \quad (3.17)$$

Inserting this into (3.12) finds the second assertion in (3.6). □

The next proposition provides a characterization of the critical set for the map from $\mathcal{H}$ to $T^*\mathcal{T}$.

Proposition 3.3 *The kernel of the differential at (g, m) of the map from $\mathcal{H}$ to $T^*\mathcal{T}$ is canonically isomorphic to the kernel of the operator $\Delta_{(g,m)} \equiv -d_C d_C - |m|^2 + \frac{1}{3}$. Moreover, the isomorphism in question sends a function $\sigma \in \mathrm{kernel}(\Delta_{(g,m)})$ to the pair $(h, n) \in T\mathcal{H}_{(g,m)}$ that are given in a local frame by*

$$\begin{aligned} h_{AB} &= -(d_A \sigma_B + d_B \sigma_A) + g_{AB}\sigma \\ n_{CB} &= -(\sigma_F d_F m_{BC} + m_{CF} d_B \sigma_F + m_{BF} d_C \sigma_F - g_{CB} m_{AF} d_A \sigma_F) \end{aligned} \quad (3.18)$$

where $\{\sigma_A\}$ are determined by requiring (h, n) to be orthogonal to the tangent space to the orbit of $\mathrm{Diff}_0(\Sigma)$ through (g, m).

Proof of Proposition 3.3 Any element in the kernel of the differential of the map to $T^*\mathcal{T}$ must have the form given above for some function σ, so the proposition follows by verifying two assertions: First, if σ is a non-trivial element of the kernel of $\Delta_{(g,m)}$, then no choice for $\{\sigma_A\}$ makes (3.6) vanish. Second, if $(h, n) \in T\mathcal{H}|_{(g,m)}$ and in the kernel of the differential of the map to $T * \mathcal{T}$, then $\sigma \in \mathrm{kernel}(\Delta_{(g,m)})$.

To verify the first assertion, suppose that the right hand side of (3.14) is zero. Now define a real-valued section (v_A, v_3) of $T^*\Sigma \oplus \mathbb{R}$ by setting $v_A \equiv \sigma_A$ and $v_3 \equiv 0$. This section then satisfies (D.6) with $\kappa = 0$ and $\sigma = d_C v_C$. Now, as a consequence of Lemma D.1, the section η as defined in (D.7) obeys $\nabla'_A \nabla_A \eta = 0$ and then Lemma 2.4 asserts that $\eta_B = 0$. Thus, $v_B = \sigma_B = 0$ and so $\sigma = 0$.

To verify the second assertion, it is sufficient to verify that the pair (h, n) in (3.14) make (2.1)'s pair (γ_B, γ_3) vanish if and only if $\sigma \in \mathrm{kernel}(\Delta_{(g,m)})$. In this regard, it is sufficient to consider the case where $\{\beta_A \equiv 0\}$. In this case, $n_{CB} \equiv 0$ and γ_B vanishes identically and the second line of (2.1) asserts that $\sigma \in \mathrm{kernel}(\Delta_{(g,m)})$. □

It is almost surely the case that neither the map from $\mathcal{H}$ to $T^*\mathcal{T}$ nor that from $\mathcal{H}$ to $\mathcal{M}$ is proper. In this regard, known results about the behavior of sequences of immersed surfaces in a fixed 3–manifold (see, e.g. [1, 16, 13]) are surely relevant. In any event, non-convergent sequences in $\mathcal{H}$ can be characterized to some extent.

4 $\mathcal{H}$ and symplectic forms on $T^*\mathcal{T}$ and $\mathcal{M}$

Propositions 3.2 and 3.3 nix certain obvious candidate for a symplectic form on $\mathcal{H}$. To elaborate, let $\mathrm{Met}(\Sigma)$ denote the space of smooth Riemannian metrics on Σ and let $T^*\mathrm{Met}(\Sigma)$ denote the bundle over $\mathrm{Met}(\Sigma)$ whose fiber is the space of symmetric, measured valued sections of $\Lambda^2 T\Sigma$. Note that this space is a linear subspace of the honest cotangent bundle of Σ, the latter being a space of distribution valued sections of $\Lambda^2 T^*\Sigma$. In any event, $T^*\mathrm{Met}(\Sigma)$ has a canonical, Diff_0–invariant symplectic form; it is defined as follows: If $(\delta g, \delta\hat{m})$ and $(\delta g', \delta\hat{m}')$ are tangent vectors to $T^*\mathrm{Met}$, then their symplectic pairing is by definition

$$\int_\Sigma (\delta g_{AB} \delta\hat{m}'^{AB} - \delta g'_{AB} \delta\hat{m}^{AB}). \tag{4.1}$$

Meanwhile, the set of pairs (g, m) that obey the conditions in (1.1) embeds in $T^*\Sigma$ by the map that sends any given (g, m) to the pair $(g, \hat{m})$ where $\hat{m}$ is given in local coordinates by the tensor $\hat{m}^{AB} \equiv \det(g)^{\frac{1}{2}} g^{AC} g^{BD} m_{CD}$. Thus, $\mathcal{H}$ embeds in the quotient of $T^*\mathrm{Met}(\Sigma)$ by the action of $\mathrm{Diff}_0(\Sigma)$. In this regard, the symplectic form in (4.1) does not descend to the whole of the latter. However, it does formally descend to the $\mathrm{Diff}_0(\Sigma)$ quotient of the zero set in $T^*\mathrm{Met}(\Sigma)$ of a certain "moment map". This map, denoted by $\wp$, is the map from $T^*\mathrm{Met}(\Sigma)$ to the dual of the space of smooth 1–forms on Σ that sends a pair $(g, \hat{m})$ to the linear functional on $C^\infty(T\Sigma)$ that assigns to any given vector field (v_A) the value

$$\wp(g, \hat{m}) \cdot v \equiv \int_\Sigma \hat{m}^{AB} d_A v_B. \tag{4.2}$$

As the zero set of $\wp$ consists of those pairs $(g, \hat{m})$ such that $d_A \hat{m}^{AB} = 0$, it follows from the first line in (1.1a) and (1.1b) that $\mathcal{H}$ lies in $\wp^{-1}(0)/\operatorname{Diff}_0(\Sigma)$. As a consequence, the pairing in (4.1) defines a closed 2–form on $\mathcal{H}$. This form is denoted here by $\omega_{\mathcal{H}}$.

To be explicit, the form $\omega_{\mathcal{H}}$ is defined on any two pair of tangent vectors at the same given point in $\mathcal{H}$ as follows: Take any pair (g, m) that projects to the given point in $\mathcal{H}$ and choose respective pairs (h, n) and (h', n') that project to the given tangent vectors. Then, the pairing of the two vectors down on $\mathcal{H}$ with the closed form is

$$\int_\Sigma (h_{AB} n'_{AB} - h'_{AB} n_{AB}). \tag{4.3}$$

As (4.1) defines a closed form on $T^* \operatorname{Met}(\Sigma)$, so (4.3) defines a closed 2–form on $\mathcal{H}$. Thus, the latter is symplectic on $\mathcal{H}$ if it is non-degenerate. However, as will now be explained, it is degenerate precisely on the critical locus for the map to $T^*\mathcal{T}$. To see why, note first that the additive group of smooth functions acts on $T^* \operatorname{Met}(\Sigma)$ where by a function σ sends a given pair $(g, \hat{m})$ to $(e^{-\sigma} g, e^{\sigma} \hat{m})$. This action is compatible with that of Diff_0 if the joint action is viewed as one of a semi-direct product group. In any event, the action of $C^\infty(\Sigma)$ also preserves the symplectic form, it preserves $\wp^{-1}(0)$, and it has a moment map, p, whose zero set is $\operatorname{Diff}_0(\Sigma)$ invariant. In particular, the latter consists of the pairs $(g, \hat{m})$ for which $g_{AB} \hat{m}^{AB} = 0$. In this regard, note that any pair (g, m) that solves (1.1) maps to the subset $\wp^{-1}(0) \cap p^{-1}(0)$ of $T^* \operatorname{Met}(\Sigma)$. In particular, this means that $\mathcal{H} \subset (\wp^{-1}(0) \cap p^{-1}(0))/\operatorname{Diff}_0(\Sigma)$.

Granted these last points, note next that the quotient of $\wp^{-1}(0) \cap p^{-1}(0)$ by the semi-direct product of $\operatorname{Diff}_0(\Sigma)$ and $C^\infty(\Sigma)$ is $T^*\mathcal{T}$. In addition, the pull-back of the form in (4.1) to $\wp^{-1}(0) \cap p^{-1}(0)$ is the pull-back from $T^*\mathcal{T}$ of the canonical cotangent bundle symplectic form.

Coupled with the remarks of the previous paragraph, this then has the following consequence: The form in (4.2) is non-degenerate on $\mathcal{H}$ at non-critical points of the projection to $T^*\mathcal{T}$. Even so, it must annihilate the tangent vectors to $\mathcal{H}$ that are mapped to 0 by the differential of the projection to $T^*\mathcal{T}$.

Meanwhile, the smooth locus in $\mathcal{M}$ also has a canonical symplectic form; this induced from the complex, symplectic pairing on $C^\infty(T^*\Sigma \otimes E)$ that assigns to an ordered pair (v, v') of sections the number

$$\int_\Sigma \operatorname{Tr}(v \wedge v'). \tag{4.4}$$

Here, Tr denotes the contraction of indices using the $\mathbb{C}$–bilinear inner product on E that is induced by the product metric on $T^*\Sigma \oplus \mathbb{R}$.

To elaborate, note that when ∇ is the covariant derivative of a flat, irreducible $SO_3(\mathbb{C})$ connection on E, then the tangent space at the image of ∇ in $\mathcal{M}$ can be identified with the quotient of the kernel of the corresponding covariant exterior derivative,

$$d^\nabla\colon\, C^\infty(T^*\Sigma \otimes E) \to C^\infty(\Lambda^2 T^*\Sigma \otimes E) \tag{4.5}$$

by the image of ∇ from $C^\infty(E)$. This granted, an appeal to Stokes' theorem explains why the symplectic pairing in (4.4) sends a pair (v, v') to zero whenever $v \in \operatorname{kernel}(d^\nabla)$ and $v' \in \operatorname{image}(\nabla)$. Thus, (4.4) descends as a closed form to $\mathcal{M}$. The desired symplectic form is obtained by first taking the imaginary part of the resulting complex-valued form and then multiplying the result by the seemingly perverse factor of $-2\sqrt{6}$. Denote this form by $\omega_\mathcal{M}$. To see that $\omega_\mathcal{M}$ is non-degenerate, recall first that the quotient of the kernel of d^∇ by the image of ∇ is isomorphic to the vector space of sections v of $T^*\Sigma \otimes E$ that obey both $\varepsilon_{AB}\nabla_A v_B = 0$ and also $\overline{\nabla}_A v_A = 0$. This vector space admits the anti-holomorphic involution that sends v to $t(v)$ where $t(v)_A \equiv i\varepsilon_{AB}\overline{v_B}$. Then $\omega_\mathcal{M}(t(v), v) = \int_\Sigma |v|^2 d\,\mathrm{vol}$.

Now, the map from $\mathcal{H}$ to $\mathcal{M}$ pulls back the form $\omega_\mathcal{M}$ to a closed form on $\mathcal{H}$ and one has:

Lemma 4.1 *The pull-backs to $\mathcal{H}$ of the symplectic forms on $T^*\mathcal{T}$ and $\mathcal{M}$ agree.*

Proof of Lemma 4.1 Let $(g, m) \in \mathcal{H}$ and let ∇ denote the resulting flat $SO_3(\mathbb{C})$ connection from (1.9) and (1.10). When (h, n) is a tangent vector to $\mathcal{H}$ at (g, m), then the differential of the map from (g, m) to $\mathcal{M}$ assigns the d^∇–closed vector v in (3.4) to (h, n). If (h, n) and (h', n') are two such vectors, then the imaginary part of the pairing in (4.4) on the corresponding ordered pair (v, v') is exactly $(-2\sqrt{6})^{-1}$ times the pairing in (4.3) between (h, n) and (h', n'). □

This last result has the following amusing corollary:

Lemma 4.2 *Let $(g,m) \in \mathcal{H}$ and let $K_{\mathcal{T}}$ and $K_{\mathcal{M}}$ denote the respective kernels in $T\mathcal{H}|_{(g,m)}$ of the differentials of the maps to $T^*\mathcal{T}$ and to $\mathcal{M}$. Then the following are true:*

- *$K_{\mathcal{T}} \cap K_{\mathcal{M}} = \{0\}$ and so the differential of the map to $T^*\mathcal{T}$ is injective on $K_{\mathcal{M}}$ while that of the map to $\mathcal{M}$ is injective on $K_{\mathcal{T}}$.*
- *The cokernel at the image of (g,m) in $T^*\mathcal{T}$ of the differential of the map from $\mathcal{H}$ is the symplectic dual to the image of $K_{\mathcal{M}}$, while the cokernel at the image of (g,m) in $\mathcal{M}$ of the map from $\mathcal{H}$ is the symplectic dual of the image of $K_{\mathcal{T}}$.*
- *With suitable signs chosen for the symplectic forms on $T^*\mathcal{T}$ and $\mathcal{M}$, the maps to these spaces immerse $\mathcal{H}$ in $T^*\mathcal{T} \times \mathcal{M}$ as a immersed, Lagrangian subvariety.*

With regards to this lemma, keep in mind that Propositions 3.2 and 3.3 respectively identify $K_{\mathcal{M}}$ and $K_{\mathcal{T}}$ with the kernel of the operator $\Delta \equiv -d_C d_C + (\frac{1}{3} - |m|^2)$.

Proof of Lemma 4.2 To prove that $K_{\mathcal{T}} \cap K_{\mathcal{M}} = \{0\}$, appeal to Lemma D.1 in the Appendix to justify the remark that any element in this intersection provides a non-zero pair of sections (η, u) of E that obey $\nabla_A \eta = \varepsilon_{AB} \nabla'_B u$ As argued in Section 2, there are no such pairs.

Granted the first point of Lemma 4.1, here is why the second point holds: Since $K_{\mathcal{M}}$ is annihilated by the differential of the map to $\mathcal{M}$, it must have zero pairing with the whole of $T\mathcal{H}|_{(g,m)}$ using the pull-back of the symplectic from from $\mathcal{M}$. Thus, it has zero pairing with respect to the pull-back of the form from $T^*\mathcal{T}$. However, as it is mapped injectively to $T(T^*\mathcal{T})$ by the differential of the map from $\mathcal{H}$, this can happen only if the symplectic dual of its image in $T(T^*\mathcal{T})$ has trivial intersection with the image of $T\mathcal{H}|_{(g,m)}$. A comparison of dimensions then establishes the assertion that the symplectic dual to the image of $K_{\mathcal{M}}$ in $T(T^*\mathcal{T})$ is isomorphic to the cokernel of the differential at (g,m) of the map to $T^*\mathcal{T}$. The analogous argument where the roles of $\mathcal{M}$ and $T^*\mathcal{T}$ are switched proves the assertion in the second point of the lemma about $K_{\mathcal{T}}$ and the cokernel of the differential of the map to $\mathcal{M}$.

The third point of the lemma follows as an immediate consequence of Lemma 4.1 and the second point. In this regard, note that this consequence was pointed out to the author by Curt McMullen. □

The nullity of an immersed, minimal surface in a three manifold is defined to be the dimension of the null space of the Hessian at zero of a certain function on the space of sections of the normal bundle. To be precise, the function assigns to any given section the area of a corresponding surface in the 3–manifold; and the corresponding surface is obtained by moving the original surface distance 1 along the geodesics that are tangent to the given section. When the ambient 3–manifold is hyperbolic, the null space of this Hessian is precisely the kernel of $-d_C d_C + (\frac{1}{3} - |m|^2)$. This understood, then Lemma 4.2 leads to

Lemma 4.3 *Let M be any hyperbolic 3–manifold and let $\Sigma \subset M$ be a compact, oriented, immersed minimal surface with Euler characteristic $-\chi$. Then the nullity of Σ is no greater than 3χ.*

5 Extending the germ as an honest hyperbolic metric

The purpose of this section is to construct from any given $(g, m) \in \mathcal{H}$ a hyperbolic metric (with scalar curvature -1) on a neighborhood of $\Sigma \times \{0\}$ in $\Sigma \times \mathbb{R}$ whose respective first and second fundamental forms on $\Sigma \times \{0\}$ are g and m. In this regard, the model for such an extension is that of the pair $(g, 0)$ where g is a metric on Σ with constant scalar curvature equal to $-\frac{1}{3}$. In particular, the latter pair comes from a complete hyperbolic metric on $\Sigma \times \mathbb{R}$, this the metric given on the product of $\mathbb{R}$ with a local coordinate chart on Σ by the line element with square

$$ds^2 = \cosh^2(\frac{1}{\sqrt{6}}t) \cdot g_{AB}(z) dz^A dz^B + dt^2. \tag{5.1}$$

Now, suppose that (g, m) is some given pair from $\mathcal{H}$. The plan is to seek a hyperbolic metric near $\Sigma \times \{0\}$ whose line element has square

$$ds^2 = \gamma_{AB}(t, z) dz^A dz^B + dt^2. \tag{5.2}$$

As it turns out, such a metric can be obtained by solving at each point in Σ the ordinary differential equation

$$\begin{aligned} \partial_t \gamma &= 2\gamma\mu \\ \partial_t \mu &= -\mu^2 + \frac{1}{6}\mathbb{I}. \end{aligned} \tag{5.3}$$

where γ and μ are 2×2 matrix functions of t with $t = 0$ values $\gamma = g$ and $\mu = g^{-1}m$. In this regard, standard techniques from the theory of ordinary differential equations can be used to prove that (5.3) with the attending $t = 0$ conditions has a unique solution on some interval about $t = 0$. Moreover, the size of this interval can be bounded from below in terms of g and m. This last point implies that there is a fixed interval about 0 in $\mathbb{R}$ on which (5.3) and its attending $t = 0$ conditions has a unique solution for each point in Σ.

The equations of Gauss–Codazzi can be used to verify that the resulting $\gamma(t, z)$ defines a metric via (5.2) on a neighborhood of $\Sigma \times \{0\}$ in $\Sigma \times \mathbb{R}$ that is hyperbolic with scalar curvature -1.

6 A neighborhood of the Fuchsian locus

The purpose of this section is to describe the structure of $\mathcal{H}$ near the locus of pairs of the form $(g_0, 0)$ where g_0 is a metric on Σ with constant curvature equal to $-\frac{1}{3}$. This is called the "Fuchsian locus" because the image such a point in $\mathcal{M}$ is a Fuchsian group. In any event, this locus is observedly a smooth submanifold of $\mathcal{H}$. Moreover, the latter has a tubular neighborhood that consists of the pairs (g, m) that obey (1.1) with the auxiliary bound

$$|m|^2 < \frac{1}{3}. \tag{6.1}$$

In fact, the bound in (6.1) insures that the operator $\Delta_{(g,m)} = -d_C d_C - |m|^2 + \frac{1}{3}$ is strictly positive, and so the map from $\mathcal{H}$ to $T^*\mathcal{T}$ identifies the open subset of $\mathcal{H}$ where (6.1) holds with an open neighborhood of the zero section in $T^*\mathcal{T}$. Let $\mathcal{U}$ denote the subset in $\mathcal{H}$ where (6.1) holds.

The set $\mathcal{U}$ has the following property:

Proposition 6.1 *Each $(g, m) \in \mathcal{U}$ consists of the respective first and second fundamental form of a surface in $\mathbb{R} \times \Sigma$ that is minimal with respect to some complete hyperbolic metric. Meanwhile, the corresponding flat $\mathrm{SO}_3(\mathbb{C})$ connection defines a homomorphism from $\pi_1(\Sigma)$ to $\mathrm{PSL}_2(\mathbb{C})$ whose image is a quasi-Fuchsian group.*

Proof of Proposition 6.1 As explained in the previous section, the equations in (5.3) have a unique solution on an open neighborhood in $\Sigma \times \mathbb{R}$ of $\Sigma \times \{0\}$ if $\gamma|_{t=0} = g$ and $\mu|_{t=0} = g^{-1}m$. This noted, suppose that this solution to (5.3) extends to the whole of $\Sigma \times \mathbb{R}$. Granted that such is the case, it then follows from the algebraic structure of (5.3) that γ is a positive definite, symmetric section over $\Sigma \times \mathbb{R}$ of $T^*\Sigma \otimes T^*\Sigma$ and thus (5.2) defines a complete, hyperbolic metric on the whole of $\Sigma \times \mathbb{R}$. Of course, by virtue of the $t = 0$ conditions, the surface $\Sigma \times \{0\}$ is minimal with respect to this metric with (g, m) being its respective first and second fundamental forms.

To prove that the solution does indeed extend to $\Sigma \times \mathbb{R}$, let x denote the trace of μ and let y denote the trace of μ^2. Since

$$\operatorname{tr}(\mu^3) = \frac{1}{2}(3 \operatorname{tr}(\mu) \operatorname{tr}(\mu^2) - (\operatorname{tr} \mu)^3), \tag{6.2}$$

the second equation implies that

$$\begin{aligned} \partial_t x &= -y + \frac{1}{3}, \quad \text{and} \\ \partial_t y &= -x(3y - x^2 - \frac{1}{3}). \end{aligned} \tag{6.3}$$

Of interest are the solutions to these equations that start at $t = 0$ with $x = 0$ and $y < \frac{1}{3}$. Take note that the given initial conditions also imply that $\frac{1}{3}x^2 \le y$ at $t = 0$.

The analysis that follows considers the $t \ge 0$ evolution. The identical analysis holds for $t \le 0$ after reversing the sign of μ. In any event, to study the $t \ge 0$ case, note first that the compact subset where

$$\begin{aligned} 0 \le x \quad &\text{and} \\ \frac{1}{2}x^2 \le y \le \frac{1}{3} \end{aligned} \tag{6.4}$$

is a trapping region for the trajectories that obey (6.3). This is to say that no trajectory can exit this region. To verify this claim, simply check the direction of motion implied by (6.3) on the boundary. In this regard, note that

$$\partial_t(y - \frac{1}{2}x^2) = -2x(y - \frac{1}{2}x^2). \tag{6.5}$$

The fact that the no trajectory can leave the region where (6.4) holds implies that all trajectories that start in this region are defined for all $t \ge 0$.

The next point to make is that the flow in (6.3) has two fixed points, both at corners of the region that is delineated by (6.4); one fixed point is $(x = 0, y = \frac{1}{3})$ and the other is the point $(x = \sqrt{\frac{2}{3}}, y = \frac{1}{3})$. Since $\partial_t x > 0$ where $y < \frac{1}{3}$, the first point is repelling where $y < \frac{1}{3}$. On the other hand, the second point is an attracting fixed point. Moreover, as $\partial_t x > 0$ where $y < \frac{1}{3}$, all trajectories that start where (6.4) holds and with $y < \frac{1}{3}$ limit as $t \to \infty$ to the $(x = \sqrt{\frac{2}{3}}, y = \frac{1}{3})$ fixed point.

To see what this all implies for solutions to (6.3), introduce $\nu \equiv \mu - \frac{1}{2}x\mathbb{I}$. Thus, ν is the traceless part of μ. Then (5.3) asserts that

$$\partial_t \nu = -x\nu. \tag{6.6}$$

Since $x \in (0, \sqrt{\frac{2}{3}})$ for all $t \geq 0$, this last equation implies that the solution $\nu(t)$ to (6.6) exists for all $t \geq 0$. Moreover,

$$|\nu(t)| \leq \text{constant}\, e^{-\sqrt{\frac{2}{3}}t}. \tag{6.7}$$

Thus, the solution μ to the second line in (5.3) exists for all $t \geq 0$.

Granted (6.7), it then follows from the first line in (5.3) that γ exists for all $t > 0$ as well. Moreover,

$$\gamma \sim \gamma_+ e^{\sqrt{\frac{2}{3}}t} \tag{6.8}$$

as $t \to \infty$, where γ_+ is a smooth metric on Σ.

With the preceding understood, it remains only to establish that the image of the pair (g, m) in $\mathcal{M}$ defines a quasi-Fuchsian representation of $\pi_1(\Sigma)$ into $\mathrm{PSL}_2(\mathbb{C})$. To this end, note that $\Sigma \times \mathbb{R}$ with the (g, m) version of the metric in (5.2) is uniformized by the hyperbolic ball, $\mathbb{B}$, with its metric of constant sectional curvature $-\frac{1}{3}$. This is to say that the space $\Sigma \times \mathbb{R}$ with its hyperbolic metric is the quotient of $\mathbb{B}$ by the image, Γ, of $\pi_1(\Sigma)$ in $\mathrm{PSL}_2(\mathbb{C})$. The inverse image of $\Sigma \times \{0\}$ via this map is a properly embedded disk in $\mathbb{B}$ with principle curvatures having absolute value less than $\frac{1}{\sqrt{6}}$. This understood, theorems of Epstein in [6] and [7] imply that Γ is quasi-Fuchsian. □

Appendices

A The curvature for the metric in (1.2)

The computation begins with the definition of the orthonormal frame

$$\begin{aligned} e^A &= dz^A + t m_{AB} dz^B + \frac{1}{12} t^2 dz^A, \quad \text{and} \\ e^3 &= dt, \end{aligned} \tag{A.1}$$

where (z^A) are Gaussian normal coordinates on Σ for the metric g. A computation then finds

$$\begin{aligned} de^A &= t d_C m_{AB} dz^C \wedge dz^B + m_{AB} dt \wedge dz^B + \frac{1}{6} t dt \wedge dz^A \\ &= -\Gamma^{AB} e^B - \Gamma^{A3} e^3 \\ de^3 &= 0 = -\Gamma^{A3} \wedge e^A. \end{aligned} \tag{A.2}$$

Thus,

$$\begin{aligned}\Gamma^{AB} &= t(d_B m_{AC} - d_A m_{BC})dz^C, \quad \text{and} \\ \Gamma^{A3} &= m_{AB}dz^B + \frac{1}{6}t dz^A.\end{aligned} \tag{A.3}$$

To continue, the computation of the curvature 2–form $\mathcal{R}^{ij} = \frac{1}{2}R^{ij}_{km}e^k \wedge e^m$ finds it at $t = 0$ equal to

$$\begin{aligned}\mathcal{R}^{AB} &= \frac{1}{4}r\varepsilon^{AB}\varepsilon_{CD}dz^C dz^D + (d_B m_{AC} - d_A m_{BC})dt \wedge dz^C \\ &\quad - m_{AC}m_{BD}dz^C \wedge dz^D, \\ \mathcal{R}^{A3} &= d_C m_{AB}dz^C \wedge dz^B + \frac{1}{6}dt \wedge dz^A.\end{aligned} \tag{A.4}$$

Thus, the curvature form at $t = 0$ is

$$\begin{aligned}R^{AB}_{CD} &= \frac{1}{2}r\varepsilon^{AB}\varepsilon_{CD} - (m_{AC}m_{BD} - m_{AD}m_{BC}), \\ R^{AB}_{C3} &= -(d_B m_{AC} - d_A m_{BC}), \\ R^{A3}_{B3} &= -\frac{1}{6}g_{AB}.\end{aligned} \tag{A.5}$$

This gives the $t = 0$ Ricci tensor with components

$$\begin{aligned}R_{AC} &= \frac{1}{2}(r - \frac{1}{3})g_{AC} + m^0_{AC}m^0_{BC} - \frac{1}{4}k^2 g_{AB}, \\ R_{A3} &= d_B m^0_{AB} - \frac{1}{2}d_A k, \\ R_{33} &= -\frac{1}{3},\end{aligned} \tag{A.6}$$

where m^0 is the traceless part of m. Thus, the $t = 0$ Ricci tensor obeys the hyperbolicity condition if and only if

$$r = -(\frac{1}{3} + |m^0|^2) + \frac{1}{2}k^2 \quad \text{and} \quad d_C m_{AB} - d_B m_{AC} = 0. \tag{A.7}$$

Note that the second fundamental form for the $t = 0$ slice can be computed as follows:

$$\begin{aligned}\kappa_{AB} &= \langle \nabla_A e^3, dz_B \rangle \\ &= -\Gamma^{3B}_A \\ &= m_{AB},\end{aligned} \tag{A.8}$$

by virtue of (A.3). Thus, the $t = 0$ slice has zero mean curvature when $k = g^{AB}m_{AB} = 0$.

B The variation of (1.1) with a conformal change of the metric

The differential of the equation of (1.1) at a pair (g,m) can be computed in the following manner: First of all, consider the change where $(g,m) \to (e^{-u}g,m)$. The change in the covariant derivative is computed as follows: Observe first that the new orthonormal frame is $\{\hat{e}^A = e^{-\frac{u}{2}}e^A\}$ and

$$d\hat{e}^A = -\frac{1}{2}d_C u e^{-\frac{u}{2}} e^C \wedge e^A - \gamma^{AB} \wedge \hat{e}^B. \tag{B.1}$$

Thus,

$$\hat{\gamma}^{AB} = \gamma^{AB} - \frac{1}{2}(e^A d_B u - e^B d_A u). \tag{B.2}$$

This then finds the new curvature 2–form

$$\hat{r}^{AB} = r^{AB} - \frac{1}{2}(d_C d_B u e^C \wedge e^A - d_C d_A u e^C \wedge e^B). \tag{B.3}$$

Hence,

$$\hat{r}^{AB}_{CD} = e^u(r^{AB}_{CD} - \frac{1}{2}(d_C d_B u g_{AD} - d_D d_B u g_{AC} - d_C d_A u g_{BD} + d_D d_A u g_{BC})). \tag{B.4}$$

Taking the necessary traces finds

$$\hat{r} = e^u(r + \Delta u). \tag{B.5}$$

In terms of the coordinate frame $\{dz^A\}$, the new Christoffel symbols are

$$\begin{aligned} \delta\gamma^D_{CB} &\equiv \hat{\gamma}^D_{CB} - \gamma^D_{CB} \\ &= \frac{1}{2}(d_B u g_{DC} + d_C u g_{DB} - d_D u g_{CB}). \end{aligned} \tag{B.6}$$

This then implies that $\hat{d}_C m_{AB} - \hat{d}_A m_{CB}$ is equal to

$$\begin{aligned} &= -\delta\gamma^D_{CB} m_{AD} + \delta\gamma^D_{AB} m_{CB} \\ &= \frac{1}{2}(d_B u m_{AC} + d_C u m_{AB} - d_D u m_{AD} g_{CB}) \\ &\quad - \frac{1}{2}(d_B u m_{CA} + d_A u m_{CB} - d_D u m_{CD} g_{AB}) \\ &= \frac{1}{2}\varepsilon_{CA}(d_D u \varepsilon_{DF} m_{FB} - d_D u m_{DF} \varepsilon_{BF}) \\ &= 0 \end{aligned} \tag{B.7}$$

by virtue of the fact that the tensor $\varepsilon_{DF} m_{FB}$ is symmetric when m is symmetric and tracefree. Thus, $(e^{-u}g,m)$ obeys the first equation in (1.1a) when (g,m) does. Needless to say, the equation in (1.1b) is also obeyed by $(e^{-u}g,m)$.

As for the second equation in (1.1a), introduce

$$f(g,m) \equiv r_g + (|m|^2 + \frac{1}{3}). \tag{B.8}$$

Then by virtue of (B.5),

$$\begin{aligned} f(e^{-u}g,m) &= e^u(r + \Delta u + e^u|m|^2 + \frac{1}{3}e^{-u}) \\ &= e^u(\Delta u + (e^u - 1)|m|^2 + \frac{1}{3}(e^{-u} - 1)). \end{aligned} \tag{B.9}$$

Note that if $f(e^{-u}g,m)$ is to vanish for small u, then

$$\Delta u + (|m|^2 - \frac{1}{3})u + O(u^2) = 0. \tag{B.10}$$

C The connection defined by (2.4)

The curvature of the covariant derivative in (2.4) at a given point in Σ is computed most easily by choosing the frame $\{e^A\}$ for $T^*\Sigma$ to have zero metric covariant derivative at the point in question. This understood, then the curvature 2–form is the $\mathrm{SO}_3(\mathbb{C})$–valued 2–form whose components are $\mathcal{R}_{AC} = [\nabla_A, \nabla_C]$. Therefore, two appeals to (2.4) find

$$\mathcal{R}_{AC}\eta = \begin{pmatrix} -r_{DBAC}\eta_D + (d_A\theta_{CB} - d_C\theta_{AB})\eta_3 - (\theta_{AB}\theta_{CD} - \theta_{CB}\theta_{AD})\eta_D \\ -(d_A\theta_{CD} - d_C\theta_{AD})\eta_D - (\theta_{AD}\theta_{CD} - \theta_{CD}\theta_{AD})\eta_3 \end{pmatrix}, \tag{C.1}$$

where $r_{DBAC} = \frac{1}{2}\varepsilon_{DB}\varepsilon_{AC}r$ is the curvature tensor for the Riemannian metric g. Now, the bottom term here is zero provided that

$$d_A\theta_{CD} - d_C\theta_{AD} = 0. \tag{C.2}$$

This noted, the top term is zero provided that

$$-r - \theta_{AB}\theta_{BA} + \theta_{AA}\theta_{BB} = 0. \tag{C.3}$$

These are the equations in (1.1).

The next task for this appendix is to verify that any two solutions to (1.1) from the same Diff_0 orbit define gauge equivalent flat $\mathrm{SO}_3(\mathbb{C})$ connections. For this purpose, it proves more convenient to work with the metric in (1.2) on $\Sigma \times \mathbb{R}$. To start, let g now denote a given metric on $\Sigma \times \mathbb{R}$. Fix an oriented, orthonormal frame, $\{e^i\}_{i=1,2,3}$, for the cotangent bundle of $\Sigma \times \mathbb{R}$ and this frame can then be used to define from g the $\mathrm{SO}_3(\mathbb{C})$ connection on the trivial $\mathbb{C}^3$ whose covariant derivative, ∇, is

$$(\nabla\eta)_i = d\eta_i - \Gamma_i^k\eta_k + \frac{i}{\sqrt{6}}\varepsilon_{ijk}\eta_k e^j. \tag{C.4}$$

Here, Γ_i^k denotes the Levi–Civita connection as defined by the given metric using the given orthonormal frame. The covariant derivative defined by (2.4) in directions along the $t = 0$ slice is the restriction of the version of (C.4) that is defined by the metric in (1.2). If a first order change in the metric g is given by $\delta g = h$ with h a symmetric, 3×3 tensor, then the associated orthonormal frame changes to first order so that

$$\delta e^k = \frac{1}{2} h_{kj} e^j. \tag{C.5}$$

Meanwhile, the metric's Levi–Civita covariant derivative changes to first order as

$$\begin{aligned}\delta\Gamma_i^k &= \frac{1}{2}(\partial_i h_{kj} - \partial_k h_{ij}) e^j \\ &= \varepsilon_{imk}(\frac{1}{2}\varepsilon_{nmp}\partial_n h_{pj} e^j)\end{aligned} \tag{C.6}$$

Here, ∂ is used to denote the covariant derivative of the metric g. Thus, the covariant derivative in (C.4) changes to first order as

$$\delta(\nabla\eta)_i = \varepsilon_{imk}(-\frac{1}{2}\varepsilon_{nmp}\partial_n h_{pj} + \frac{i}{\sqrt{6}} h_{mj}) e^j \eta_k. \tag{C.7}$$

As a parenthetical remark, note that such a change preserves the flatness condition to first order if and only if the $\mathbb{C}^3$–valued 1–form

$$v_i \equiv (-\frac{1}{2}\varepsilon_{nip}\partial_n h_{pj} + \frac{i}{\sqrt{6}} h_{ij}) e^j \tag{C.8}$$

obeys $D^\nabla v = 0$ where D^∇ denotes the exterior covariant derivative that is defined by ∇.

By comparison, a change in the covariant derivative comes from a first order change in the connection in a direction tangent to its gauge orbit if and only if the change in ∇ has the form

$$\delta(\nabla\eta)_i = \varepsilon_{imk}(\nabla\sigma)_m \eta_k \tag{C.9}$$

where σ is some $\mathbb{C}^3$–valued function.

Granted the preceding, suppose now that the variation, h, in the metric g comes from the action by Lie derivative of a vector field, v, on $\Sigma \times \mathbb{R}$. This being the case, then $h_{ij} = \partial_i v_j + \partial_j v_i$ and so (C.8) reads

$$\begin{aligned}v_i &= -\frac{1}{2}\varepsilon_{nip}\partial_n(\partial_p v_j + \partial_j v_p) e^j + \frac{i}{\sqrt{6}}(\partial_i v_j + \partial_j v_i) e^j \\ &= \nabla(-\frac{1}{2}\varepsilon_{nip}\partial_n v_p + \frac{i}{\sqrt{6}} v_i).\end{aligned} \tag{C.10}$$

As this has the form given in (C.9), the infinitesimal action of $\mathrm{Diff}(\Sigma \times \mathbb{R})$ on the metric results in an infinitesimal action of the gauge group on the resulting connection.

To finish the story, note that this first order result can then by "integrated" to establish that any two solutions to (1.1) that lie on the same $\mathrm{Diff}_0(\Sigma)$ orbit define gauge equivalent, flat $\mathrm{SO}_3(\mathbb{C})$–connections along Σ.

D The cokernel of the operator $L_{(g,m)}$

A pair (σ_B, σ_3) is in the cokernel of $L_{(g,m)}$ if and only if it is L^2–orthogonal to all pairs (γ_B, γ_3) that can be written as in (2.1). This then implies that the pair satisfy the following system of equations:

$$-\frac{1}{2}(\varepsilon^{CA} d_C \sigma_B + \varepsilon^{CB} d_C \sigma_A - g_{AB} \varepsilon_{EF} d_E \sigma_F) + 2m_{AB}\sigma_3 = 0 \tag{D.1}$$
$$\left(\frac{1}{2}(\frac{1}{3} - |m|^2)\sigma_3 - d_C d_C \sigma_3\right) g_{AB} + d_A d_B \sigma_3 - \frac{1}{4}\varepsilon^{EC} d_C(\sigma_B m_{EA} + \sigma_A m_{EB})$$
$$-\frac{1}{4}\left(\varepsilon^{EA} d_E(\sigma_C m_{CB}) + \varepsilon^{EB} d_E(\sigma_C m_{CA})\right) - \frac{1}{2}\varepsilon^{CD} d_C \sigma_D m_{AB} = 0.$$

Now write $\sigma_B = \varepsilon_{BC} v_C$ and $\sigma_3 = -\frac{1}{2} v_3$. As is explained next, the equations just written are equivalent to the following:

$$d_A v_B + d_B v_A - g_{AB} d_C v_C + 2m_{AB} v_3 = 0 \tag{D.2}$$
$$\frac{1}{2}(\frac{1}{3} + |m|^2) v_3 g_{AB} - d_A d_B v_3 + v^C d_C m_{AB} + d_B v_C m_{CA} + d_A v_C m_{CB} = 0$$

To start the explanation, make the indicated substitution and then up to a constant factor, the following is the result:

$$d_A v_B + d_B v_A - g_{AB} d_C v_C + 2m_{AB} v_3 = 0 \tag{D.3}$$
$$\left(d_C d_C v_3 - \frac{1}{2}(\frac{1}{3} - |m|^2) v_3\right) g_{AB} - d_A d_B v_3$$
$$-\frac{1}{2}(2 d_C v_C m_{AB} - d_B v_C m_{CA} - d_A v_C m_{CB})$$
$$-\frac{1}{2}(d_C v_A m_{CB} + d_C v_B m_{CA} - 2 d_C v_C m_{AB} - 2 v^C d_C m_{AB})$$
$$+ d_D v_D m_{AB} = 0.$$

Note that the first equation in (D.2) is identical to its mate in (D.3). To simplify the second equation, begin by taking its trace to deduce that

$$d_C d_C v_3 - (\frac{1}{3} - |m|^2) v_3 = 0. \tag{D.4}$$

The second equation in (D.3) can now be rewritten using (D.4) and the first equation in (D.3) to read

$$\frac{1}{2}(\frac{1}{3} - |m|^2) v_3 g_{AB} - d_A d_B v_3 - \frac{1}{2}(2 d_C v_C m_{AB} - 2 d_B v_C m_{CA} - 2 d_A v_C m_{CB})$$
$$-\frac{1}{2}\Big((g_{CA} d_D v_D - 2m_{CA} v_3) m_{CB} + (g_{CB} d_D v_D - 2 m_{CB} v_3) m_{CA}$$
$$- 2 d_C v_C m_{AB} - 2 v^C d_C m_{AB}\Big) + d_D v_D m_{AB} = 0. \tag{D.5}$$

The elimination of canceling terms and the use of the identity $m_{AB}m_{BC} = \frac{1}{2}|m|^2 g_{AC}$ makes (D.5) into the second equation in (D.2).

The next part of this appendix explains why (D.2) is equivalent to (2.5). As the next lemma points out, a more general version of (D.2) and (2.4) together imply (2.5).

Lemma D.1 *Suppose that (v_A, v_3) define a real-valued section of $T^*\Sigma \oplus \mathbb{R}$, that κ is a real-valued constant, and that the following equations hold:*

$$\begin{aligned} &d_A v_B + d_B v_A - g_{AB}(d_C v_C + \kappa) + 2m_{AB}v_3 = 0 \\ &\frac{1}{2}(\frac{1}{3} + |m|^2)v_3 g_{AB} - d_A d_B v_3 + v^C d_C m_{AB} + d_B v_C m_{CA} + d_A v_C m_{CB} = 0. \end{aligned} \tag{D.6}$$

Then the complex-valued sections η and u of $(T^*\Sigma \oplus \mathbb{R})_{\mathbb{C}}$ given by

$$\begin{aligned} &\eta_B \equiv v_B + i\sqrt{6}(-\varepsilon_{CB} d_C v_3 + v_E \varepsilon_{CB} m_{EC}), \\ &\eta_3 \equiv v_3 + i\sqrt{\frac{3}{2}}\varepsilon_{EF} d_E v_F, \\ &u_B \equiv 0, \quad \text{and} \\ &v_3 = -i\sqrt{\frac{3}{2}}(d_C v_C + \kappa) \end{aligned} \tag{D.7}$$

obey the equation $\nabla_A \eta = \varepsilon_{AB} \nabla'_B u$

Proof of Lemma D.1 Note first that the vanishing of the first term in (D.6) implies that

$$d_A v_B - \frac{1}{2}\varepsilon_{AB}\varepsilon_{EF} d_E v_F + m_{AB} v_3 = \frac{1}{2} g_{AB}(d_C v_C + \kappa). \tag{D.8}$$

Then, since

$$\frac{1}{2} g_{AB}(d_C v_C + \kappa) = \varepsilon_{AC}(-\frac{i}{\sqrt{6}}\varepsilon_{CB} u_3), \tag{D.9}$$

this last equation is the real part of the equation $d_A \eta_B + \theta_{AB}\eta_3 = \varepsilon_{AC}(-\theta_{CB} u_3)$.

To continue, the vanishing of the second term in (D.6) can be rewritten using (D.8) so as to read

$$\begin{aligned} - d_A d_B v_3 + \frac{1}{2}(\frac{1}{3} - |m|^2) g_{AB} v_3 + v_C d_C m_{AB} \\ + m_{AC}\varepsilon_{BC}(\varepsilon_{EF} d_E v_F) = -m_{AB}(d_C v_C + \kappa). \end{aligned} \tag{D.10}$$

Moreover, this is equivalent to

$$-d_A d_B v_3 + \frac{1}{6} g_{AB} v_3 + d_A(v_C m_{CB}) + \frac{1}{2} m_{AC}\varepsilon_{BC}(\varepsilon_{EF} d_E v_F) = -\frac{1}{2} m_{AB}(d_C v_C + \kappa). \tag{D.11}$$

Finally, the latter gives

$$d_A\sqrt{6}(-\varepsilon_{BD}d_Bv_3 + v_C\varepsilon_{BD}m_{CB}) + \frac{1}{\sqrt{6}}\varepsilon_{AD}v_3 + m_{AD}\left(\sqrt{\frac{3}{2}}\varepsilon_{EF}d_Ev_F\right)$$
$$= -\sqrt{\frac{3}{2}}\varepsilon_{BD}m_{AB}(d_Cv_C + \kappa). \quad \text{(D.12)}$$

The latter equation is the imaginary part of the equation

$$d_A\eta_D + \theta_{AD}\eta_3 = \varepsilon_{AC}(-\theta_{CD}u_3).$$

As will now be explained, it is also the case that $d_A\eta_3 - \theta_{AB}\eta_B = \varepsilon_{AC}d_Cu_3$. To this end, consider that the real part of $\theta_{AB}\eta_B$ can be written as follows:

$$\begin{aligned} \mathrm{re}(\theta_{AB}\eta_B) &= m_{AB}v_B + \varepsilon_{AB}\varepsilon_{CB}d_Cv_3 - \varepsilon_{AB}\varepsilon_{CB}m_{EC}v_E \\ &= d_Av_3 \\ &= \mathrm{re}(d_A\eta_3). \end{aligned} \quad \text{(D.13)}$$

Meanwhile, the imaginary part of $\theta_{AB}\eta_B$ can be written to read

$$\begin{aligned} \mathrm{im}(\theta_{AB}\eta_B) &= \frac{1}{\sqrt{6}}\varepsilon_{AB}v_B + \sqrt{6}(-m_{AB}\varepsilon_{CB}d_Cv_3 + m_{AB}\varepsilon_{CB}v_Em_{EC}) \\ &= \varepsilon_{AB}\left(\frac{1}{\sqrt{6}}v_B + \sqrt{\frac{3}{2}}|m|^2v_B - \sqrt{6}d_C(m_{CB}v_3)\right) \\ &= -\sqrt{\frac{3}{2}}\varepsilon_{AB}\left(rv_B + d_C(2m_{CB}v_3)\right). \end{aligned} \quad \text{(D.14)}$$

On the other hand, commuting derivatives and invoking (D.8) allows the imaginary part of $d_A\eta_3$ to be written as

$$\begin{aligned} \mathrm{im}(d_A\eta_3) &= \sqrt{\frac{3}{2}}\varepsilon_{EF}(d_Ad_Ev_F) \\ &= \sqrt{\frac{3}{2}}(\varepsilon_{EA}\frac{1}{2}rv_E + \varepsilon_{EF}d_Ed_Av_F) \\ &= \sqrt{\frac{3}{2}}\left(\varepsilon_{BA}\frac{1}{2}rv_B - \varepsilon_{EF}d_Ed_Fv_A - \varepsilon_{EF}d_E(2m_{AF}v_3) + \varepsilon_{EA}d_Ed_Cv_C\right) \\ &= -\sqrt{\frac{3}{2}}\varepsilon_{AB}\left(rv + d_C(2m_{BC}v_3)\right) + \varepsilon_{AE}d_E(-(d_Cv_C + \kappa)). \end{aligned} \quad \text{(D.15)}$$

Together, (D.11) and (D.13) imply the claim that $d_A\eta_3 - \theta_{AB}\eta_B = \varepsilon_{AC}d_Cu_3$. □

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