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A BECKENBACH–DRESHER TYPE INEQUALITY IN UNIFORMLY COMPLETE f-ALGEBRAS

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To the memory of Gleb Akilov on the occasion of the 90th anniversary of his birth

A general form Beckenbach–Dresher inequality in uniformly complete f-algebras is given.

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An easy modification of the *continuous functional calculus* on unitary f-algebras as defined in [3] makes it possible to translate the Fenchel–Moreau duality to f-algebra setting and to produce some *envelope representations* results, see [8]. This machinery, often called *quasi-linearization* (see [2, 9]), yields the validity of some classical inequalities in every uniformly complete vector lattice [4, 5]. The aim of this note is to give general forms of Peetre–Persson and Beckenbach–Dresher inequalities in uniformly complete f-algebras.

The unexplained terms of use below can be found in [1] and [6].

1°. We need a slightly improved version of continuous functional calculus on uniformly complete f-algebras constructed in [3, Theorem 5.2].

Denote by $\mathscr{B}(\mathbb{R}^N_+)$ the f-algebra of continuous functions on $\mathbb{R}^N_+$ with polynomial growth; i. e., $\varphi \in \mathscr{B}(\mathbb{R}^N_+)$ if and only if $\varphi \in C(\mathbb{R}^N_+)$ and there are $n \in \mathbb{N}$ and $M \in \mathbb{R}_+$ satisfying $|\varphi(\mathbf{t})| \leqslant M(\mathbf{1} + w(\mathbf{t}))^n$ ($\mathbf{t} \in \mathbb{R}^N_+$), where $\mathbf{t} := (t_1, \dots, t_N)$, $w(\mathbf{t}) := |t_1| + \ldots + |t_N|$ and $\mathbf{1}$ is the function identically equal to 1 on $\mathbb{R}^N_+$. Denote by $\mathscr{B}_0(\mathbb{R}^N_+)$ the set of all functions in $\mathscr{B}(\mathbb{R}^N_+)$ vanishing at zero. Let $\mathscr{A}(\mathbb{R}^N_+)$ stands for the set of all $\varphi \in \mathscr{B}(\mathbb{R}^N_+)$ such that $\lim_{\alpha\downarrow 0} \alpha^{-1}\varphi(\alpha\mathbf{t})$ exists uniformly on bounded subsets of $\mathbb{R}^N_+$. Evidently, $\mathscr{A}(\mathbb{R}^N_+) \subset \mathscr{B}_0(\mathbb{R}^N_+)$. Finally, let $\mathscr{H}(\mathbb{R}^N_+)$ denotes the set of all continuous positively homogeneous functions on $\mathbb{R}^N_+$.

Lemma 1. *The sets $\mathscr{B}(\mathbb{R}^N_+)$, $\mathscr{B}_0(\mathbb{R}^N_+)$, and $\mathscr{A}(\mathbb{R}^N_+)$ are uniformly complete f-algebras with respect to pointwise operations and ordering. Any $\varphi \in \mathscr{A}(\mathbb{R}^N_+)$ admits a unique decomposition $\varphi = \varphi_1 + w\varphi_2$ with $\varphi_1 \in \mathscr{H}(\mathbb{R}^N_+)$ and $\varphi_2 \in \mathscr{B}_0(\mathbb{R}^N_+)$, i. e.*

$$\mathscr{A}(\mathbb{R}^N_+) = \mathscr{H}(\mathbb{R}^N_+) \oplus w\mathscr{B}_0(\mathbb{R}^N_+).$$

Moreover, $\varphi_1(\mathbf{t}) = \varphi'(0)\mathbf{t} := \lim_{\alpha\downarrow 0} \alpha^{-1}\varphi(\alpha\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^N$.

$\lhd$ See [3, Lemma 4.8, Section 5]. $\rhd$

2°. Consider an f-algebra E. Denote by $H(E)$ the the set of all nonzero $\mathbb{R}$-valued lattice homomorphisms on E and by $H_m(E)$ the subset of $H(E)$ consisting of multiplicative functionals. We say that $\omega \in H(E)$ is *singular* if $\omega(xy) = 0$ for all $x, y \in E$. Let $H_s(E)$ denotes the set of singular members of $H(E)$. Given a finite tuple $\mathbf{x} = (x_1, \dots, x_N) \in E^N$, denote by $\langle\!\langle\, \mathbf{x} \,\rangle\!\rangle := \langle\!\langle x_1, \dots, x_N \rangle\!\rangle$ the f-subalgebra of E generated by $\{x_1, \dots, x_N\}$.

Definition. Let E be a uniformly complete f-algebra and $x_1, \dots, x_N \in E_+$. Take a continuous function $\varphi : \mathbb{R}^N_+ \to \mathbb{R}$. Say that the element $\widehat{\varphi}(x_1, \dots, x_N)$ *exists* or is *well-defined* in E provided that there is $y \in E$ satisfying

$$\begin{aligned} \omega(y) &= \varphi(\omega(x_1), \dots, \omega(x_N)) \quad (\omega \in H_m(\langle\!\langle x_1, \dots, x_N, y \rangle\!\rangle), \\ \omega(y) &= \varphi_1(\omega(x_1), \dots, \omega(x_N)) \quad (\omega \in H_s(\langle\!\langle x_1, \dots, x_N, y \rangle\!\rangle), \end{aligned} \tag{1}$$

cp. [3, Remark 5.3 (ii)]. This is written down as $y = \widehat{\varphi}(x_1, \dots, x_N)$.

Lemma 2. *Assume that E is a uniformly complete f-algebra and $x_1, \dots, x_N \in E_+$, and $\mathbf{x} := (x_1, \dots, x_N)$. Then $\widehat{\mathbf{x}}(\varphi) := \widehat{\varphi}(x_1, \dots, x_N)$ exists for every $\varphi \in \mathscr{A}(\mathbb{R}^N_+)$, and the mapping $\widehat{\mathbf{x}} : \varphi \mapsto \widehat{\mathbf{x}}(\varphi) = \widehat{\varphi}(x_1, \dots, x_N)$ is the unique multiplicative lattice homomorphism from $\mathscr{A}(\mathbb{R}^N_+)$ to E such that $\widehat{dt}_j(x_1, \dots, x_N) = x_j$ for all $j := 1, \dots, N$. Moreover, $\widehat{\mathbf{x}}(\mathscr{A}(\mathbb{R}^N_+)) = \overline{\langle\!\langle x_1, \dots, x_N \rangle\!\rangle}$.*

$\lhd$ Take $\varphi \in \mathscr{A}(\mathbb{R}^N_+)$. In view of Lemma 1 $\varphi = \varphi_1 + w\varphi_2$ with $\varphi_1 \in \mathscr{H}(\mathbb{R}^N_+)$, $\varphi_2 \in \mathscr{B}_0(\mathbb{R}^N_+)$, and $w(\mathbf{t}) = |t_1| + \ldots + |t_N|$. For $x \in E$ denote by $\dot{x} \in \mathrm{Orth}(E)$ the multiplication operator $y \mapsto xy$ $(x \in E)$. According to [5, Theorem 3.3] and [8, Theorem 2.10] we can define correctly $\widehat{\varphi}_1(x_1, \dots, x_N)$ in E and $\widehat{\varphi}_2(\dot{x}_1, \dots, \dot{x}_N)$ in $\mathrm{Orth}(E)$, respectively. Now, it remains to put $\widehat{\varphi}(x_1, \dots, x_N) := \widehat{\varphi}_1(x_1, \dots, x_N) + \widehat{\varphi}_2(\dot{x}_1, \dots, \dot{x}_N)w(x_1, \dots, x_N)$ and check the soundness of this definition. Closer examination of the proof can be carried out as in the case of $\varphi \in \mathscr{A}(\mathbb{R}^N)$, see [3]. $\rhd$

Lemma 3. *Assume that $\varphi \in \mathscr{A}(\mathbb{R}^N_+)$ is convex. Then for all $\mathbf{x} := (x_1, \dots, x_N) \in E^N$, $\mathbf{y} := (y_1, \dots, y_N) \in E^N$, and $\pi, \rho \in \mathrm{Orth}(E)_+$ with $\pi + \rho = I_E$ we have $\widehat{\varphi}(\pi\mathbf{x} + \rho\mathbf{y}) \leqslant \pi\widehat{\varphi}(\mathbf{x}) + \rho\widehat{\varphi}(\mathbf{y})$, where $\pi\mathbf{x} := (\pi x_1, \dots, \pi x_N)$. The reverse inequality holds whenever φ is concave.*

$\lhd$ Let L be the order ideal generated by $\overline{\langle\!\langle x_1, \dots, y_N \rangle\!\rangle}$. Clearly, L is an f-subalgebra of E. If $\pi_0 := \pi|_L$ and $\rho_0 := \rho|_L$ then $\pi_0, \rho_0\, \mathrm{Orth}(L)$. For any $\omega \in H(L)$ there exists a unique $\widetilde{\omega} \in H_m(\mathrm{Orth}(L))$ such that $\omega(\pi x) = \widetilde{\omega}(\pi)\omega(x)$ for all $x \in L$ and $\pi \in \mathrm{Orth}(L)$, [3, Proposition 2.2 (i)]. If ω is nonsingular then $\alpha\omega$ is multiplicative for some $\alpha > 0$ [3, Corollary 2.5 (i)], and thus we may assume without loss of generality that $\omega \in H_m(L)$. By using (1), the convexity of φ, and the relation $\widetilde{\omega}(\pi) + \widetilde{\omega}(\rho) = 1$ we deduce

$$\begin{aligned} \omega(\widehat{\varphi}(\pi\mathbf{x} + \rho\mathbf{y})) &= \varphi\big(\widetilde{\omega}(\pi_0)\omega(\mathbf{x}) + \widetilde{\omega}(\rho_0)\omega(\mathbf{y})\big) \leqslant \widetilde{\omega}(\pi_0)\varphi(\omega(\mathbf{x})) + \widetilde{\omega}(\rho_0)\varphi(\omega(\mathbf{y})) \\ &= \widetilde{\omega}(\pi_0)\omega(\widehat{\varphi}(\mathbf{x})) + \widetilde{\omega}(\rho_0)\omega(\widehat{\varphi}(\mathbf{y})) = \omega(\pi\widehat{\varphi}(\mathbf{x}) + \rho\widehat{\varphi}(\mathbf{y})), \end{aligned}$$

where $\omega(\mathbf{x}) := (\omega(x_1), \dots, \omega(x_N))$. If ω is singular then by above definition we have $\omega(\widehat{\varphi}(\mathbf{x})) = \omega(\widehat{\varphi}_1(\mathbf{x}))$, $\omega(\widehat{\varphi}(\mathbf{y})) = \omega(\widehat{\varphi}_1(\mathbf{y}))$, and $\omega(\widehat{\varphi}(\pi\mathbf{x} + \rho\mathbf{y})) = \omega(\widehat{\varphi}_1(\pi\mathbf{x} + \rho\mathbf{y}))$. At the same time φ_1 is sublinear, since it coincides with the directional derivative of the convex function φ at zero, see Lemma 3. Thus, by replacing φ by φ_1 in the above arguments we again obtain $\omega(\widehat{\varphi}(\pi\mathbf{x} + \rho\mathbf{y})) \leqslant \omega(\pi\widehat{\varphi}(\mathbf{x}) + \rho\widehat{\varphi}(\mathbf{y}))$. It remains to observe that every $\omega_0 \in H(\langle\!\langle x_1, \dots, x_N \rangle\!\rangle)$ admits an extension to $\omega \in H(L)$ and thus $H(L)$ separates the points of $\langle\!\langle x_1, \dots, x_N \rangle\!\rangle$. $\rhd$

Lemma 4. *If $\varphi \in \mathscr{A}(\mathbb{R}^N_+)$ is isotonic, then $\widehat{\varphi}$ is also isotonic, i. e. $\mathbf{x} \leqslant \mathbf{y}$ implies $\widehat{\varphi}(\mathbf{x}) \leqslant \widehat{\varphi}(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in E^N_+$. (The order in E^N is defined componentwise.)*

$\lhd$ Follows immediately from the above definition (1). $\rhd$

3°. Everywhere below $(G,+)$ is a commutative semigroup, while E is a uniformly complete f-algebra and $f_1,\dots,f_N : G \to E_+$. Let $\mathscr{P}(M)$ stands for the power set of M. Assume that some set-valued map $\mathscr{F} : G \to \mathscr{P}(\mathrm{Orth}(E)_+)$ meets the following three conditions:

(i) π^{-1} exists in $\mathrm{Orth}(E)$ for every $\pi \in \mathscr{F}(u)$,

(ii) $\mathscr{F}(u) + \mathscr{F}(v) \subset \mathscr{F}(u+v) - \mathrm{Orth}(E)_+$ for all $u,v \in G$, and

(iii) the infimum (the supremum) of $\{\pi\widehat{\varphi}(\pi^{-1}\mathbf{f}(u)) : \pi \in \mathscr{F}(u)\}$ exists in E for each $u \in G$, where $\mathbf{f}(u) := (f_1(u),\dots,f_N(u)) \in E_+^N$ and $\pi^{-1}\mathbf{f}(u) := (\pi^{-1}f_1(u),\dots,\pi^{-1}f_N(u))$.

Lemma 5. *Given a function* $\varphi : \mathscr{A}(\mathbb{R}_+^N)$ *and a set-valued map* $\mathscr{F} : G \to \mathscr{P}(\mathrm{Orth}(E)_+)$ *satisfying* 3 (i–iii), we have the operator $g : G \to E$ $(h : G \to E)$ well defined as

$$g(u) := \inf_{\pi\in\mathscr{F}(u)} \big\{\pi\widehat{\varphi}\big(\pi^{-1}\mathbf{f}(u)\big)\big\}, \quad \Big(h(u) := \sup_{\pi\in\mathscr{F}(u)} \big\{\pi\widehat{\varphi}\big(\pi^{-1}\mathbf{f}(u)\big)\big\}\Big). \tag{2}$$

$\lhd$ By 3 (i) and Lemma 2 $\widehat{\varphi}\big(\pi^{-1}\mathbf{f}(u)\big)$ exists in E and by 3 (iii) g and h are well defined. $\rhd$

4°. Now we are able to state and prove our main result. A function $g : G \to F$ is said to be *subadditive* if $g(u+v) \leqslant g(u) + g(v)$ for all $u,v \in G$ and *superadditive* if the reversed inequality holds for all $u,v \in G$.

Theorem. *Suppose that the operators* $g,h : G \to E$ *are defined as in* (2). *Then:*

(1) *g is subadditive whenever $f_1,\dots,f_N$ are subadditive and $\varphi \in \mathscr{A}(\mathbb{R}_+^{\mathbb{N}})$ is increasing and convex;*

(2) *h is superadditive whenever $f_1,\dots,f_N$ are superadditive and $\varphi \in \mathscr{A}(\mathbb{R}_+^{\mathbb{N}})$ is increasing and concave.*

$\lhd$ We restrict ourselves to the subadditivity of g. The superadditivity of h can be proved in a similar way. Take $u,v \in G$ and let $\pi \in \mathscr{F}(u)$ and $\rho \in \mathscr{F}(v)$. By 3 (ii) we can choose $\sigma \in \mathscr{F}(u+v)$ with $\sigma \geqslant \pi + \rho$. In view of 3 (i) π, ρ, and σ are invertible. Taking subadditivity of $\mathbf{f} : G \to E^N$ and some elementary properties of orthomorphisms into account we have

$$\sigma^{-1}\mathbf{f}(u+v) \leqslant \sigma^{-1}\big(\mathbf{f}(u) + \mathbf{f}(v)\big) \leqslant \pi\sigma^{-1}\big(\pi^{-1}\mathbf{f}(u)\big) + \rho\sigma^{-1}\big(\rho^{-1}\mathbf{f}(v)\big).$$

Putting $\tau := \sigma - \pi - \rho$ and making use of Lemmas 3, 4 and 5 we deduce

$$\begin{aligned} g(u+v) &\leqslant \sigma\widehat{\varphi}\big(\sigma^{-1}\mathbf{f}(u+v)\big) \leqslant \sigma\widehat{\varphi}\big(\pi\sigma^{-1}(\pi^{-1}\mathbf{f}(u))\big) + \rho\sigma^{-1}\big(\rho^{-1}\mathbf{f}(v) + \tau\sigma^{-1}0\big) \\ &\leqslant \pi\widehat{\varphi}(\pi^{-1}\mathbf{f}(u)) + \rho\widehat{\varphi}(\rho^{-1}\mathbf{f}(v)) + \sigma^{-1}\tau\widehat{\varphi}(0) = \pi\widehat{\varphi}(\pi^{-1}\mathbf{f}(u)) + \rho\widehat{\varphi}(\rho^{-1}\mathbf{f}(v)). \end{aligned}$$

By taking infimum over $\pi \in \mathscr{F}(u)$ and $\rho \in \mathscr{F}(v)$ we come to the required inequality. $\rhd$

Remark 1. Suppose that the hypotheses of 3 (i–iii) are fulfilled for some fixed $u,v \in G$. Then the inequality $g(u+v) \leqslant g(u) + g(v)$ $(h(u+v) \geqslant h(u) + h(v))$ holds.

Remark 2. An f-algebra E can be identified with $\mathrm{Orth}(E)$ if and only if E has a unit element. Thus, above theorem remains true if E is a uniformly complete unitary f-algebra and the map $\mathscr{F} : G \to \mathscr{P}(E_+)$ satisfies the condition 3 (i–iii) with $\mathrm{Orth}(E)$ replaced by E.

5°. For a single-valued map $\mathscr{F}(x) = \{f_0(x)\}$ $(x \in G)$ with $f_0 : G \to \mathrm{Orth}(E)_+$ we have the following particular case of the above Theorem, see [8].

Corollary 1. *Suppose that $f_1,\dots,f_N$ are subadditive, $f_0 : G \to \mathrm{Orth}(E)_+$ is superadditive, and $f_0(u)$ is invertible in $\mathrm{Orth}(E)$ for every $u \in G$. Then, given an increasing continuous convex function $\varphi \in \mathscr{A}(\mathbb{R}_+^N)$, the Peetre–Persson inequality holds:*

$$f_0(u+v)\widehat{\varphi}\bigg(\frac{\mathbf{f}(u+v)}{f_0(u+v)}\bigg) \leqslant f_0(u)\widehat{\varphi}\bigg(\frac{\mathbf{f}(u)}{f_0(u)}\bigg) + f_0(v)\widehat{\varphi}\bigg(\frac{\mathbf{f}(v)}{f_0(v)}\bigg). \tag{3}$$

The reverse inequality holds in (3) *whenever $f_0,f_1,\dots,f_N$ are superadditive, and φ is an increasing concave function.*

REMARK 3. The above theorem in the particular case of $E = \mathbb{R}$ was obtained by Persson [12, Theorems 1 and 2], while Corollary 2 covers the "single-valued case" by Peetre and Persson [11]. A short history of the Beckenbach–Dresher inequality is presented in [13]. Some instances of the inequality are also addressed in [9, 10].

6°. We need two more auxiliary facts. First of them is a generalized Minkowski inequality.

Lemma 6. *Let E and F be uniformly complete vector lattices, $f : E_+ \to F$ an increasing sublinear operator. If either and $0 < \alpha \leqslant 1$ or $\alpha < 0$, then for all $x_1, \dots, x_N \in E$ we have*

$$f\left(\left(\sum_{i=1}^{N} |x_i|^\alpha\right)^{1/\alpha}\right) \leqslant \left(\sum_{i=1}^{N} f(|x_i|)^\alpha\right)^{1/\alpha}. \tag{4}$$

The reverse inequality holds if $f : E_+ \to F$ is superlinear and $\alpha \geqslant 1$.

$\lhd$ The function $\phi_\alpha(\mathbf{t}) = \left(t_1^\alpha + \dots + t_N^\alpha\right)^{1/\alpha}$ $(\mathbf{t} \in \mathbb{R}_+^N)$ is superlinear if $0 < \alpha < 1$ and sublinear if $\alpha \geqslant 1$. In case $\alpha < 0$ we define $\phi_\alpha(\mathbf{t}) = 0$ whenever $t_1 \cdot \ldots \cdot t_N = 0$ and then ϕ_α is superlinear on $\operatorname{int}(\mathbb{R}_+^N)$. In all cases $\phi_\alpha \in \mathscr{H}(\mathbb{R}_+^N)$ and (4) follows from the generalized Jensen inequality in vector lattices, see [4, Theorem 5.2] and [7, Theorem 4.2]. $\rhd$

Let A and B be uniformly complete unitary f-algebras, while $E \subset A$ is a vector sublattice. For every $x \in A_+$ and $0 < p \in \mathbb{R}$ the p-power x^p is well defined in A, see [3, Theorem 4.12]. If $x \in A_+$ is invertible and $p < 0$, then we can also define $x^p := (x^{-1})^{-p}$. It can be easily seen that $\omega(x^p) = \omega(x)^p$ for any $\omega \in H_m(A_0)$ with an f-subalgebra $A_0 \subset$ containing x. Assume that $R : E \to B$ is a positive operator. Given $x \in A$ with $x^p \in E$, we define $R_p(x) := R(x^p)^{\frac{1}{p}}$. This definition is sound provided that x is invertible in A and $R(x^p)$ is invertible in B.

Lemma 7. *If $p \geqslant 1$ and $x_1, \dots, x_N \in A_+$ are such that $x_1^p, \dots, x_N^p \in E$ and $(x_1 + \dots + x_N)^p \in E$, then the inequality holds:*

$$R_p(x_1 + \dots + x_N) \leqslant R_p(x) + \dots + R_p(x_N). \tag{5}$$

The reversed inequality is true whenever $p \leqslant 1$, $p \neq 0$. (In case $p < 0$ the positive elements x_i and $R(x_i^p)$ are assumed to be invertible in A.)

$\lhd$ Denote $u_i := x_i^p$, $\alpha := 1/p$, and observe that $\left(u_1^\alpha + \dots + u_N^\alpha\right)^{\frac{1}{\alpha}} = \phi_\alpha(u_1, \dots, u_N)$ where $\phi_\alpha(u_1, \dots, u_N)$ is understood in the sense of homogeneous functional calculus. In particular, $(x_1 + \dots + x_N)^p = \left(u_1^\alpha + \dots + u_N^\alpha\right)^{\frac{1}{\alpha}} \in E$ for every $p \neq 0$. We need consider three cases. If $p \geqslant 0$ then by applying Lemma 6 to the right-hand side of the equality

$$R_p(x_1 + \dots + x_N) = R\Big(\left(u_1^\alpha + \dots + u_N^\alpha\right)^{\frac{1}{\alpha}}\Big)^\alpha = \big(R(\phi_\alpha(u_1, \dots, u_N))\big)^\alpha$$

with $u_i^\alpha \in E$ replaced by x_i and making use of $R_p(x_i) = R(u_i)^\alpha$ $(i := 1, \dots, N)$, we arrive immediately at the desired inequality (5). The same arguments involving the reversed version of (4) leads to the reversed inequality in (5) whenever $0 < p < 1$. Finally, in the case $p < 0$, again by Lemma 6, we have $R\big((u_1^\alpha + \dots + u_N^\alpha)^{\frac{1}{\alpha}}\big) \leqslant \big(R(u_1)^\alpha + \dots + R(u_N)^\alpha\big)^{\frac{1}{\alpha}}$ and rising both sides of this inequality to the αth power we get the reversed inequality (5). $\rhd$

7°. Now, we can deduce a generalization of one more Beckenbach–Dresher type inequality due to Peetre and Persson [11].

Corollary 2. *Let $S : E \to F$ and $T : E \to \operatorname{Orth}(F)$ be positive operators. Take $x_1, \dots, x_N \in A_+$ such that $x_i^\alpha, x_i^\beta, \left(\sum_{i=1}^N x_i\right)^\alpha, \left(\sum_{i=1}^N x_i\right)^\beta \in E$ $(i := 1, \dots, N)$. If $p \geqslant 1$,*

$\beta \leqslant 1 \leqslant \alpha$, $\beta \neq 0$, and $T(x_i^\beta)$ are invertible in $\mathrm{Orth}(F)$ *whenever $\beta < 0$, then*

$$\frac{\Big(S\big(\big(\sum_{i=1}^N x_i\big)^\alpha\big)\Big)^{p/\alpha}}{\Big(T\big(\big(\sum_{i=1}^N x_i\big)^\beta\big)\Big)^{(p-1)/\beta}} \leqslant \sum_{i=1}^N \frac{\big(S\big(x_i^\alpha\big)\big)^{p/\alpha}}{\big(T\big(x_i^\beta\big)\big)^{(p-1)/\beta}}. \tag{6}$$

$\lhd$ Put $G = E$, $f(x) := \mathbf{f}(x) := S(x^\alpha)^{1/\alpha}$, $f_0(x) := T(x^\beta)^{1/\beta}$, $N = 1$, and $\varphi(t) = t^p$ in Corollary 1. By Lemma 7 f is subadditive, f_0 is superadditive, and $f_0(x_i)$ is invertible in $\mathrm{Orth}(F)$. Moreover, $\varphi \in \mathscr{A}(\mathbb{R}_+)$ is convex and increasing whenever $p \geqslant 1$. Now, the desired inequality is deduced by induction. $\rhd$

Remark 4. If $0 < p < 1$ then the concave function $\varphi(t) = t^p$ is not in $\mathscr{A}(\mathbb{R}_+)$ and we cannot guarantee the reversed inequality in (6). Nevertheless, in the case that F is a unitary f-algebra one can take $\varphi \in \mathscr{B}(\mathbb{R}_+^N)$ in Peetre–Persson's inequality (3) and thus the reversed inequality is true in (6) whenever $0 < p \leqslant 1$, $\alpha, \beta \leqslant 1$, and $\alpha, \beta \neq 0$.

References

1. *Aliprantis C. D., Burkinshaw O.* Positive operators.—London etc.: Acad. Press Inc., 1985.—xvi+367 p.
2. *Beckenbach E. F., Bellman R.* Inequalities.—Berlin: Springer, 1983.
3. *Buskes G., de Pagter B., van Rooij A.* Functional calculus on Riesz spaces // Indag. Math.—1991.—Vol. 4, № 2.—P. 423–436.
4. *Kusraev A. G.* Homogeneous functional calculus on vector lattices.—Vladikavkaz: Southern Math. Inst. VSC RAS, 2008.—34 p.—(Preprint № 1).
5. *Kusraev A. G.* Functional calculus and Minkowski duality on vector lattices // Vladikavkaz Math. J.—2009.—Vol. 11, № 2.—P. 31–42.
6. *Kusraev A. G., Kutateladze S. S.* Subdifferentials: Theory and applications.—Dordrecht: Kluwer Academic Publ., 1995.—ix+398 p.
7. *Kusraev A. G.* Inequalities in vector lattices // Studies on Mathematical Analysis, Differential Equation, and their Applications / Eds. Korobeĭnik Yu. F., Kusraev A. G.—Vladikavkaz: SMI VSC RAS, 2010.—P. 82–96.—(Review of Science: The South of Russia).
8. *Kusraev A. G., Kutateladze S. S.* Envelopes and inequalities in vector lattices.—(to appear).
9. *Mitrinović D. S., Pečarić J. E., Fink A. M.* Classical and new inequalities in analysis.—Dordrecht: Kluwer, 1993.—xvii+740 p.
10. *Pečarić J. E., Proschan F., Tong Y. L.* Convex functions, partial orderings, and statistical application.—Boston a. o.: Academic Press, 1992.—xiii+469 p.
11. *Peetre J., Persson L.-E.* A general Beckenbach's inequality with applications // Function Spaces, Differential Operators and Nonlinear Analysis. Pitman Res. Notes Math. Ser. 211.—1989.—P. 125–139.
12. *Persson L.-E.* Generalizations of some classical inequalities and their applications // Nonlinear Analysis, Function Spaces and Applications / Eds. Krbec M., Kufner A., Opic B., Rákosník J.— Leipzig: Teubner, 1990.—P. 127–148.
13. *Varošanec S.* A generalized Beckenbach–Dresher inequality // Banach J. Math. Anal.—2010.—Vol. 4, № 1.—13–20.

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НЕРАВЕНСТВО ТИПА БЕККЕНБАХА — ДРЕШЕРА В РАВНОМЕРНО ПОЛНЫХ f-АЛГЕБРАХ

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Установлено общее неравенство типа Беккенбаха — Дрешера в равномерно полных f-алгебрах.

Ключевые слова: f-алгебра, векторная решетка, решеточный гомоморфизм, положительный оператор.