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QUASI-COMPLETE Q-GROUPS ARE BOUNDED

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We prove that any p-torsion quasi-complete abelian Q-group is bounded. This extends a recent statement of ours in [6, Corollary 8] to an arbitrary large cardinality, thus also answering in the negative a conjecture from [6]. Some other related assertions are established as well.

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From the frontier of this paper, unless specified something else, let it be agreed that all groups into consideration are p-primary abelian for some arbitrary but fixed prime p written additively as is the custom when dealing with abelian group theory. The present short note is a contribution to a recent flurry of our results in [6]. Standardly, all notions and notations are essentially the same as those from [7]. For instance, A^1 denotes the first Ulm subgroup of a group A. If $A^1 = 0$, then A is termed separable. We shall also assume throughout that the *Continuum Hypothesis* (abbreviated as CH) holds fulfilled whenever we deal with torsion-complete groups of cardinality $\aleph_1$.

Following [9], a separable group A is said to be a *Q-group* if for all $G \leqslant A$ with $|G| \geqslant \aleph_0$ the inequality $|(A/G)^1| \leqslant |G|$ holds. It is a routine technical exercise to verify that a subgroup of a Q-group is also a Q-group (see, e. g., [9]). Direct sums of cyclics are obviously Q-groups.

Moreover, imitating [7], a reduced group A is called *quasi-complete* if for all pure $G \leqslant A$ the quotient $(A/G)^1$ is divisible. It is easily observed that these groups are also separable as well as they are closed with respect to direct summands.

In [6] we obtained the following.

Theorem [6]**.** *Quasi-complete Q-groups of cardinality $\aleph_1$ are precisely the bounded ones.*

The goal here is to strengthen this claim by ignoring the cardinal restriction. First, we need the following preliminary technicality.

Proposition 1. *Each torsion-complete thin group is bounded.*

$\lhd$ Follows from a simple argument given in [12] and [14], respectively. $\rhd$

We are now ready to attack

Theorem 2 [2]**.** *Every quasi-complete Q-group is bounded.*

$\lhd$ If for such a group A we have $|A| \leqslant \aleph_1$, the result was argued by us in [6] (see also the Theorem alluded to above). If now $|A| > \aleph_1$, we may without loss of generality assume that fin $r(A) > \alpha_1$. So, it follows by virtue of [7, Theorem 74.8] that A is torsion-complete. On the other hand, A being a Q-group must be fully starred whence thin [9, 14]. Henceforth, the affirmation from previous Proposition 1 works. $\rhd$

Remark. Theorem 2 resolves in a negative way the Conjecture in [6] for Q-groups.

As an immediate consequence we derive the following.

Corollary 3 [2]**.** *A Q-group is a direct sum of quasi-complete groups if and only if it is a direct sum of cyclics.*

$\lhd$ Write $A = \oplus_{i\in I} A_i$, where, for each index $i \in I$, the summand A_i is quasi-complete. Since a subgroup of a Q-group is again a Q-group [9], the foregoing theorem leads us to that every component A_i is bounded. Therefore, A is a direct sum of cyclics. $\rhd$

Conforming with [9], a group A is said to be *Fuchs five* if every infinite subset of A can be embedded in a direct summand of A with the same cardinality, itself.

Corollary 4. *Each quasi-complete Fuchs five-group is bounded.*

$\lhd$ It follows from [9] that separable Fuchs five-groups are themselves Q-groups. Hereafter, the preceding theorem works. $\rhd$

Remark. In [10] was constructed a Q-group which is not Fuchs five. Inspired by the last result, it is of necessity not quasi-complete.

We recollect that a group A is *essentially finitely indecomposable* whenever $A = B \oplus C$ with C a direct sum of cyclics implies that C is bounded. Likewise, a group A is known as $\aleph_1$*-separable* provided that any countable subgroup of A is contained in a direct summand of A which is a countable direct sum of cyclics. Apparently, $\aleph_1$-separable groups are separable and separable Fuchs five-groups are $\aleph_1$-separable.

We are now concerned with some other similar assertions of this type presented.

Proposition 5. *Each essentially finitely indecomposable $\aleph_1$-separable group is bounded.*

$\lhd$ Suppose that C is a countable subgroup of such a group A. Then C can be embedded in a direct summand of A which is a direct sum of cyclics, thus bounded. Hence C is bounded as well. Therefore all subgroups of A are bounded by a fixed positive integer and thereby A is bounded too (compare with the proof of Proposition from [6]). In fact, if A is countable we are finished. If not, i. e. A is uncountable, it possesses an uncountable number of countable subgroups whereas the number of positive integers is countable. This discrepancy allows us to conclude that A is bounded, indeed. $\rhd$

According to ([8] or [13]) a group A is called *essentially indecomposable* if $A = B \oplus C$ implies that either B or C is bounded. It is noteworthy that we have proved in [6] that $p^{\omega+1}$-projective essentially indecomposable groups are direct sums of cyclics; we also showed in [5] that $p^{\omega+1}$-projective essentially finitely indecomposable groups are bounded (notice that even C-decomposable $p^{\omega+n}$-projective essentially indecomposable groups are direct sums of cyclics for each other $n \geqslant 2$ while in [1] was constructed a $p^{\omega+n}$-projective essentially finitely indecomposable group which is not bounded — however it is clear that C-decomposable essentially finitely indecomposable groups are bounded).

We are now concentrated on a more limited class of the so-called *Crawley groups* [3, 4] than the class of essentially indecomposable groups. These are groups for which every direct decomposition involves a finite direct summand (for example, see also cf. [8]).

Proposition 6. *Every Crawley starred group is a direct sum of cyclics.*

$\lhd$ Owing to [11], for such a group A we may write $A = C \oplus H$ where C is a direct sum of cyclics with $|C| = |A|$. If C is finite, then the same is true for A and we are done. Otherwise, if H is finite, it is obviously seen that A has to be a direct sum of cyclics. $\rhd$

In closing, we state the following three problems of interest.

Question 1. Does it follow that each pure-complete (in particular, quasi-complete)

(a) thin group

or

(b) starred group (with $\neg CH$)

or

(c) weakly $\aleph_1$-separable group

is a direct sum of cyclics (in particular, bounded)?

Question 2. Does it follow that each essentially finitely indecomposable (in particular, thick)

(a) thin group

or

(b) starred group (with $\neg CH$)

or

(c) weakly $\aleph_1$-separable group

is bounded?

Question 3. What is the structure of $p^{\omega+n}$-projective essentially finitely indecomposable Q-groups? Whether or not they are bounded?

About other questions on the same theme discussed, we refer to [6].

In accordance with [2], we shall say that the arbitrary (not necessarily p-primary) reduced abelian group K is *quasi-closed* if for all pure subgroups P of K the factor-group $(K/P)^1$ is divisible. Notice that reduced algebraically compact groups are themselves quasi-closed [7].

So, we can state our final

Question 4. Determine the structure of quasi-closed Q-groups. Decide whether or not they are bounded.

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