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Algebraic and combinatorial codimension–1 transversality

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Abstract The Waldhausen construction of Mayer–Vietoris splittings of chain complexes over an injective generalized free product of group rings is extended to a combinatorial construction of Seifert–van Kampen splittings of CW complexes with fundamental group an injective generalized free product.

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Dedicated to Andrew Casson

Introduction

The close relationship between the topological properties of codimension–1 submanifolds and the algebraic properties of groups with a generalized free product structure first became apparent with the Seifert–van Kampen Theorem on the fundamental group of a union, the work of Kneser on 3–dimensional manifolds with fundamental group a free product, and the topological proof of Grushko's theorem by Stallings.

This paper describes two abstractions of the geometric codimension–1 transversality properties of manifolds (in all dimensions):

(1) the algebraic transversality construction of Mayer–Vietoris splittings of chain complexes of free modules over the group ring of an injective generalized free product,

(2) the combinatorial transversality construction of Seifert–van Kampen splittings of CW complexes with fundamental group an injective generalized free product.

By definition, a group G is a *generalized free product* if it has one of the following structures:

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(A) $G = G_1 *_H G_2$ is the amalgamated free product determined by group morphisms $i_1\colon H \to G_1$, $i_2\colon H \to G_2$, so that there is defined a pushout square of groups

$$\begin{tikzcd} H \arrow[r, "i_1"] \arrow[d, "i_2"'] & G_1 \arrow[d, "j_1"] \\ G_2 \arrow[r, "j_2"] & G \end{tikzcd}$$

The amalgamated free product is *injective* if i_1, i_2 are injective, in which case so are j_1, j_2, with

$$G_1 \cap G_2 = H \subseteq G.$$

An injective amalgamated free product is *nontrivial* if the morphisms $i_1\colon H \to G_1$, $i_2\colon H \to G_2$ are not isomorphisms, in which case the group G is infinite, and G_1, G_2, H are subgroups of infinite index in G.

The amalgamated free product is *finitely presented* if the groups G_1, G_2, H are finitely presented, in which case so is G. (If G is finitely presented, it does not follow that G_1, G_2, H need be finitely presented).

(B) $G = G_1 *_H \{t\}$ is the HNN extension determined by group morphisms $i_1, i_2\colon H \to G_1$

$$\begin{tikzcd} H \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] & G_1 \arrow[r, "j_1"] & G \end{tikzcd}$$

with $t \in G$ such that

$$j_1 i_1(h) t = t j_1 i_2(h) \in G \quad (h \in H).$$

The HNN extension is *injective* if i_1, i_2 are injective, in which case so is j_1, with

$$G_1 \cap tG_1t^{-1} = i_1(H) = ti_2(H)t^{-1} \subseteq G$$

and G is an infinite group with the subgroups G_1, H of infinite index in $G = G_1 *_H \{t\}$.

The HNN extension is *finitely presented* if the groups G_1, H are finitely presented, in which case so is G. (If G is finitely presented, it does not follow that G_1, H need be finitely presented).

A subgroup $H \subseteq G$ is *2–sided* if G is either an injective amalgamated free product $G = G_1 *_H G_2$ or an injective HNN extension $G = G_1 *_H \{t\}$. (See Stallings [18] and Hausmann [5] for the characterization of 2–sided subgroups in terms of bipolar structures.)

A CW pair $(X, Y \subset X)$ is *2–sided* if Y has an open neighbourhood $Y \times \mathbb{R} \subset X$. The pair is *connected* if X and Y are connected. By the Seifert–van Kampen Theorem $\pi_1(X)$ is a generalized free product:

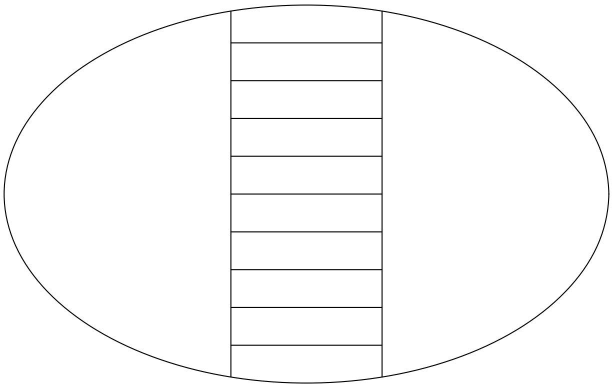
(A) if Y separates X then $X - Y$ has two components, and

$$X = X_1 \cup_Y X_2$$

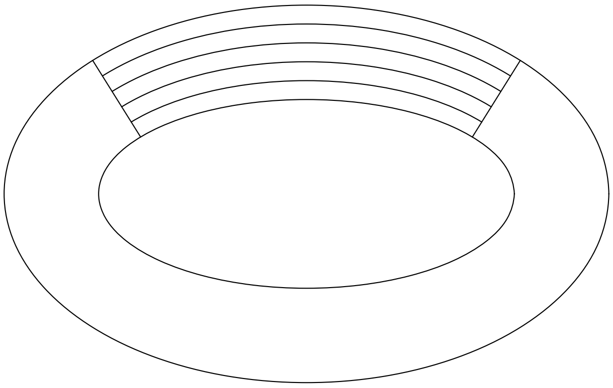
for connected $X_1, X_2 \subset X$ with

$$\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2)$$

the amalgamated free product determined by the morphisms $i_1 \colon \pi_1(Y) \to \pi_1(X_1)$, $i_2 \colon \pi_1(Y) \to \pi_1(X_2)$ induced by the inclusions $i_1 \colon Y \to X_1$, $i_2 \colon Y \to X_2$.



induced by the inclusions $i_1, i_2\colon Y \to X_1$.



with

$$\begin{cases} \pi_1(X_1) = G_1, \pi_1(X_2) = G_2 \\ \pi_1(X_1) = G_1 \end{cases}, \ \pi_1(Y) = H.$$

The splitting is *finite* if the complexes W, X, Y are finite, and *infinite* otherwise.

A connected CW complex W with $\pi_1(W) = G = \begin{cases} G_1 *_H G_2 \\ G_1 *_H \{t\} \end{cases}$ an injective generalized free product is a homotopy pushout

$$\left\{\begin{array}{l}
\begin{tikzcd}
\widetilde{W}/H \arrow[r, "i_1"] \arrow[d, "i_2"'] & \widetilde{W}/G_1 \arrow[d, "j_1"] \\
\widetilde{W}/G_2 \arrow[r, "j_2"] & W
\end{tikzcd} \\
\begin{tikzcd}
\widetilde{W}/H \times \{0,1\} \arrow[r, "i_1 \cup i_2"] \arrow[d] & \widetilde{W}/G_1 \arrow[d, "j_1"] \\
\widetilde{W}/H \times [0,1] \arrow[r] & W
\end{tikzcd}
\end{array}\right.$$

with $\widetilde{W}$ the universal cover of W and $\begin{cases} i_1, i_2, j_1, j_2 \\ i_1, i_2, j_1 \end{cases}$ the covering projections. (See Proposition $\begin{cases} 3.6 \\ 3.14 \end{cases}$ for proofs). Thus W has a canonical infinite injective Seifert–van Kampen splitting $(X(\infty), Y(\infty))$ with

$$\begin{cases} Y(\infty) = \widetilde{W}/H \times \{1/2\} \subset X(\infty) = \widetilde{W}/G_1 \cup_{i_1} \widetilde{W}/H \times [0,1] \cup_{i_2} \widetilde{W}/G_2 \\ Y(\infty) = \widetilde{W}/H \times \{1/2\} \subset X(\infty) = \widetilde{W}/G_1 \cup_{i_1 \cup i_2} \widetilde{W}/H \times [0,1] \end{cases}.$$

For finite W with $\pi_1(W)$ a finitely presented injective generalized free product it is easy to obtain finite injective Seifert–van Kampen splittings by codimension–1 manifold transversality. In fact, there are two somewhat different ways of doing so:

(i) Consider a regular neighbourhood $(M, \partial M)$ of $W \subset S^n$ (n large), apply codimension–1 manifold transversality to a map

$$\begin{cases} f\colon M \to BG = BG_1 \cup_{BH \times \{0\}} BH \times [0,1] \cup_{BH \times \{1\}} BG_2 \\ f\colon M \to BG = BG_1 \cup_{BH \times \{0,1\}} BH \times [0,1] \end{cases}$$

inducing the identification $\pi_1(M) = G$ to obtain a finite Seifert–van Kampen splitting (M^n, N^{n-1}) with $N = f^{-1}(BH \times \{1/2\}) \subset M$, and then make the splitting injective by low-dimensional handle exchanges.

(ii) Replace the 2–skeleton $W^{(2)}$ by a homotopy equivalent manifold with boundary $(M, \partial M)$, so $\pi_1(M) = \pi_1(W)$ is a finitely presented injective generalized free product and M has a finite injective Seifert–van Kampen splitting by manifold transversality (as in (i)). Furthermore, (W, M) is a finite CW pair and

$$W = M \cup \bigcup_{n \geqslant 3} (W, M)^{(n)}$$

with the relative n–skeleton $(W, M)^{(n)}$ a union of n–cells D^n attached along maps $S^{n-1} \to M \cup (W, M)^{(n-1)}$. Set $(W, M)^{(2)} = \emptyset$, and assume inductively that for some $n \geqslant 3$ $M \cup (W, M)^{(n-1)}$ already has a finite Seifert–van Kampen splitting (X, Y). For each n–cell $D^n \subset (W, M)^{(n)}$ use manifold transversality to make the composite

$$S^{n-1} \to M \cup (W, M)^{(n-1)} \simeq X$$

transverse at $Y \subset X$, and extend this transversality to make the composite

$$f\colon\ D^n \to M \cup (W, M)^{(n)} \to BG$$

transverse at $BH \subset BG$. The transversality gives D^n a finite CW structure in which $N^{n-1} = f^{-1}(BH) \subset D^n$ is a subcomplex, and

$$(X', Y') = \Big(X \cup \bigcup_{D^n \subset (W,M)^{(n)}} D^n\,,\ Y \cup \bigcup_{D^n \subset (W,M)^{(n)}} N^{n-1}\Big)$$

is an extension to $M \cup (W, M)^{(n)}$ of the finite Seifert–van Kampen splitting.

However, the geometric nature of manifold transversality does not give any insight into the CW structures of the splittings (X, Y) of W obtained as above, let alone into the algebraic analogue of transversality for $\mathbb{Z}[G]$–module chain complexes. Here, we obtain Seifert–van Kampen splittings combinatorially, in the following converse of the Seifert–van Kampen Theorem.

Combinatorial Transversality Theorem *Let W be a finite connected CW complex with* $\pi_1(W) = G = \begin{cases} G_1 *_H G_2 \\ G_1 *_H \{t\} \end{cases}$ *an injective generalized free product.*

(i) *The canonical infinite Seifert–van Kampen splitting* $(X(\infty),Y(\infty))$ *of* W *is a union of finite Seifert–van Kampen splittings* $(X,Y)\subset(X(\infty),Y(\infty))$

$$(X(\infty),Y(\infty))=\bigcup(X,Y).$$

In particular, there exist finite Seifert–van Kampen splittings (X,Y) *of* W*.*

(ii) *If the injective generalized free product structure on* $\pi_1(W)$ *is finitely presented then for any finite Seifert–van Kampen splitting* (X,Y) *of* W *it is possible to attach finite numbers of 2– and 3–cells to* X *and* Y *to obtain an injective finite Seifert–van Kampen splitting* (X',Y') *of* W*, such that* $(X,Y)\subset(X',Y')$ *with the inclusion* $X\to X'$ *a homotopy equivalence and the inclusion* $Y\to Y'$ *a* $\mathbb{Z}[H]$*–coefficient homology equivalence.*

The Theorem is proved in section 3. The main ingredient of the proof is the construction of a finite Seifert–van Kampen splitting of W from a finite *domain* of the universal cover $\widetilde{W}$, as given by finite subcomplexes $\begin{cases}W_1,W_2\subseteq\widetilde{W}\\ W_1\subseteq\widetilde{W}\end{cases}$ such that

$$\begin{cases}G_1W_1\cup G_2W_2=\widetilde{W}\\ G_1W_1=\widetilde{W}\ .\end{cases}$$

Algebraic transversality makes much use of the induction and restriction functors associated to a ring morphism $i\colon A\to B$

$$i_!\colon\ \{A\text{-modules}\}\to\{B\text{-modules}\};\ M\mapsto i_!M=B\otimes_A M,$$

$$i^!\colon\ \{B\text{-modules}\}\to\{A\text{-modules}\};\ N\mapsto i^!N=N.$$

These functors are adjoint, with

$$\mathrm{Hom}_B(i_!M,N)=\mathrm{Hom}_A(M,i^!N).$$

Let $G=\begin{cases}G_1*_H G_2\\ G_1*_H\{t\}\end{cases}$ be a generalized free product. By definition, a *Mayer–Vietoris splitting* (or *presentation*) $\mathcal{E}$ of a $\mathbb{Z}[G]$–module chain complex C is:

(A) an exact sequence of $\mathbb{Z}[G]$–module chain complexes

$$\mathcal{E}\colon\ 0\to k_!D\xrightarrow{\begin{pmatrix}1\otimes e_1\\ 1\otimes e_2\end{pmatrix}}(j_1)_!C_1\oplus(j_2)_!C_2\to C\to 0$$

with C_1 a $\mathbb{Z}[G_1]$–module chain complex, C_2 a $\mathbb{Z}[G_2]$–module chain complex, D a $\mathbb{Z}[H]$–module chain complex, $e_1\colon\ (i_1)_!D\to C_1$ a $\mathbb{Z}[G_1]$–module chain map and $e_2\colon\ (i_2)_!D\to C_2$ a $\mathbb{Z}[G_2]$–module chain map,

(B) an exact sequence of $\mathbb{Z}[G]$–module chain complexes

$$\mathcal{E}\colon\ 0 \to (j_1 i_1)_! D \xrightarrow{1\otimes e_1 - t\otimes e_2} (j_1)_! C_1 \to C \to 0$$

with C_1 a $\mathbb{Z}[G_1]$–module chain complex, D a $\mathbb{Z}[H]$–module chain complex, and $e_1\colon (i_1)_! D \to C_1$, $e_2\colon (i_2)_! D \to C_1$ $\mathbb{Z}[G_1]$–module chain maps.

A Mayer–Vietoris splitting $\mathcal{E}$ is *finite* if every chain complex in $\mathcal{E}$ is finite f.g. free, and *infinite* otherwise. See section 1 for the construction of a (finite) Mayer–Vietoris splitting of the cellular $\mathbb{Z}[\pi_1(X)]$–module chain complex $C(\widetilde{X})$ of the universal cover $\widetilde{X}$ of a (finite) connected CW complex X with a 2–sided connected subcomplex $Y \subset X$ such that $\pi_1(Y) \to \pi_1(X)$ is injective.

For any injective generalized free product $G = \begin{cases} G_1 *_H G_2 \\ G_1 *_H \{t\} \end{cases}$ every free $\mathbb{Z}[G]$–module chain complex C has a canonical infinite Mayer–Vietoris splitting

$$\text{(A)}\quad \mathcal{E}(\infty)\colon\ 0 \to k_! k^! C \to (j_1)_! j_1^! C \oplus (j_2)_! j_2^! C \to C \to 0$$

$$\text{(B)}\quad \mathcal{E}(\infty)\colon\ 0 \to k_! k^! C \to (j_1)_! j_1^! C \to C \to 0.$$

For finite C we shall obtain finite Mayer–Vietoris splittings in the following converse of the Mayer–Vietoris Theorem.

Algebraic Transversality Theorem *Let* $G = \begin{cases} G_1 *_H G_2 \\ G_1 *_H \{t\} \end{cases}$ *be an injective generalized free product. For a finite f.g. free* $\mathbb{Z}[G]$*–module chain complex* C *the canonical infinite Mayer–Vietoris splitting* $\mathcal{E}(\infty)$ *of* C *is a union of finite Mayer–Vietoris splittings* $\mathcal{E} \subset \mathcal{E}(\infty)$

$$\mathcal{E}(\infty) = \bigcup \mathcal{E}.$$

In particular, there exist finite Mayer–Vietoris splittings $\mathcal{E}$ *of* C*.*

The existence of finite Mayer–Vietoris splittings was first proved by Waldhausen [19], [20]. The proof of the Theorem in section 2 is a simplification of the original argument, using chain complex analogues of the CW domains.

Suppose now that (X, Y) is the finite 2–sided CW pair defined by a (compact) connected n–dimensional manifold X^n together with a connected codimension–1 submanifold $Y^{n-1} \subset X$ with trivial normal bundle. By definition, a homotopy equivalence $f\colon M^n \to X$ from an n–dimensional manifold *splits* at $Y \subset X$ if f is homotopic to a map (also denoted by f) which is transverse at Y, such that the restriction $f|\colon N^{n-1} = f^{-1}(Y) \to Y$ is also a homotopy equivalence.

In general, homotopy equivalences do not split: it is not possible to realize the Seifert–van Kampen splitting X of M by a codimension–1 submanifold $N \subset M$. For (X,Y) with injective $\pi_1(Y) \to \pi_1(X)$ there are algebraic K– and L–theory obstructions to splitting homotopy equivalences, involving the Nil-groups of Waldhausen [19], [20] and the UNil-groups of Cappell [2], and for $n \geqslant 6$ these are the complete obstructions to splitting. As outlined in Ranicki [9, section 7.6], [10, section 8], algebraic transversality for chain complexes is an essential ingredient for a systematic treatment of both the algebraic K– and L–theory obstructions. The algebraic analogue of the combinatorial approach to CW transversality worked out here will be used to provide such a treatment in Ranicki [13].

Although the algebraic K– and L–theory of generalized free products will not actually be considered here, it is worth noting that the early results of Higman [6], Bass, Heller and Swan [1] and Stallings [17] on the Whitehead groups of polynomial extensions and free products were followed by the work of the dedicatee on the Whitehead group of amalgamated free products (Casson [4]) prior to the general results of Waldhausen [19], [20] on the algebraic K–theory of generalized free products.

The algebraic K–theory spectrum $A(X)$ of a space (or simplicial set) X was defined by Waldhausen [21] to be the K–theory

$$A(X) = K(\mathcal{R}_f(X))$$

of the category $\mathcal{R}_f(X)$ of retractive spaces over X, and also as

$$A(X) = K(S \wedge G(X)_+)$$

with S the sphere spectrum and $G(X)$ the loop group of X. See Hüttemann, Klein, Vogell, Waldhausen and Williams [7], Schwänzl and Staffeldt [14], Schwänzl, Staffeldt and Waldhausen [15] for the current state of knowledge concerning the Mayer–Vietoris-type decomposition of $A(X)$ for a finite 2–sided CW pair (X,Y). The A–theory splitting theorems obtained there use the second form of the definition of $A(X)$. The Combinatorial Transversality Theorem could perhaps be used to obtain A–theory splitting theorems directly from the first form of the definition, at least for injective $\pi_1(Y) \to \pi_1(X)$.

I am grateful to Bob Edwards, Dirk Schütz and the referee for useful suggestions.

1 The Seifert–van Kampen and Mayer–Vietoris Theorems

Following some standard material on covers and fundamental groups we recall the well-known Bass–Serre theory relating injective generalized free products and groups acting on trees. The Seifert–van Kampen theorem for the fundamental group $\pi_1(X)$ and the Mayer–Vietoris theorem for the cellular $\mathbb{Z}[\pi_1(X)]$–module chain complex $C(\widetilde{X})$ of the universal cover $\widetilde{X}$ of a connected CW complex X with a connected 2–sided subcomplex $Y\subset X$ and injective $\pi_1(Y)\to\pi_1(X)$ are then deduced from the construction of the universal cover $\widetilde{X}$ of X by cutting along Y, using the tree T on which $\pi_1(X)$ acts.

1.1 Covers

Let X be a connected CW complex with fundamental group $\pi_1(X)=G$ and universal covering projection $p\colon \widetilde{X}\to X$, with G acting on the left of $\widetilde{X}$. Let $C(\widetilde{X})$ be the cellular free (left) $\mathbb{Z}[G]$–module chain complex. For any subgroup $H\subseteq G$ the covering $Z=\widetilde{X}/H$ of X has universal cover $\widetilde{Z}=\widetilde{X}$ with cellular $\mathbb{Z}[H]$–module chain complex

$$C(\widetilde{Z})=k^!C(\widetilde{X})$$

with $k\colon \mathbb{Z}[H]\to\mathbb{Z}[G]$ the inclusion. For a connected subcomplex $Y\subseteq X$ the inclusion $Y\to X$ induces an injection $\pi_1(Y)\to\pi_1(X)=G$ if and only if the components of $p^{-1}(Y)\subseteq\widetilde{X}$ are copies of the universal cover $\widetilde{Y}$ of Y. Assuming this injectivity condition we have

$$p^{-1}(Y)=\bigcup_{g\in[G;H]} g\widetilde{Y}\subset\widetilde{X}$$

with $H=\pi_1(Y)\subseteq G$ and $[G;H]$ the set of right H–cosets

$$g=xH\subseteq G\quad(x\in G).$$

The cellular $\mathbb{Z}[G]$–module chain complex of $p^{-1}(Y)$ is induced from the cellular $\mathbb{Z}[H]$–module chain complex of $\widetilde{Y}$

$$C(p^{-1}(Y))=k_!C(\widetilde{Y})=\mathbb{Z}[G]\otimes_{\mathbb{Z}[H]}C(\widetilde{Y})=\bigoplus_{g\in[G;H]}C(g\widetilde{Y})\subseteq C(\widetilde{X}).$$

The inclusion $Y\to X$ of CW complexes induces an inclusion of $\mathbb{Z}[H]$–module chain complexes

$$C(\widetilde{Y})\to C(Z)=k^!C(\widetilde{X})$$

adjoint to the inclusion of $\mathbb{Z}[G]$–module chain complexes

$$C(p^{-1}(Y)) = k_! C(\widetilde{Y}) \to C(\widetilde{X}).$$

1.2 Amalgamated free products

Theorem 1.1 (Serre [16]) *A group G is (isomorphic to) an injective amalgamated free product $G_1 *_H G_2$ if and only if G acts on a tree T with*

$$T/G = [0,1].$$

Idea of proof Given an injective amalgamated free product $G = G_1 *_H G_2$ let T be the tree defined by

$$T^{(0)} = [G;G_1] \cup [G;G_2],\ T^{(1)} = [G;H].$$

The edge $h \in [G;H]$ joins the unique vertices $g_1 \in [G;G_1]$, $g_2 \in [G;G_2]$ with

$$g_1 \cap g_2 = h \subset G.$$

The group G acts on T by

$$G \times T \to T;\ (g,x) \mapsto gx$$

with $T/G = [0,1]$. Conversely, if a group G acts on a tree T with $T/G = [0,1]$ then $G = G_1 *_H G_2$ is an injective amalgamated free product with $G_i \subseteq G$ the isotropy subgroup of $G_i \in T^{(0)}$ and $H \subseteq G$ the isotropy subgroup of $H \in T^{(1)}$. □

If the amalgamated free product G is nontrivial the tree T is infinite.

Theorem 1.2 *Let*

$$X = X_1 \cup_Y X_2$$

be a connected CW complex which is a union of connected subcomplexes such that the morphisms induced by the inclusions $Y \to X_1$, $Y \to X_2$

$$i_1\colon\ \pi_1(Y) = H \to \pi_1(X_1) = G_1,\quad i_2\colon\ \pi_1(Y) = H \to \pi_1(X_2) = G_2$$

are injective, and let

$$G = G_1 *_H G_2$$

with tree T.

(i) *The universal cover $\widetilde{X}$ of X is the union of translates of the universal covers $\widetilde{X}_1, \widetilde{X}_2$ of X_1, X_2*

$$\widetilde{X} = \bigcup_{g_1 \in [G;G_1]} g_1\widetilde{X}_1 \cup_{\bigcup_{h \in [G;H]} h\widetilde{Y}} \bigcup_{g_2 \in [G;G_2]} g_2\widetilde{X}_2.$$

with intersections translates of the universal cover $\widetilde{Y}$ *of* Y

$$g_1\widetilde{X}_1 \cap g_2\widetilde{X}_2 = \begin{cases} h\widetilde{Y} & \textit{if } g_1 \cap g_2 = h \in [G;H] \\ \emptyset & \textit{otherwise} . \end{cases}$$

(ii) (Seifert–van Kampen) *The fundamental group of* X *is the injective amalgamated free product*

$$\pi_1(X) = G = G_1 *_H G_2 .$$

(iii) (Mayer–Vietoris) *The cellular* $\mathbb{Z}[\pi_1(X)]$*–module chain complex* $C(\widetilde{X})$ *has a Mayer–Vietoris splitting*

$$0 \to k_!C(\widetilde{Y}) \xrightarrow{\begin{pmatrix} 1\otimes e_1 \\ 1\otimes e_2 \end{pmatrix}} (j_1)_!C(\widetilde{X}_1) \oplus (j_2)_!C(\widetilde{X}_2)$$
$$\xrightarrow{(f_1 \ -f_2)} C(\widetilde{X}) \to 0$$

with $e_1 \colon Y \to X_1$, $e_2 \colon Y \to X_2$, $f_1 \colon X_1 \to X$, $f_2 \colon X_2 \to X$ *the inclusions.*

Proof (i) Consider first the special case $G_1 = G_2 = H = \{1\}$. Every map $S^1 \to X = X_1 \cup_Y X_2$ is homotopic to one which is transverse at $Y \subset X$ (also denoted f) with $f(0) = f(1) \in Y$, so that $[0,1]$ can be decomposed as a union of closed intervals

$$[0,1] = \bigcup_{i=0}^{n} [a_i, a_{i+1}] \quad (0 = a_0 < a_1 < \cdots < a_{n+1} = 1)$$

with

$$f(a_i) \in Y,\ f[a_i, a_{i+1}] \subseteq \begin{cases} X_1 & \text{if } i \text{ is even} \\ X_2 & \text{if } i \text{ is odd} . \end{cases}$$

Choosing paths $g_i \colon [0,1] \to Y$ joining a_i to a_{i+1} and using $\pi_1(X_1) = \pi_1(X_2) = \{1\}$ on the loops $f|_{[a_i,a_{i+1}]} \cup g_i \colon S^1 \to X_1$ (resp. X_2) for i even (resp. odd) there is obtained a contraction of $f \colon S^1 \to X$, so that $\pi_1(X) = \{1\}$ and X is its own universal cover.

In the general case let

$$p_j \colon \widetilde{X}_j \to X_j \ (j = 1, 2),\ q \colon \widetilde{Y} \to Y$$

be the universal covering projections. Since $i_j \colon H \to G_j$ is injective

$$(p_j)^{-1}(Y) = \bigcup_{h_j \in [G_j;H]} h_j\widetilde{Y} .$$

The CW complex defined by

$$\widetilde{X} = \bigcup_{g_1\in[G;G_1]} g_1\widetilde{X}_1 \cup \bigcup_{h\in[G;H]} h\widetilde{Y} \bigcup_{g_2\in[G;G_2]} g_2\widetilde{X}_2$$

is simply-connected by the special case, with a free G–action such that $\widetilde{X}/G = X$, so that $\widetilde{X}$ is the universal cover of X and $\pi_1(X) = G$.

(ii) The vertices of the tree T correspond to the translates of $\widetilde{X}_1$, $\widetilde{X}_2 \subset \widetilde{X}$, and the edges correspond to the translates of $\widetilde{Y} \subset \widetilde{X}$. The free action of G on $\widetilde{X}$ determines a (non-free) action of G on T with $T/G = [0,1]$, and $\pi_1(X) = G = G_1 *_H G_2$ by Theorem 1.1.

(iii) Immediate from the expression of $\widetilde{X}$ in (i) as a union of copies of $\widetilde{X}_1$ and $\widetilde{X}_2$. □

Moreover, in the above situation there is defined a G–equivariant map $\widetilde{f}\colon \widetilde{X} \to T$ with quotient a map

$$f\colon \widetilde{X}/G = X \to T/G = [0,1]$$

such that

$$X_1 = f^{-1}([0,1/2]),\ X_2 = f^{-1}([1/2,1]),\ Y = f^{-1}(1/2) \subset X.$$

1.3 HNN extensions

Theorem 1.3 *A group G is (isomorphic to) an injective HNN extension $G_1 *_H \{t\}$ if and only if G acts on a tree T with*

$$T/G = S^1.$$

Idea of proof Given an injective HNN extension $G = G_1 *_H \{t\}$ let T be the infinite tree defined by

$$T^{(0)} = [G;G_1],\ T^{(1)} = [G;H],$$

identifying $H = i_1(H) \subseteq G$. The edge $h \in [G;H]$ joins the unique vertices $g_1, g_2 \in [G;G_1]$ with

$$g_1 \cap g_2t^{-1} = h \subset G.$$

The group G acts on T by

$$G \times T \to T;\ (g,x) \mapsto gx$$

with $T/G = S^1$, $G_1 \subseteq G$ the isotropy subgroup of $G_1 \in T^{(0)}$ and $H \subseteq G$ the isotropy subgroup of $H \in T^{(1)}$.

Conversely, if a group G acts on a tree T with $T/G = S^1$ then $G = G_1 *_H \{t\}$ is an injective HNN extension with $G_1 \subset G$ the isotropy group of $G_1 \in T^{(0)}$ and $H \subset G$ the isotropy group of $H \in T^{(1)}$. □

Theorem 1.4 *Let*

$$X = X_1 \cup_{Y \times \{0,1\}} Y \times [0,1]$$

be a connected CW complex which is a union of connected subcomplexes such that the morphisms induced by the inclusions $Y \times \{0\} \to X_1$, $Y \times \{1\} \to X_1$

$$i_1, i_2 \colon \pi_1(Y) = H \to \pi_1(X_1) = G_1$$

are injective, and let

$$G = G_1 *_H \{t\}$$

with tree T.

(i) *The universal cover* $\widetilde{X}$ *of* X *is the union of translates of the universal cover* $\widetilde{X}_1$ *of* X_1

$$\widetilde{X} = \bigcup_{g_1 \in [G:G_1]} g_1 \widetilde{X}_1 \cup_{\bigcup\limits_{h \in [G_1;H]} (h\widetilde{Y} \cup ht\widetilde{Y})} \bigcup_{h \in [G_1;H]} h\widetilde{Y} \times [0,1]$$

with $\widetilde{Y}$ *the universal cover* $\widetilde{Y}$.

(ii) (Seifert–van Kampen) *The fundamental group of* X *is the injective HNN extension*

$$\pi_1(X) = G = G_1 *_H \{t\}.$$

(iii) (Mayer–Vietoris) *The cellular* $\mathbb{Z}[\pi_1(X)]$*–module chain complex* $C(\widetilde{X})$ *has a Mayer–Vietoris splitting*

$$\mathcal{E} \colon\ 0 \to k_! C(\widetilde{Y}) \xrightarrow{1 \otimes e_1 - t \otimes e_2} (j_1)_! C(\widetilde{X}_1) \xrightarrow{f_1} C(\widetilde{X}) \to 0$$

with $e_1, e_2 \colon Y \to X_1$, $f_1 \colon X_1 \to X$ *the inclusions.*

Proof (i) Consider first the special case $G_1 = H = \{1\}$, so that $G = \mathbb{Z} = \{t\}$. The projection $\widetilde{X} \to X$ is a simply-connected regular covering with group of covering translations $\mathbb{Z}$, so that it is the universal covering of X and $\pi_1(X) = \mathbb{Z}$.

In the general case let

$$p_1 \colon \widetilde{X}_1 \to X_1,\ q \colon \widetilde{Y} \to Y$$

be the universal covering projections. Since $i_j\colon H \to G_1$ is injective

$$\begin{aligned}(p_1)^{-1}(Y \times \{0\}) &= \bigcup_{g_1 \in [G;H]} g_1\widetilde{Y},\\ (p_1)^{-1}(Y \times \{1\}) &= \bigcup_{g_2 \in [G;tHt^{-1}]} g_2\widetilde{Y}.\end{aligned}$$

The CW complex defined by $\widetilde{X} = \bigcup\limits_{g_1 \in [G:G_1]} g_1\widetilde{X}_1$ is simply-connected and with a free G–action such that $\widetilde{X}/G = X$, so that $\widetilde{X}$ is the universal cover of X and $\pi_1(X) = G$.

(ii) The vertices of the tree T correspond to the translates of $\widetilde{X}_1 \subset \widetilde{X}$, and the edges correspond to the translates of $\widetilde{Y} \times [0,1] \subset \widetilde{X}$. The free action of G on $\widetilde{X}$ determines a (non-free) action of G on T with $T/G = S^1$, and $\pi_1(X) = G = G_1 *_H \{t\}$ by Theorem 1.3.

(iii) It is immediate from the expression of $\widetilde{X}$ in (i) as a union of copies of $\widetilde{X}_1$ that there is defined a short exact sequence

$$\begin{aligned}0 \to k_!C(\widetilde{Y}) \oplus k_!C(\widetilde{Y}) \xrightarrow{\begin{pmatrix}1 \otimes e_1 & t \otimes e_2\\ 1 & 1\end{pmatrix}} & (j_1)_!C(\widetilde{X}_1) \oplus k_!C(\widetilde{Y})\\ & \xrightarrow{(f_1 - f_1(1 \otimes e_1))} C(\widetilde{X}) \to 0\end{aligned}$$

which gives the Mayer–Vietoris splitting. □

Moreover, in the above situation there is defined a G–equivariant map $\widetilde{f}\colon \widetilde{X} \to T$ with quotient a map

$$f\colon \widetilde{X}/G = X \to T/G = [0,1]/(0=1) = S^1$$

such that

$$X_1 = f^{-1}[0,1/2],\ Y \times [0,1] = f^{-1}[1/2,1] \subset X.$$

2 Algebraic transversality

We now investigate the algebraic transversality properties of $\mathbb{Z}[G]$–module chain complexes, with G an injective generalized free product. The Algebraic Transversality Theorem stated in the Introduction will now be proved, treating the cases of an amalgamated free product and an HNN extension separately.

2.1 Algebraic transversality for amalgamated free products

Let

$$G = G_1 *_H G_2$$

be an injective amalgamated free product. As in the Introduction write the injections as

$$\begin{aligned}
&i_1\colon\ H \to G_1,\ i_2\colon\ H \to G_2,\\
&j_1\colon\ G_1 \to G,\ j_2\colon\ G_2 \to G,\\
&k = j_1 i_1 = j_2 i_2\colon\ H \to G.
\end{aligned}$$

Definition 2.1 (i) A *domain* (C_1, C_2) of a $\mathbb{Z}[G]$–module chain complex C is a pair of subcomplexes $(C_1 \subseteq j_1^! C, C_2 \subseteq j_2^! C)$ such that the chain maps

$$\begin{aligned}
&e_1\colon\ (i_1)_!(C_1 \cap C_2) \to C_1;\ b_1 \otimes y_1 \mapsto b_1 y_1,\\
&e_2\colon\ (i_2)_!(C_1 \cap C_2) \to C_2;\ b_2 \otimes y_2 \mapsto b_2 y_2,\\
&f_1\colon\ (j_1)_! C_1 \to C;\ a_1 \otimes x_1 \mapsto a_1 x_1,\\
&f_2\colon\ (j_1)_! C_2 \to C;\ a_2 \otimes x_2 \mapsto a_2 x_2
\end{aligned}$$

fit into a Mayer–Vietoris splitting of C

$$\mathcal{E}(C_1, C_2)\colon\ 0 \to k_!(C_1 \cap C_2) \xrightarrow{e} (j_1)_! C_1 \oplus (j_2)_! C_2 \xrightarrow{f} C \to 0$$

with $e = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$, $f = (f_1 - f_2)$. (ii) A domain (C_1, C_2) is *finite* if C_i $(i = 1, 2)$ is a finite f.g. free $\mathbb{Z}[G_i]$–module chain complex, $C_1 \cap C_2$ is a finite f.g. free $\mathbb{Z}[H]$–module chain complex, and *infinite* otherwise.

Proposition 2.2 *Every free $\mathbb{Z}[G]$–module chain complex C has a canonical infinite domain $(C_1, C_2) = (j_1^! C, j_2^! C)$ with*

$$C_1 \cap C_2 = k^! C,$$

so that C has a canonical infinite Mayer–Vietoris splitting

$$\mathcal{E}(\infty) = \mathcal{E}(j_1^! C, j_2^! C)\colon\ 0 \to k_! k^! C \to (j_1)_! j_1^! C \oplus (j_2)_! j_2^! C \to C \to 0.$$

Proof It is enough to consider the special case $C = \mathbb{Z}[G]$, concentrated in degree 0. The pair

$$(C_1, C_2) = (j_1^! \mathbb{Z}[G], j_2^! \mathbb{Z}[G]) = (\bigoplus_{[G;G_1]} \mathbb{Z}[G_1], \bigoplus_{[G;G_2]} \mathbb{Z}[G_2])$$

is a canonical infinite domain for C, with

$$\mathcal{E}(\infty) = \mathcal{E}(C_1, C_2)\colon\ 0 \to k_!k^!\mathbb{Z}[G] \to (j_1)_!j_1^!\mathbb{Z}[G] \oplus (j_2)_!j_2^!\mathbb{Z}[G] \to \mathbb{Z}[G] \to 0$$

the simplicial chain complex $\Delta(T \times G) = \Delta(T) \otimes_{\mathbb{Z}} \mathbb{Z}[G]$, along with its augmentation to $H_0(T \times G) = \mathbb{Z}[G]$. □

Definition 2.3 (i) For a based f.g. free $\mathbb{Z}[G]$–module $B = \mathbb{Z}[G]^b$ and a subtree $U \subseteq T$ define a domain for B (regarded as a chain complex concentrated in degree 0)

$$(B(U)_1, B(U)_2) = (\sum_{U_1^{(0)}} \mathbb{Z}[G_1]^b, \sum_{U_2^{(0)}} \mathbb{Z}[G_2]^b)$$

with

$$\begin{aligned} &U_1^{(0)} = U^{(0)} \cap [G; G_1],\ U_2^{(0)} = U^{(0)} \cap [G; G_2],\\ &B(U)_1 \cap B(U)_2 = \sum_{U^{(1)}} \mathbb{Z}[H]^b. \end{aligned}$$

The associated Mayer–Vietoris splitting of B is the subobject $\mathcal{E}(U) \subseteq \mathcal{E}(\infty)$ with

$$\mathcal{E}(U)\colon\ 0 \to k_! \sum_{U^{(1)}} \mathbb{Z}[H]^b \to (j_1)_! \sum_{U_1^{(0)}} \mathbb{Z}[G_1]^b \oplus (j_2)_! \sum_{U_2^{(0)}} \mathbb{Z}[G_2]^b \to B \to 0$$

the simplicial chain complex $\Delta(U \times G)^b = \Delta(U) \otimes_{\mathbb{Z}} B$, along with its augmentation to $H_0(U \times G)^b = B$. If $U \subset T$ is finite then $(B(U)_1, B(U)_2)$ is a finite domain.

(ii) Let C be an n–dimensional based f.g. free $\mathbb{Z}[G]$–module chain complex, with $C_r = \mathbb{Z}[G]^{c_r}$. A sequence $U = \{U_n, U_{n-1}, \ldots, U_1, U_0\}$ of subtrees $U_r \subseteq T$ is *realized* by C if the differentials $d_C\colon\ C_r \to C_{r-1}$ are such that

$$d(C_r(U_r)_i) \subseteq C_{r-1}(U_{r-1})_i \quad (1 \leqslant r \leqslant n,\ i = 1, 2),$$

so that there is defined a Mayer–Vietoris splitting of C

$$\mathcal{E}(U)\colon\ 0 \to k_! \sum_{U^{(1)}} C(U)_1 \cap C(U)_2 \to (j_1)_! \sum_{U_1^{(0)}} C(U)_1 \oplus (j_2)_! \sum_{U_2^{(0)}} C(U)_2 \to C \to 0$$

with $C(U)_i$ the free $\mathbb{Z}[G_i]$–module chain complex defined by

$$d_{C(U)} = d_C|\colon\ (C(U)_i)_r = C_r(U_r)_i \to (C(U)_i)_{r-1} = C_{r-1}(U_{r-1})_i.$$

The sequence U is *finite* if each subtree $U_r \subseteq T$ is finite, in which case $\mathcal{E}(U)$ is finite.

Proposition 2.4 *For a finite based f.g. free $\mathbb{Z}[G]$–module chain complex C the canonical infinite domain is a union of finite domains*

$$(j_1^! C, j_2^! C) = \bigcup_U (C(U)_1, C(U)_2),$$

with U running over all the finite sequences which are realized by C. The canonical infinite Mayer–Vietoris splitting of C is thus a union of finite Mayer–Vietoris splittings

$$\mathcal{E}(\infty) = \bigcup_U \mathcal{E}(U).$$

Proof The proof is based on the following observations:

(a) for any subtrees $V \subseteq U \subseteq T$

$$\mathcal{E}(V) \subseteq \mathcal{E}(U) \subseteq \mathcal{E}(T) = \mathcal{E}(\infty)$$

(b) the infinite tree T is a union

$$T = \bigcup U$$

of the finite subtrees $U \subset T$,

(c) for any finite subtrees $U, U' \subset T$ there exists a finite subtree $U'' \subset T$ such that $U \subseteq U''$ and $U' \subseteq U''$,

(d) for every $d \in \mathbb{Z}[G]$ the $\mathbb{Z}[G]$–module morphism

$$d \colon \mathbb{Z}[G] \to \mathbb{Z}[G];\ x \mapsto xd$$

is resolved by a morphism $d_* \colon \mathcal{E}(T) \to \mathcal{E}(T)$ of infinite Mayer–Vietoris splittings, and for any finite subtree $U \subset T$ there exists a finite subtree $U' \subset T$ such that

$$d_*(\mathcal{E}(U)) \subseteq \mathcal{E}(U')$$

and $d_*| \colon \mathcal{E}(U) \to \mathcal{E}(U')$ is a resolution of d by a morphism of finite Mayer–Vietoris splittings (cf. Proposition 1.1 of Waldhausen [19]).

Assume C is n–dimensional, with $C_r = \mathbb{Z}[G]^{c_r}$. Starting with any finite subtree $U_n \subseteq T$ let

$$U = \{U_n, U_{n-1}, \dots, U_1, U_0\}$$

be a sequence of finite subtrees $U_r \subset T$ such that the f.g. free submodules

$$\begin{aligned}
C_r(U)_1 &= \sum_{U_{r,1}^{(0)}} \mathbb{Z}[G_1]^{c_r} \subset j_1^! C_r = \sum_{T_1^{(0)}} \mathbb{Z}[G_1]^{c_r}, \\
C_r(U)_2 &= \sum_{U_{r,2}^{(0)}} \mathbb{Z}[G_2]^{c_r} \subset j_2^! C_r = \sum_{T_2^{(0)}} \mathbb{Z}[G_2]^{c_r}, \\
D(U)_r &= \sum_{U_r^{(1)}} \mathbb{Z}[H]^{c_r} \subset k^! C_r = \sum_{T^{(1)}} \mathbb{Z}[H]^{c_r}
\end{aligned}$$

define subcomplexes

$$C(U)_1 \subset j_1^! C,\ C(U)_2 \subset j_2^! C,\ D(U) \subset k^! C.$$

Then $(C(U)_1, C(U)_2)$ is a domain of C with

$$C(U)_1 \cap C(U)_2 = D(U),$$

and U is realized by C. □

Remark 2.5 (i) The existence of finite Mayer–Vietoris splittings was first proved by Waldhausen [19],[20], using essentially the same method. See Quinn [8] for a proof using controlled algebra. The construction of generalized free products by noncommutative localization (cf. Ranicki [12]) can be used to provide a different proof.

(ii) The construction of the finite Mayer–Vietoris splittings $\mathcal{E}(U)$ in 2.4 as subobjects of the universal Mayer–Vietoris splitting $\mathcal{E}(T) = \mathcal{E}(\infty)$ is taken from Remark 8.7 of Ranicki [10].

This completes the proof of the Algebraic Transversality Theorem for amalgamated free products.

2.2 Algebraic transversality for HNN extensions

The proof of algebraic transversality for HNN extensions proceeds exactly as for amalgamated free products, so only the statements will be given.

Let

$$G = G_1 *_H \{t\}$$

be an injective HNN extension. As in the Introduction, write the injections as

$$i_1, i_2 \colon H \to G_1,\ j \colon G_1 \to G,\ k = j_1 i_1 = j_1 i_2 \colon G_1 \to G.$$

Definition 2.6 (i) A *domain* C_1 of a $\mathbb{Z}[G]$–module chain complex C is a subcomplex $C_1 \subseteq j_1^! C$ such that the chain maps

$$\begin{aligned} e_1 &\colon (i_1)_!(C_1 \cap tC_1) \to C_1;\ b_1 \otimes y_1 \mapsto b_1 y_1, \\ e_2 &\colon (i_2)_!(C_1 \cap tC_1) \to C_1;\ b_2 \otimes y_2 \mapsto b_2 t^{-1} y_2, \\ f &\colon (j_1)_! C_1 \to C;\ a \otimes x \mapsto ax \end{aligned}$$

fit into a Mayer–Vietoris splitting of C

$$\mathcal{E}(C_1) \colon\ 0 \to k_!(C_1 \cap tC_1) \xrightarrow{1 \otimes e_1 - t \otimes e_2} (j_1)_! C_1 \xrightarrow{f} C \to 0.$$

(ii) A domain C_1 is *finite* if C_1 is a finite f.g. free $\mathbb{Z}[G_1]$–module chain complex and $C_1 \cap tC_1$ is a finite f.g. free $\mathbb{Z}[H]$–module chain complex.

Proposition 2.7 *Every free* $\mathbb{Z}[G]$*–module chain complex* C *has a canonical infinite domain* $C_1 = j_1^! C$ *with*

$$C_1 \cap tC_1 = k^! C_1,$$

so that C *has a canonical infinite Mayer–Vietoris splitting*

$$\mathcal{E}(\infty) = \mathcal{E}(j_1^! C)\colon\ 0 \to k_! k^! C \to (j_1)_! j_1^! C \to C \to 0. \qquad \square$$

Definition 2.8 For any subtree $U \subseteq T$ define a domain for $\mathbb{Z}[G]$

$$C(U)_1 = \sum_{U^{(0)}} \mathbb{Z}[G_1]$$

with

$$C(U)_1 \cap tC(U)_1 = \sum_{U^{(1)}} \mathbb{Z}[H].$$

The associated Mayer–Vietoris splitting of $\mathbb{Z}[G]$ is the subobject $\mathcal{E}(U) \subseteq \mathcal{E}(\infty)$ with

$$\mathcal{E}(U)\colon\ 0 \to k_! \sum_{U^{(1)}} \mathbb{Z}[H] \to (j_1)_! \sum_{U^{(0)}} \mathbb{Z}[G_1] \to \mathbb{Z}[G] \to 0.$$

If $U \subset T$ is finite then $C(U)_1$ is finite.

Proposition 2.9 *For a finite f.g. free* $\mathbb{Z}[G]$*–module chain complex* C *the canonical infinite domain is a union of finite domains*

$$j_1^! C = \bigcup C_1.$$

The canonical infinite Mayer–Vietoris splitting of C *is thus a union of finite Mayer–Vietoris splittings*

$$\mathcal{E}(\infty) = \bigcup \mathcal{E}(C_1). \qquad \square$$

This completes the proof of the Algebraic Transversality Theorem for HNN extensions.

3 Combinatorial transversality

We now investigate the algebraic transversality properties of CW complexes X with $\pi_1(X) = G$ an injective generalized free product. The Combinatorial Transversality Theorem stated in the Introduction will now be proved, treating the cases of an amalgamated free product and an HNN extension separately.

3.1 Mapping cylinders

We review some basic mapping cylinder constructions.

The *mapping cylinder* of a map $e\colon V \to W$ is the identification space

$$\mathcal{M}(e) = (V \times [0,1] \cup W)/\{(x,1) \sim e(x) \,|\, x \in V\}$$

such that $V = V \times \{0\} \subset \mathcal{M}(e)$. As ever, the projection

$$p\colon \mathcal{M}(e) \to W;\ \begin{cases} (x,s) \mapsto e(x) & \text{for } x \in V,\ s \in [0,1] \\ y \mapsto y & \text{for } y \in W \end{cases}$$

is a homotopy equivalence.

If e is a cellular map of CW complexes then $\mathcal{M}(e)$ is a CW complex. The cellular chain complex $C(\mathcal{M}(e))$ is the *algebraic mapping cylinder* of the induced chain map $e\colon C(V) \to C(W)$, with

$$\begin{aligned} d_{C(\mathcal{M}(e))} = \begin{pmatrix} d_{C(W)} & (-1)^r e & 0 \\ 0 & d_{C(V)} & 0 \\ 0 & (-1)^{r-1} & d_{C(V)} \end{pmatrix} :\ & \\ C(\mathcal{M}(e))_r = C(W)_r \oplus C(V)_{r-1} \oplus C(V)_r & \\ \to C(\mathcal{M}(e))_{r-1} = C(W)_{r-1} \oplus C(V)_{r-2} \oplus C(V)_{r-1}. & \end{aligned}$$

The chain equivalence $p\colon C(\mathcal{M}(e)) \to C(W)$ is given by

$$p = (1\ 0\ e)\colon C(\mathcal{M}(e))_r = C(W)_r \oplus C(V)_{r-1} \oplus C(V)_r \to C(W)_r.$$

The *double mapping cylinder* $\mathcal{M}(e_1,e_2)$ of maps $e_1\colon V \to W_1$, $e_2\colon V \to W_2$ is the identification space

$$\begin{aligned} \mathcal{M}(e_1,e_2) &= \mathcal{M}(e_1) \cup_V \mathcal{M}(e_2) \\ &= W_1 \cup_{e_1} V \times [0,1] \cup_{e_2} W_2 \\ &= (W_1 \cup V \times [0,1] \cup W_2)/\{(x,0) \sim e_1(x), (x,1) \sim e_2(x) \,|\, x \in V\}. \end{aligned}$$

Given a commutative square of spaces and maps

$$\begin{tikzcd} V \arrow[r, "e_1"] \arrow[d, "e_2"'] & W_1 \arrow[d, "f_1"] \\ W_2 \arrow[r, "f_2"] & W \end{tikzcd}$$

define the map

$$f_1 \cup f_2\colon \mathcal{M}(e_1,e_2) \to W;\ \begin{cases} (x,s) \mapsto f_1e_1(x) = f_2e_2(x) & (x \in V,\ s \in [0,1]) \\ y_i \mapsto f_i(y_i) & (y_i \in W_i,\ i = 1,2)\ . \end{cases}$$

The square is a *homotopy pushout* if $f_1 \cup f_2 \colon \mathcal{M}(e_1, e_2) \to W$ is a homotopy equivalence.

If $e_1 \colon V \to W_1$, $e_2 \colon V \to W_2$ are cellular maps of CW complexes then $\mathcal{M}(e_1, e_2)$ is a CW complex, such that cellular chain complex $C(\mathcal{M}(e_1, e_2))$ is the algebraic mapping cone of the chain map

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} : C(V) \to C(W_1) \oplus C(W_2)$$

with

$$\begin{aligned} d_{C(\mathcal{M}(e_1,e_2))} = \begin{pmatrix} d_{C(W_1)} & (-1)^r e_1 & 0 \\ 0 & d_{C(V)} & 0 \\ 0 & (-1)^r e_2 & d_{C(W_2)} \end{pmatrix} : \\ C(\mathcal{M}(e_1, e_2))_r = C(W_1)_r \oplus C(V)_{r-1} \oplus C(W_2)_r \\ \to C(\mathcal{M}(e_1, e_2))_{r-1} = C(W_1)_{r-1} \oplus C(V)_{r-2} \oplus C(W_2)_{r-1} . \end{aligned}$$

3.2 Combinatorial transversality for amalgamated free products

In this section W is a connected CW complex with fundamental group an injective amalgamated free product

$$\pi_1(W) = G = G_1 *_H G_2$$

with tree T. Let $\widetilde{W}$ be the universal cover of W, and let

$$\begin{tikzcd} \widetilde{W}/H \arrow[r, "i_1"] \arrow[d, "i_2"'] & \widetilde{W}/G_1 \arrow[d, "j_1"] \\ \widetilde{W}/G_2 \arrow[r, "j_2"] & W \end{tikzcd}$$

be the commutative square of covering projections.

Definition 3.1 (i) Suppose given subcomplexes $W_1, W_2 \subseteq \widetilde{W}$ such that

$$G_1 W_1 = W_1 ,\ G_2 W_2 = W_2$$

so that

$$H(W_1 \cap W_2) = W_1 \cap W_2 \subseteq \widetilde{W} .$$

Define a commutative square of CW complexes and cellular maps

$$\begin{tikzcd}
(W_1 \cap W_2)/H \arrow[r, "e_1"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\Phi"] & W_1/G_1 \arrow[d, "f_1"] \\
W_2/G_2 \arrow[r, "f_2"] & W
\end{tikzcd}$$

with

$$(W_1 \cap W_2)/H \subseteq \widetilde{W}/H,\ W_1/G_1 \subseteq \widetilde{W}/G_1,\ W_2/G_2 \subseteq \widetilde{W}/G_2,$$
$$e_1 = i_1| \colon (W_1 \cap W_2)/H \to W_1/G_1,\ e_2 = i_2| \colon (W_1 \cap W_2)/H \to W_2/G_2,$$
$$f_1 = j_1| \colon W_1/G_1 \to W,\ f_2 = j_2| \colon W_2/G_2 \to W.$$

(ii) A *domain* (W_1, W_2) for the universal cover $\widetilde{W}$ of W consists of connected subcomplexes $W_1, W_2 \subseteq \widetilde{W}$ such that $W_1 \cap W_2$ is connected, and such that for each cell $D \subseteq \widetilde{W}$ the subgraph $U(D) \subseteq T$ defined by

$$\begin{aligned}
U(D)^{(0)} &= \{g_1 \in [G;G_1] \,|\, g_1 D \subseteq W_1\} \cup \{g_2 \in [G;G_2] \,|\, g_2 D \subseteq W_2\} \\
U(D)^{(1)} &= \{h \in [G;H] \,|\, hD \subseteq W_1 \cap W_2\}
\end{aligned}$$

is a tree.

(iii) A domain (W_1, W_2) for $\widetilde{W}$ is *fundamental* if the subtrees $U(D) \subseteq T$ are either single vertices or single edges, so that

$$\begin{aligned}
g_1 W_1 \cap g_2 W_2 &= \begin{cases} h(W_1 \cap W_2) & \text{if } g_1 \cap g_2 = h \in [G;H] \\ \emptyset & \text{if } g_1 \cap g_2 = \emptyset, \end{cases} \\
W &= (W_1/G_1) \cup_{(W_1 \cap W_2)/H} (W_2/G_2).
\end{aligned}$$

Proposition 3.2 *For a domain (W_1, W_2) of $\widetilde{W}$ the pair of cellular chain complexes $(C(W_1), C(W_2))$ is a domain of the cellular chain complex $C(\widetilde{W})$.*

Proof The union of $GW_1, GW_2 \subseteq \widetilde{W}$ is

$$GW_1 \cup GW_2 = \widetilde{W}$$

since for any cell $D \subseteq \widetilde{W}$ there either exists $g_1 \in [G;G_1]$ such that $g_1 D \subseteq W_1$ or $g_2 \in [G;G_2]$ such that $g_2 D \subseteq W_2$. The intersection of $GW_1, GW_2 \subseteq \widetilde{W}$ is

$$GW_1 \cap GW_2 = G(W_1 \cap W_2) \subseteq \widetilde{W}.$$

The Mayer–Vietoris exact sequence of cellular $\mathbb{Z}[G]$–module chain complexes

$$0 \to C(GW_1 \cap GW_2) \to C(GW_1) \oplus C(GW_2) \to C(\widetilde{W}) \to 0$$

is the Mayer–Vietoris splitting of $C(\widetilde{W})$ associated to $(C(W_1), C(W_2))$

$$0 \to k_!C(W_1 \cap W_2) \to (j_1)_!C(W_1) \oplus (j_2)_!C(W_2) \to C(\widetilde{W}) \to 0$$

with $C(W_1 \cap W_2) = C(W_1) \cap C(W_2)$. □

Example 3.3 W has a canonical infinite domain $(W_1, W_2) = (\widetilde{W}, \widetilde{W})$ with $(W_1 \cap W_2)/H = \widetilde{W}/H$, and $U(D) = T$ for each cell $D \subseteq \widetilde{W}$.

Example 3.4 (i) Suppose that $W = X_1 \cup_Y X_2$, with $X_1, X_2, Y \subseteq W$ connected subcomplexes such that the isomorphism

$$\pi_1(W) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \xrightarrow{\cong} G = G_1 *_H G_2$$

preserves the amalgamated free structures. Thus (W, Y) is a Seifert–van Kampen splitting of W, and the morphisms

$$\pi_1(X_1) \to G_1,\ \pi_1(X_2) \to G_2,\ \pi_1(Y) \to H$$

are surjective. (If $\pi_1(Y) \to \pi_1(X_1)$ and $\pi_1(Y) \to \pi_1(X_2)$ are injective these morphisms are isomorphisms, and the splitting is injective). The universal cover of W is

$$\widetilde{W} = \bigcup_{g_1 \in [G;G_1]} g_1\widetilde{X}_1 \cup \bigcup_{h \in [G;H]} h\widetilde{Y} \bigcup_{g_2 \in [G;G_2]} g_2\widetilde{X}_2$$

with $\widetilde{X}_i$ the regular cover of X_i corresponding to $\ker(\pi_1(X_i) \to G_i)$ $(i = 1, 2)$ and $\widetilde{Y}$ the regular cover of Y corresponding to $\ker(\pi_1(Y) \to H)$ (which are the universal covers of X_1, X_2, Y in the case $\pi_1(X_1) = G_1$, $\pi_1(X_2) = G_2$, $\pi_1(Y) = H$). The pair

$$(W_1, W_2) = (\widetilde{X}_1, \widetilde{X}_2)$$

is a fundamental domain of $\widetilde{W}$ such that

$$\begin{aligned} &(W_1 \cap W_2)/H = Y, \\ &g_1W_1 \cap g_2W_2 = (g_1 \cap g_2)\widetilde{Y} \subseteq \widetilde{W} \quad (g_1 \in [G;G_1], g_2 \in [G;G_2]). \end{aligned}$$

For any cell $D \subseteq \widetilde{W}$

$$U(D) = \begin{cases} \{g_1\} & \text{if } g_1D \subseteq \widetilde{X}_1 - \bigcup\limits_{h_1 \in [G_1;H]} h_1\widetilde{Y} \text{ for some } g_1 \in [G;G_1] \\ \{g_2\} & \text{if } g_2D \subseteq \widetilde{X}_2 - \bigcup\limits_{h_2 \in [G_2;H]} h_2\widetilde{Y} \text{ for some } g_2 \in [G;G_1] \\ \{g_1, g_2, h\} & \text{if } hD \subseteq \widetilde{Y} \text{ for some } h = g_1 \cap g_2 \in [G;H]. \end{cases}$$

(ii) If (W_1, W_2) is a fundamental domain for any connected CW complex W with $\pi_1(W) = G = G_1 *_H G_2$ then $W = X_1 \cup_Y X_2$ as in (i), with

$$X_1 = W_1/G_1,\ X_2 = W_2/G_2,\ Y = (W_1 \cap W_2)/H.$$

Definition 3.5 Suppose that W is n–dimensional. Lift each cell $D^r \subseteq W$ to a cell $\widetilde{D}^r \subseteq \widetilde{W}$. A sequence $U = \{U_n, U_{n-1}, \dots, U_1, U_0\}$ of subtrees $U_r \subseteq T$ is *realized* by W if the subspaces

$$W(U)_1 = \bigcup_{r=0}^{n} \bigcup_{D^r \subset W} \bigcup_{g_1 \in U^{(0)}_{r,1}} g_1\widetilde{D}^r,\ W(U)_2 = \bigcup_{r=0}^{n} \bigcup_{D^r \subset W} \bigcup_{g_2 \in U^{(0)}_{r,2}} g_2\widetilde{D}^r \subseteq \widetilde{W}$$

are connected subcomplexes, in which case $(W(U)_1, W(U)_2)$ is a domain for $\widetilde{W}$ with

$$W(U)_1 \cap W(U)_2 = \bigcup_{r=0}^{n} \bigcup_{D^r \subset W} \bigcup_{h \in U^{(1)}_r} h\widetilde{D}^r \subseteq \widetilde{W}$$

a connected subcomplex. Thus U is realized by $C(\widetilde{W})$ and

$$(C(W(U)_1), C(W(U)_2)) = (C(\widetilde{W})(U)_1, C(\widetilde{W})(U)_2) \subseteq (C(\widetilde{W}), C(\widetilde{W}))$$

is the domain for $C(\widetilde{W})$ given by $(C_r(\widetilde{W})_1(U_r), C_r(\widetilde{W})(U)_2)$ in degree r.

If a sequence $U = \{U_n, U_{n-1}, \dots, U_1, U_0\}$ realized by W is finite (ie if each $U_r \subseteq T$ is a finite subtree) then $(W(U)_1, W(U)_2)$ is a finite domain for $\widetilde{W}$.

Proposition 3.6 (i) *For any domain (W_1, W_2) there is defined a homotopy pushout*

$$\begin{tikzcd}
(W_1 \cap W_2)/H \arrow[r, "e_1"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\Phi"] & W_1/G_1 \arrow[d, "f_1"] \\
W_2/G_2 \arrow[r, "f_2"] & W
\end{tikzcd}$$

with $e_1 = i_1|$, $e_2 = i_2|$, $f_1 = j_1|$, $f_2 = j_2|$. The connected 2–sided CW pair

$$(X, Y) = (\mathcal{M}(e_1, e_2), (W_1 \cap W_2)/H \times \{1/2\})$$

is a Seifert–van Kampen splitting of W, with a homotopy equivalence

$$f = f_1 \cup f_2 \colon X = \mathcal{M}(e_1, e_2) \xrightarrow{\simeq} W.$$

(ii) *The commutative square of covering projections*

$$\begin{tikzcd}
\widetilde{W}/H \arrow[r, "i_1"] \arrow[d, "i_2"'] & \widetilde{W}/G_1 \arrow[d, "j_1"] \\
\widetilde{W}/G_2 \arrow[r, "j_2"] & W
\end{tikzcd}$$

is a homotopy pushout. The connected 2–sided CW pair

$$(X(\infty), Y(\infty)) = (\mathcal{M}(i_1, i_2), \widetilde{W}/H \times \{1/2\})$$

is a canonical injective infinite Seifert–van Kampen splitting of W*, with a homotopy equivalence* $j = j_1 \cup j_2 \colon X(\infty) \to W$ *such that*

$$\pi_1(Y(\infty)) = H \subseteq \pi_1(X(\infty)) = G_1 *_H G_2 .$$

(iii) *For any (finite) sequence* $U = \{U_n, U_{n-1}, \ldots, U_0\}$ *of subtrees of* T *realized by* W *there is defined a homotopy pushout*

$$\begin{tikzcd}
Y(U) \arrow[r, "e_1"] \arrow[d, "e_2"'] & X(U)_1 \arrow[d, "f_1"] \\
X(U)_2 \arrow[r, "f_2"] & W
\end{tikzcd}$$

with

$$\begin{aligned}
&X(U)_1 = W(U)_1/G_1,\ X(U)_2 = W(U)_2/G_2, \\
&Y(U) = (W(U)_1 \cap W(U)_2)/H, \\
&e_1 = i_1|,\ e_2 = i_2|,\ f_1 = j_1|,\ f_2 = j_2| .
\end{aligned}$$

Thus

$$(X(U), Y(U)) = (\mathcal{M}(e_1, e_2), Y(U) \times \{1/2\})$$

is a (finite) Seifert–van Kampen splitting of W*.*

(iv) *The canonical infinite domain of a finite CW complex* W *with* $\pi_1(W) = G_1 *_H G_2$ *is a union of finite domains*

$$(\widetilde{W}, \widetilde{W}) = \bigcup_U (W(U)_1, W(U)_2)$$

with U *running over all the finite sequences realized by* W*. The canonical infinite Seifert–van Kampen splitting of* W *is thus a union of finite Seifert–van Kampen splittings*

$$(X(\infty), Y(\infty)) = \bigcup_U (X(U), Y(U)).$$

Proof (i) Given a cell $D \subseteq W$ let $\widetilde{D} \subseteq \widetilde{W}$ be a lift. The inverse image of the interior $\mathrm{int}(D) \subseteq W$

$$f^{-1}(\mathrm{int}(D)) = U(\widetilde{D}) \times \mathrm{int}(D) \subseteq \mathcal{M}(i_1, i_2) = T \times_G \widetilde{W}$$

is contractible. In particular, point inverses are contractible, so that $f \colon X \to W$ is a homotopy equivalence. (Here is a more direct proof that $f \colon X \to W$ is a

$\mathbb{Z}[G]$–coefficient homology equivalence. The Mayer–Vietoris Theorem applied to the union $\widetilde{W} = GW_1 \cup GW_2$ expresses $C(\widetilde{W})$ as the cokernel of the $\mathbb{Z}[G]$–module chain map

$$e = \begin{pmatrix} 1\otimes e_1 \\ 1\otimes e_2 \end{pmatrix} : \mathbb{Z}[G]\otimes_{\mathbb{Z}[H]}C(W_1\cap W_2) \to \mathbb{Z}[G]\otimes_{\mathbb{Z}[G_1]}C(W_1)\oplus\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_2]}C(W_2)$$

with a Mayer–Vietoris splitting

$$\begin{aligned} 0 \to \mathbb{Z}[G]\otimes_{\mathbb{Z}[H]} C(W_1\cap W_2) \xrightarrow{\ e\ } \mathbb{Z}[G]\otimes_{\mathbb{Z}[G_1]} C(W_1) \oplus Z[G]\otimes_{\mathbb{Z}[G_2]} C(W_2)& \\ \longrightarrow C(\widetilde{W}) \to 0.& \end{aligned}$$

The decomposition $X = \mathcal{M}(e_1,e_2) = X_1 \cup_Y X_2$ with

$$X_i = \mathcal{M}(e_i)\ (i=1,2),\ Y = X_1\cap X_2 = (W_1\cap W_2)/H \times \{1/2\}$$

lifts to a decomposition of the universal cover as

$$\widetilde{X} = \bigcup_{g_1\in[G;G_1]} g_1\widetilde{X}_1 \cup_{\bigcup\limits_{h\in[G;H]} h\widetilde{Y}} \bigcup_{g_2\in[G;G_2]} g_2\widetilde{X}_2 .$$

The Mayer–Vietoris splitting

$$0 \to \mathbb{Z}[G]\otimes_{\mathbb{Z}[H]}C(\widetilde{Y}) \longrightarrow \mathbb{Z}[G]\otimes_{\mathbb{Z}[G_1]}C(\widetilde{X}_1)\oplus\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_2]}C(\widetilde{X}_2)\to C(\widetilde{X}) \to 0,$$

expresses $C(\widetilde{X})$ as the algebraic mapping cone of the chain map e

$$C(\widetilde{X}) = \mathcal{C}(e\colon\ \mathbb{Z}[G]\otimes_{\mathbb{Z}[H]}C(W_1\cap W_2) \to \mathbb{Z}[G]\otimes_{\mathbb{Z}[G_1]}C(W_1)\oplus\mathbb{Z}[G]\otimes_{\mathbb{Z}[G_2]}C(W_2)).$$

Since e is injective the $\mathbb{Z}[G]$–module chain map

$$\widetilde{f} = \text{projection}\colon\ C(\widetilde{X}) = \mathcal{C}(e) \to C(\widetilde{W}) = \text{coker}(e)$$

induces isomorphisms in homology.)

(ii) Apply (i) to $(W_1,W_2) = (\widetilde{W},\widetilde{W})$.

(iii) Apply (i) to the domain $(W(U)_1, W(U)_2)$.

(iv) Assume that W is n–dimensional. Proceed as for the chain complex case in the proof of Proposition 2.4 for the existence of a domain for $C(\widetilde{W})$, but use only the sequences $U = \{U_n, U_{n-1}, \dots, U_0\}$ of finite subtrees $U_r \subset T$ realized by W. An arbitrary finite subtree $U_n \subset T$ extends to a finite sequence U realized by W since for $r \geqslant 2$ each r–cell $\widetilde{D}^r \subset \widetilde{W}$ is attached to an $(r-1)$–dimensional finite connected subcomplex, and every 1–cell $\widetilde{D}^1 \subset \widetilde{W}$ is contained in a 1–dimensional finite connected subcomplex. Thus finite sequences U realized by

W exist, and can be chosen to contain arbitrary finite collections of cells of $\widetilde{W}$, with

$$(\widetilde{W},\widetilde{W}) = \bigcup_{U}(W(U)_1, W(U)_2). \qquad \square$$

This completes the proof of part (i) of the Combinatorial Transversality Theorem, the existence of finite Seifert–van Kampen splittings. Part (ii) deals with existence of finite injective Seifert–van Kampen splittings: the adjustment of fundamental groups needed to replace $(X(U), Y(U))$ by a homology-equivalent finite injective Seifert–van Kampen splitting will use the following rudimentary version of the Quillen plus construction.

Lemma 3.7 *Let K be a connected CW complex with a finitely generated fundamental group $\pi_1(K)$. For any surjection $\phi\colon\ \pi_1(K) \to \Pi$ onto a finitely presented group Π it is possible to attach a finite number n of 2– and 3–cells to K to obtain a connected CW complex*

$$K' = K \cup \bigcup_{n} D^2 \cup \bigcup_{n} D^3$$

such that the inclusion $K \to K'$ is a $\mathbb{Z}[\Pi]$–coefficient homology equivalence inducing $\phi\colon\ \pi_1(K) \to \pi_1(K') = \Pi$.

Proof The kernel of $\phi\colon\ \pi_1(K) \to \Pi$ is the normal closure of a finitely generated subgroup $N \subseteq \pi_1(K)$ by Lemma I.4 of Cappell [3]. (Here is the proof. Choose finite generating sets

$$g = \{g_1, g_2, \dots, g_r\} \subseteq \pi_1(K),\ h = \{h_1, h_2, \dots, h_s\} \subseteq \Pi$$

and let $w_k(h_1, h_2, \dots, h_s)$ $(1 \leqslant k \leqslant t)$ be words in h which are relations for Π. As ϕ is surjective, can choose $h'_j \in \pi_1(K)$ with $\phi(h'_j) = h_j$ $(1 \leqslant j \leqslant s)$. As h generates Π $\phi(g_i) = v_i(h_1, h_2, \dots, h_s)$ $(1 \leqslant i \leqslant r)$ for some words v_i in h. The kernel of ϕ is the normal closure $N = \langle N' \rangle \lhd \pi_1(K)$ of the subgroup $N' \subseteq \pi_1(K)$ generated by the finite set $\{v_i(h'_1, \dots, h'_s)g_i^{-1}, w_k(h'_1, \dots, h'_s)\}$.) Let $x = \{x_1, x_2, \dots, x_n\} \subseteq \pi_1(K)$ be a finite set of generators of N, and set

$$L = K \cup_x \bigcup_{i=1}^{n} D^2.$$

The inclusion $K \to L$ induces

$$\phi\colon\ \pi_1(K) \to \pi_1(L) = \pi_1(K)/\langle x_1, x_2, \dots, x_n\rangle = \pi_1(K)/\langle N\rangle = \Pi.$$

Let $\widetilde{L}$ be the universal cover of L, and let $\widetilde{K}$ be the pullback cover of K. Now

$$\pi_1(\widetilde{K}) = \ker(\phi) = \langle x_1, x_2, \dots, x_n\rangle = \langle N\rangle$$

so that the attaching maps $x_i\colon S^1 \to K$ of the 2–cells in $L - K$ lift to null-homotopic maps $\widetilde{x}_i\colon S^1 \to \widetilde{K}$. The cellular chain complexes of $\widetilde{K}$ and $\widetilde{L}$ are related by

$$C(\widetilde{L}) = C(\widetilde{K}) \oplus \bigoplus_n (\mathbb{Z}[\Pi], 2)$$

where $(\mathbb{Z}[\Pi], 2)$ is just $\mathbb{Z}[\Pi]$ concentrated in degree 2. Define

$$x^* = \{x_1^*, x_2^*, \dots, x_n^*\} \subseteq \pi_2(L)$$

by

$$x_i^* = (0, (0, \dots, 0, 1, 0, \dots, 0)) \in \pi_2(L) = H_2(\widetilde{L}) = H_2(\widetilde{K}) \oplus \mathbb{Z}[\Pi]^n \ (1 \leqslant i \leqslant n),$$

and set

$$K' = L \cup_{x^*} \bigcup_{i=1}^{n} D^3.$$

The inclusion $K \to K'$ induces $\phi\colon \pi_1(K) \to \pi_1(K') = \pi_1(L) = \Pi$, and the relative cellular $\mathbb{Z}[\Pi]$–module chain complex is

$$C(\widetilde{K'}, \widetilde{K})\colon \ \cdots \to 0 \to \mathbb{Z}[\Pi]^n \xrightarrow{1} \mathbb{Z}[\Pi]^n \to 0 \to \cdots$$

concentrated in degrees 2,3. In particular, $K \to K'$ is a $\mathbb{Z}[\Pi]$–coefficient homology equivalence. □

Proposition 3.8 *Let (X, Y) be a finite connected 2–sided CW pair with $X = X_1 \cup_Y X_2$ for connected X_1, X_2, Y, together with an isomorphism*

$$\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \xrightarrow{\cong} G = G_1 *_H G_2$$

preserving amalgamated free product structures, with the structure on G injective. It is possible to attach a finite number of 2– and 3–cells to (X, Y) to obtain a finite injective Seifert–van Kampen splitting (X', Y') with $X' = X_1' \cup_{Y'} X_2'$ such that

- (i) $\pi_1(X') = G$, $\pi_1(X_i') = G_i$ $(i = 1, 2)$, $\pi_1(Y') = H$,
- (ii) *the inclusion $X \to X'$ is a homotopy equivalence,*
- (iii) *the inclusion $X_i \to X_i'$ $(i = 1, 2)$ is a $\mathbb{Z}[G_i]$–coefficient homology equivalence,*
- (iv) *the inclusion $Y \to Y'$ is a $\mathbb{Z}[H]$–coefficient homology equivalence.*

Proof Apply the construction of Lemma 3.7 to the surjections $\pi_1(X_1) \to G_1$, $\pi_1(X_2) \to G_2$, $\pi_1(Y) \to H$, to obtain

$$\begin{aligned} X'_i &= (X_i \cup_{x_i} \bigcup_{m_i} D^2) \cup_{x_i^*} \bigcup_{m_i} D^3 \ (i = 1,2), \\ Y' &= (Y \cup_y \bigcup_n D^2) \cup_{y^*} \bigcup_n D^3 \end{aligned}$$

for any $y = \{y_1, y_2, \dots, y_n\} \subseteq \pi_1(Y)$ such that $\ker(\pi_1(Y) \to H)$ is the normal closure of the subgroup of $\pi_1(Y)$ generated by y, and any

$$x_i = \{x_{i,1}, x_{i,2}, \dots, x_{i,m_i}\} \subseteq \pi_1(X_i)$$

such that $\ker(\pi_1(X_i) \to G_i)$ is the normal closure of the subgroup of $\pi_1(X_i)$ generated by x_i $(i = 1,2)$. Choosing x_1, x_2 to contain the images of y, we obtain the required 2–sided CW pair (X', Y') with $X' = X'_1 \cup_{Y'} X'_2$. □

This completes the proof of the Combinatorial Transversality Theorem for amalgamated free products.

3.3 Combinatorial transversality for HNN extensions

The proof of combinatorial transversality for HNN extensions proceeds exactly as for amalgamated free products, so only the statements will be given.

In this section W is a connected CW complex with fundamental group an injective HNN extension

$$\pi_1(W) = G = G_1 *_H \{t\}$$

with tree T. Let $\widetilde{W}$ be the universal cover of W, and let

$$\begin{tikzcd} \widetilde{W}/H \arrow[r, shift left, "i_1"] \arrow[r, shift right, "i_2"'] & \widetilde{W}/G_1 \arrow[r, "j_1"] & W \end{tikzcd}$$

be the covering projections, and define a commutative square

$$\begin{tikzcd} \widetilde{W}/H \times \{0,1\} \arrow[r, "i_1 \cup i_2"] \arrow[d, "i_3"'] & \widetilde{W}/G_1 \arrow[d, "j_1"] \\ \widetilde{W}/H \times [0,1] \arrow[r, "j_2"] & W \end{tikzcd}$$

where

$$\begin{aligned} &i_3 = \text{inclusion}\colon\ \widetilde{W}/H \times \{0,1\} \to \widetilde{W}/H \times [0,1], \\ &j_2\colon\ \widetilde{W}/H \times [0,1] \to W;\ (x,s) \mapsto j_1 i_1(x) = j_1 i_2(x). \end{aligned}$$

Definition 3.9 (i) Suppose given a subcomplex $W_1 \subseteq \widetilde{W}$ with

$$G_1 W_1 = W_1$$

so that

$$H(W_1 \cap tW_1) = W_1 \cap tW_1 \subseteq \widetilde{W}.$$

Define a commutative square of CW complexes and cellular maps

$$\begin{tikzcd}
(W_1 \cap tW_1)/H \times \{0,1\} \arrow[r, "e_1"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\Phi"] & W_1/G_1 \arrow[d, "f_1"] \\
(W_1 \cap tW_1)/H \times [0,1] \arrow[r, "f_2"] & W
\end{tikzcd}$$

with

$$\begin{aligned}
&(W_1 \cap tW_1)/H \subseteq \widetilde{W}/H,\ W_1/G_1 \subseteq \widetilde{W}/G_1, \\
&e_1 = (i_1 \cup i_2)|: (W_1 \cap tW_1)/H \times \{0,1\} \to W_1/G_1, \\
&e_2 = i_3|: (W_1 \cap tW_1)/H \times \{0,1\} \to (W_1 \cap tW_1)/H \times [0,1], \\
&f_1 = j_1|: W_1/G_1 \to W,\ f_2 = j_2|: (W_1 \cap tW_1)/H \times [0,1] \to W.
\end{aligned}$$

(ii) A *domain* W_1 for the universal cover $\widetilde{W}$ of W is a connected subcomplex $W_1 \subseteq \widetilde{W}$ such that $W_1 \cap tW_1$ is connected, and such that for each cell $D \subseteq \widetilde{W}$ the subgraph $U(D) \subseteq T$ defined by

$$\begin{aligned}
U(D)^{(0)} &= \{g_1 \in [G;G_1] \,|\, g_1 D \subseteq W_1\} \\
U(D)^{(1)} &= \{h \in [G_1;H] \,|\, hD \subseteq W_1 \cap tW_1\}
\end{aligned}$$

is a tree.

(iii) A domain W_1 for $\widetilde{W}$ is *fundamental* if the subtrees $U(D) \subseteq T$ are either single vertices or single edges, so that

$$g_1 W_1 \cap g_2 W_1 = \begin{cases} h(W_1 \cap tW_1) & \text{if } g_1 \cap g_2 t^{-1} = h \in [G_1;H] \\ g_1 W_1 & \text{if } g_1 = g_2 \\ \emptyset & \text{if } g_1 \neq g_2 \text{ and } g_1 \cap g_2 t^{-1} = \emptyset, \end{cases}$$

$$W = (W_1/G_1) \cup_{(W_1 \cap tW_1)/H \times \{0,1\}} (W_1 \cap tW_1)/H \times [0,1].$$

Proposition 3.10 *For a domain W_1 of $\widetilde{W}$ the cellular chain complex $C(W_1)$ is a domain of the cellular chain complex $C(\widetilde{W})$.* □

Example 3.11 W has a canonical infinite domain $W_1 = \widetilde{W}$ with

$$(W_1 \cap tW_1)/H = \widetilde{W}/H$$

and $U(D) = T$ for each cell $D \subseteq \widetilde{W}$.

Example 3.12 (i) Suppose that $W = X_1 \cup_{Y\times\{0,1\}} Y \times [0,1]$, with $X_1, Y \subseteq W$ connected subcomplexes such that the isomorphism

$$\pi_1(W) = \pi_1(X_1) *_{\pi_1(Y)} \{t\} \xrightarrow{\cong} G = G_1 *_H \{t\}$$

preserves the HNN extensions. The morphisms $\pi_1(X_1) \to G_1$, $\pi_1(Y) \to H$ are surjective. (If $i_1, i_2\colon \pi_1(Y) \to \pi_1(X_1)$ are injective these morphisms are also injective, allowing identifications $\pi_1(X_1) = G_1$, $\pi_1(Y) = H$). The universal cover of W is

$$\widetilde{W} = \bigcup_{g_1\in[G:G_1]} g_1\widetilde{X}_1 \cup \bigcup_{h\in[G_1;H]} (h\widetilde{Y} \cup ht\widetilde{Y}) \bigcup_{h\in[G_1;H]} h\widetilde{Y} \times [0,1]$$

with $\widetilde{X}_1$ the regular cover of X_1 corresponding to $\ker(\pi_1(X_1) \to G_1)$ and $\widetilde{Y}$ the regular cover of Y corresponding to $\ker(\pi_1(Y) \to H)$ (which are the universal covers of X_1, Y in the case $\pi_1(X_1) = G_1$, $\pi_1(Y) = H$). Then $W_1 = \widetilde{X}_1$ is a fundamental domain of $\widetilde{W}$ such that

$$\begin{aligned}
&(W_1 \cap tW_1)/H = Y,\ W_1 \cap tW_1 = \widetilde{Y},\\
&g_1W_1 \cap g_2W_1 = (g_1 \cap g_2t^{-1})\widetilde{Y} \subseteq \widetilde{W} \quad (g_1 \neq g_2 \in [G : G_1]).
\end{aligned}$$

For any cell $D \subseteq \widetilde{W}$

$$U(D) = \begin{cases} \{g_1\} & \text{if } g_1D \subseteq \widetilde{X}_1 - \bigcup\limits_{h\in[G_1;H]} (h\widetilde{Y} \cup ht\widetilde{Y}) \text{ for some } g_1 \in [G : G_1] \\ \{g_1, g_2, h\} & \text{if } hD \subseteq \widetilde{Y} \times [0,1] \text{ for some } h = g_1 \cap g_2t^{-1} \in [G_1; H]. \end{cases}$$

(ii) If W_1 is a fundamental domain for any connected CW complex W with $\pi_1(W) = G = G_1 *_H \{t\}$ then $W = X_1 \cup_{Y\times\{0,1\}} Y \times [0,1]$ as in (i) , with

$$X_1 = W_1/G_1,\ Y = (W_1 \cap tW_1)/H.$$

Definition 3.13 Suppose that W is n–dimensional. Lift each cell $D^r \subseteq W$ to a cell $\widetilde{D}^r \subseteq \widetilde{W}$. A sequence $U = \{U_n, U_{n-1}, \ldots, U_1, U_0\}$ of subtrees $U_r \subseteq T$ is *realized* by W if the subspace

$$W(U)_1 = \bigcup_{r=0}^{n} \bigcup_{D^r\subset W} \bigcup_{g_1\in U_r^{(0)}} g_1\widetilde{D}^r \subseteq \widetilde{W}$$

is a connected subcomplex, in which case $W(U)_1$ is a domain for $\widetilde{W}$ with

$$W(U)_1 \cap tW(U)_1 = \bigcup_{r=0}^{n} \bigcup_{D^r \subset W} \bigcup_{h \in U_r^{(1)}} h\widetilde{D}^r \subseteq \widetilde{W}$$

a connected subcomplex. Thus U is realized by $C(\widetilde{W})$ and

$$C(W(U)_1) = C(\widetilde{W}(U)_1 \subseteq j_1^! C(\widetilde{W})$$

is the domain for $C(\widetilde{W})$ given by $C_r(\widetilde{W})_1(U_r)$ in degree r.

Proposition 3.14

(i) *For any domain* W_1 *there is defined a homotopy pushout*

$$\begin{tikzcd}
(W_1 \cap tW_1)/H \times \{0,1\} \arrow[r, "e_1"] \arrow[d, "e_2"'] \arrow[dr, phantom, "\Phi"] & W_1/G_1 \arrow[d, "f_1"] \\
(W_1 \cap tW_1)/H \times [0,1] \arrow[r, "f_2"] & W
\end{tikzcd}$$

with $e_1 = i_1 \cup i_2|$, $e_2 = i_3|$, $f_1 = j_1|$, $f_2 = j_2|$. *The connected 2–sided CW pair*

$$(X,Y) = (\mathcal{M}(e_1,e_2), (W_1 \cap tW_1)/H \times \{1/2\})$$

is a Seifert–van Kampen splitting of W, *with a homotopy equivalence*

$$f = f_1 \cup f_2 \colon\ X = \mathcal{M}(e_1,e_2) \xrightarrow{\simeq} W.$$

(ii) *The commutative square of covering projections*

$$\begin{tikzcd}
\widetilde{W}/H \times \{0,1\} \arrow[r, "i_1 \cup i_2"] \arrow[d, "i_3"'] & \widetilde{W}/G_1 \arrow[d, "j_1"] \\
\widetilde{W}/H \times [0,1] \arrow[r, "j_2"] & W
\end{tikzcd}$$

is a homotopy pushout. The connected 2–sided CW pair

$$(X(\infty), Y(\infty)) = (\mathcal{M}(i_1 \cup i_2, i_3), \widetilde{W}/H \times \{0\})$$

is a canonical injective infinite Seifert–van Kampen splitting of W, *with a homotopy equivalence* $j = j_1 \cup j_2 \colon X(\infty) \to W$ *such that*

$$\pi_1(Y(\infty)) = H \subseteq \pi_1(X(\infty)) = G_1 *_H \{t\}.$$

(iii) *For any (finite) sequence $U = \{U_n, U_{n-1}, \dots, U_0\}$ of subtrees of T realized by W there is defined a homotopy pushout*

$$\begin{tikzcd}
Y(U) \times \{0,1\} \arrow[r, "e_1"] \arrow[d, "e_2"'] & X(U)_1 \arrow[d, "f_1"] \\
Y(U) \times [0,1] \arrow[r, "f_2"] & W
\end{tikzcd}$$

with

$$Y(U) = (W(U)_1 \cap tW(U)_1)/H,\ X(U)_1 = W(U)_1/G_1,$$
$$e_1 = i_1 \cup i_2|,\ e_2 = i_3|,\ f_1 = j_1|,\ f_2 = j_2|.$$

Thus

$$(X(U), Y(U)) = (\mathcal{M}(e_1, e_2), Y(U) \times \{1/2\})$$

is a (finite) Seifert–van Kampen splitting of W.

(iv) *The canonical infinite domain of a finite CW complex W with $\pi_1(W) = G_1 *_H \{t\}$ is a union of finite domains $W(U)_1$*

$$\widetilde{W} = \bigcup_U W(U)_1$$

with U running over all the finite sequences realized by W. The canonical infinite Seifert–van Kampen splitting is thus a union of finite Seifert–van Kampen splittings

$$(X(\infty), Y(\infty)) = \bigcup_U (X(U), Y(U)). \qquad \square$$

Proposition 3.15 *Let (X, Y) be a finite connected 2–sided CW pair with $X = X_1 \cup_{Y \times \{0,1\}} Y \times [0,1]$ for connected X_1, Y, together with an isomorphism*

$$\pi_1(X) = \pi_1(X_1) *_{\pi_1(Y)} \{t\} \xrightarrow{\cong} G = G_1 *_H \{t\}$$

preserving the HNN structures, with the structure on G injective. It is possible to attach a finite number of 2– and 3–cells to the finite Seifert–van Kampen splitting (X, Y) of X to obtain a finite injective Seifert–van Kampen splitting (X', Y') with $X' = X'_1 \cup_{Y' \times \{0,1\}} Y' \times [0,1]$ such that

(i) $\pi_1(X') = G$, $\pi_1(X'_1) = G_1$, $\pi_1(Y') = H$,

(ii) *the inclusion $X \to X'$ is a homotopy equivalence,*

(iii) *the inclusion $X_1 \to X'_1$ is a $\mathbb{Z}[G_1]$–coefficient homology equivalence,*

(iv) *the inclusion $Y \to Y'$ is a $\mathbb{Z}[H]$–coefficient homology equivalence.* $\square$

This completes the proof of the Combinatorial Transversality Theorem for HNN extensions.

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