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ON NON-COMMUTATIVE ERGODIC TYPE THEOREMS FOR FREE FINITELY GENERATED SEMIGROUPS

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In the paper the authors generalize Bufetov's Ergodic Type Theorems to the case of the actions of free finitely generated semigroups on von Neumann algebras.

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1. Introduction

First Ergodic Theorems for actions of arbitrary countable groups were obtained by Oseledets [26], who followed an idea of Kakutani [17]. For actions of free groups Guivarc'h [14] considered uniform averages over spheres of increasing radii in a group and proved the related mean ergodic theorem. Grigorchuk [12] announced the Pointwise Ergodic Theorem for Česaro averages of the spherical averages. Nevo [24] and Nevo and Stein [25] published a proof of the Pointwise Ergodic Theorem. In [13] Grigorchuk announced an Ergodic Theorem for Actions of Free Semigroups. In [3] Bufetov generalized classical and recent Ergodic Theorems of Kakutani, Oseledets, Guivarc'h, Grigorchuk, Nevo and Nevo and Stein for measure-preserving actions of free semigroups and groups.

The first results in the field of non-commutative Ergodic Theorems were obtained by Sinai and Anshelevich [29] and Lance [22]. Developments of the subject are reflected in the monographs of Jajte [15] and Krengel [21].

Majorant ergodic theorem for the operators affiliated to tracial von Neumann algebras was proved in [6].

The aim of the present paper is to generalize Bufetov's results from [3] to the non-commutative case to obtain non-commutative Ergodic Theorems for the actions of free finitely generated semigroups on von Neumann algebras.

Remark1. The paper extends results presented by the authors in [7] and [8].

2. Non-commutative Operator Ergodic Theorems

Let the pair (M, τ) be a *non-commutative probability space*, where M is a von Neumann algebra with a faithful, normal tracial state τ.

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Let $\alpha_1, \alpha_2, \ldots, \alpha_m : M \to M$ be *positive kernels* or linear maps satisfying following conditions: ($\alpha_i(M_+) \subset M_+$; $\alpha_i \mathbf{1} \leqslant \mathbf{1}$; $\tau \circ \alpha_i \leqslant \tau$).

All the $\{\alpha_i\}$'s could be extended to operators $L_1(M,\tau) \to L_1(M,\tau)$, which we will also call $\{\alpha_i\}$ without loss of generality.

Let $\Omega_m = \{\omega = \omega_1\omega_2 \ldots \omega_n \ldots : \omega_i = 1, \ldots, m\}$ be the space of all one-sided infinite sequences in the symbols $1, \ldots, m$.

We denote by σ_m the shift on Ω_m, defined by the formula $(\sigma_m \omega)_i = \omega_{i+1}$.

Consider the set $W_m = \{w = w_1 w_2 \ldots w_n : w_i = 1, \ldots, m\}$ of all finite words in the symbols $1, \ldots, m$.

Denote by $|w|$ the length of the word w. For each $w \in W_m$, let $C(w) \subset \Omega_m$ be the set of all sequences starting with the word w. For an arbitrary Borel measure μ on Ω_m, set $\mu(w) = \mu(C(w))$.

Measure μ on Ω_m invariant with respect to shift σ_m we call σ_m-invariant measure.

For each $w \in W_m$, introduce the operator

$$\alpha_w = \alpha_{w_n} \alpha_{w_{n-1}} \ldots \alpha_{w_1}. \tag{2.1}$$

Let μ be a Borel σ_m-invariant probability measure on Ω_m. Consider the words w with $|w| = l$, and the sum of the corresponding operators α_w with the weights $\mu(w)$,

$$s_l^\mu(\alpha) = \sum_{|w|=l} \mu(w) \alpha_w.$$

Average $s_l^\mu(\alpha)$ over $l = 0, \ldots, n-1$,

$$c_n^\mu(\alpha) = \frac{1}{n} \sum_{l=0}^{n-1} s_l^\mu(\alpha).$$

Definition 2.1. A sequence $\{X_n\} \subset L_1(M,\tau)$ is said to converge to $X_0 \in L_1(M,\tau)$ doubleside almost everywhere if for every $\epsilon \geqslant 0$ and $\delta \geqslant 0$ there exists $N \in \mathbb{N}$ and projection $E \in M$ such that $\tau(\mathbb{I} - E) < \delta$ and $E(X_n - X_0)E \in M$ and $\|E(X_n - X_0)E\|_\infty \leqslant \epsilon$ for $n \geqslant N$.

Suppose μ is a σ_m-invariant Markov measure on Ω_m. We will show that the averages $c_n^\mu(\alpha)\varphi$ converge both doubleside almost everywhere and in $L_1(M,\tau)$ for any operator $\varphi \in L_1(A,\tau)$.

Definition 2.2. A matrix Q with non-negative entries is said to be irreducible if, for some $n > 0$, all entries of the matrix $Q + Q^2 + \ldots + Q^n$ are positive (if Q is stochastic, then this is equivalent to saying that in the corresponding Markov chain any state is attainable from any other state).

Definition 2.3. A matrix P with non-negative entries is said to be strictly irreducible if P and PP^T are irreducible (here P^T stands for the transpose of the matrix P).

Definition 2.4. A Markov chain is said to be strictly irreducible if the corresponding transition matrix is strictly irreducible.

Let (M,τ) be a non-commutative probability space, $\alpha_1, \ldots, \alpha_m : M \to M$ positive kernels, and $\alpha_1, \ldots, \alpha_m : L_1(M,\tau) \to L_1(M,\tau)$ their corresponding extensions. Let μ be a σ_m-invariant Markov measure on Ω_m. Then, for any element $\varphi \in L_1(M,\tau)$, there exists an element $\overline{\varphi} \in L_1(M,\tau)$, such that $c_n^\mu(\alpha)\varphi \to \overline{\varphi}$ in $L_1(M,\tau)$ norm as $n \to \infty$.

Theorem 2.1. *The following equality holds whenever $\alpha_1, \ldots, \alpha_m$ preserve the state τ: $\tau(\varphi) = \tau(\overline{\varphi})$. If the measure μ is strictly irreducible, then $\alpha_j \overline{\varphi} = \overline{\varphi}$, for $j = 1, \ldots, m$. If*

$\varphi \in L_p(M,\tau)$, $p \geqslant 1$, then $c_n^\mu(\alpha)\varphi \to \overline{\varphi}$ both doubleside almost everywhere and in $L_p(M,\tau)$ norm as well.

Remark 2. Theorem 2.1 generalizes Ergodic Theorems of Grigorchuk [13], Nevo [24], Nevo and Stein [25], and Bufetov [3] to the non-commutative case.

Now we discuss an operator version of Theorem 2.1.

Let (M,τ) be a non-commutative probability space and $\alpha_1,\dots,\alpha_m : L_1(M,\tau) \to L_1(M,\tau)$ be linear operators. The operators α_w, $s_l^\mu(\alpha)$, and $c_n^\mu(\alpha)$ are introduced as above.

Recall the standard terminology. A linear operator on a Banach space is called a contraction, if its norm is not greater than one.

Definition 2.5. A linear operator $\alpha : L_1(M,\tau) \to L_1(M,\tau)$ is said to be positive, if the image of each non-negative element is a non-negative element.

Definition 2.6. A linear operator $\alpha : L_1(M,\tau) \to L_1(M,\tau)$ is called an L_1-L_∞-contraction, if $\|\alpha\|_{L_1} \leqslant 1$ and $\|\alpha\|_{L_\infty} \leqslant 1$.

Definition 2.7. A linear operator $\alpha : L_1(M,\tau) \to L_1(M,\tau)$ is said to be τ-preserving, if $\tau(\varphi) = \tau(\alpha(\varphi))$ for any $\varphi \in L_1(M,\tau)$.

The following is a non-commutative Operator Ergodic Theorem:

Theorem 2.2. *Let μ be a σ_m-invariant Markov measure on Ω_m, let (M,τ) be a non-commutative probability space, and let $\alpha_1,\dots,\alpha_m$ be positive L_1-L_∞-contractions. Then for each $\varphi \in L_1(M,\tau)$, there exists $\overline{\varphi} \in L_1(M,\tau)$, such that $c_n^\mu(\alpha)\varphi \to \overline{\varphi}$, as $n \to \infty$ both doubleside almost everywhere and in $L_1(M,\tau)$. If the measure μ is strictly irreducible, then $\alpha_i\overline{\varphi} = \overline{\varphi}$ for all $i = 1,\dots,m$. If the operators $\alpha_1,\dots,\alpha_m$ preserve the state τ, then $\tau(\varphi) = \tau(\overline{\varphi})$. If $p \geqslant 1$, then $c_n^\mu(\alpha)\varphi \to \overline{\varphi}$, in $L_p(M,\tau)$ norm as well (modulo the definition of the actions in $L_p(M,\tau)$).*

It is easy to see that Theorem 2.1 is a consequence of Theorem 2.2.

The following is a generalized version of the Mean Ergodic Theorem for operators on Hilbert spaces.

Theorem 2.3. *Let μ be a σ_m-invariant Markov measure on Ω_m, let $H = L_2(M,\tau)$, be the Hilbert space constructed using non-commutative probability space (M,τ), and let the linear operators $\alpha_1,\dots,\alpha_m : L_2(M,\tau) \to L_2(M,\tau)$, be contractions. Then for each operator $h \in L_2(M,\tau)$, there exists an operator $\overline{h} \in L_2(M,\tau)$, such that $c_n^\mu(\alpha)h \to \overline{h}$, in $L_2(M,\tau)$ as $n \to \infty$. If the measure μ is strictly irreducible, then $\alpha_i\overline{h} = \overline{h}$ for all $i = 1,\dots,m$.*

The following is a non-commutative version of the Ergodic Theorem for operators on $L_p(M,\tau)$.

Theorem 2.4. *Let μ be a σ_m-invariant Markov measure on Ω_m. Let (M,τ) be a non-commutative probability space and let $p > 1$. Suppose that all operators $\alpha_1,\dots,\alpha_m : L_p(M,\tau) \to L_p(M,\tau)$, are positive contractions. Then for each $\varphi \in L_p(M,\tau)$, there exists $\overline{\varphi} \in L_p(M,\tau)$, such that $c_n^\mu(\alpha)\varphi \to \overline{\varphi}$, as $n \to \infty$ both doubleside almost everywhere and in $L_p(A,\tau)$. If the measure μ is strictly irreducible, then $\alpha_i\overline{\varphi} = \overline{\varphi}$ for all $i = 1,\dots,m$.*

3. Convergence of Multiparametric Česaro Averages

In this section we discuss the convergence of time averages in Theorems 2.1–2.4. The main idea here is to use the operator α_μ introduced later in this section.

Let L be a Real or Complex linear space, let $\alpha_1,\dots,\alpha_m : L \to L$, be linear operators, and let μ be a σ_m-invariant Markov measure on Ω_m with initial distribution $(p_1,\dots,p_m)$ and transition probability matrix $P = (p_{ij})$. We always assume in what follows that $p_i > 0$ for any $i = 1,\dots,m$.

Consider the weighted sum of operators α_w over all words of length l with given last symbol,

$$s_l^{\mu,i}(\alpha) = \sum_{\{w:|w|=l,w_l=i\}} \mu(w)\alpha_w. \tag{3.1}$$

For the sake of convenience, we set α_w with $|w| = 0$ equal to an identical operator on M.

Now we average $s_l^{\mu,i}(\alpha)$ over $l = 1, \dots, n-1$,

$$c_n^{\mu,i}(\alpha) = \frac{1}{n}\sum_{l=0}^{n-1} s_l^{\mu,i}(\alpha).$$

The following lemma describes relation between $s_l^{\mu,i}(\alpha)$ and $s_{l+1}^{\mu,j}(\alpha)$.

Lemma 3.1. *For any positive integer l and any $j \in \{1, \dots, m\}$, we have*

$$s_{l+1}^{\mu,j}(\alpha) = \sum_{l=1}^{m} p_{ij}\alpha_j s_l^{\mu,i}(\alpha).$$

◁ The proof of the lemma follows directly from definition (3.1) of $s_l^{\mu,i}(\alpha)$ and α_w. ▷

We can rewrite expression from the above lemma as follows:

$$\frac{s_{l+1}^{\mu,j}(\alpha)}{p_j} = \sum_{i=1}^{m} \frac{p_i p_{ij}}{p_j}\alpha_j \left(\frac{s_l^{\mu,i}(\alpha)}{p_i}\right).$$

Now we consider the space L^m, i. e., the m-th Cartesian power of L. We introduce operators $\alpha_\mu : L^m \to L^m$ defined by the formula

$$\alpha_\mu(v_1, \dots, v_m) = (\widetilde{v}_1, \dots, \widetilde{v}_m), \tag{3.2}$$

where

$$\widetilde{v}_j = \sum_{i=1}^{m} \frac{p_i p_{ij}}{p_j}\, \alpha_j v_i.$$

Lemma 3.2. *For any $v \in L$ and $n \in \mathbb{N}$ or $(n \geqslant 1)$, we have*

$$\alpha_\mu^n(v, \dots, v) = \left(\frac{s_n^{\mu,1}(\alpha)v}{p_1}, \dots, \frac{s_n^{\mu,m}(\alpha)v}{p_m}\right).$$

◁ Follows by induction from the formulae 3.2 above. ▷

Corollary 3.3. *For any $v \in L$ and $n \in \mathbb{N}$,*

$$\frac{1}{n}\sum_{l=0}^{n-1} \alpha_\mu^l(v, \dots, v) = \left(\frac{c_n^{\mu,1}(\alpha)v}{p_1}, \dots, \frac{c_n^{\mu,m}(\alpha)v}{p_m}\right).$$

◁ Follows from the previous lemma. ▷

Applying the classical non-commutative Individual Ergodic Theorem of Goldstein [5], Theorem 1.1 to the operator α_μ and using Corollary 3.3, we obtain statements on the convergence of the averages $c_n^m(\alpha)$.

Lemma 3.4. *Let μ be a σ_m-invariant Markov measure on Ω_m, let $H = L_2(M,\tau)$ be the Hilbert space constructed using non-commutative probability space (M,τ), and let linear operators $\alpha_1,\dots,\alpha_m : L_2(M,\tau) \to L_2(M,\tau)$ be contractions. Then for any operator $h \in L_2(M,\tau)$ and $i \in \{1,\dots,m\}$, the sequence $\left(\frac{1}{p_i}\right) c_n^{\mu,i}(\alpha)h \to \overline{h}_i$ in H as $n \to \infty$, where $\overline{h}_i \in L_2(M,\tau)$, and $\alpha_\mu(\overline{h}_1,\dots,\overline{h}_m) = (\overline{h}_1,\dots,\overline{h}_m)$.*

$\lhd$ If $\alpha_1,\dots,\alpha_m$ are contractions on $L_2(M,\tau)$, then α_μ is a contraction on $(L_2(A,\tau))^m$. Corollary 1 and the Mean Ergodic Theorem for α_μ complete the proof. $\rhd$

The relation $c_n^\mu(\alpha) = c_n^{\mu,1}(\alpha) + \ldots + c_n^{\mu,m}(\alpha)$ yields the following assertion.

Corollary 3.5. *Under the assumptions of previous Lemma, for any $h \in L_2(M,\tau)$, the sequence $c_n^\mu(\alpha)h$ converges in $L_2(M,\tau)$ norm as $n \to \infty$.*

Corollary 3.5 proves the convergence of time averages in Theorem 2.3.

Similarly, the following results are valid.

Theorem 3.6. *Let μ be a σ_m-invariant Markov measure on Ω_m. Let (M,τ) be a non-commutative probability space and let $p > 1$. Suppose that all operators $\alpha_1,\dots,\alpha_m : L_p(M,\tau) \to L_p(M,\tau)$ are contractions. Then, for any $v \in L_p(M,\tau)$ and $i \in \{1,\dots,m\}$, the sequence $\left(\frac{1}{p_i}\right) c_n^{\mu,i}(\alpha)v \to \overline{v}_i$ in $L_p(M,\tau)$ as $n \to \infty$, where the operator $\overline{v}_i \in L_p(M,\tau)$, and $\alpha_\mu(\overline{v}_1,\dots,\overline{v}_m) = (\overline{v}_1,\dots,\overline{v}_m)$.*

$\lhd$ If $\alpha_1,\dots,\alpha_m$ are contractions on $L_p(M,\tau)$, then α_μ is a contraction on $(L_p(M,\tau))^m$. The result follows from Corollary 3.3 and Lorch's Ergodic Theorem applied to the contraction α_μ (see [3] or [21, p. 73, Theorem 1.2]). $\rhd$

Corollary 3.7. *Under assumptions of the previous Theorem, for any $v \in L_p(M,\tau)$, the sequence $c_n^\mu(\alpha)v$ converges in $L_p(M,\tau)$ as $n \to \infty$.*

Now let (M,τ) be a non-commutative probability space as above, and $\alpha_1,\dots,\alpha_m : L_1(M,\tau) \to L_1(M,\tau)$, be linear operators.

Now we specialize the construction of α_μ from condition of Corollary 3.2 to the case of $L_1(M,\tau)$.

Let μ be a σ_m-invariant Markov measure on Ω_m with initial distribution $p = (p_1,\dots,p_m)$, and transition probability matrix $P = (p_{ij})$, and let $\alpha_\mu : (L_1(M,\tau))^m \to (L_1(M,\tau))^m$, be the operator defined as before.

The space $(L_1(M,\tau))^m$, can be identified with the space $L_1(M \times \{1,\dots,m\}, \tau \times p)$, where $\tau \times p$ is the product of the state τ on the algebra A and the probability distribution $p = (p_1,\dots,p_m)$ on $\{1,\dots,m\}$. Now the operator α_μ becomes an operator on the space $L_1(M \times \{1,\dots,m\}, \tau \times p)$. It is clear that, if $\alpha_1,\dots,\alpha_m$ are positive, then so is α_μ; if $\alpha_1,\dots,\alpha_m$ are $L_1(A,\tau)$-contractions, then so is α_μ; if $\alpha_1,\dots,\alpha_m$ are $L_\infty(M,\tau)$-contractions then so is α_μ; if $\alpha_1,\dots,\alpha_m$ preserve the state τ, then α_μ preserves the measure $\tau \times p$.

Lemma 3.8. *Let (M,τ) be a non-commutative probability space and let $\alpha_1,\dots,\alpha_m$ be positive L_1-L_∞-contractions. Then, for any operator $\varphi \in L_1(M,\tau)$, and $i = 1,\dots,m$, sequence $c_n^{\mu,i}(\alpha)\varphi$ converges as $n \to \infty$ both doubleside almost everywhere and in $L_1(M,\tau)$. If $\overline{\varphi}_i = \lim\limits_{n\to\infty} \left(\frac{1}{p_i}\right) c_n^{\mu,i}(\alpha)\varphi$, then $\alpha_\mu(\overline{\varphi}_1,\dots,\overline{\varphi}_m) = (\overline{\varphi}_1,\dots,\overline{\varphi}_m)$.*

To prove the lemma, we use the following standard fact [5]:

Theorem 3.9. *If α is a positive L_1-L_∞-contraction on the non-commutative probability space (M,τ), then, for any $\varphi \in L_1(M,\tau)$, there exists an operator $\overline{\varphi} \in L_1(M,\tau)$ such that $\frac{1}{n}(\varphi + \alpha\varphi + \ldots + \alpha^{n-1}\varphi) \to \overline{\varphi}$ as $n \to \infty$ both doubleside almost everywhere and in $L_1(M,\tau)$. The operator $\overline{\varphi}$ satisfies the relation $\alpha\overline{\varphi} = \overline{\varphi}$.*

$\triangleleft$ (of lemma 3.8) Applying Theorem 3.9 to the operator α_μ, and using Corollary 3.7, we obtain statement of the Lemma. $\triangleright$

The Lemma 3.8 proves the convergence of time averages in Theorem 2.2.

The doubleside almost everywhere convergence in the Theorem above also holds for spaces of infinite measure; therefore, we have the following Lemma.

Lemma 3.10. *Let (M,τ) be a von Neumann algebra with faithful normal semifinite trace τ and let $\alpha_1,\ldots,\alpha_m$ be positive L_1-L_∞-contractions. Then for any operator $\varphi \in L_1(M,\tau)$, and $i = 1,\ldots,m$, the sequence $c_n^{\mu,i}(\alpha)\varphi$ converges doubleside almost everywhere as $n \to \infty$.*

Now consider contractions on $L_p(M,\tau)$, for $p > 1$.

Lemma 3.11. *Let (M,τ) be a non-commutative probability space, let $p > 1$, and let $\alpha_1,\ldots,\alpha_m$ be positive $L_p(M,\tau)$-contractions. For any operator $\varphi \in L_p(M,\tau)$, and $i = 1,\ldots,m$, the sequence $(\frac{1}{p_i})c_n^{\mu,i}(\alpha)\varphi$, converges as $n \to \infty$ both doubleside almost everywhere and in $L_p(M,\tau)$ to an operator $\overline{\varphi}_i \in L_p(M,\tau)$. We have $\alpha_\mu(\overline{\varphi}_1,\ldots,\overline{\varphi}_m) = (\overline{\varphi}_1,\ldots,\overline{\varphi}_m)$.*

$\triangleleft$ If $\alpha_1,\ldots,\alpha_m$ are contractions, then so is α_μ. Applying Theorem 2.2 from [33] (see also [3] or [21, p. 73]) to the operator α_μ and using Corollary 3.7, we obtain the result. $\triangleright$

Corollary 3.12. *Under the assumptions of the previous Lemma, for any operator $\varphi \in L_p(M,\tau)$, the sequence $c_n^\mu(\alpha)\varphi$ converges both doubleside almost everywhere and in $L_p(M,\tau)$ norm.*

Corollary 3.12 completes the proof of the convergence of time averages in Theorem 2.4.

4. Invariance of the Limit

In this section we establish the invariance of the limit in Theorems 2.1–2.4 and complete the proof of these theorems.

The following theorem allows to conclude invariance of the limit in Theorem 2.2 from the Lemma 3.8 and as consequence invariance of the limit in the Theorem 4.1.

Theorem 4.1. *Let (M,τ) be a non-commutative probability space and let $\alpha_1,\ldots,\alpha_m$ be positive L_1-L_∞-contractions on $L_1(M,\tau)$. Let μ be a strictly σ_m-invariant Markov measure on Ω_m. Suppose that the operators $\varphi_1,\ldots,\varphi_m \in L_1(M,\tau)$, satisfy the condition*

$$\alpha_\mu(\varphi_1,\ldots,\varphi_m) = (\varphi_1,\ldots,\varphi_m). \tag{4.1}$$

Then $\varphi_1 = \ldots = \varphi_m = \varphi$ and $\alpha_i\varphi = \varphi$ for all $i = 1,\ldots,m$.

In order to prove Theorem 4.1 we first establish a similar result for contractions on the Hilbert space $H = L_2(M,\tau)$.

Theorem 4.2. *Let (M,τ) be a non-commutative probability space, and let the linear operators $\alpha_1,\ldots,\alpha_m : L_2(M,\tau) \to L_2(M,\tau)$, be contractions. Let μ be a σ_m-invariant Markov measure on Ω_m, and let $h_1,\ldots,h_m \in L_2(M,\tau)$, be such that $\alpha_\mu(h_1,\ldots,h_m) = (h_1,\ldots,h_m)$. If measure μ is strictly irreducible, then $h_1 = \ldots = h_m = h$, and $\alpha_i h = h$, for each $i = 1,\ldots,m$.*

The main idea of the proof is just this: if v_1, v_2, and v_3 are operators from $L_2(M,\tau)$ such that $\|v_1\| = \|v_2\| = \|v_3\|$, and $v_1 = \frac{(v_2+v_3)}{2}$, then $v_1 = v_2 = v_3$. Y. Guivarc'h used this observation in [14] to prove the invariance of the limit function in his ergodic theorem.

$\triangleleft$ Let $(p_1,\ldots,p_m)$ be initial distribution of the measure μ, and let $P = (p_{ij})$ be the transition probability matrix of μ. For any $i,j \in \{1,\ldots,m\}$ and $n \in \mathbb{N}$, denote by $p_{ij}^{(n)}$ the n-step transition probability from i to j (in other words, $p_{ij}^{(n)} = (P^n)_{ij}$).

We partition proof of the theorem 4.2 into series of steps.

Proposition 4.3. *Let (M,τ) be a non-commutative probability space, and let linear operators $\alpha_1,\dots,\alpha_m : L_2(M,\tau) \to L_2(M,\tau)$ be contractions. Let μ be a σ_m-invariant Markov measure on Ω_m, such that the corresponding Markov chain is irreducible. Suppose that operators $h_1,\dots,h_m \in L_2(M,\tau)$ satisfy the relation*

$$\alpha_\mu(h_1,\dots,h_m) = (h_1,\dots,h_m). \tag{4.2}$$

Then there is an $r \in \mathbb{R}$, such that $\|h_1\| = \ldots = \|h_m\| = r$ and, if $p_{ij} > 0$, then $\|\alpha_j h_i\| = r$.

$\lhd$ Assume that $\|h_1\| \geqslant \|h_i\|$, for any $i = 1,\dots,m$. Since equality 4.11 implies

$$h_1 = \sum_{i=1}^{m} \left(\frac{p_i p_{i1}}{p_1}\right) \alpha_1 h_i,$$

and invariancy of initial vector $(p_1,\dots,p_m)$ with respect to transition matrix P^T implies $1 = \sum\limits_{i=1}^{m} \frac{p_i p_{i1}}{p_1}$. It follows from the triangle inequality that $\|h_1\| = \|\alpha_1 h_i\| = \|h_i\|$, if $p_{i1} > 0$. Similarly, $\|h_1\| = \|h_i\|$, for any i such that $p_{i1}^{(2)} > 0$, and so on. The Markov chain corresponding to the measure μ is irreducible; hence, $\|h_1\| = \ldots = \|h_m\|$, and $\|h_j\| = \|\alpha_j h_i\|$ if $p_{ij} > 0$. $\rhd$

Proposition 4.4. *Suppose that $h_1,\dots,h_n,h \in L_2(M,\tau)$ satisfy the condition $\|h_1\| = \|h_2\| = \ldots = \|h_n\| = \|h\|$. Let $h = c_1 h_1 + \ldots + c_n h_n$ for some $c_1 > 0,\dots,c_n > 0$ such that $c_1 + \ldots + c_n = 1$. Then $h_1 = h_2 = \ldots = h_n = h$.*

$\lhd$ This immediately follows from equality condition for the Cauchy–Bunyakowsky–Schwartz inequality in the Hilbert space. $\rhd$

Proposition 4.5. *Let (M,τ) be a non-commutative probability space, $\alpha : L_2(M,\tau) \to L_2(M,\tau)$, be a contraction, and let operators $h_1,h_2 \in L_2(M,\tau)$, satisfy the relations $\|h_1\| = \|h_2\| = \|\alpha h_1\| = \|\alpha h_2\|$. Then $\alpha h_1 = \alpha h_2$, implies $h_1 = h_2$.*

$\lhd$ Indeed, if $h_1 \neq h_2$, then $\left\|\frac{(h_1+h_2)}{2}\right\| < \|h_1\|$ by Proposition 4.4. Since $\left\|\alpha(\frac{(h_1+h_2)}{2})\right\| = \|h_1\|$ and α is a contraction, we arrive at a contradiction. $\rhd$

In what follows, $(PP^T)_{ij}$ stands for the (i,j)-entry of the matrix PP^T.

Proposition 4.6. *Let (M,τ) be a non-commutative probability space, and let linear operators $\alpha_1,\dots,\alpha_m : L_2(M,\tau) \to L_2(M,\tau)$, be contractions. Let μ be a σ_m-invariant Markov measure on Ω_m, and let $h_1,\dots,h_m \in L_2(M,\tau)$, be such that $\alpha_\mu(h_1,\dots,h_m) = (h_1,\dots,h_m)$. Let transition matrix P of μ be irreducible. Then $(PP^T)_{ij} > 0$ implies $h_i = h_j$.*

$\lhd$ By Proposition 4.3, if P is irreducible, then $\|h_1\| = \ldots = \|h_m\|$. Note that $(PP^T)_{ij} > 0$ if and only, if there is a k for which $p_{ik} > 0$, and $p_{jk} > 0$. Since $h_k = \sum\limits_{l=1}^{m} \left(\frac{p_l p_{lk}}{p_k}\right) \alpha_k h_l$, it follows from Proposition 4.4 and 2.4 that $h_k = \alpha_k h_i = \alpha_k h_j$, and $\|h_k\| = \|h_i\| = \|h_j\|$, by Proposition 4.3. By Proposition 4.5 this yields $h_i = h_j$, which completes the proof. $\rhd$

Combination of statements of the propositions 4.3–4.6 finishes the proof of the theorem 4.2. $\rhd$

Let us return to the proof of Theorem 4.1.

Suppose that there exist φ_i and $\varphi_j \in L_1(M,\tau)$ with $i,j \in \{1,\dots,m\}$ and $\|\varphi_i - \varphi_j\|_{L_1} \geqslant \epsilon > 0$, and satisfying equality 4.2. Since $L_2(M,\tau)$ is dense in the $L_1(M,\tau)$ in $L_1(M,\tau)$ norm, we can find $h_j \in L_2(M,\tau)$ satisfying $\|h_j - \varphi_j\|_{L_1} < \epsilon/3$, for each $j \in \{1,\dots,m\}$. Let

$$(\overline{h_1},\dots,\overline{h_m}) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \alpha_\mu^n(h_1,\dots,h_m). \tag{4.3}$$

The limit in equation 4.3 exists in L_1 and L_2 norm. Hence, $\overline{h_l} \in L_2(M,\tau)$. Since α_μ is contraction in $L_1(M,\tau)$, then $\|\overline{h_l} - \varphi_l\|_{L_1} \leqslant \epsilon/3$. In addition, the following equality holds $\alpha_\mu(\overline{h_1},\ldots,\overline{h_m}) = (\overline{h_1},\ldots,\overline{h_m})$. Hence, from Theorem 4.2 the following equality holds: $\overline{h_1} = \overline{h_2} = \ldots = \overline{h_m}$. The latter equality implies that $\epsilon \leqslant \|\varphi_i - \varphi_j\|_{L_1} \leqslant \|\varphi_i - \overline{h_i}\|_{L_1} + \|\overline{h_j} - \varphi_j\|_{L_1} \leqslant 2\epsilon/3$. We came to contradiction with the suggestion that $\epsilon \leqslant \|\varphi_i - \varphi_j\|_{L_1}$. Theorem 4.1 is established.

Theorem 4.7. *Let (M,τ) be a non-commutative probability space, let $p > 1$ and let $\alpha_1,\ldots,\alpha_m : L_p(M,\tau) \to L_p(M,\tau)$, be contractions. Let μ be a σ_m-invariant Markov measure on Ω_m, and let operators $\varphi_1,\ldots,\varphi_m \in L_p(M,\tau)$, be such that $\alpha_\mu(\varphi_1,\ldots,\varphi_m) = (\varphi_1,\ldots,\varphi_m)$. If the measure μ is strictly irreducible, then $\varphi_1 = \ldots = \varphi_m = \varphi$, and $\alpha_i\varphi = \varphi$, for any $i = 1,\ldots,m$.*

$\lhd$ The proof of the latter Theorem reproduces that of Theorem 4.2 above. The key observation is that Proposition 4.4 holds for the space $L_p(M,\tau)$ since $L_p(M,\tau)$ is a strictly convex space (see for example [27]). $\rhd$

Theorems 4.7 and 4.1 imply Theorem 2.4.

5. Ergodic type theorem for the action of finitely generated locally free semigroups

Definition 5.1. A locally free semigroup (see [31] and references there) $\mathscr{LFS}_{m+1}$ with m generators is defined as a semigroup determined by generators satisfying the following relations: $\mathscr{LFS}_{m+1} = \{g_1,\ldots,g_m : g_ig_j = g_jg_i;\ i,j \in \{1,\ldots,m\}, |i-j| > 1\}$.

Semigroup $\mathscr{LFS}_{m+1}$ is associated with a topological Markov chain with states $\{1,\ldots,m\}$ and transition matrix

$$m = (m_{i,j}), \quad m_{i,j} = \begin{cases} 1, & \text{if } |i-j| \leqslant 1 \text{ or } i \leqslant j; \\ 0, & \text{otherwise.} \end{cases}$$

The set of admissible words in the chain corresponds to the W_m, the set of admissible one-sided sequences corresponds to Ω_m and left shift σ_m corresponds to shift on Ω_m. Each word $\omega_1\ldots\omega_n$ corresponds to $g_\omega = g_{\omega_1}\ldots g_{\omega_n}$.

The correspondence $\omega \mapsto g_\omega$ defines a bijection between W_m and $\mathscr{LFS}_{m+1}$, and from (4.1) it follows that system (Ω_m,σ_m) mixes topologically, hence ergodic measure has a positive measure on cylinders corresponding to the words W_m.

Now we assume that semigroup $\mathscr{LFS}_{m+1}$ acts as a semigroup with generators g_i mapped to the kernels α_i acting on a tracial von Neumann Algebra (M,τ). Applying Theorem 2.1, we obtain an ergodic theorem for the action of $\mathscr{LFS}_{m+1}$.

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