Владикавказский математический журнал
апрель–июнь, 2007, Том 9, Выпуск 2

UDC 512.742

ON A DECOMPOSITION EQUALITY IN MODULAR GROUP RINGS

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Let G be an abelian group such that $A \leqslant G$ with p-component A_p and $B \leqslant G$, and let R be a commutative ring with 1 of prime characteristic p with nil-radical $N(R)$. It is proved that if $A_p \nsubseteq B_p$ or $N(R) \neq 0$, then $S(RG) = S(RA)(1 + I_p(RG; B)) \iff G = AB$ and $G_p = A_pB_p$. In particular, if $A_p \neq 1$ or $N(R) \neq 0$, then $S(RG) = S(RA) \times (1 + I_p(RG; B)) \iff G = A \times B$. So, the question concerning the validity of this formula is completely exhausted. The main statement encompasses both the results of this type established by the author in (Hokkaido Math. J., 2000) and (Miskolc Math. Notes, 2005). We also point out and eliminate in a concrete situation an error in the proof of a statement due to T. Zh. Mollov on a decomposition formula in commutative modular group rings (Proceedings of the Plovdiv University-Math., 1973).

Mathematics Subject Classification (2000): 16S34, 16U60, 20K10, 20K21.

Key words: direct factors, decompositions, normed unit groups, homomorphisms.

1. Introduction

Traditionally, suppose RG is the group ring (often regarded as an R-algebra) of an abelian group G over a commutative ring with identity of prime characteristic, for instance, p. As usual, $V(RG)$ denotes the group of normalized units in RG, and $S(RG)$ is its Sylow p-component. For any subgroup C of G, the symbol $I(RG; C)$ will designate the relative augmentation ideal of RG with respect to C, and $I_p(RG; C)$ designates its nil-radical.

For an abelian group G, the letter G_p will denote its p-torsion part and for a commutative unitary ring R, the letter $N(R)$ denotes its nil-radical.

In a series of our investigations (e. g. [1–4, 6, 7]), we study how the direct decomposition of G can be translated on $S(RG)$; another treatment but of G_p was demonstrated in [5].

In [9] we generalized the foregoing results of this direction by considering an ordinary, not necessarily direct, decomposition and by exploring how such a decomposing of $S(RG)$ implies a corresponding one of G and G_p.

The aim of this article is to strengthen the aforementioned results from [9] and to settle completely the existence of such a decomposition formula for $S(RG)$. In doing that, we use some helpful facts that are of independent interest as well.

In closing, we correct a proof by Mollov [11] in a rather special case. The generalized version is wide-open yet. Likewise, we demonstrate that some assertions from [13] are not original and can be simplified in a more convenient form not as they stand.

2. Direct Factors and Decompositions in Modular Abelian Group Rings

Our point of departure here is to proceed by proving the following two main assertions, motivated the writing up of the paper presented.

Theorem (Decomposition). *Suppose G is an abelian group with proper subgroups A and B, and suppose R is a commutative ring with unity of prime characteristic p. If $A_p \not\subseteq B_p$ or $N(R) \neq 0$, then the following equivalence holds:*

$$S(RG) = S(RA)(1 + I_p(RG; B)) \iff G = AB \text{ and } G_p = A_p B_p. \tag{1}$$

As an immediate consequence, we obtain the following.

Corollary (Direct Decomposition). *Suppose that G is an abelian group with proper subgroups A and B, and suppose that R is a commutative ring with unity of prime characteristic p. If $A_p \neq 1$ or $N(R) \neq 0$, then the following relationship is fulfilled:*

$$S(RG) = S(RA) \times (1 + I_p(RG; B)) \iff G = A \times B. \tag{2}$$

Before starting with the detailed proofs, we give a brief comment. As indicated in [9], the report [12] was false. The incorrectness arises from the fact that Mollov [12], without any concrete arguments, said that our proof of a statement from [1] is incorrect. In [9] we have shown that the Mollov's sentence is really inadequate. In this work we once again illustrate this. To avoid such purported reviewer's reports in the future, we shortly represent some trivial, but crucial, facts concerning the inner construction of the basis group members from the canonical record of any element of the group algebra.

If $r_1g_1+\ldots+r_tg_t \in S(RG)$ is in canonical form, then there is an index, say j, $1 \leqslant j \leqslant t \in \mathbb{N}$, such that $g_j \in G_p$. Besides, if $f_1b_1 + \ldots + f_sb_s \in 1 + I(RG; B)$ is in canonical form, then there exists an index, say i, $1 \leqslant i \leqslant s \in \mathbb{N}$, so that $b_i \in B$.

However, unfortunately, even when $f_1b_1 + \ldots + f_sb_s \in 1 + I_p(RG; B)$, for all $i \in [1, s]$ the relation $b_i \notin B_p$ may occur. For example, such is the element $1 + (1 - g_p)(b - 1) = g_p - g_pb + b$ where $g_p \in G_p \setminus B_p$ and $b \in B \setminus B_p$.

Now, we have at our disposal all the information necessary for

$\lhd$ Proof of the Theorem. What we need to argue is that $G = AB$ holds under the additional restrictions that $A_p \not\subseteq B_p$ or $N(R) \neq 0$, because the other ratio $G_p = A_pB_p$ plus the converse part half were done in [9].

In order to do this, we first presume that $A_p \not\subseteq B_p$. In the case when $G = G_p$ everything is made. Henceforth, $G \neq G_p$ and there exist $g \in G \setminus G_p$ and $g_p \in A_p \setminus B$. We easily see that $1 + g - gg_p = 1 + g(1 - g_p) \in S(RG)$ is in canonical type. Furthermore, we write $1+g-gg_p-(r_1a_1+\ldots+r_ta_t) = f_1b_1+\ldots+f_sb_s \in I_p(RG; B)$ where $r_1a_1+\ldots+r_ta_t \in S(RA)$. Since by what we have already shown in [9], $g_p \in A_pB_p$ (even $g_p \in AB$ is completely enough), via the previous commentary it is only a technical matter to detect by comparing the canonical forms of the left and the right hand-sides that $g \in AB$, as required.

In the situation when ($G_p = 1$ but) $N(R) \neq 0$, we construct the element $1 + r(1 - g)$ where $r \in N(R) \setminus \{0\}$ and $g \in G \setminus \{1\}$. Observe that $1 + r - rg$ is in canonical type. Then the same argument applies to this element to infer once again that $g \in AB$.

Combining these two derivations, we conclude that $G = AB$, as expected.

We also wish to give two new independent confirmations of the implication that $S(RG) = S(RA)(1 + I_p(RG; B))$ forces $G_p = A_pB_p$, the first of which is similar to the presented above, but requires the extra limitation that $N(R) = 0$ which insures that $S(RG) = 1 + I(RG; G_p)$ and $S(RA) = 1 + I(RA; A_p)$.

In fact, for each $g_p \in G_p \subseteq S(RG)$ we can analogously write as above that $g_p(r_1a_1 + \ldots + r_ta_t) = f_1b_1 + \ldots + f_tb_t \in 1 + I_p(RG; B)$. By comparison of the two canonical forms we have $g_pa_j = b_j \ \forall\ j$: $1 \leqslant j \leqslant t$. Since t is always an odd natural number, multiplying all equalities in a way so that $b_{k-1}b_k^{-1} \in B$ and $b_{l-1}b_l^{-1} \in A_p$ for all appropriate chosen indices $k, l \in [1, t]$ and taking into account the preliminary discussion alluded to above, it is simple to check that $g_p \in A_pB_p$. Therefore, $G_p = A_pB_p$, as promised.

Next, we shall continue with the second smooth verification of the implication mentioned; here the condition $N(R) = 0$ is not necessary. This method is universal and can be successfully employed even to prove that $S(RG) = S(RA)(1 + I_p(RG; B))$ implies $G = AB$ whenever $A_p \nsubseteq B_p$ or $N(R) \neq 0$; see, for example, the Claim stated below. And so, consider the natural map $\phi : G \to G/B$ and its induced restrictions ϕ_{G_p}: $G_p \to G_pB/B$ and ϕ_A: $A \to AB/B$. The first one can be linearly extended to Φ: $RG \to R(G/B)$ with kernel $I(RG; B)$. Thus Φ: $S(RG) \to S(R(G/B))$ and Φ: $S(RA) \to S(R(AB/B))$. Next, bearing Φ in both sides of $g_p \in S(RG) = S(RA)(1 + I_p(RG; B))$, we have that $g_pB = \sum_{i=1}^{n} r_ia_iB$ since $\Phi(1 + I_p(RG; B)) \subseteq \Phi(1 + I(RG; B)) = 1$. The key moment here is that ϕ: $A_p \to A_pB/B$ and thereby Φ: $S(RA_p) \to S(R(A_pB/B))$. Thus, in view of the preliminary comments, the writing of $\sum_{i=1}^{n} r_ia_iB$ in canonical form yields that $g_pB = a_mB$ for some index m, $1 \leqslant m \leqslant n$ such that $r_m = 1$ and such that $a_m \in A_pB$. That is why $g_p \in A_pB$, i. e. $g_p \in A_pB_p$. Consequently, $G_p = A_pB_p$, as claimed and we are finished.

If now $1 + g - gg_p$ is the already constructed above element from $S(RG)$, the action of Φ gives that $B + gB - gg_pB = u_1a_1B + \ldots + u_sa_sB$ for some $u_1, \ldots, u_s \in R$ and $a_1, \ldots, a_s \in A$. This equality obviously allows us to conclude that $g \in AB$ as required. $\rhd$

Remark. If $A_p \subseteq B_p = G_p$ and $N(R) = 0$, the theorem is no longer true. Indeed, $S(RA) = 1 + I(RA; A_p) \subseteq 1 + I(RG; B_p) \subseteq 1 + I_p(RG; B)$ and $S(RG) = 1 + I(RG; G_p) = 1 + I(RG; B_p) = 1 + I_p(RG; B) = S(RA)(1 + I_p(RG; B))$ without $G = AB$.

The following technicality is listed only for the convenience of the reader.

Lemma [8, Lemma 1]**.** *Let $C \leqslant G$ be an abelian group and R a commutative ring with identity of prime $char(R) = p$. Then*

$$I_p(RG; C) = 0 \iff a) \quad G_p = 1,\ N(R) = 0,\ C \neq 1;\ b)\ C = 1,\ N(R) \neq 0.$$

Now, we are ready with

$\lhd$ Proof of the Corollary. With point (1) in hand, what suffices to show is that $A \cap B = 1$. In fact, referring to the second dependence of Lemma (Intersection) from [3], we establish that $S(RA) \cap (1 + I_p(RG; B)) = 1 + I_p(RA; A \cap B)$. Thus $I_p(RA; A \cap B) = 0$ and since either $A_p \neq 1$ or $N(R) \neq 0$, we deduce by the foregoing Lemma that $A \cap B = 1$, as wanted. On the other hand, the identity $G_p = A_p \times B_p$ is a plain consequence of $G = A \times B$, hence it may be dropped off. $\rhd$

Before examining the special direct decompositions in $V(RG)$, it will be convenient to recollect some facts about certain direct decompositions of subgroups in $S(RG)$, and to quote an easy technical claim for later use.

Firstly, we notice the well-known classical fact mainly due to May that $V(RG) = GS(RG)$, whenever G is a p-mixed abelian group (that is the only torsion is p-torsion) and R is a commutative unital ring without nontrivial idempotents (in particular, without nontrivial zero divisors) of $char(R) = p$.

Is is worthwhile noticing that the following lemma is already derived as a consequence of our main result (see, e. g., Corollary (Direct Decomposition)).

Lemma [3]**.** *Given that $G = A \times B$ is an abelian group and R is a commutative ring with identity of prime $char(R) = p$. Then*

$$S(RG) = S(RA) \times (1 + I_p(RG; B)).$$

We now list two inclusions which allow one to discover the intersection between certain components of a product in $V(RG)$.

Lemma [10]**.** *Assume that G is an abelian group with $A \leqslant G$ and $B \leqslant G$ and that R is a commutative ring with identity of prime $char(R) = p$. Then*

$$(B \cap A)(1 + I(RA; A \cap B)) \supseteq [B(1 + I_p(RG; B))] \cap V(RA) \supset (B \cap A)(1 + I_p(RA; A \cap B)).$$

We are now able to state and attack the following.

Proposition. *Let G be a p-mixed abelian group and R a commutative ring with* 1 *with no nontrivial idempotent elements of $char(R) = p$. Then*

$$V(RG) = V(RA) \times [B(1 + I_p(RG; B))] \iff G = A \times B.$$

$\lhd$ Foremost, we deal with the sufficiency and so we may apply the first Lemma to get that $S(RG) = S(RA) \times (1 + I_p(RG; B))$. But by the observation preceding this Lemma, we find that $V(RG) = GS(RG)$ and by symmetry $V(RA) = AS(RA)$. Knowing this, it is now an elementary exercise to demonstrate that $V(RG) = V(RA)[B(1 + I_p(RG; B))]$. With the aid of the second Lemma we are in position to check that $V(RA) \cap [B(1 + I_p(RG; B))] = 1$. Indeed, since $A \cap B = 1$, we derive at once that this holds true. So, the desired decomposition now follows and thereby we are finished.

As for the necessity, it is simple observed that $A \cap B \subseteq V(RA) \cap [B(1 + I_p(RG; B))] = 1$, whence $A \cap B = 1$. Moreover, since each element of $1 + I_p(RG; B)$ contains in its support an element from B, it is a plain technical exercise to see that the given direct decomposition for $V(RG)$ implies a decomposition for its subgroup G. $\rhd$

Remark. The crux of that decomposing problem in the general case, namely provided that G is not p-mixed, is whether among the subgroups of $V(RG)$ there exist appropriate members so that they are in interrelation with $GS(RG)$ or its sections.

We close with the correction of a big flaw in the proof of Mollov [11, Theorem 2] who has proposed the following assertion-[sic]:

Proposition (Mollov, 1973)**.** *Let H be a subgroup of the abelian p-group G and L a commutative ring with* 1 *of prime characteristic p. The subgroup $S(LH)$ is a direct factor of the group $S(LG)$ $\iff$ the subgroup H is a direct factor of the group G.*

Unfortunately, as was already noted in [5], the Mollov's proof of the necessity is wrong and contains a simple mistake. In fact, he wrote: «Suppose $S(LG) = S(LH) \times Q$ and $G \cap Q = B$. Then $H \cap B = 1$. If $g \in G$, the given above direct decomposition for $S(LG)$ implies $g = (\sum_i \alpha_i h_i)(\sum_j \beta_j b_j)$, where $h_i \in H$, hence $g = \alpha h b = 1hb$, where $h \in H$ and $b \in G$ (our commentary: $g = hb$ is always true for some $h \in H$ and $b \in G$ which is a clear fact since $g = h.(h^{-1}g)$). Because $g \in S(LG)$ and $h \in S(LH)$, it follows that $b \in Q$ (our commentary: this is the main defect in the Mollov's conclusion), whence $b \in B$. Therefore $G = HB$ and $G = H \times B$».

We observe as in [5] that the *Direct Factor Conjecture*, which says that the abelian p-group G is a direct factor of the group $S(LG)$ of normalized units in the commutative group ring LG of prime characteristic p, settles the part of «necessity» of the Mollov's affirmation. However, this conjecture is still in question, but nevertheless in some concrete cases we can obtain a particular confirmation provided that the complementary factor Q is explicitly described. For instance, $Q = 1 + I(LG; C)$ whenever $C \leqslant G$ is p-primary.

In the next lines we give a simpler proof of the incorrect part in this special variant than the proofs in [1, 2, 4, 6], where the attainment below was used intensively many times.

Claim. *Assume L is a commutative unitary ring and G is an abelian group with $M \leqslant G$ and $C \leqslant G$. Then*

$$V(LG) = V(LM) \times [(1 + I(LG; C)) \cap V(LG)] \Rightarrow G = M \times C.$$

$\lhd$ Consider the natural map $\phi : G \to G/C$. It is well-known that it may be linearly extended to a group homomorphism Φ: $V(LG) \to V(L(G/C))$ with kernel $(1 + I(LG; C)) \cap V(LG)$. Thus $\Phi([1+I(LG; C)]\cap V(LG)) = 1$ and $\Phi(V(LM)) \subseteq V(L(MC/C))$. Furthermore, if g lies in G then by assumption g belongs to $V(LM)([1+I(LG; C)]\cap V(LG))$ and by taking Φ in both sides, we deduce $gC = \sum_i r_i(m_iC)$ where $r_i \in L$ and $m_i \in C$. Consequently, the canonical record implies that $gC \in (MC)/C$ whence $g \in MC$. Finally, this yields that $G = MC$.

Moreover, $M\cap C \subseteq V(LM)\cap([1+I(LG; C)]\cap V(LG)) = 1$, hence $G = M\times C$ as asserted, and so we are done. $\rhd$

It is also noteworthy that Lemmas 5.2, 5.3 and 5.4 as well as Propositions 5.5 and 5.7 from [13] are well-known classical facts. Moreover, their formulation can be simplified. Indeed, Proposition 5.5 claims that if $G = A \times G_p$ with $char(R) = p$, then $S(RG) = [1 + N(R).I(RA; A)] \times (1 + I(RG; G_p))$. The proof in [13] is too long and unnecessary intricated. It follows easily like this: It is well-known that $S(RG) = S(RA) \times (1 + I(RG; G_p))$. But it is a folklore fact that $S(RA)$ can be generally represented as $S(RA) = (1+I(RA; A_p))\times M[N(R); \prod(A/A_p)]$, where $M[N(R); \prod(A/A_p)] = \{1 + \sum_{a\in\prod(A/A_p)} r_a(a-1)|r_a \in N(R)\} \subseteq 1 + N(R).I(RA; A) = 1 + I(N(R)A; A) \subseteq S(RA)$ with $\prod(A/A_p)$ a complete set of representatives of A with respect to A_p containing the identity element of A. Hence $S(RA) = M[N(R); \prod(A/\{1\})]$ since $A_p = 1$. That is why, it is clear that the last inclusion is tantamount to the equality $S(RA) = 1 + N(R).I(RA; A)$ and the claim sustained.

On the other hand, Proposition 5.7 asserts that if R is indecomposable of $char(R) = p$ and G is p-mixed such that $G = F \times G_p$, then $V(RG) \cong (G/G_p) \times S(RG)$. Again the proof in [13] is complicated and can be simplified as follows: It is well-known that $V(RG) = V(RF)\times(1+I(RG; G_p))$ and hence $S(RG) = S(RF)\times(1+I(RG; G_p))$. But it is readily seen that $V(RF) = S(RF) \times F$ because $V((R/N(R))F) = F$ since $R/N(R)$ is indecomposable reduced (a classical result for trivial units in commutative group rings due to Karpilovsky). Finally, combining these identities, we infer that $V(RG) = F \times S(RG) \cong (G/G_p) \times S(RG)$, as stated.

Besides, we would like to emphasize that there is no a novelty in the approach used in [13]; only the idea from our paper in Glasgow Math. J. (2001) is imitated, although that this paper is not cited there.

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Received July 3, 2006.

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