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INFINITELY FINE PARTITIONS OF MEASURES SPACES

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1. Infinitely fine partitions

Let $(\mathcal{X}, \mathcal{A})$ be a measurable space, i. e. $\mathcal{X}$ is a set and $\mathcal{A}$ is an algebra of its subsets. Let Q be the Stone space of the algebra $\mathcal{A}$ whose points are the ultrafilters of measurable sets, and let ι be the canonical boolean isomorphism between $\mathcal{A}$ and the algebra $\operatorname{Clop} Q$ of all clopen subsets of Q, namely, $\iota(A) = \{q \in Q \mid A \in q\}$.

Fix a nonstandard extension ${}^*\mathcal{X}$ of the set $\mathcal{X}$. We assume the nonstandard model be k-saturated, where k is the cardinality of $\mathcal{A}$.

Definition 1.1. A measurable partition P of ${}^*\mathcal{X}$ is said to be an *infinitely fine partition (ifp)*, if ${}^*A = \bigcup\{p \in P \mid p \subset {}^*A\}$ for every measurable set A.

The saturation principle implies the existence of a hyperfinite ifp for any measurable space $(\mathcal{X}, \mathcal{A})$ (see, for example, [1]) and even for a Boolean algebra. All the following results except section 3 are valid for arbitrary Boolean algebras, not only for algebras of subsets.

Thus, in what follows, we denote by P some ifp of $(\mathcal{X}, \mathcal{A})$. Denote by P_s and P_n respectively the totalities of standard and nonstandard elements of P. Note that if all singletons are measurable, then $P_s \supset \{\{x\} \mid x \in \mathcal{X}\}$.

Definition 1.2. Let $p_1, p_2 \in P$. We say that p_1 is equivalent to p_2 and denote this relation by $p_1 \sim p_2$ if $P \setminus \{p_1, p_2\} \cup \{p_1 \cup p_2\}$ is an ifp. An element of the partition p_1 is called *joinable* if there exists another element $p_2 \in P$ such that $p_1 \sim p_2$. In such situation we say that p_2 *integrates* p_1.

Theorem 1.3. *An element of an ifp is nonjoinable if and only if it is standard.*

Proof. Obviously all the elements of P_s nonjoinable. We must show that any element $p_0 \in P_n$ is joinable. Consider the measurable ultrafilter $q = \{A \in \mathcal{A} \mid p_0 \subset {}^*A\}$. Let $\mathcal{S}$ be a finite subset of q and let $S = \bigcap \mathcal{S}$. Evidently $p_0 \subset {}^*S$, but since $p_0 \in P_n$, it follows that $p_0 \neq {}^*S$. Therefore there exists $p \in P$ such that $p \subset {}^*S$ and $p \neq p_0$.

By the saturation principle, this implies that there exists an element $p \in P$ other than p_0 contained in the extension of every set belonging to q. This means that the element p_0 is joinable.

In [1] it was shown that a standard bounded measurable function is approximately equal to a constant on every element of ifp. This has the following consuquence.

Lemma 1.4. *Let $p_1, p_2 \in P$, then $p_1 \sim p_2$ if and only if the values of the extension of any standard bounded measurable function $f : X \to \mathbb{R}$ on p_1 and p_2 are approximately equal.*

Proof. The implication from left to right follows from the fact that $P \setminus \{p_1, p_2\} \cup \{p_1 \cup p_2\}$ is an ifp. The converse can be easily obtained by considering characteristic functions of standard measurable sets.

2. Nonatomicity

In this section we present some interpretations of nonatomicity in the language of infinitely fine partitions.

Let a standard measurable space $(\mathcal{X}, \mathcal{A})$ be endowed with a standard finitely additive measure μ. Set $P_+ = \{p \in P \mid {}^*\mu(p) > 0\}$; $P_0 = \{p \in P \mid {}^*\mu(p) = 0\}$; $\mathcal{A}_+ = \{A \in \mathcal{A} \mid \mu(A) > 0\}$; $\mathcal{A}_0 = \{A \in \mathcal{A} \mid \mu(A) = 0\}$. We call the elements of P_+ the essential elements of the ifp P. Notice that all the points of elements of P_+ are random (recall that an element $x \in {}^*\mathcal{X}$ is random if $x \notin {}^*N$ whenever N is μ-null ($N \subset \mathcal{X}$)).

Definition 2.1. Let $p \in P_+$. We say that p is *essentially joinable* if there exists $p' \in P_+$ integrating p. We call p *essentially divisible* if there exist disjoint $p_1, p_2 \in {}^*\mathcal{A}_+$ such that $p = p_1 \cup p_2$.

Definition 2.2. A measurable set F is said to be an *atom* of measure μ if $\mu(F) > 0$ and $E \subset F$ implies either $\mu(E) = 0$ or $\mu(E) = \mu(F)$ for any measurable set E. A measure is said to be *nonatomic* if it has no atoms. A measure is said to be *strongly continuous* if for any positive ε, there exists a finite measurable partition of $\mathcal{X}$ such that the measure of each of its elements is less than ε. A measure μ is said to be *strongly nonatomic* if for every measurable set F and real number $c \in [0, \mu(F)]$ there exists a measurable set $E \subset F$ such that $\mu(E) = c$.

It was shown in [2] that the strong nonatomicity of a measure implies its strong continuity, strong continuity implies nonatomicity, and in case of a σ-additive measure, all the three properties are equivalent.

Theorem 2.3. *The following statements are equivalent:*
E very essential element of an ifp is essentially divisible;
E very essential element of an ifp is essentially joinable;
T he measure μ is nonatomic.

Proof. To prove the implication from (1) to (2) consider an essential element p_0. By the assumption, it is essentially divisible. Define $q = \{A \in \mathcal{A} \mid p_0 \subset {}^*A\}$; let $\mathcal{S}$ be a finite subset of q, and let $S = \bigcap \mathcal{S}$. Evidently, $p_0 \subset {}^*S$. Essential divisibility of p_0 implies that there exists $p_1 \in {}^*\mathcal{A}_+$ such that $0 < {}^*\mu(p_1) < {}^*\mu(p_0) \leqslant \mu(S)$. By the transfer principle there exists a standard set $B \subset S$ such that the both B and $S \setminus B$ belong to $\mathcal{A}_+$. Apparently, either $p_0 \subset {}^*B$ or $p_0 \subset {}^*S \setminus {}^*B$; let $p_0 \subset {}^*B$. It is easily verified that there is an essential element $p \subset {}^*S \setminus {}^*B$. If we set $F_A = \{p \in P_+ \mid p \subset {}^*A,\ p \neq p_0\}$ where $A \in q$ then this collection has the finite intersection property, and, by the saturation principle, there exists an essential element of ifp P, integrating p_0.

Let us prove the implication (2) $\Rightarrow$ (3) now. Take an arbitrary set $A \in \mathcal{A}_+$. There exists $p \in P_+$ such that $p \subset {}^*A$. Since p is essentially joinable, then there is $p' \in P_+$ such that $p' \subset {}^*A \setminus p$; it follows ${}^*\mu(p) < \mu(A)$. By using the transfer principle we obtain the result.

The implication (3) $\Rightarrow$ (1) is transparent.

Theorem 2.4. *A measure is strongly continuous if and only if the measure of every element of an ifp is infinitesimal.*

Proof. This s transparent.

Theorem 2.5. *A measure μ is strongly nonatomic if and only if* $(\forall p \in P_+)(\forall \lambda \in {}^*[0,1])(\exists p' \in {}^*\mathcal{A}_+)(p' \subset p \ \& \ \frac{{}^*\mu(p')}{{}^*\mu(p)} = \lambda)$.

Proof. The implication from left to right is obvious. To prove the converse implication, for any $F \in \mathcal{A}_+$ and $c \in [0, \mu(F))$. Consider an internal subset E of ${}^*F$ such that

(1) $E = \cup\{p \in P \,|\, p \subset E\}$;

(2) ${}^*\mu(E) \leqslant c$;

(3) $\forall p \not\subseteq E \quad {}^*\mu(E) + {}^*\mu(p) > c$.

It is easily verified that such a set exists (since ${}^*F$ consists of a hyperfinite number of p's). Then, if ${}^*\mu(E) < c$, take any $p \not\subseteq E$, divide it into p_1 and p_2 so that ${}^*\mu(E) + {}^*\mu(p_1) = c$ and append p_1 to E. The transfer principle completes the proof.

3. Representation of L_∞

In the following section we develop one of the results obtained by P.Loeb in his paper [1]. For this subsection, let μ be a standard σ-additive measure and P a hyperfinite ifp for a standard measurable space $(\mathcal{X}, \mathcal{A})$. In [1] Loeb introduced the map $T_0 : L_\infty \to \mathbb{R}^P$ by the following rule: in every $p \in P_+$ choose a point c_p; for $f \in L_\infty$ set $(T_0(f))_p = {}^*f(c_p)$ for $p \in P_+$ and $(T_0(f))_p = 0$ for $p \in P_0$. Loeb proved in [1] that a vector $\rho \in \mathbb{R}^P$ is equal to $T_0(f)$ for some $f \in L_\infty$ if and only if the following three conditions hold:

(1) $\rho_p = 0$ for all $p \in P_0$;

(2) The value $\max\limits_{p \in P} |\rho_p|$ is nearstandard;

(3) $\forall p \in P_+$ and $\forall \varepsilon > 0$ in $\mathbb{R}$, $\exists A \in \mathcal{A}$ such that $p \subset {}^*A$ and $|\rho_p - \rho_{p'}| < \varepsilon$ for every essential $p' \subset {}^*A$.

Lemma 3.1. *The condition* (3) *is equivalent to the following statement:*

(3′) *For any $p_1, p_2 \in P_+$ if $p_1 \sim p_2$ then $\rho_{p_1} \approx \rho_{p_2}$.*

Proof. (3′) follows from (3) since $(\forall \varepsilon > 0 \,\text{in}\, \mathbb{R} \quad |\rho_{p_1} - \rho_{p_2}| < \varepsilon)$ implies $\rho_{p_1} \approx \rho_{p_2}$. We the converse using reductio ad absurdum. Assume that $(\exists \varepsilon \in \mathbb{R}^+)(\forall A \in \mathcal{A})(p \subset {}^*A \to (\exists p' \in P_+ \quad p' \subset {}^*A \quad \& \quad |\rho_p - \rho_{p'}| \geqslant \varepsilon))$. Let $q = \{A \in \mathcal{A} \,|\, p \subset {}^*A\}$; Let $\mathcal{S}$ be a finite subset of q and $S = \bigcap \mathcal{S}$. By the assumption, there exists an essential $p' \subset {}^*S$ such that $|\rho_p - \rho_{p'}| \geqslant \varepsilon$. By the saturation principle, there exists an element $p' \in P_+$, such that $(\forall A \in q)(p' \subset {}^*A \ \& \ |\rho_p - \rho_{p'}| \geqslant \varepsilon)$, but this contradicts to (3′).

4. Monads of ifp

Let P be an ifp for a standard measurable space $(\mathcal{X}, \mathcal{A})$.

Definition 4.1. A collection of equivalent elements of P is called a *monad*: $m_p = \{p' \in P \,|\, p' \sim p\}$.

Since «$\sim$» is an equivalence relation, $\mathfrak{M} = \{\bigcup m_p\}_{p \in P}$ is an partition of ${}^*\mathcal{X}$. Let $\mathcal{A}_* = \{{}^*\mathcal{A} \,|\, A \in \mathcal{A}\}$. Clearly, every standard measurable set is exactly the union of the elements of all monads, contained in this set.

Lemma 4.2. *The partition $\mathfrak{M}$ coincides with the partition generated by the algebra $\mathcal{A}_*$.*

Proof. On the one hand, if $x_1, x_2 \in \bigcup m \in \mathfrak{M}$ for some monad m, then there exist $p_1, p_2 \in m$ such that $x_i \in p_i \in P$ for $i = 1, 2$. Since $p_1 \sim p_2$, it follows that $P \setminus \{p_1, p_2\} \cup \{p_1 \cup p_2\}$ is an ifp; and, therefore, for any set $A \in \mathcal{A}$ either $p_1 \cup p_2 \subset {}^*A$ or $(p_1 \cup p_2) \cap {}^*A = \emptyset$; i. e. x_1 and x_2 cannot be separated by sets from $\mathcal{A}_*$.

On the other hand, if x_1 and x_2 are not separated by the extensions of standard measurable sets, then the elements of P containing them are, obviously, equivalent.

There are one-to-one correspondeces between the monads of an ifp, the ultrafilters of $\mathcal{A}$, the zero-one measures on $\mathcal{A}$, and the Stone space Q. Consider a monad m of an ifp P. The corresponding ultrafilter q_m is $\{A \in \mathcal{A} \mid \bigcup m \subset {}^*A\}$, i. e., the elements of q_m are exactly the standard measurable sets containing the monad m. On the other hand, $m = \{p \in P \mid (\forall A \in q_m)(p \subset {}^*A)\}$, i. e., $\bigcup m = \bigcap q_m$. Any monad m corresponds to some zero-one measure on $\mathcal{A}$ defined by

$$\delta_m(A) = \begin{cases} 1, & \text{if } \bigcup m \subset {}^*A, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $q_m = \{A \in \mathcal{A} \mid \delta_m(A) = 1\}$. Finally, for any monad m we can consider the ultrafilter q_m as a point of the Stone space Q of $\mathcal{A}$. Notice that $\bigcup\{\iota(p) \mid p \in m\}$, where ι is the canonical boolean isomorphism between $\mathcal{A}$ and $\operatorname{Clop} Q$, is the usual topological monad of q_m in Q. Notice also, that $\{q_m \mid \bigcup m \subset A\} = \iota(A)$ for any A in $\mathcal{A}$.

Among the elements of a monad there is one distinguished (we shall call it *central*). It will be shown that properties of measures are essentially determined by their values at the central elements. The only element of m which belongs to ${}^*q_m$ is called *central* central and is denoted p_m. On the other hand, p_m is the only element of m such that ${}^*\delta_m$ takes value 1 on it. Finally, p_m is the only element of m such that ${}^*\iota(p)$ contains a standard point of ${}^*Q$, namely, $q_m \in {}^*\iota(p)$. Denote P_c the collection of all central elements of ifp P.

The following is some interesting and usefull tools on monads, central elements, and measures on $\mathcal{A}$. Let μ be a standard finitely additive measure on $\mathcal{A}$. We assume for simplicity that μ is finite.

Theorem 4.3. *Let $p \in P_c$ and ${}^*\mu(p) \in \mathbb{R}$. Then*
(1) *${}^*\mu(p') = 0$ for any noncentral p' equivalent to p.*
(2) *There exists a set $A \in \mathcal{A}$ such that $p \subset {}^*A$ and $\mu(A) = {}^*\mu(p)$.*

Proof. Let $p = p_m$ for some monad m. Since $p \in {}^*q_m$ and ${}^*\mu(p) \in \mathbb{R}$, then by the transfer principle we can find a standard $A \in q_m$ such that $\mu(A) = {}^*\mu(p)$. Thus, we have (2). (1) follows apparently from (2).

Corollary 4.4. If $p \in P_c$, ${}^*\mu(p) = 0$, and $p' \sim p$ then, ${}^*\mu(p') = 0$.

Theorem 4.5. *If m is a monad of an ifp then for every standard positive ε there exists a standard measurable set A such that ${}^*A$ contains m and $\mu(A) \leqslant {}^*\mu(p_m) + \varepsilon$.*

Proof. Assume that for any $A \in q_m$ holds $\mu(A) > {}^\circ({}^*\mu(p_m)) + \varepsilon$. Then, by the transfer principle, ${}^*\mu(p_m) > {}^\circ({}^*\mu(p_m)) + \varepsilon$. The obtained contradiction gives the desired result.

Corollary 4.6. *Measure of any noncentral element is infinitesimal. Moreover, measure of any measurable subset of $\bigcup m$, ${}^*\iota$-image of which does not contain q_m, is infinitesimal.*

Lemma 4.7. *For any $p \in P_c$ the inequality $\mu(p) \geqslant {}^\circ\mu(p)$ holds.*

Proof. For any $A \in q_{m_p}$ we have $p \subset A$, and, therefore, ${}^\circ\mu(p) \leqslant \mu(A)$. By the transfer principle, ${}^\circ\mu(p) \leqslant \mu(p)$.

5. Sobczik–Hammer Decomposition Theorem

The constructions developed above give us an opportunity to present a simple proof of the Sobczik–Hammer Decomposition Theorem.

Theorem 5.1 (Sobczik–Hammer Decomposition Theorem). *Let μ be a finite finitely additive measure on a measurable space $(\mathcal{X}, \mathcal{A})$. Then there exists a sequence $(\delta_n)_{n\in\mathbb{N}}$ of distinct zero-one measures on $\mathcal{A}$, a sequence $(a_n)_{n\in\mathbb{N}}$ of nonnegative real numbers, and a strongly continuous measure μ_0 on $\mathcal{A}$, such that $\sum_{n=1}^{\infty} a_n < \infty$ and $\mu = \mu_0 + \sum_{n=1}^{\infty} a_n\delta_n$. Further, this decomposition is unique.*

Proof. Let P be an ifp for $(\mathcal{X}, \mathcal{A})$. Take as p_1 an element of P of maximum measure. Let $a_1 = \operatorname{st}({}^*\mu(p_1))$ and $\delta_1 = \delta^{p_1}$. If μ is strongly continuous, then $a_1 = 0$. Otherwise, p_1 is central and, by Lemma 4.7, we have $a_1 \leqslant {}^*\mu(p_1)$. It follows that $\mu_1 = \mu - a_1\delta_1$ is a nonnegative standard measure. Now we can apply this procedure to μ_1 and obtain $\mu_2 = \mu_1 - a_2\delta_2$, etc. Iterating this process, we obtain the nonincreasing sequence $(a_n)_{n\in\mathbb{N}}$ of standard positive real numbers, the sequence $(\delta_n)_{n\in\mathbb{N}}$ of standard distinct zero-one measures, and the sequence $(\mu_n)_{n\in\mathbb{N}}$ of standard measures, such that $\mu_n = \mu - \sum_{i=1}^{n} a_i\delta_i$ for every $n \in \mathbb{N}$. In particular, $\mu(\mathcal{X}) \geqslant \sum_{i=1}^{n} a_i\delta_i(\mathcal{X}) = \sum_{i=1}^{n} a_i$ for every natural n; it follows that $\sum_{i=1}^{\infty} a_i < \infty$.

It is easily verified that μ_0 defined by $\mu_0 = \mu - \sum_{n=1}^{\infty} a_n\delta_n$ is a standard nonnegative measure, and $\mu_0 \leqslant \mu_n$ for all $n \in \mathbb{N}$. Assume that μ_0 is not strongly continuous; then there exists $p \in P$ such that $\operatorname{st}({}^*\mu(p)) > 0$. Since a_n converges to zero, we can find $n \in \mathbb{N}$ such that $a_n < \operatorname{st}({}^*\mu(p))$; by the definition of a_n it follows that $\operatorname{st}({}^*\mu_n(p)) < \operatorname{st}({}^*\mu(p))$, but this contradicts to the fact that $\mu \leqslant \mu_n$.

We see that the numbers a_n and measures δ_n are determined by the values of μ on the central elements of P up to the order. Thus, the constructed decomposition is unique.

The concept of an infinitely fine partition also gives us an opportunity to prove the Sobczik-Hammer Decomposition Theorem for the case of a vector measure. Let X be a standard Banach space, and let $F : \mathcal{A} \to X$ be a standard X-valued measure on $(\mathcal{X}, \mathcal{A})$. Recall that the variation of F is a real-valued measure given by $|F|(A) = \sup_{\pi} \sum_{B\in\pi} \|F(B)\|$, where sup is taken over all finite measurable partitions π of A. A vector measure is said to be nonatomic (strongly continuous, strongly nonatomic) if this is true for its variation.

To prove the vector analogue of Sobczik–Hammer Theorem, we need the following lemma:

Lemma 5.2. *Let F be a standard vector measure of bounded variation on $(\mathcal{X}, \mathcal{A})$; let P be an ifp for $(\mathcal{X}, \mathcal{A})$, and $p \in P$. Then $|{}^*F|(p) \approx \|{}^*F(p)\|$.*

Proof. Evidently, $|F|(A) \geqslant \|F(A)\|$ for any measurable A. If $p \notin P_c$ then $|{}^*F|(p) \approx 0$ by Corollary 4.6. Therefore, $0 \leqslant \operatorname{st}(\|{}^*F(p)\|) \leqslant \operatorname{st}((|{}^*F|(p)) = 0$, and we conclude $\|{}^*F(p)\| \approx |{}^*F|(p) \approx 0$.

Let $p \in P_c$. Fix an infinitesimal $\varepsilon > 0$, consider a finite measurable partition π of p such that $|{}^*F|(p) \leqslant \sum_{p'\in\pi} \|{}^*F(p')\| + \varepsilon$. Denote by p_π the element of π such that $\varphi(m_p) \in {}^*\iota(p_\pi)$.

Then, by Corollary 4.6, $|^*F|(p \setminus p_\pi) \approx 0$. Therefore,

$$\|^*F(p_\pi)\| \leqslant |^*F|(p) \leqslant \sum_{p' \in \pi} \|^*F(p')\| + \varepsilon \leqslant$$

$$\leqslant \sum_{p' \in \pi, p' \neq p_\pi} |^*F|(p') + \|^*F(p_\pi)\| + \varepsilon =$$

$$|^*F|(p \setminus p_\pi) + \|^*F(p_\pi)\| + \varepsilon \approx \|^*F(p_\pi)\|.$$

Thus, $\|^*F(p_\pi)\| \approx |^*F|(p)$. On the other hand,

$$\|^*F(p_\pi)\| = \|^*F(p) - {}^*F(p \setminus p_\pi)\| \leqslant \|^*F(p)\| + \|^*F(p \setminus p_\pi)\| \approx \|^*F(p)\|,$$

and

$$\|^*F(p)\| = \|^*F(p_\pi) + {}^*F(p \setminus p_\pi)\| \leqslant \|^*F(p_\pi)\| + \|^*F(p \setminus p_\pi)\| \approx \|^*F(p_\pi)\|,$$

so that $\|^*F(p_\pi)\| \approx \|^*F(p)\|$.

Now we can proceed to the vector version of the Sobczik–Hammer Theorem. A similar result can be found in [4], however, the proof presented there is based on absolutely different concepts.

Theorem 5.3 (Sobczik–Hammer Decomposition Theorem for vector measure). *Let F be a standard X-valued measure of bounded variation on $(\mathcal{X}, \mathcal{A})$ such that the range of F is contained in some compact set. Then there exists a strongly additive vector measure F_0, a sequence $(x_n)_{n\in\mathbb{N}} \subset X$, and a sequence $(\delta_n)_{n\in\mathbb{N}}$ of distinct zero-one measures on $(\mathcal{X}, \mathcal{A})$, such that $F = F_0 + \sum_{n=1}^{\infty} x_n \delta_n$. Further, this decomposition is unique up to the order of addenda.*

Proof. is analogous to the proof of Theorem 5.1. Let P be an ifp for $(\mathcal{X}, \mathcal{A})$; let p_1 be an element of P of maximum value of $|^*F|$. Since ${}^*F(p_1)$ belongs to some compact set, it is nearstandard; let $x_1 = \operatorname{st}({}^*F(p_1))$, $\delta_1 = \delta^{p_1}$, $F_1 = F - x_1\delta_1$. Then we have $|^*F_1|(p) \approx \|^*F_1(p_1)\| = \|^*F(p_1) - \operatorname{st}(F(p_1))\| \approx 0$. It is simple to check, using the transfer principle, that $|F_1| \leqslant |F|$.

Iterating this process we obtain the sequences $(F_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}}$, $(\delta_n)_{n\in\mathbb{N}}$, consisting of standard elements with the properties $F_n = F - \sum_{i=1}^{n} x_i\delta_i$, $|F_{n+1}| \leqslant |F_n| \leqslant |F|$, and $\|x_{n+1}\| \leqslant \|x_n\|$. Let $F_0 = F - \sum_{i=1}^{\infty} x_i\delta_1$; the examination of the strong continuity of $|F_0|$ is the same as in Theorem 5.1.

6. Horn–Tarsky Theorem

Theorem 6.1 (Horn–Tarsky). *Let $\mathcal{A}$ be an algebra of subsets of a set $\mathcal{X}$, $\mathcal{C}$ a subalgebra of $\mathcal{A}$, and μ a finitely additive measure on $\mathcal{C}$. Then μ can be extended to $\mathcal{A}$.*

Proof. Let $P_{\mathcal{A}}$ and $P_{\mathcal{C}}$ be hiperfinite ifp's for $\mathcal{A}$ and $\mathcal{C}$ respectively. Without loss of generality we can assume $P_{\mathcal{A}}$ is a refinement of $P_{\mathcal{C}}$. Take any $p \in P_{\mathcal{C}}$, then some $p_1, p_2, \ldots$,

p_N from $P_{\mathcal{A}}$ form a partition of p (N is hyperfinite). Distribute ${}^*\mu(p)$ among $p_1, p_2, \dots,$ p_N arbitrarily, i. e. assign some pseudoweight $w(p_i)$ to each p_i $(i = 1, \dots, N)$ such that

$$\sum_{i=1}^{N} w(p_i) = {}^*\mu(p).$$

Apply this procedure to every $p \in P_{\mathcal{C}}$. Now each element of $P_{\mathcal{A}}$ is assigned some pseudoweight.

Let $A \in \mathcal{A}$, then ${}^*A = p'_1 \cup p'_2 \cup \cup p'_M$ for some hyperfinite M and $p'_j \in P_{\mathcal{A}}$. Let $\lambda(A) = {}^\circ\big(\sum\limits_{j=1}^{M} w(p'_j)\big)$. It can be easily verified that λ is a standard finitely additive measure and $\lambda_{|}\mathcal{C} = \mu$.

Obvioulsly, the same reasoning can be used to prove Horn–Tarsky Theorem for a Banach-valued measure, but we need its range lie inside some compact set (to be able to take standard part when defining λ).

Referances

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