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STRONG INSERTION OF AN α-CONTINUOUS FUNCTION[1]

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Necessary and sufficient conditions in terms of lower cut sets are given for the strong insertion of an α-continuous function between two comparable real-valued functions.

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1. Introduction

The concept of a preopen set in a topological space was introduced by H. H. Corson and E. Michael in 1964 [3]. A subset A of a topological space (X, τ) is called *preopen* or *locally dense* or *nearly open* if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A))$. A set A is called *preclosed* if its complement is preopen or equivalently if $\mathrm{Cl}(\mathrm{Int}(A)) \subseteq A$. The term, preopen, was used for the first time by A. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deeb [11], while the concept of a, locally dense, set was introduced by H. H. Corson and E. Michael [3].

The concept of a semi-open set in a topological space was introduced by N. Levine in 1963 [10]. A subset A of a topological space (X, τ) is called *semi-open* [10] if $A \subseteq \mathrm{Cl}(\mathrm{Int}(A))$. A set A is called *semi-closed* if its complement is semi-open or equivalently if $\mathrm{Int}(\mathrm{Cl}(A)) \subseteq A$.

Recall that a subset A of a topological space (X, τ) is called *α-open* if A is the difference of an open and a nowhere dense subset of X. A set A is called *α-closed* if its complement is α-open or equivalently if A is union of a closed and a nowhere dense set.

We have a set is α-open if and only if it is semi-open and preopen.

Recall that a real-valued function f defined on a topological space X is called A-continuous [13] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subset of X. Most of the definitions used throughout this paper are consequences of the definition of A-continuity. However, for unknown concepts the reader may refer to [4, 5].

Hence, a real-valued function f defined on a topological space X is called precontinuous (resp. semi-continuous or α-continuous) if the preimage of every open subset of $\mathbb{R}$ is preopen (resp. semi-open or α-open) subset of X.

Precontinuity was called by V. Pták nearly continuity [14]. Nearly continuity or precontinuity is known also as almost continuity by T. Husain [6]. Precontinuity was studied for real-valued functions on Euclidean space by Blumberg back in 1922 [1].

Results of Katětov [7, 8] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give necessary and sufficient conditions for the strong insertion of an α-continuous function between two comparable real-valued functions.

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If g and f are real-valued functions defined on a space X, we write $g \leqslant f$ in case $g(x) \leqslant f(x)$ for all x in X.

The following definitions are modifications of conditions considered in [9].

A property P defined relative to a real-valued function on a topological space is an *α-property* provided that any constant function has property P and provided that the sum of a function with property P and any α-continuous function also has property P. If P_1 and P_2 are α-property, the following terminology is used:

(i) A space X has the *weak α-insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leqslant f, g$ has property P_1 and f has property P_2, then there exists an α-continuous function h such that $g \leqslant h \leqslant f$.

(ii) A space X has the *strong α-insertion property* for (P_1, P_2) if and only if for any functions g and f on X such that $g \leqslant f, g$ has property P_1 and f has property P_2, then there exists an α-continuous function h such that $g \leqslant h \leqslant f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$.

In this paper, is given a sufficient condition for the weak α-insertion property. Also for a space with the weak α-insertion property, we give necessary and sufficient conditions for the space to have the strong α-insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Result

Before giving a sufficient condition for insertability of an α-continuous function, the necessary definitions and terminology are stated.

The abbreviations *pc*, *sc* and αc are used for precontinuous, semicontinuous and α-continuous, respectively.

Let (X, τ) be a topological space, the family of all α-open, α-closed, semi-open, semi-closed, preopen and preclosed will be denoted by $\alpha O(X,\tau)$, $\alpha C(X,\tau)$, $sO(X,\tau)$, $sC(X,\tau)$, $pO(X,\tau)$ and $pC(X,\tau)$, respectively.

Definition 2.1. Let A be a subset of a topological space (X, τ). Respectively, we define the *α-closure, α-interior, s-closure, s-interior, p-closure* and *p-interior* of a set A, denoted by $\alpha\,\mathrm{Cl}(A)$, $\alpha\,\mathrm{Int}(A)$, $s\,\mathrm{Cl}(A)$, $s\,\mathrm{Int}(A)$, $p\,\mathrm{Cl}(A)$ and $p\,\mathrm{Int}(A)$ as follows:

$\alpha\,\mathrm{Cl}(A) = \cap\{F : F \supseteq A, F \in \alpha C(X,\tau)\}$,

$\alpha\,\mathrm{Int}(A) = \cup\{O : O \subseteq A, O \in \alpha O(X,\tau)\}$,

$s\,\mathrm{Cl}(A) = \cap\{F : F \supseteq A, F \in sC(X,\tau)\}$,

$s\,\mathrm{Int}(A) = \cup\{O : O \subseteq A, O \in sO(X,\tau)\}$,

$p\,\mathrm{Cl}(A) = \cap\{F : F \supseteq A, F \in pC(X,\tau)\}$ and

$p\,\mathrm{Int}(A) = \cup\{O : O \subseteq A, O \in pO(X,\tau)\}$.

Respectively, we have $\alpha\,\mathrm{Cl}(A), s\,\mathrm{Cl}(A), p\,\mathrm{Cl}(A)$ are α-closed, semi-closed, preclosed and $\alpha\,\mathrm{Int}(A)$, $s\,\mathrm{Int}(A)$, $p\,\mathrm{Int}(A)$ are α-open, semi-open, preopen.

The following first two definitions are modifications of conditions considered in [7, 8].

Definition 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x\bar{\rho}y$ if and only if $y\rho\nu$ implies $x\rho\nu$ and $u\rho x$ implies $u\rho y$ for any u and v in S.

Definition 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

1) If $A_i\rho B_j$ for any $i \in \{1,\dots,m\}$ and for any $j \in \{1,\dots,n\}$, then there exists a set C in $P(X)$ such that $A_i\rho C$ and $C\rho B_j$ for any $i \in \{1,\dots,m\}$ and any $j \in \{1,\dots,n\}$.

2) If $A \subseteq B$, then $A\bar{\rho}B$.

3) If $A\rho B$, then $\alpha\,\mathrm{Cl}(A) \subseteq B$ and $A \subseteq \alpha\,\mathrm{Int}(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

DEFINITION 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f,\ell) \subseteq \{x \in X : f(x) \leqslant \ell\}$ for a real number ℓ, then $A(f,\ell)$ is called a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main result:

Theorem 2.1. *Let g and f be real-valued functions on a topological space X with $g \leqslant f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\rho A(g,t_2)$, then there exists an α-continuous function h defined on X such that $g \leqslant h \leqslant f$.*

$\lhd$ Let g and f be real-valued functions defined on X such that $g \leqslant f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\rho A(g,t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f,t)$ and $G(t) = A(g,t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\bar{\rho}F(t_2)$, $G(t_1)\bar{\rho}G(t_2)$, and $F(t_1)\rho G(t_2)$. By Lemmas 1 and 2 of [8] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\rho H(t_2)$, $H(t_1)\rho H(t_2)$ and $H(t_1)\rho G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leqslant h \leqslant f$. If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g,t')$ implies that $g(x) \leqslant t'$, it follows that $g(x) \leqslant t$. Hence $g \leqslant h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f,t')$ implies that $f(x) > t'$, it follows that $f(x) \geqslant t$. Hence $h \leqslant f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = \alpha\,\mathrm{Int}(H(t_2)) \setminus \alpha\,\mathrm{Cl}(H(t_1))$. Hence $h^{-1}(t_1,t_2)$ is an α-open subset of X, i.e., h is an α-continuous function on X. $\rhd$

The above proof used the technique of proof of Theorem 1 of [7].

If a space has the strong α-insertion property for (P_1,P_2), then it has the weak α-insertion property for (P_1,P_2). The following result uses lower cut sets and gives a necessary and sufficient condition for a space satisfies that weak α-insertion property to satisfy the strong α-insertion property.

Theorem 2.2. *Let P_1 and P_2 be α-property and X be a space that satisfies the weak α-insertion property for (P_1,P_2). Also assume that g and f are functions on X such that $g \leqslant f$, g has property P_1 and f has property P_2. The space X has the strong α-insertion property for (P_1,P_2) if and only if there exist lower cut sets $A(f-g,2^{-n})$ and there exists a sequence $\{F_n\}$ of subsets of X such that (i) for each n, F_n and $A(f-g,2^{-n})$ are completely separated by α-continuous functions, and (ii) $\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$.*

$\lhd$ Theorem 3.1, of [12]. $\rhd$

Theorem 2.3. *Let P_1 and P_2 be α-properties and assume that the space X satisfied the weak α-insertion property for (P_1,P_2). The space X satisfies the strong α-insertion property for (P_1,P_2) if and only if X satisfies the strong α-insertion property for $(P_1,\alpha c)$ and for $(\alpha c,P_2)$.*

$\lhd$ Theorem 3.2, of [12]. $\rhd$

3. Applications

Corollary 3.1. *If for each pair of disjoint preclosed (resp. semi-closed) sets F_1, F_2 of X, there exist α-open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then X has the weak α-insertion property for (pc, pc) (resp. (sc, sc)).*

$\lhd$ Let g and f be real-valued functions defined on the X, such that f and g are pc (resp. sc), and $g \leqslant f$. If a binary relation ρ is defined by $A\rho B$ in case $p\,\mathrm{Cl}(A) \subseteq p\,\mathrm{Int}(B)$ (resp. $s\,\mathrm{Cl}(A) \subseteq s\,\mathrm{Int}(B)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leqslant t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leqslant t_1\}$ is a preclosed (resp. semi-closed) set and since $\{x \in X : g(x) < t_2\}$ is a preopen (resp. semi-open) set, it follows that $p\,\mathrm{Cl}(A(f, t_1)) \subseteq p\,\mathrm{Int}(A(g, t_2))$ (resp. $s\,\mathrm{Cl}(A(f, t_1)) \subseteq s\,\mathrm{Int}(A(g, t_2))$). Hence $t_1 < t_2$ implies that $A(f, t_1)\rho A(g, t_2)$. The proof follows from Theorem 2.1. $\rhd$

Corollary 3.2. *If for each pair of disjoint preclosed (resp. semi-closed) sets F_1, F_2, there exist α-open sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then every precontinuous (resp. semi-continuous) function is α-continuous.*

$\lhd$ Let f be a real-valued precontinuous (resp. semi-continuous) function defined on the X. Set $g = f$, then by Corollary 3.1, there exists an α-continuous function h such that $g = h = f$. $\rhd$

Corollary 3.3. *If for each pair of disjoint preclosed (resp. semi-closed) sets F_1, F_2 of X, there exist α-open sets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then X has the strong α-insertion property for (pc, pc) (resp. (sc, sc)).*

$\lhd$ Let g and f be real-valued functions defined on the X, such that f and g are pc (resp. sc), and $g \leqslant f$. Set $h = (f + g)/2$, thus $g \leqslant h \leqslant f$ and if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$. Also, by Corollary 3.2, since g and f are α-continuous functions hence h is an α-continuous function. $\rhd$

Corollary 3.4. *If for each pair of disjoint subsets F_1, F_2 of X, such that F_1 is preclosed and F_2 is semi-closed, there exist α-open subsets G_1 and G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$ then X have the weak α-insertion property for (pc, sc) and (sc, pc).*

$\lhd$ Let g and f be real-valued functions defined on the X, such that g is pc (resp. sc) and f is sc (resp. pc), with $g \leqslant f$. If a binary relation ρ is defined by $A\rho B$ in case $s\,\mathrm{Cl}(A) \subseteq p\,\mathrm{Int}(B)$ (resp. $p\,\mathrm{Cl}(A) \subseteq s\,\mathrm{Int}(B)$), then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leqslant t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leqslant t_1\}$ is a semi-closed (resp. preclosed) set and since $\{x \in X : g(x) < t_2\}$ is a preopen (resp. semi-open) set, it follows that $s\,\mathrm{Cl}(A(f, t_1)) \subseteq p\,\mathrm{Int}(A(g, t_2))$ (resp. $p\,\mathrm{Cl}(A(f, t_1)) \subseteq s\,\mathrm{Int}(A(g, t_2))$). Hence $t_1 < t_2$ implies that $A(f, t_1)\rho A(g, t_2)$. The proof follows from Theorem 2.1. $\rhd$

Before stating the consequences of Theorems 2.2, and 2.3, we state and prove the necessary lemmas.

Lemma 3.1. *The following conditions on the space X are equivalent:*

(i) *For each pair of disjoint subsets F_1, F_2 of X, such that F_1 is preclosed and F_2 is semi-closed, there exist α-open subsets G_1, G_2 of X such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \varnothing$.*

(ii) *If F is a semi-closed (resp. preclosed) subset of X which is contained in a preopen (resp. semi-open) subset G of X, then there exists an α-open subset H of X such that $F \subseteq H \subseteq \alpha\,\mathrm{Cl}(H) \subseteq G$.*

$\lhd$ (i) $\Rightarrow$ (ii): Suppose that $F \subseteq G$, where F and G are semi-closed (resp. preclosed) and preopen (resp. semi-open) subsets of X, respectively. Hence, G^c is a preclosed (resp. semi-closed) and $F \cap G^c = \varnothing$.

By (i) there exists two disjoint α-open subsets G_1, G_2 of X s. t., $F \subseteq G_1$ and $G^c \subseteq G_2$. But $G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G$, and $G_1 \cap G_2 = \varnothing \Rightarrow G_1 \subseteq G_2^c$ hence $F \subseteq G_1 \subseteq G_2^c \subseteq G$ and since G_2^c is an α-closed set containing G_1 we conclude that $\alpha\,\mathrm{Cl}(G_1) \subseteq G_2^c$, i. e.,

$$F \subseteq G_1 \subseteq \alpha\,\mathrm{Cl}(G_1) \subseteq G.$$

By setting $H = G_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i): Suppose that F_1, F_2 are two disjoint subsets of X, such that F_1 is preclosed and F_2 is semi-closed.

This implies that $F_2 \subseteq F_1^c$ and F_1^c is a preopen subset of X. Hence by (ii) there exists an α-open set H s. t., $F_2 \subseteq H \subseteq \alpha\,\mathrm{Cl}(H) \subseteq F_1^c$. But

$$H \subseteq \alpha\,\mathrm{Cl}(H) \Rightarrow H \cap (\alpha\,\mathrm{Cl}(H))^c = \varnothing$$

and

$$\alpha\,\mathrm{Cl}(H) \subseteq F_1^c \Rightarrow F_1 \subseteq (\alpha\,\mathrm{Cl}(H))^c.$$

Furthermore, $(\alpha\,\mathrm{Cl}(H))^c$ is an α-open set of X. Hence $F_2 \subseteq H, F_1 \subseteq (\alpha\,\mathrm{Cl}(H))^c$ and $H \cap (\alpha\,\mathrm{Cl}(H))^c = \varnothing$. This means that condition (i) holds. $\rhd$

Lemma 3.2. *Suppose that X is a topological space. If each pair of disjoint subsets F_1, F_2 of X, where F_1 is preclosed and F_2 is semi-closed, can separate by α-open subsets of X then there exists an α-open continuous function $h : X \to [0,1]$ s. t., $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$.*

$\lhd$ Suppose F_1 and F_2 are two disjoint subsets of X, where F_1 is preclosed and F_2 is semi-closed. Since $F_1 \cap F_2 = \varnothing$, hence $F_2 \subseteq F_1^c$. In particular, since F_1^c is a preopen subset of X containing semi-closed subset F_2 of X, by Lemma 3.1, there exists an α-open subset $H_{1/2}$ of X s. t.,

$$F_2 \subseteq H_{1/2} \subseteq \alpha\,\mathrm{Cl}(H_{1/2}) \subseteq F_1^c.$$

Note that $H_{1/2}$ is also a preopen subset of X and contains F_2, and F_1^c is a preopen subset of X and contains a semi-closed subset $\alpha\,\mathrm{Cl}(H_{1/2})$ of X. Hence, by Lemma 3.1, there exists α-open subsets $H_{1/4}$ and $H_{3/4}$ s. t.,

$$F_2 \subseteq H_{1/4} \subseteq \alpha\,\mathrm{Cl}(H_{1/4}) \subseteq H_{1/2} \subseteq \alpha\,\mathrm{Cl}(H_{1/2}) \subseteq H_{3/4} \subseteq \alpha\,\mathrm{Cl}(H_{3/4}) \subseteq F_1^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain α-open subsets H_t of X with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin F_1$ and $h(x) = 1$ for $x \in F_1$.

Note that for every $x \in X$, $0 \leqslant h(x) \leqslant 1$, i. e., h maps X into $[0,1]$. Also, we note that for any $t \in D$, $F_2 \subseteq H_t$; hence $h(F_2) = \{0\}$. Furthermore, by definition, $h(F_1) = \{1\}$. It remains only to prove that h is an α-continuous function on X. For every $\beta \in \mathbb{R}$, we have if $\beta \leqslant 0$ then $\{x \in X : h(x) < \beta\} = \varnothing$ and if $0 < \beta$ then $\{x \in X : h(x) < \beta\} = \cup\{H_t : t < \beta\}$, hence, they are α-open subsets of X. Similarly, if $\beta < 0$ then $\{x \in X : h(x) > \beta\} = X$ and if $0 \leqslant \beta$ then $\{x \in X : h(x) > \beta\} = \cup\{(\alpha\,\mathrm{Cl}(H_t))^c : t > \beta\}$ hence, every of them is an α-open subset of X. Consequently h is an α-continuous function. $\rhd$

Lemma 3.3. *Suppose that X is a topological space. If each pair of disjoint subsets F_1, F_2 of X, where F_1 is preclosed and F_2 is semi-closed, can separate by α-open subsets of X, and F_1 (resp. F_2) is a countable intersection of α-open subsets of X, then there exists an α-continuous function $h : X \to [0,1]$ s. t., $h^{-1}(0) = F_1$ (resp. $h^{-1}(0) = F_2$) and $h(F_2) = \{1\}$ (resp. $h(F_1) = \{1\}$).*

$\lhd$ Suppose that $F_1 = \bigcap_{n=1}^{\infty} G_n$ (resp. $F_2 = \bigcap_{n=1}^{\infty} G_n$), where G_n is an α-open subset of X. We can suppose that $G_n \cap F_2 = \varnothing$ (resp. $G_n \cap F_1 = \varnothing$), otherwise we can substitute G_n by $G_n \setminus F_2$ (resp. $G_n \setminus F_1$). By Lemma 3.2, for every $n \in \mathbb{N}$, there exists an α-continuous function $h_n : X \to [0,1]$ s.t., $h_n(F_1) = \{0\}$ (resp. $h_n(F_2) = \{0\}$) and $h_n(X \setminus G_n) = \{1\}$. We set $h(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x)$.

Since the above series is uniformly convergent, it follows that h is an α-continuous function from X to $[0,1]$. Since for every $n \in \mathbb{N}$, $F_2 \subseteq X \setminus G_n$ (resp. $F_1 \subseteq X \setminus G_n$), therefore $h_n(F_2) = \{1\}$ (resp. $h_n(F_1) = \{1\}$) and consequently $h(F_2) = \{1\}$ (resp. $h(F_1) = \{1\}$). Since $h_n(F_1) = \{0\}$ (resp. $h_n(F_2) = \{0\}$), hence $h(F_1) = \{0\}$ (resp. $h(F_2) = \{0\}$). It suffices to show that if $x \notin F_1$ (resp. $x \notin F_2$), then $h(x) \neq 0$.

Now if $x \notin F_1$ (resp. $x \notin F_2$), since $F_1 = \bigcap_{n=1}^{\infty} G_n$ (resp. $F_2 = \bigcap_{n=1}^{\infty} G_n$), therefore there exists $n_0 \in \mathbb{N}$ s. t., $x \notin G_{n_0}$, hence $h_{n_0}(x) = 1$, i. e., $h(x) > 0$. Therefore $h^{-1}(0) = F_1$ (resp. $h^{-1}(0) = F_2$). $\rhd$

Lemma 3.4. *Suppose that X is a topological space such that every two disjoint semi-closed and preclosed subsets of X can be separated by α-open subsets of X. The following conditions are equivalent:*

(i) *For every two disjoint subsets F_1 and F_2 of X, where F_1 is preclosed and F_2 is semi-closed, there exists an α-continuous function $h : X \to [0,1]$ s. t., $h^{-1}(0) = F_1$ (resp. $h^{-1}(0) = F_2$) and $h^{-1}(1) = F_2$ (resp. $h^{-1}(1) = F_1$).*

(ii) *Every preclosed (resp. semi-closed) subset of X is a countable intersection of α-open subsets of X.*

(iii) *Every preopen (resp. semi-open) subset of X is a countable union of α-closed subsets of X.*

$\lhd$ (i) $\Rightarrow$ (ii). Suppose that F is a preclosed (resp. semi-closed) subset of X. Since $\varnothing$ is a semi-closed (resp. preclosed) subset of X, by (i) there exists an α-continuous function $h : X \to [0,1]$ s. t., $h^{-1}(0) = F$. Set $G_n = \left\{x \in X : h(x) < \frac{1}{n}\right\}$. Then for every $n \in \mathbb{N}$, G_n is an α-open subset of X and $\bigcap_{n=1}^{\infty} G_n = \{x \in X : h(x) = 0\} = F$.

(ii) $\Rightarrow$ (i). Suppose that F_1 and F_2 are two disjoint subsets of X, where F_1 is preclosed and F_2 is semi-closed. By Lemma 3.3, there exists an α-continuous function $f : X \to [0,1]$ s. t., $f^{-1}(0) = F_1$ and $f(F_2) = \{1\}$. Set $G = \left\{x \in X : f(x) < \frac{1}{2}\right\}$, $F = \{x \in X : f(x) = \frac{1}{2}\}$, and $H = \left\{x \in X : f(x) > \frac{1}{2}\right\}$. Then $G \cup F$ and $H \cup F$ are two α-closed subsets of X and $(G \cup F) \cap F_2 = \varnothing$. By Lemma 3.3, there exists an α-continuous function $g : X \to \left[\frac{1}{2}, 1\right]$ s. t., $g^{-1}(1) = F_2$ and $g(G \cup F) = \left\{\frac{1}{2}\right\}$. Define h by $h(x) = f(x)$ for $x \in G \cup F$, and $h(x) = g(x)$ for $x \in H \cup F$. Then h is well-defined and an α-continuous function, since $(G \cup F) \cap (H \cup F) = F$ and for every $x \in F$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(G \cup F) \cup (H \cup F) = X$, hence h defined on X and maps to $[0,1]$. Also, we have $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$.

(ii) $\Leftrightarrow$ (iii): By De Morgan law and noting that the complement of every α-open subset of X is an α-closed subset of X and complement of every α-closed subset of X is an α-open subset of X, the equivalence is hold. $\rhd$

Corollary 3.5. *If for every two disjoint subsets F_1 and F_2 of X, where F_1 is preclosed (resp. semi-closed) and F_2 is semi-closed (resp. preclosed), there exists an α-continuous* *function $h : X \to [0,1]$ s. t., $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$ then X has the strong α-insertion property for (pc, sc) (resp. (sc, pc)).*

$\lhd$ Since for every two disjoint subsets F_1 and F_2 of X, where F_1 is preclosed (resp. semi-closed) and F_2 is semi-closed (resp. preclosed), there exists an α-continuous function $h : X \to [0,1]$ s. t., $h^{-1}(0) = F_1$ and $h^{-1}(1) = F_2$, define $G_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $G_2 = \{x \in X : h(x) > \frac{1}{2}\}$. Then G_1 and G_2 are two disjoint α-open subsets of X that contain F_1 and F_2, respectively. Hence by Corollary 3.4, X has the weak α-insertion property for (pc, sc) and (sc, pc). Now, assume that g and f are functions on X such that $g \leqslant f$, g is pc (resp. sc) and f is αc. Since $f - g$ is pc (resp. sc), therefore the lower cut set $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leqslant 2^{-n}\}$ is a preclosed (resp. semi-closed) subset of X. By Lemma 3.4, we can choose a sequence $\{F_n\}$ of α-closed subsets of X s. t., $\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$ and for every $n \in \mathbb{N}$, F_n and $A(f - g, 2^{-n})$ are disjoint subsets of X. By Lemma 3.2, F_n and $A(f - g, 2^{-n})$ can be completely separated by α-continuous functions. Hence by Theorem 2.2, X has the strong α-insertion property for $(pc, \alpha c)$ (resp. $(sc, \alpha c)$).

By an analogous argument, we can prove that X has the strong α-insertion property for $(\alpha c, sc)$ (resp. $(\alpha c, pc)$). Hence, by Theorem 2.3, X has the strong α-insertion property for (pc, sc) (resp. (sc, pc)). $\rhd$

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СТРОГОЕ ВЛОЖЕНИЕ α-НЕПРЕРЫВНЫХ ФУНКЦИЙ

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В терминах верхнего сечения даны необходимые и достаточные условия для строгой вложимости α-непрерывных функций между двумя сравнимыми вещественнозначными функциями.

Ключевые слова: строгое вложение, строгое бинарное отношение, предоткрытое множество, полуоткрытое множество, α-открытое множество, верхнее сечение.