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INFINITESIMALS IN ORDERED VECTOR SPACES

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Dedicated to Professor Anatoly Kusraev
on the occasion of his 60th birthday

An infinitesimal approach to ordered spaces is proposed. Archimedean property and Dedekind completeness in ordered spaces are discussed from a nonstandard point of view.

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1. Introduction

Nonstandard analysis has contributed to various branches of mathematics (see, for example, [1, 5, 6]). The infinitesimal methods are rather useful in problems related with compactness and ultrafilters. In the present paper we apply nonstandard methods to ordered vector spaces. We consider some questions on the order/uniform convergence, the Archimedean property, and the order completeness in ordered spaces. Our terminology and notations follow to [2, 7, 9, 10], and [8] in what is related to ordered vector spaces and to [1, 5], and [4] in what is related to nonstandard analysis.

Here we collect some basic preliminary facts that will be used throughout the paper. We consider a superstructure $V(S) = \bigcup_n V_n(S)$ over a set S which includes several algebraic systems such as real numbers, necessarily vector spaces, etc. (cf. [4, p. 164–165]). Dealing with some superstructure in the sequel, we do not specify the basic set over which it is constructed. This set is usually chosen to be sufficiently substantive for questions under consideration. We denote the superstructure under consideration by M. We suppose that for our enlargement ${}^*S$ of S, the natural embedding $* : V(S) \hookrightarrow V({}^*S)$ (or, simply, $* : M \hookrightarrow {}^*M$) satisfies the following principles (cf. [4, p. 165–166]).

Extension Principle. The set S is a proper subset of ${}^*S$ in the sense that ${}^*x = x$ for every $x \in S$. Moreover, ${}^*S$ is equipped with the same family of operations and relations as S is.

Transfer Principle. Let $\psi(x_1, x_2, \ldots, x_n)$ be a bounded formula of the superstructure M (i. e., a formula of the language L_M [4, p. 165]), and let $A_1, A_2, \ldots, A_n$ be elements of the superstructure M.

Then the assertion $\psi(A_1, A_2, \ldots, A_n)$ about elements of M is true if and only if the assertion $\psi({}^*A_1, {}^*A_2, \ldots, {}^*A_n)$ about elements of ${}^*M$ is true.

Let ${}^*M$ be a nonstandard enlargement of a superstructure M. An element $x \in {}^*M$ is called: *standard* if $x = {}^*X$ for some $X \in M$; *internal* if $x \in {}^*X$ for some $X \in M$; *external* if x is not internal. Note that every standard element is internal, and every element of an internal set is internal too (cf. [1, 1.2.5]). The following technical principle is an easy consequence of the transfer principle:

Internal Definition Principle. Let $\psi(x, x_1, \ldots, x_n)$ be a formula of L_M, and let $A, A_1, \ldots, A_n$ be internal sets. Then the set $\{x \in A : \psi(x, A_1, \ldots, A_n)\}$ is internal too.

It is well known (cf. [1, 2.1.4]) that a nonstandard enlargement ${}^*M$ of the superstructure M can be chosen so that the following principle (our last one) holds:

General Saturation Principle. For every family $\{X_\gamma\}_{\gamma\in\Gamma}$ of internal sets which has standard cardinality (i. e., $\mathrm{card}(\Gamma) < \mathrm{card}(M)$) and has the finite intersection property, the condition $\bigcap_{\gamma\in\Gamma} X_\gamma \neq \varnothing$ is valid.

In the sequel, we deal only with nonstandard enlargements satisfying the general saturation principle. These nonstandard enlargements are called *poly-saturated*. The following assertion (see, for example, [4, Lemma 4.0.5]) yields in poly-saturated enlargements:

Proposition 1. *For every directed set* $(\Theta, \prec) \in M$, *there is an* (infinitely) remote element $a \in {}^a\Theta := \{\xi \in {}^*\Theta : (\forall \tau \in \Theta)\, \tau \prec \xi\}$.

An equivalent form of this proposition is that in a poly-saturated enlargement of M, any set X of $\mathrm{card}(X) < \mathrm{card}(M)$ is a subset of a *hyperfinite* set Y (i. e. Y is in one-to-one internal correspondence with some $\nu \in {}^*\mathbb{N}$).

In the paper we deal with real vector spaces only. A nonempty subset K of a vector space V is said to be a *cone* if $K+K \subseteq K$, $\lambda K \subseteq K$ for all scalars $\lambda \geqslant 0$, and $K \cap (-K) = \{0\}$. A cone K is said to be *generating* if $K - K = V$. A cone K defines the order relation $\geqslant_K$ on V by $x \geqslant_K y$ if $x - y \in K$. The pair (V, K) is called *ordered vector space*, or simply *ordered space*. An ordered space (V, K) is denoted also by $(V, \geqslant_K)$, or by $(V, \geqslant)$, or just by V when there is no chance of ambiguity. We denote the cone of V by V_+. By $[x, y]$ for $x, y \in V$ we denote the *order interval* $\{z \in V : y \geqslant z \geqslant x\}$. An ordered space V is called *Archimedean* if for any $x \in V_+$, $\inf_{n\in\mathbb{N}}(n^{-1}x) = 0$. The Archimedean property is equivalent (see, for example, [10, Theorem I.3.1]) to the fact that $[(x, y \in V) \wedge ((\forall\, n \in \mathbb{N})(y \geqslant nx))] \Rightarrow [0 \geqslant x]$. An ordered space V is called *almost Archimedean* [2, p. 12] if it follows from $y \in \frac{1}{n}[-x, x]$ for all $n \in \mathbb{N}$ and some $x, y \in V$ that $y = 0$. It can be shown that any Archimedean ordered space is almost Archimedean, and that any almost Archimedean vector lattice is Archimedean, but in general the Archimedean property is stronger then almost Archimedean one (see, for example, [9, p. 254]). The following very useful criterion of the Archimedean property is well known for vector lattices (see, for example, [8, Theorem 22.5]). Since we did not find an appropriate reference for this criterion in the setting of ordered spaces, we include it with a proof.

Proposition 2. *Let V be an ordered space. The following conditions are equivalent*:

(a) *V is Archimedean*;

(b) *Given any decreasing net $x_\tau \downarrow\geqslant d$ in V, and writing $L = \{y \in V : (\forall\, \tau \in \{\tau\})\, [x_\tau \geqslant y]\}$, then* $\inf_{\substack{\tau\in\{\tau\},\\ y\in L}}(x_\tau - y) = 0$.

$\lhd$ (a) $\Rightarrow$ (b): Let $x_\tau \downarrow\geqslant d$ and $L = \{y \in V : (\forall \tau \in \{\tau\})\, [x_\tau \geqslant y]\}$. Assume that $x_\tau - y \geqslant z$ for all τ and $y \in L$. Since $x_\tau - y \geqslant 0$, the proof will be complete if we show that $0 \geqslant z$. As $x_\tau \geqslant y + z$ holds for all τ and all $y \in L$, we get $y + z \in L$ for every $y \in L$. It follows by induction that $y + nz \in L$ for every $y \in L$ and $n \in \mathbb{N}$, in particular, $x_{\tau_0} \geqslant d + nz$ (and hence, $x_{\tau_0} - d \geqslant nz$) for some $\tau_0 \in \{\tau\}$ and all $n \in \mathbb{N}$. Since V is Archimedean, the last condition implies that $0 \geqslant z$, what is required.

$(b) \Rightarrow (a)$: Given arbitrary $x \geqslant 0$, we have to show that $\inf_{n\in\mathbb{N}}(n^{-1}x) = 0$. Let y be such that $n^{-1}x \geqslant y \geqslant 0$ for all $n \in \mathbb{N}$. Then $x \geqslant ny$ for all $n \in \mathbb{N}$, and hence, $-ny \downarrow\geqslant -x$. Denote $L = \{z \in V : (\forall n \in \mathbb{N})\,[-ny - z \geqslant 0]\}$. Let $u \in L$, $n \in \mathbb{N}$, then $-ny - u = (-(n+1)y - u) + y \geqslant y$. Since $u \in L$, $n \in \mathbb{N}$ are arbitrary, and $\inf\limits_{n\in\mathbb{N},u\in L}(-ny - u) = 0$ by the hypothesis, we get $0 = \inf\limits_{n\in\mathbb{N},u\in L}(-ny - u) \geqslant y \geqslant 0$. Thus $y = 0$, and hence $\inf\limits_{n\in\mathbb{N}}(n^{-1}x) = 0$. $\rhd$

An ordered space for which every decreasing bounded from below net has an infimum is called *Dedekind complete* [2, p. 10] (shortly, $x_\tau \downarrow\geqslant y \Rightarrow x_\tau \downarrow z$). Any Dedekind complete ordered space is Archimedean (for example, by Proposition 2). For further information on ordered spaces we refer to [2, 8, 10].

2. Some external vector spaces associated with an ordered space

Let $V = (V, V_+)$ be an ordered space. Given $\kappa \in {}^*V$, we denote by $U(\kappa)$ the set $\{x \in V : x \geqslant \kappa\}$ of standard upper bounds of κ, and by $L(\kappa)$ the set $-U(-\kappa) = \{x \in V : \kappa \geqslant x\}$ of standard lower bounds of κ. Consider the following external sets (cf. [4, p. 184], [3] for the vector lattice setting):

$$\mathrm{fin}({}^*V) := \big\{\kappa \in {}^*V : \ (\exists\, x, y \in V)\ \kappa \in [x, y]\big\},$$

$$\text{o-pns}({}^*V) := \big\{\kappa \in {}^*V : \inf_V (U(\kappa) - L(\kappa)) = 0\big\},$$

$$\eta({}^*V) := \big\{\kappa \in {}^*V : \inf_V U(\kappa) = \sup_V L(\kappa) = 0\big\},$$

$$\lambda({}^*V) := \big\{\kappa \in {}^*V : \ (\exists\, y \in V)(\forall\, n \in \mathbb{N})\ n\kappa \in [-y, y]\big\}.$$

Here, by $\inf_V / \sup_V$ we denote the infimum/supremum calculated in V. It is easy to see that above defined sets are external vector spaces *over* $\mathbb{R}$ satisfying $\eta({}^*V) \subseteq \text{o-pns}({}^*V) \subseteq \mathrm{fin}({}^*V)$, $V \subseteq \text{o-pns}({}^*V)$, and $\lambda({}^*V) \subseteq \mathrm{fin}({}^*V)$. In general, $V \cap \lambda({}^*V)$ may contain nonzero elements (take $\mathbb{R}^2$ with lexicographical ordering), and $V + \eta({}^*V)$ may be proper subspace of $\text{o-pns}({}^*V)$ (cf. [4, Theorem 4.4.2]). Moreover, the sets above are external ordered spaces with respect to the ordering inherited from ${}^*V$. Clearly a subset $A \subseteq V$ is order bounded iff ${}^*A \subseteq \mathrm{fin}({}^*V)$ iff $(\exists\, \nu \in {}^*\mathbb{N} \setminus \mathbb{N})\frac{1}{\nu}{}^*A \subset \lambda({}^*V)$ iff $(\exists\, \nu \in {}^*\mathbb{N})\frac{1}{\nu}{}^*A \subset \mathrm{fin}({}^*V)$. For the order convergence (the (o)-convergence, sf. [10, p. 14]) of monotone nets in V, there is the following simple nonstandard condition (cf. [4, 4.3.2], [3] in the case of vector lattice V). Since its proof is slightly different, we include it below.

Proposition 3. *Let $(x_\alpha)_{\alpha\in\Xi}$ be monotone net in an ordered space V. Then the following conditions are equivalent:*

(a) *$(x_\alpha)_\alpha$ order converges to* 0;

(b) *$x_\beta \in \eta({}^*V)$ for all remote elements $\beta \in {}^a\Xi$*;

(c) *$x_\beta \in \eta({}^*V)$ for some remote element $\beta \in {}^a\Xi$.*

$\lhd$ We consider only the case of decreasing net, $x_\alpha \downarrow$. For $(a) \Rightarrow (b)$, assume $x_\alpha \downarrow 0$. Then, $x_\alpha \geqslant x_\beta \geqslant 0$ for every $\beta \in {}^a\Xi$, $\alpha \in \Xi$. Hence, $\inf_V U(x_\beta) = 0$. But clearly, $\sup_V L(x_\beta) \geqslant 0$. Consequently, $\sup_V L(x_\beta) = 0$, and $x_\beta \in \eta({}^*V)$. The implication $(b) \Rightarrow (c)$ is obvious. For $(c) \Rightarrow (a)$, take some $\beta \in {}^a\Xi$, $x_\beta \in \eta({}^*V)$ (by Proposition 1). In particular, $\inf_V U(x_\beta) = 0$. Since $x_\alpha \geqslant x_\beta$ for every $\alpha \in \Xi$, one gets $x_\alpha \downarrow\geqslant 0$. Let $y \in V$ be such that $x_\alpha \geqslant y \geqslant 0$ for all $\alpha \in \Xi$. Then, by the transfer principle, $x_\alpha \geqslant y \geqslant 0$ for all $\alpha \in {}^*\Xi$, in particular, $x_\beta \geqslant y$. It is possible only if $y = 0$. So, $x_\alpha \downarrow 0$. $\rhd$

It can be seen easily that implications $(a) \Rightarrow (b) \Rightarrow (c)$ are true for an arbitrary net $(x_\alpha)_{\alpha\in\Xi}$. The implication $(c) \Rightarrow (a)$ may be false without monotonicity condition (see, for example, [4, p. 185]). For the uniform convergence (i. e. (r)-convergence [8, 10]) of monotone sequences, the following proposition (cf. [4, 4.3.3.]) is true. Its proof is quite similar, so we omit it (a remark similar to that one after Proposition 3 holds too).

Proposition 4. *Let $(x_n)_n$ be monotone sequence in an ordered space V. Then the following conditions are equivalent:*

(a) *$(x_n)_n$ (r)-converges to 0;*

(b) *$x_\nu \in \lambda({}^*V)$ for all $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$;*

(c) *$x_\nu \in \lambda({}^*V)$ for some $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$.*

We continue with the following auxiliary assertion (cf. [3, Lemma 4.3.4.]).

Lemma 1. *Let V be an ordered space, $u \in V$, and $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$. Then either $u = 0$ or $\nu u \notin \text{o-pns}({}^*V)$.*

$\lhd$ Let $u \neq 0$. Take $x \in U(\nu u)$ and $y \in L(\nu u)$. Then $x \geqslant \nu u \geqslant y$. By the transfer principle, there exists $n \in \mathbb{N}$ such that $x \geqslant nu \geqslant y$. By the convexity of order intervals, $x \geqslant mu \geqslant y$ for all $m \in {}^*\mathbb{N}$ satisfying $n \leqslant m \leqslant \nu$ (indeed, $mu = \alpha nu + (1-\alpha)\nu u$, for $0 \leqslant \alpha = \frac{\nu-m}{\nu-n} \leqslant 1$). In particular, $x \geqslant (n+1)u \geqslant y$. Thus, $nu - y \in [0, x-y] \subseteq [y-x, x-y]$ and $(n+1)u - y \in [0, x-y] \subseteq [y-x, x-y]$. Therefore $nu - y \in [y-x, x-y]$ & $y - (n+1)u \in [y-x, x-y]$. By the convexity of order intervals again, we get

$$-\frac{1}{2}u = -\frac{1}{2}\left((n+1) - n\right)u = \frac{1}{2}(y - (n+1)\,u) + \frac{1}{2}\,(nu - y) \in [y-x, x-y]. \tag{1}$$

It follows from (1), that

$$\pm\frac{1}{2}\,u \in [y-x, x-y]. \tag{2}$$

Suppose now that $\nu u \in \text{o-pns}({}^*V)$, that is $\inf_V(U(\nu u) - L(\nu u)) = 0$. Since $x \in U(\nu u)$ and $y \in L(\nu u)$ in (2) were chosen arbitrary, we obtain $0 \geqslant \pm\frac{1}{2}u$, that is $u = 0$, which is a contradiction to our initial assumption. Hence, $\nu u \notin \text{o-pns}({}^*V)$, what is required. $\rhd$

Clearly, it may be happened for some $u \neq 0$ in a non-Archimedean space V that $\nu u \in \text{fin}({}^*V)$ for all $\nu \in {}^*\mathbb{N}$. The following theorem (cf. [3, Theorem 3.4.5] in the vector lattice setting) characterizes the almost Archimedean property in terms of external ordered spaces associated with our space.

Theorem 1. *Let V be an ordered space. Then the following conditions are equivalent:*

(1) *V is almost Archimedean;*

(2) *$\lambda({}^*V) \cap V = \{0\}$;*

(3) *$\lambda({}^*V) \subseteq \eta({}^*V)$;*

(4) *$\lambda({}^*V) \subseteq \text{o-pns}({}^*V)$.*

$\lhd$ (1) $\Rightarrow$ (2): It follows directly from the definition of $\lambda({}^*V)$.

(2) $\Rightarrow$ (3): Suppose that $\kappa \in \lambda({}^*V) \setminus \eta({}^*V)$. Then $n\kappa \in [-y, y]$ holds for some $y \in V$ and all $n \in \mathbb{N}$. Hence, for every $n \in \mathbb{N}$, $-\frac{1}{n}y \in L(\kappa)$, and $\frac{1}{n}y \in U(\kappa)$. Since $\kappa \notin \eta({}^*V)$, there exists an element $0 \neq z \in V$ such that $U(\kappa) \geqslant z \geqslant L(\kappa)$. In particular, $z \in \lambda({}^*V)$, which contradicts (2).

(3) $\Rightarrow$ (4): It follows from $\eta({}^*V) \subseteq \text{o-pns}({}^*V)$.

(4) $\Rightarrow$ (1): Take $u, v \in V$ such that $u \geqslant nv \geqslant -u$ for all $n \in \mathbb{N}$. Let $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$. By the transfer principle, $u \geqslant m\nu v \geqslant -u$ for all $m \in \mathbb{N}$, and hence, $\nu v \in \lambda({}^*V)$. Then, by hypothesis, $\nu v \in \text{o-pns}({}^*V)$. Lemma 1 implies that $v = 0$. So, V is almost Archimedean. $\rhd$

Our second theorem gives a nonstandard condition for an Archimedean ordered space to be Dedekind complete. Remark that for Archimedean vector lattices this condition is also necessary [4, Theorem 4.4.2] for the Dedekind completeness.

Theorem 2. *Let V be an Archimedean ordered space satisfying* $\text{o-pns}(^*V) = V + \eta(^*V)$, *then V is Dedekind complete.*

$\lhd$ It suffices to show that every decreasing bounded from below net in V is order convergent. Let $(x_\alpha)_\alpha \downarrow\geqslant d \in V$. By Proposition 2,

$$\inf_{\substack{\alpha\in\{\alpha\},\\ z\in L}} (x_\alpha - z) = 0, \tag{3}$$

where $L = \{y \in V : (\forall\, \alpha \in \{\alpha\})\,[x_\alpha \geqslant y]\}$. Fix a remote element $\beta \in {}^a\{\alpha\}$. Since $\{u \in V : x_\alpha \downarrow\geqslant u\} = L(x_\beta)$ and $\{x_\alpha : \alpha \in \{\alpha\}\} \subseteq U(x_\beta)$, it follows from (3) that $\inf_V(U(x_\beta) - L(x_\beta)) = 0$, or, in other words, $x_\beta \in \text{o-pns}(^*V) = V + \eta(^*V)$. Let $x \in V$ be such that $x_\beta - x \in \eta(^*V)$. Thus, by Proposition 3, the decreasing net $(x_\alpha - x)_\alpha$ is order convergent to 0, that is $x = \inf_{\alpha\in\{\alpha\}} x_\alpha$. Hence, V is Dedekind complete. $\rhd$

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БЕСКОНЕЧНО МАЛЫЕ В УПОРЯДОЧЕННЫХ ВЕКТОРНЫХ ПРОСТРАНСТВАХ

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Предложен инфинитезимальный подход к упорядоченным пространствам. Архимедовость и порядковая полнота в упорядоченных пространствах исследованы с позиции нестандартного анализа.

Ключевые слова: упорядоченное векторное пространство, нестандартный анализ.