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GEODESIC ORBIT RIEMANNIAN METRICS ON SPHERES[1]

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In this paper, a complete classification of geodesic orbit Riemannian metrics on spheres S^n is obtained. We also construct some explicit examples of geodesic vectors for $Sp(n+1)U(1)$-invariant metrics on S^{4n+3}.

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Introduction

A Riemannian manifold (M, g) is called *a geodesic orbit manifold* (*GO-manifold*) if every its geodesic is an orbit of a one-parameter group of isometries of (M, g). Every such manifold is homogeneous and can be identified with a coset space $M = G/H$ of a transitive Lie group G of isometries. A Riemannian homogeneous space $(M = G/H, g)$ of a Lie group G is called *a space with homogeneous geodesics* (or *a geodesic orbit space*, shortly, *GO-space*), if any its geodesic is an orbit of a one-parameter subgroup of the group G. This terminology was introduced by O. Kowalski and L. Vanhecke in the paper [17]. We discuss some properties of geodesic orbit Riemannian manifolds and related results in Section 1.

The main goal of this paper is a complete classification of geodesic orbit Riemannian metrics on spheres S^n. The classification of all transitive and effective actions of connected compact Lie groups on spheres is obtained in [18]. We collect in Table 1 all variants to represent S^n as a homogeneous space G/H. In this table, by $\dim(G/H)$ we denote the dimension of the corresponding space (and the corresponding sphere), $\dim(\mathscr{M})$ (respectively, $\dim(\mathscr{M}_{GO})$) means the dimension of the space of G-invariant (the space of G-invariant geodesic orbit) Riemannian metrics on the homogeneous space G/H.

Note that Riemannian metrics of constant sectional curvature constitute a one-parameter family of metrics on each sphere S^n, $n \geqslant 2$ (there is no such notion for the trivial case $n = 1$). These are exactly the metrics invariant under the action of the orthogonal group $O(n+1)$ and under its connected unit component $SO(n+1)$. These groups are respectively the full isometry group and the full connected isometry group of each metric of constant curvature on S^n.

We list all inclusions between the isometry groups in Table 1: $G_2 \subset SO(7)$, $SU(k) \subset U(k) \subset SO(2k)$, $Sp(k) \subset Sp(k)U(1) \subset Sp(k)Sp(1) \subset SO(4k)$, $Sp(k) \subset Sp(k)U(1) \subset U(2k)$, $SU(4) \subset Spin(7) \subset SO(8)$, $Spin(9) \subset SO(16)$ (see details e. g. in Chapter 4 of [20]).

It should be noted that geodesic orbit metrics on some spaces in Table 1 are well known. Below we describe all known results and emphasize the cases that should be studied.

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Table 1

Invariant metrics on spheres

	G	H	$\dim(G/H)$	$\dim(\mathscr{M})$	$\dim(\mathscr{M}_{GO})$	Cond.
1	$SO(n+1)$	$SO(n)$	n	1	1	$n \geqslant 1$
2	G_2	$SU(3)$	6	1	1	
3	$Spin(7)$	G_2	7	1	1	
4	$SU(2)$	$\{e\}$	3	6	1	
5	$SU(n+1)$	$SU(n)$	$2n+1$	2	2	$n \geqslant 2$
6	$U(n+1)$	$U(n)$	$2n+1$	2	2	$n \geqslant 1$
7	$Spin(9)$	$Spin(7)$	15	2	2	
8	$Sp(n+1)Sp(1)$	$Sp(n)\operatorname{diag}(Sp(1))$	$4n+3$	2	2	$n \geqslant 1$
9	$Sp(n+1)U(1)$	$Sp(n)\operatorname{diag}(U(1))$	$4n+3$	3	3	$n \geqslant 1$
10	$Sp(n+1)$	$Sp(n)$	$4n+3$	7	2	$n \geqslant 1$

Case 1). The homogeneous space $SO(n+1)/SO(n)$ is irreducible symmetric, all $SO(n+1)$-invariant Riemannian metrics are $SO(n+1)$-normal homogeneous (hence, geodesic orbit) and constitute the set of Riemannian metrics of constant sectional curvature on S^n. This set is a part of any other family of invariant metrics from Table 1.

Cases 2) and 3). The spaces $G_2/SU(3)$ and $Spin(7)/G_2$ are isotropy irreducible. All invariant metrics on these spaces are normal homogeneous (hence, GO-metrics) and have constant sectional curvature (see Section 7 in [12]).

Case 4). All left-invariant metrics on a compact Lie group G, that are geodesic orbit with respect to G, should be biinvariant (see Proposition 8 in [2]). Since the group $SU(2)$ is simple, then all biinvariant Riemannian metrics on $SU(2)$ constitute a one-parameter family of metrics. Since $SU(2)^2/\operatorname{diag}(SU(2)) = SO(4)/SO(3)$, then these metrics are exactly metrics of constant curvature on $S^3 = SU(2)$.

Cases 5) and 6). Note that the set of $U(n+1)$-invariant metrics on S^{2n+1} coincides with the set of $SU(n+1)$-invariant metrics and constitutes a 2-parametric family of metrics. Every such metric is naturally reductive and weakly symmetric [26, 27, 28]. Therefore, in both these cases we have a two-parameter family of geodesic orbit metrics.

Case 7). The family of invariant metrics on $Spin(9)/Spin(7)$ is 2-parametric. All these metrics are weakly symmetric but not naturally reductive [27, 28]. Therefore, we have a two-parameter family of geodesic orbit metrics.

Case 8). The family of invariant metrics on $Sp(n+1)Sp(1)/Sp(n)\operatorname{diag}(Sp(1))$ is 2-parametric. Every such metric is naturally reductive and weakly symmetric [27, 28]. Therefore, in both these cases we have a 2-parameter family of geodesic orbit metrics. More details on this case could be found in Sections 2.

Case 9). Note that the previous family is a part of this one. The family of $Sp(n+1)U(1)$-invariant metrics on $Sp(n+1)U(1)/Sp(n)\operatorname{diag}(U(1)) = S^{4n+3}$ is 3-parametric. Every such metric is weakly symmetric (see e. g. 12.9.2 in [24] or Table 1 in [25]). Therefore, in this case we have a three-parameter family of geodesic orbit metrics. Details on the normality and the natural reductivity of $Sp(n+1)U(1)$-invariant metrics could be found in Remark 1. Some explicit form of geodesic vectors for $Sp(n+1)\times U(1)$-invariant metric could be found in Section 4.

Case 10). In the last case we get a 7-dimensional space of $Sp(n+1)$-invariant metrics on $Sp(n+1)/Sp(n) = S^{4n+3}$. This is a unique case that we should study in details. We deal with this case in Sections 2 and 3. By Theorem 1, a homogeneous Riemannian space $(S^{4n+3} = Sp(n+1)/Sp(n), g)$ is geodesic orbit if and only if the metric g is invariant under $Sp(n+1)Sp(1)$.

The structure of the paper is the following. We recall in Section 1 some useful facts on the class of geodesic orbit Riemannian manifolds and some related classes of Riemannian manifolds. In Section 2 we discuss actions of the groups $Sp(n+1)$, $Sp(n+1)U(1)$, $Sp(n+1)Sp(1)$ (and corresponding invariant metrics) on the sphere S^{4n+3}. In the next section we classify $Sp(n+1)$-invariant geodesic orbit metrics on S^{4n+3}. The final section is devoted to an explicit description of geodesic vectors for $Sp(n+1)U(1)$-invariant metrics on the sphere S^{4n+3}.

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1. On geodesic orbit manifolds

There are some important subclasses of geodesic orbit manifolds. Indeed, GO-spaces may be considered as a natural generalization of Riemannian symmetric spaces, introduced and classified by È. Cartan in [13]. On the other hand, the class of GO-spaces is much larger than the class of symmetric spaces. Any homogeneous space $M = G/H$ of a compact Lie group G admits a Riemannian metric g such that (M, g) is a GO-space. It suffices to take the metric g induced by a biinvariant Riemannian metric g_0 on the Lie group G such that $(G, g_0) \to (M = G/H, g)$ is a Riemannian submersion with totally geodesic fibres. Such geodesic orbit space $(M = G/H, g)$ is called a *normal homogeneous space* (in the sense of M. Berger [10]). It should be noted also that any naturally reductive Riemannian manifold is geodesic orbit. Recall that a Riemannian manifold (M, g) is *naturally reductive* if it admits a transitive Lie group G of isometries with a biinvariant pseudo-Riemannian metric g_0, which induces the metric g on $M = G/H$ (see [12] and [16]).

An important class of GO-spaces consists of weakly symmetric spaces, introduced by A. Selberg [21]. A homogeneous Riemannian manifold $(M = G/H, g)$ is a *weakly symmetric space* if any two points $p, q \in M$ can be interchanged by an isometry $a \in G$. This property does not depend on the particular G-invariant metric g. Weakly symmetric spaces $M = G/H$ have many interesting properties and are closely related with spherical spaces, commutative spaces, Gelfand pairs etc. (see papers [3, 25] and book [24] by J. A. Wolf). The classification of weakly symmetric reductive homogeneous Riemannian spaces was given by O. S. Yakimova [25] on the base of the paper [3] (see also [24]). It is very important that *weakly symmetric Riemannian manifolds are geodesic orbit* by a result of J. Berndt, O. Kowalski, and L. Vanhecke [11].

Note that *generalized normal homogeneous Riemannian manifolds* (δ*-homogeneous manifold*, in another terminology) constitute another important subclass of geodesic orbit manifold. All metrics from this subclass are of non-negative sectional curvature and have some other interesting properties (see details in [4–6]). In the paper [9], a classification of generalized normal homogeneous metrics on spheres and projective spaces is obtained. Finally, we notice that *Clifford-Wolf homogeneous Riemannian manifolds* constitute a partial subclass of generalized normal homogeneous Riemannian manifolds [7].

Many interesting results about GO-spaces one can find in [2, 11, 14, 15, 19, 22, 23, 28], where there are also extensive references.

Now we recall some important properties of homogeneous Riemannian spaces and geodesic orbit Riemannian spaces in particular.

Let $M = G/H$ be a homogeneous space of a compact connected Lie group G. Let us denote by $\langle \cdot, \cdot \rangle$ a fixed $\mathrm{Ad}(G)$-invariant Euclidean metric on the Lie algebra $\mathfrak{g}$ of G (for example, the minus Killing form if G is semisimple) and by

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} \tag{1}$$

the associated $\langle \cdot, \cdot \rangle$-orthogonal reductive decomposition, where $\mathfrak{h}$ is the Lie algebra of the group H. An invariant Riemannian metric g on M is determined by an $\mathrm{Ad}(H)$-invariant inner product $g_o = (\cdot, \cdot)$ on the space $\mathfrak{p}$ which is identified with the tangent space M_o at the initial point $o = eH$.

Recall that $X \in \mathfrak{g}$ is called *a geodesic vector* if the orbit of the point $o = eH$ under the action of the one-parameter group $\gamma(t) = \exp(tX)$, $t \in \mathbb{R}$, is a geodesic in $(M = G/H, g)$, see details in the paper [17] or in Section 5 of the book [8].

For a given inner product $(\cdot, \cdot)$, we consider *a metric endomorphism* $A : \mathfrak{p} \to \mathfrak{p}$ that is defined by the equality $(X, Y) = \langle AX, Y \rangle$ for all $X, Y \in \mathfrak{p}$. Obviously, A is $\mathrm{Ad}(H)$-equivariant, positive definite and symmetric operator (with respect to $\langle \cdot, \cdot \rangle$). It is clear also that a metric endomorphism determines a corresponding invariant Riemannian metric uniquely. The following lemma is very useful.

Lemma 1 [1]**.** *A compact homogeneous Riemannian manifold* $(M = G/H, g)$ *with reductive decomposition* (1) *and metric endomorphism A is GO-space if and only if for any $X \in \mathfrak{p}$ there is $Z \in \mathfrak{h}$ such that $[Z + X, AX] \in \mathfrak{h}$. The latter condition is equivalent to the property of $Z + X \in \mathfrak{g}$ to be a geodesic vector.*

2. On invariant metrics and transitive actions of groups $Sp(n+1)$, $Sp(n+1)U(1)$, and $Sp(n+1)Sp(1)$ on S^{4n+3}

Let $\mathbb{H}$ be the field of quaternions. Denote by $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the quaternionic units in $\mathbb{H}$ ($\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$, $\mathbf{jk} = -\mathbf{kj} = \mathbf{i}$, $\mathbf{ki} = -\mathbf{ik} = \mathbf{j}$, $\mathbf{ii} = \mathbf{jj} = \mathbf{kk} = -1$). For $X = x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4$, $x_i \in \mathbb{R}$, define $\mathrm{Re}(X) = x_1$ (the real part of X), $\overline{X} = x_1 - \mathbf{i}x_2 - \mathbf{j}x_3 - \mathbf{k}x_4$ and $\|X\| = \sqrt{X\overline{X}}$. If $\mathrm{Re}(X) = x_1 = 0$, then the quaternion X is called pure imaginary.

Let us consider a (left) vector space $\mathbb{H}^{n+1}$ over $\mathbb{H}$. For $X = (X_1, X_2, \dots, X_{n+1}) \in \mathbb{H}^{n+1}$ and $Y = (Y_1, Y_2, \dots, Y_{n+1}) \in \mathbb{H}^{n+1}$ we define $(X, Y)_1 = \sum_{s=1}^{n+1} X_s \overline{Y}_s$. Then $Sp(n+1)$ is the group of all $\mathbb{H}$-linear operators $A : \mathbb{H}^{n+1} \to \mathbb{H}^{n+1}$ with the property $(A(X), A(Y))_1 = (X, Y)_1$ for every $X, Y \in \mathbb{H}^{n+1}$. If we choose some $(\cdot, \cdot)_1$-orthonormal quaternionic basis in $\mathbb{H}^{n+1}$, then we can identify $Sp(n+1)$ with a group of matrices $A = (a_{ij})$, $a_{ij} \in \mathbb{H}$, with the property $A^{-1} = A^*$, where $a^*_{ij} = \overline{a}_{ji}$ for $1 \leqslant i, j \leqslant n+1$. In this case $\mathfrak{sp}(n+1)$ consists of quaternionic $\big((n+1) \times (n+1)\big)$-matrices A with the property $A^* = -A$. Later on we shall use these identifications.

We have a natural embedding $\mathbb{H} \mapsto \mathbb{R}^4$ via $x_1 + \mathbf{i}x_2 + \mathbf{j}x_3 + \mathbf{k}x_4 \mapsto (x_1, x_2, x_3, x_4)$ and the induced embedding $\mathbb{H}^{n+1} \mapsto \mathbb{R}^{4n+4}$. It is well known that the group $G := Sp(n+1)$ acts transitively on the sphere

$$S^{4n+3} = \Big\{ \big(X_1, X_2, \dots, X_{n+1}\big) \in \mathbb{H}^{n+1} : \|X_1\|^2 + \|X_2\|^2 + \dots + \|X_{n+1}\|^2 = 1 \Big\}.$$

Let us consider natural embedding $\mathrm{diag}(Sp(1), Sp(n)) \subset Sp(n+1)$, and let K and H be the images of $Sp(1)$ and $Sp(n)$ respectively under this embedding. Then H is the isotropy subgroup of a point $(1, 0, \dots, 0) \in \mathbb{H}^{n+1}$ under the above action of $Sp(n+1)$. Since $K = Sp(1)$

is a normal subgroup in the group $\mathrm{diag}(Sp(1), Sp(n))$, then we have the following (almost effective) transitive action of $G \times K$ on $S^{4n+3} = G/H$:

$$(a, b)(cH) = acHb^{-1} = acb^{-1}H, \quad a, c \in G,\ b \in K. \tag{2}$$

The isotropy group of this action at the point $(1, 0, \dots, 0) \in \mathbb{H}^n$ is

$$Sp(n-1) \times Sp(1) = Sp(n-1) \times \mathrm{diag}(Sp(1)) \subset Sp(n) \times Sp(1) = G \times K.$$

We also get an effective representation $S^{4n+3} = Sp(n+1)Sp(1)/Sp(n)\,\mathrm{diag}(Sp(1))$ (after dividing by the noneffectiveness kernel).

Let L be any subgroup $U(1) = S^1$ in $K = Sp(1)$. Then we get transitive (and almost effective) action of $G \times L$ on $S^{4n+3} = G/H$:

$$(a, b)(cH) = acHb^{-1} = acb^{-1}H, \quad a, c \in G,\ b \in L, \tag{3}$$

that is a part of the action (2). In this case we get the following isotropy group at the point $(1, 0, \dots, 0) \in \mathbb{H}^{n+1}$:

$$Sp(n) \times U(1) = Sp(n) \times \mathrm{diag}(U(1)) \subset U(1) \times Sp(n) \times U(1) \subset Sp(n+1) \times U(1) = G \times L.$$

We also get an effective representation $S^{4n+3} = Sp(n+1)U(1)/Sp(n)\,\mathrm{diag}(U(1))$.

For $A, B \in \mathfrak{sp}(n+1)$ we define

$$\langle A, B \rangle = \frac{1}{2}\big(\mathrm{Re}(AB^*)\big). \tag{4}$$

It is easy to see that $\langle \cdot, \cdot \rangle$ is a $\mathrm{Ad}(Sp(n+1))$-invariant inner product on the Lie algebra $\mathfrak{g} = \mathfrak{sp}(n+1)$.

We write E_{ij} for the skew-symmetric matrix with 1 in the ij-th entry and -1 in the ji-th entry, and zeros elsewhere. We denote by F_{ij} the symmetric matrix with 1 in both the ij-th and ji-th entries, and zeros elsewhere. Denote also by G_i the matrix with $\sqrt{2}$ in ii-th entry, and zeros elsewhere.

It is easy to check that the matrices $\mathbf{i}G_i$, $\mathbf{j}G_i$, $\mathbf{k}G_i$, E_{ij}, $\mathbf{i}F_{ij}$, $\mathbf{j}F_{ij}$, $\mathbf{k}F_{ij}$, where $1 \leqslant i, j \leqslant n+1$ and $i < j$, constitute a $\langle \cdot, \cdot \rangle$-orthonormal (see (4)) basis in $\mathfrak{sp}(n+1)$.

Let us consider the following $\langle \cdot, \cdot \rangle$-orthogonal decomposition:

$$\mathfrak{sp}(n+1) = \mathfrak{k} \oplus \mathfrak{sp}(n) \oplus \mathfrak{p}_1 = \mathfrak{sp}(n) \oplus \mathfrak{p}, \quad \mathfrak{k} = \mathfrak{l} \oplus \mathfrak{p}_2, \tag{5}$$

where $\mathfrak{k}$ and $\mathfrak{l}$ are the Lie algebras of the Lie subgroups K and L (see above). Therefore, the embedding of $\mathfrak{k} \oplus \mathfrak{sp}(n) = \mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ in $\mathfrak{sp}(n+1)$ is defined by $(A, B) \mapsto \mathrm{diag}(A, B)$, where $A \in \mathfrak{sp}(1)$ and $B \in \mathfrak{sp}(n)$.

Without loss of generality we may suppose that the Lie subalgebra $\mathfrak{l} = \mathfrak{u}(1)$ ($\mathfrak{u}(1) \oplus \mathfrak{sp}(n) \subset \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \subset \mathfrak{sp}(n+1)$) is spanned on the vector $\mathbf{i}G_1$. Then $\mathfrak{p}_2 = \mathrm{Lin}\{\mathbf{j}G_1, \mathbf{k}G_1\}$.

Any $Sp(n+1)$-invariant metric on S^{4n+3} is defined by an $\mathrm{Ad}(Sp(n))$-invariant inner product $(\cdot, \cdot)$ on $\mathfrak{p}$. Note that $\mathrm{Ad}(Sp(n))$ acts irreducibly on $\mathfrak{p}_1$ and trivially on $\mathfrak{k}$. Therefore, any such inner product is generated by the metric endomorphism of the type

$$A = \widetilde{A} \oplus x_1 \,\mathrm{Id}|_{\mathfrak{p}_1} \tag{6}$$

for some $x_1 > 0$ and some symmetric and positive definite operator $\widetilde{A} : \mathfrak{k} \to \mathfrak{k}$. In particular, it implies that the space of $Sp(n+1)$-invariant Riemannian metric on S^{4n+3} is 7-dimensional.

If $\widetilde{A} = x_2 \operatorname{Id}|_{\mathfrak{p}_2} \oplus x_3 \operatorname{Id}|_{\mathfrak{l}}$, then the inner product $(\cdot,\cdot)$ generates $Sp(n+1) \times L$-invariant metric on S^{4n+3} (see Section 4). If $\widetilde{A} = x_2 \operatorname{Id}|_{\mathfrak{k}}$, then the inner product $(\cdot,\cdot)$ generates $Sp(n+1) \times K$-invariant metric on S^{4n+3}, see e. g. [27].

Remark 1. Let us note that for $x_1 = x_2 \neq x_3$ we get $Sp(n+1)U(1)$-naturally reductive metrics on the sphere S^{4n+3} (they are even $Sp(n+1)U(1)$-normal homogeneous for $x_3 < x_1 = x_2$). All these metrics are also $U(2n+2)$-invariant and $U(2n+2)$-naturally reductive. By analogy, for $x_3 = x_2 \neq x_1$ we get $Sp(n+1)Sp(1)$-naturally reductive metrics on the sphere S^{4n+3} (for $x_3 = x_2 < x_1$ these metrics are even $Sp(n+1)Sp(1)$-normal homogeneous). Obviously, for $x_1 = x_2 = x_3$ we get $Sp(n+1)$-normal homogeneous metrics. For all other values of parameters x_i, $i = 1,2,3$, $Sp(n+1)U(1)$-invariant metrics are not naturally reductive. See details in [27] and [28].

Remark 2. Consider the homogeneous spaces $Sp(n+1)/Sp(n)\cdot U(1)$, where $n \geqslant 1$, $U(1) \subset Sp(1)$, and $Sp(1)$ is the first factor in the group $Sp(1) \times Sp(n) \subset Sp(n+1)$. The Lie algebra of the group $Sp(n)\cdot U(1)$ is $\mathfrak{l} \oplus \mathfrak{sp}(n) \subset \mathfrak{sp}(n+1)$ in the decomposition (5). It is known that the homogeneous space $Sp(n+1)/Sp(n)\cdot U(1)$ is diffeomorphic to $\mathbb{C}P^{2n+1}$, hence we get a representation of an odd-dimensional complex projective space. It is also known that the space $Sp(n+1)/Sp(n)\cdot U(1)$ admits a two-parameter family of $Sp(n+1)$-invariant Riemannian metrics [27]. All these metrics are weakly symmetric [28]. It is interesting that only $Sp(n+1)$-normal and $SU(2n+2)$-normal metrics in this family are naturally reductive. Note that explicit expressions of geodesic vectors for $Sp(n+1)$-invariant Riemannian metrics on $\mathbb{C}P^{2n+1} = Sp(n+1)/Sp(n)\cdot U(1)$ could be found in paper [4].

3. $Sp(n+1)$-invariant geodesic orbit metrics on the sphere S^{4n+3}

Here we classify all geodesic orbit metrics on the sphere S^{4n+3} with respect to the group $Sp(n+1)$. At first, we should establish some auxiliary results. By direct calculation we get the following lemma (see the definitions of E_{ij}, F_{ij}, and G_i in the previous section).

Lemma 2. *For any $1 \leqslant i < j \leqslant n+1$ the following relations are fulfilled:*

$$[\alpha G_k, E_{ij}] = [\alpha G_k, \beta F_{ij}] = 0 \quad (\forall\, k \notin \{i,j\},\ \forall\, \alpha, \beta \in \{\mathbf{i},\mathbf{j},\mathbf{k}\});$$

$$[\alpha G_i, E_{ij}] = \sqrt{2}\alpha F_{ij}, \quad [\alpha G_j, E_{ij}] = -\sqrt{2}\alpha F_{ij} \quad (\forall\, \alpha \in \{\mathbf{i},\mathbf{j},\mathbf{k}\});$$

$$[\alpha G_i, \alpha F_{ij}] = -\sqrt{2}E_{ij}, \quad [\alpha G_j, \alpha F_{ij}] = \sqrt{2}E_{ij} \quad (\forall\, \alpha \in \{\mathbf{i},\mathbf{j},\mathbf{k}\});$$

$$[\alpha G_i, \beta F_{ij}] = [\alpha G_j, \beta F_{ij}] = \sqrt{2}(\alpha\cdot\beta)F_{ij} \quad (\forall\, \alpha, \beta \in \{\mathbf{i},\mathbf{j},\mathbf{k}\},\ \beta \neq \alpha),$$

where $\cdot$ means the quaternion multiplication.

The following lemma is very important for our goals.

Lemma 3. *For any $2 \leqslant s \leqslant n+1$ and given vectors $U = a_1\mathbf{i}G_1 + a_2\mathbf{j}G_1 + a_3\mathbf{k}G_1$ and $V = b_1E_{1s} + b_2\mathbf{i}F_{1s} + b_3\mathbf{j}F_{1s} + b_4\mathbf{k}F_{1s}$, there is a vector $W = c_1\mathbf{i}G_s + c_2\mathbf{j}G_s + c_3\mathbf{k}G_s$ such that $[U,V] = [W,V]$.*

$\lhd$ All is clear when V is trivial. Now we suppose that $V \neq 0$. Using Lemma 2, we see that the equality $[U,V] = [W,V]$ is equivalent to $u\cdot v = v\cdot w$ for the quaternions $u = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, $v = b_1 + b_2\mathbf{i} + b_3\mathbf{j} + b_4\mathbf{k}$, and $w = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$. But now we can define w by the formula $w = v^{-1}\cdot u\cdot v$. This definition is correct, since for a given nonzero quaternion q, a map of the type $a \mapsto q\cdot a\cdot q^{-1}$ (that is an automorphism of the field of quaternions) preserves a subspace of pure imaginary quaternions. Indeed, if $a \neq 0$

is a pure imaginary, then $a^{-1} = \|a\|^{-2}\overline{a} = -\|a\|^{-2}a$. Therefore, if $b = q \cdot a \cdot q^{-1}$, then $\|b\|^{-2}\overline{b} = b^{-1} = q \cdot a^{-1} \cdot q^{-1} = -\|a\|^{-2}b$, and b is also pure imaginary. $\rhd$

Theorem 1. *A homogeneous Riemannian space* $(S^{4n+3} = Sp(n+1)/Sp(n), g)$ *is geodesic orbit with respect to* $Sp(n+1)$, *if and only if the metric* g *is invariant under* $Sp(n+1) \times Sp(1)$.

$\lhd$ Suppose that a metric endomorphism A (see (6)) generates a Riemannian geodesic orbit metric g with respect to $Sp(n+1)$. Then by (6) and Lemma 1 we get that for any $X \in \mathfrak{k} = \mathfrak{sp}(1)$ there is $Z \in \mathfrak{sp}(n)$ such that $[Z + X, \widetilde{A}X] \in \mathfrak{sp}(n)$. But $[Z, \widetilde{A}X] \in [\mathfrak{sp}(n), \mathfrak{k}] = 0$, therefore, $[X, \widetilde{A}X] \in \mathfrak{sp}(n)$. On the other hand, $[X, \widetilde{A}X] \in [\mathfrak{k}, \mathfrak{k}] = \mathfrak{k}$. Hence, we get $[X, \widetilde{A}X] = 0$ for all $X \in \mathfrak{k} = \mathfrak{sp}(1)$. Now, it is easy to see that $\widetilde{A} = x_2 \operatorname{Id}|_{\mathfrak{k}}$ for some $x_2 > 0$ and $(\cdot,\cdot)$ generates $Sp(n+1) \times Sp(1)$-invariant metric on S^{4n+3} (see the discussion in the previous section). Indeed, the centralizer of any nontrivial $X \in \mathfrak{sp}(1)$ in $\mathfrak{sp}(1)$ is exactly $\mathbb{R}X$. Therefore, $\widetilde{A}$ preserves all 1-dimensional subspaces in $\mathfrak{sp}(1)$ and should be a multiple of Id on $\mathfrak{k} = \mathfrak{sp}(1)$.

Now, consider any $Sp(n+1) \times Sp(1)$-invariant metric on S^{4n+3}. It is generated by a metric endomorphism of the type $A = x_2 \operatorname{Id}|_{\mathfrak{k}} \oplus x_1 \operatorname{Id}|_{\mathfrak{p}_1}$. For any $X = X_1 + X_2$ ($X_1 \in \mathfrak{p}_1$, $X_2 \in \mathfrak{k}$) we get $AX = x_1X_1 + x_2X_2$. In order to prove that this metric is geodesic orbit with respect to $Sp(n+1)$, it suffices to find $Z \in \mathfrak{sp}(n)$ such that $[Z, X_1] = (x_2/x_1 - 1)[X_2, X_1]$ (see Lemma 1). If $x_1 = x_2$ then we can choose $Z = 0$. Consider now the case $x_1 \neq x_2$.

Let $X_1 = X_1^2 + X_1^3 + \cdots + X_1^{n+1}$, where $X_1^s \in \operatorname{Lin}\{E_{1s}, \mathbf{i}F_{1s}, \mathbf{j}F_{1s}, \mathbf{k}F_{1s}\}$, $2 \leqslant s \leqslant n+1$. By Lemma 3, there is a vector $U_s \in \operatorname{Lin}\{\mathbf{i}G_s, \mathbf{j}G_s, \mathbf{k}G_s\} \subset \mathfrak{sp}(n)$ such that $[U_s, X_1^s] = (x_2/x_1 - 1)[X_2, X_1^s]$. Now by Lemma 2, we get

$$(x_2/x_1 - 1)[X_2, X_1] = \sum_{s=2}^{n+1}(x_2/x_1 - 1)[X_2, X_1^s]$$
$$= \sum_{s=2}^{n+1}[U_s, X_1^s] = \sum_{s=2}^{n+1}[U_s, X_1] = \left[\sum_{s=2}^{n+1} U_s, X_1\right].$$

Therefore, we can choose $Z = \sum_{s=2}^{n+1} U_s$. By Lemma 1 we get that $A = x_2 \operatorname{Id}|_{\mathfrak{k}} \oplus x_1 \operatorname{Id}|_{\mathfrak{p}_1}$ does generate a GO-metric with respect to $Sp(n+1)$. $\rhd$

4. Geodesic vectors for $Sp(n+1)U(1)$-invariant metrics on the sphere S^{4n+3}

We already know (see Case 9) in Introduction) that all $Sp(n+1)U(1)$-invariant metrics (they constitute a 3-parametric family) on $Sp(n+1)U(1)/Sp(n)\operatorname{diag}(U(1)) = S^{4n+3}$ are geodesic orbit because of the weak symmetry. On the other hand, sometimes it is useful to have explicit forms of suitable geodesic vectors. In this section, we get such description for all $Sp(n+1)U(1)$-invariant metrics on S^{4n+3}.

At first, we give more details on $Sp(n+1)U(1)$-invariant metrics on the sphere S^{4n+3}. We suppose that $\mathfrak{l}$ is supplied with the inner product $\langle\cdot,\cdot\rangle$ as a subalgebra of $\mathfrak{sp}(n+1)$ (see the decomposition (5)). Then we can extend $\langle\cdot,\cdot\rangle$ to the Lie algebra $\mathfrak{sp}(n+1) \oplus \mathfrak{l}$ assuming $\langle\mathfrak{sp}(n+1), \mathfrak{l}\rangle = 0$. Let us consider the following $\langle\cdot,\cdot\rangle$-orthogonal decompositions:

$$\mathfrak{sp}(n+1) \oplus \mathfrak{l} = \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{p}}_1 \oplus \widetilde{\mathfrak{p}}_2 \oplus \widetilde{\mathfrak{p}}_3, \quad \widetilde{\mathfrak{h}} = \widetilde{\mathfrak{h}}_1 \oplus \widetilde{\mathfrak{h}}_2,$$

where $\widetilde{\mathfrak{p}}_1 = \{(X,0) \in \mathfrak{sp}(n+1) \oplus \mathfrak{l} \,|\, X \in \mathfrak{p}_1\}$, $\widetilde{\mathfrak{p}}_2 = \{(X,0) \in \mathfrak{sp}(n+1) \oplus \mathfrak{l} \,|\, X \in \mathfrak{p}_2\}$, $\widetilde{\mathfrak{p}}_3 = \{(X,-X) \in \mathfrak{sp}(n+1) \oplus \mathfrak{l} \,|\, X \in \mathfrak{l}\}$, $\widetilde{\mathfrak{h}}_1 = \{(X,0) \in \mathfrak{sp}(n+1) \oplus \mathfrak{l} \,|\, X \in \mathfrak{sp}(n)\}$, $\widetilde{\mathfrak{h}}_2 =$

$\{(X,X) \in \mathfrak{sp}(n+1)\oplus\mathfrak{l} \,|\, X\in\mathfrak{l}\}$. It is easy to see that the modules $\widetilde{\mathfrak{p}}_i$, $i=1,2,3$, are $\operatorname{ad}(\widetilde{\mathfrak{h}})$-irreducible. Then every $Sp(n+1)U(1)$-invariant metric g on S^{4n+3} is determined by the metric endomorphism

$$A = x_1 \operatorname{Id}|_{\widetilde{\mathfrak{p}}_1} \oplus x_2 \operatorname{Id}|_{\widetilde{\mathfrak{p}}_2} \oplus x_3 \operatorname{Id}|_{\widetilde{\mathfrak{p}}_3}$$

for some positive x_1, x_2, x_3.

Now, we shall find for every $X\in\widetilde{\mathfrak{p}}_1\oplus\widetilde{\mathfrak{p}}_2\oplus\widetilde{\mathfrak{p}}_3$ a vector $Z\in\widetilde{h}$ such that $X+Z$ is a geodesic vector on the homogeneous Riemannian space $(Sp(n+1)U(1)/Sp(n)\operatorname{diag}(U(1)) = S^{4n+3}, g)$.

Let us consider any $X = X_1+X_2+X_3$, where $X_1\in\widetilde{\mathfrak{p}}_1$, $X_2\in\widetilde{\mathfrak{p}}_2$, $X_3\in\widetilde{\mathfrak{p}}_3$. Then $AX = x_1X_1+x_2X_2+x_3X_3$ and

$$[AX,X] = (x_1-x_2)[X_1,X_2]+(x_1-x_3)[X_1,X_3]+(x_2-x_3)[X_2,X_3].$$

Obviously, $[X_1,X_2]\in\widetilde{\mathfrak{p}}_1$, $[X_1,X_3]\in\widetilde{\mathfrak{p}}_1$, $[X_2,X_3]\in\widetilde{\mathfrak{p}}_2$.

By Lemma 1, it suffices to find a vector $Z\in\widetilde{\mathfrak{h}}$ such that $[Z,AX]=[AX,X]$. Consider $Z=Z_1+Z_2$, where $Z_1\in\widetilde{\mathfrak{h}}_1$ and $Z_2\in\widetilde{\mathfrak{h}}_2$. Since $[Z_1,X_1]\in\widetilde{\mathfrak{p}}_1$, $[Z_1,X_2]=[Z_1,X_3]=0$, $[Z_2,X_1]\in\widetilde{\mathfrak{p}}_1$, $[Z_2,X_2]\in\widetilde{\mathfrak{p}}_2$ and $[Z_2,X_3]=0$ we get that $[Z,AX]=[AX,X]$ is equivalent to the following two equations:

$$\begin{gathered} x_2[Z_2,X_2] = (x_2-x_3)[X_2,X_3], \\ x_1[Z_1,X_1]+x_1[Z_2,X_1] = (x_1-x_2)[X_1,X_2]+(x_1-x_3)[X_1,X_3]. \end{gathered} \tag{7}$$

It is clear that $X_1=(Y,0)\in\widetilde{\mathfrak{p}}_1$ for some $Y\in\mathfrak{p}_1$, $Z_1=(U,0)\in\widetilde{\mathfrak{h}}_1$ for some $U\in\mathfrak{sp}(n)$,

$$X_3 = (\alpha\mathbf{i}G_1, -\alpha\mathbf{i}G_1)\in\widetilde{\mathfrak{p}}_3,\quad X_2 = (\beta\mathbf{j}G_1+\gamma\mathbf{k}G_1, 0)\in\widetilde{\mathfrak{p}}_2,\quad Z_2 = (\delta\mathbf{i}G_1,\delta\mathbf{i}G_1)\in\widetilde{\mathfrak{h}}_2$$

for some real numbers α, β, γ, δ.

Since $[Z_2,X_2] = (-\gamma\delta\mathbf{j}G_1+\beta\delta\mathbf{k}G_1,0)\in\widetilde{\mathfrak{p}}_2$ and $[X_2,X_3] = (\alpha\gamma\mathbf{j}G_1-\alpha\beta\mathbf{k}G_1,0)\in\widetilde{\mathfrak{p}}_2$, then for $\delta=(x_3/x_2-1)\alpha$ the equality (7) holds.

Substituting $\delta=(x_3/x_2-1)\alpha$ into equality (7) and using the inclusions

$$\begin{gathered} [Z_2,X_1] = \big((x_3/x_2-1)\alpha[\mathbf{i}G_1,Y],0\big)\in\widetilde{\mathfrak{p}}_1, \\ [X_2,X_1] = \big([\beta\mathbf{j}G_1+\gamma\mathbf{k}G_1,Y],0\big)\in\widetilde{\mathfrak{p}}_1, \\ [X_3,X_1] = \big(\alpha[\mathbf{i}G_1,Y],0\big)\in\widetilde{\mathfrak{p}}_1, \end{gathered}$$

we see that the equality (7) is equivalent to the following one:

$$[U,Y] = \big[(x_3/x_1-x_3/x_2)\alpha\mathbf{i}G_1+(x_2/x_1-1)\beta\mathbf{j}G_1+(x_2/x_1-1)\gamma\mathbf{k}G_1, Y\big]. \tag{8}$$

Consider the sum $Y = Y_2+Y_3+\cdots+Y_{n+1}$, where $Y_s\in\operatorname{Lin}\{E_{1s},\mathbf{i}F_{1s},\mathbf{j}F_{1s},\mathbf{k}F_{1s}\}$, $2\leqslant s\leqslant n+1$. By Lemma 3 there is a vector $U_s\in\operatorname{Lin}\{\mathbf{i}G_s,\mathbf{j}G_s,\mathbf{k}G_s\}\subset\mathfrak{sp}(n)$ such that

$$[U_s,Y_s] = \big[(x_3/x_1-x_3/x_2)\alpha\mathbf{i}G_1+(x_2/x_1-1)\beta\mathbf{j}G_1+(x_2/x_1-1)\gamma\mathbf{k}G_1, Y_s\big].$$

Now, by Lemma 2, we get

$$\begin{aligned} &\big[(x_3/x_1-x_3/x_2)\alpha\mathbf{i}G_1+(x_2/x_1-1)\beta\mathbf{j}G_1+(x_2/x_1-1)\gamma\mathbf{k}G_1, Y\big] \\ &= \sum_{s=2}^{n}\big[(x_3/x_1-x_3/x_2)\alpha\mathbf{i}G_1+(x_2/x_1-1)\beta\mathbf{j}G_1+(x_2/x_1-1)\gamma\mathbf{k}G_1, Y_s\big] \\ &= \sum_{s=2}^{n}[U_s,Y_s] = \sum_{s=2}^{n}[U_s,Y] = \Big[\sum_{s=2}^{n}U_s, Y\Big]. \end{aligned}$$

Hence, if we choose $U = \sum_{s=2}^{n+1} U_s$, then the equality (8) holds. Therefore, the vector $X + Z_1 + Z_2$, where $Z_1 = \Big(U = \sum_{s=2}^{n+1} U_s, 0\Big)$ and $Z_2 = (x_3/x_2 - 1)\alpha\,(\mathbf{i}G_1, \mathbf{i}G_1)$, is a geodesic vector by Lemma 1. In particular, the metric endomorphism $A = x_1 \operatorname{Id}|_{\widetilde{\mathfrak{p}}_1} \oplus x_2 \operatorname{Id}|_{\widetilde{\mathfrak{p}}_2} \oplus x_3 \operatorname{Id}|_{\widetilde{\mathfrak{p}}_3}$ does generate a GO-metric with respect to $Sp(n+1) \times U(1)$ for every positive x_1, x_2, x_3.

Remark 3. In particular, this proves that every $Sp(n+1)U(1)$-invariant metric on the sphere S^{4n+3} is geodesic orbit with respect to the group $Sp(n+1)U(1)$.

The conclusion

It is clear that Theorem 1 completes the classification for Case 10) in Introduction. Therefore, we have verified completely all data from Table 1.

All geodesic orbit Riemannian metrics from Table 1 induce geodesic orbit Riemannian homogeneous metrics on corresponding real projective spaces $\mathbb{R}P^n$. The metrics obtained in such a way, metrics from Remark 2 together with the normal homogeneous metrics on the projective spaces $\mathbb{C}P^n = SU(n+1)/S(U(n) \times U(1))$, $\mathbb{H}P^n = Sp(n+1)/Sp(n) \times Sp(1)$, and $\mathbb{C}aP^2 = F_4/Spin(9)$ exhaust all geodesic orbit Riemannian metrics on projective spaces (see details in [27] and [28]).

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ГЕОДЕЗИЧЕСКИ ОРБИТАЛЬНЫЕ МЕТРИКИ НА СФЕРАХ

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В данной работе получена полная классификация геодезически орбитальных римановых метрик на сферах S^n. Также найдены явные выражения геодезических векторов для $Sp(n+1)U(1)$-инвариантных метрик на S^{4n+3}.

Ключевые слова: однородные пространства, однородные римановы многообразия, естественно редуктивные римановы многообразия, нормальные однородные римановы многообразия, геодезически орбитальные пространства, симметрические пространства, слабо симметрические пространства.