Differential Geometry and Its Applications
Proc. Conf. Opava (Czechoslovakia), August 24–28, 1992
Silesian University, Opava, 1993, 151–156

GEODESIC AND HOLOMORPHICALLY–PROJECTIVE MAPPINGS OF CONFORMALLY–KÄHLERIAN SPACES

Želko Radulovič and Josef Mikeš

Abstract. It is proved that conformally Kählerian spaces do not admit nontrivial geodesic and holomorphically-projective mappings, preserving the structure, onto almost Hermitian spaces. Examples of conformally Kählerian spaces admitting nontrivial geodesic mapping onto a conformally Kählerian spaces are found.

Key words: Geodesic mapping, holomorphically-projective mapping, conformally-Kählerian space, almost Hermitian space.
MS classification: 53C35.

Geodesic and holomorphically-projective mappings were studied by many authors [1],[14]. W.Y. Westlake [13], K. Yano and T. Nagano [15] have shown that between Kählerian space one cannot establish nontrivial geodesic mappings preserving the structure. These works were further developed by A.V. Karmasina and I.N. Kurbatova [4], and it has been shown that K-spaces do not admit nontrivial and structure-preserving geodesic mappings onto almost Hermitian spaces.

In works of J. Mikeš [7]–[10] results are given on geodesic mappings of Kählerian spaces not preserving the structure.

Conformally Kählerian spaces are a natural generalization of Kählerian spaces. The interest in investigation of these spaces recently grows due to the possibility of using them as a model of Kaluza–Klein theory [3]. Investigations of geometric properties of conformally-Kählerian space are carried out, for example in [2],[5],[6].

1. Introduction

For the beginning we shall give basic definitions. A Riemannian space H_n is called *almost Hermitian* if both metric tensor $g_{ij}(x)$ and almost Hermitian structure $F_i^h(x)$ are determined in it, and $F_i^h(x)$ satisfies the conditions

$$F_\alpha^h F_i^\alpha = -\delta_i^h; \quad F_{(i}^\alpha g_{j)\alpha} = 0, \tag{1}$$

δ_i^h being the Kronecker symbol, (i,j) denoting symmetrization without division.

An almost Hermitian space with covariantly constant structure is called *Kählerian space*. Riemannian space that can be conformally mapped onto Kählerian space is called *conformally-Kählerian space* K_n. Evidently, a conformally-Kählerian space K_n is almost

This paper is in final form and no version of it will be submitted for publication elsewhere.

Hermitian. Conformally-Kählerian spaces K_n are characterized by the existence of almost Hermitian structure (1), satisfying the following conditions [2]:

$$F^h_{i,j} = \delta^h_j \varphi_i - \varphi^h g_{ij} + F^h_j \bar{\varphi}_i + F_{ij} \bar{\varphi}^h \tag{2}$$

where comma denotes the covariant derivative with respect to the connection on K_n,

$$\varphi_i \equiv \frac{1}{n-2} F^\alpha_{i,\alpha}; \quad \varphi^h \equiv g^{h\alpha} \varphi_\alpha; \quad \bar{\varphi}_i \equiv \varphi_\alpha F^\alpha_i; \quad \bar{\varphi}^h \equiv \varphi^\alpha F^h_\alpha; \quad F_{ij} \equiv g_{i\alpha} F^\alpha_j.$$

From the integrability conditions of relations (2) the following formula, for the Riemann tensor follows:

$$R_{hi\alpha\beta} F^\alpha_j F^\beta_k - R_{hijk} = g_{h[k} \bar{\varphi}_{j]i} - g_{i[k} \bar{\varphi}_{j]h} + F_{h[k} \varphi_{j]i} - F_{i[k} \varphi_{j]h}, \tag{3}$$

where

$$\bar{\varphi}_{ij} \equiv \varphi_{\alpha i} F^\alpha_j; \quad \varphi_{ij} \equiv \varphi_{i,j} - \bar{\varphi}_i \bar{\varphi}_j + \alpha F_{ij},$$

α is an invariant, expressible in terms of g_{ij} and F^h_i, $[j,k]$ denoting alternation without division.

A diffeomorphism of a Riemannian space V_n into $\bar{V}_n$ which maps all geodesics in V_n into geodesics in $\bar{V}_n$ is called a *geodesic mapping* [13]. A geodesic mapping which is not affine is said to be *nontrivial.*

V_n admits a geodesic mappings onto $\bar{V}_n$ if and only if one of the following conditions is satisfied [12]:

$$\bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_i \bar{g}_{jk} + \psi_j \bar{g}_{ik}, \tag{4}$$

where ψ_i is a gradient vector, $\bar{g}_{ij}$ is a metric tensor of space $\bar{V}_n$. In [12] it is proved, that V_n admits geodesic mappings if V_n there exists a solution of the following equations

$$a_{ij,k} = \lambda_i g_{jk} + \lambda_j g_{ik} \tag{5}$$

on an unknown symmetric regular tensor a_{ij} and a vector λ_i.

When $\psi_i \not\equiv 0$ ($\iff$ $\lambda_i \not\equiv 0$) the geodesic mapping is nontrivial. The conditions of integrability of these equations have the form

$$a_{\alpha(i} R^\alpha_{j)kl} = \lambda_{l,(i} g_{j)k} - \lambda_{k,(i} g_{j)l}, \tag{6}$$

where $\lambda_{ij} \equiv \lambda_{i,j}$.

2. Geodesic mappings of conformally-Kählerian spaces onto almost Hermitian spaces

Let H_n and $\bar{H}_n$ are almost Hermitian spaces with structures F^h_i and $\bar{F}^h_i$, respectively, and let a geodesic mapping be established between them. One can choose, in the neighborhoods $U_p \subset H_n$, $U_{\bar{p}} \subset \bar{H}_n$ of each pair of corresponding points, a "common" coordinate system. If, with respect to such a coordinate system, the condition

$$F^h_i(x) \equiv \bar{F}^h_i(x), \tag{7}$$

is satisfied, then the mapping is said to be structure-preserving. As we have mentioned above, the geodesic mappings preserving the structure of almost Hermitian spaces has been investigated in [4],[7],[13],[15]. In the paper of A.V. Karmasina and I.N. Kurbatova [7] geodesic mappings of conformally-Kählerian space onto almost Hermitian spaces structure preserving were studied. There investigations were completed by the following theorem:

Theorem 1. *A conformally-Kählerian space K_n $(n > 2)$ does not admit nontrivial structure-preserving geodesic mappings onto an almost Hermitian space.*

Proof. Let us suppose the opposite. Let K_n (g_{ij}, F_i^h) admits a nontrivial geodesic structure-preserving mapping onto almost an Hermitian space $\bar{H}_n(\bar{g}_{ij}, \bar{F}_i^h)$.

The condition $\bar{g}_{\alpha(i}\bar{F}_{j)}^{\alpha} = 0$, that takes place in $\bar{H}_n$ can be covariantly differentiated in K_n. After taking into account (2), (4) and (7), as a result we have

$$\chi_i \bar{g}_{jk} + \chi_j \bar{g}_{ki} + \chi_k \bar{g}_{ij} + \Theta_i g_{jk} + \Theta_j g_{ki} + \Theta_k g_{ij} = 0, \tag{8}$$

where $\chi_i \equiv \varphi + \psi_\alpha F_i^\alpha$; $\Theta \equiv \bar{g}_{i\alpha}\varphi^\alpha$.

From $n > 2$ one can see, that (8) implies either $\bar{g}_{ij} = \alpha\, g_{ij}$ or $\chi_i = \Theta_i = 0$. The first case means that K_n admits conformal mapping onto H_n in contradiction with nontriviality of the geodesic mapping. The second case is reduced to $\varphi_i = 0$, which means that K_n is Kählerian space, and for this case an analogous theorem is proved in [7]. Thus the theorem was proved.

3. Geodesic mappings of conformally-Kählerian spaces onto Riemannian spaces

A Riemannian space is called a $V_n(B)$*–space*, if it admits a geodesic mapping satisfying the equations (5) and

$$\lambda_{i,j} = \mu g_{ij} + B a_{ij} \tag{9}$$

where μ and B are functions.

It takes place the following

Theorem 2. *If a conformally–Kählerian space K_n $(n > 10)$ admits a geodesic mapping onto a Riemannian space, then either space K_n is a $V_n(B)$-space, or the solution of equations (5) satisfies one of the conditions:*

$$(a) \quad Rg\|a_{ij} - \rho g_{ij}\| \leq 3, \quad \textit{or} \quad (b) \quad a_{ij} = \beta\varphi_{(ij)} + \gamma\varphi_{\alpha(i}F_{j)}^{\alpha} + \delta g_{ij} \tag{10}$$

where $\rho, \beta, \gamma, \delta$ are invariants.

Proof. Let a conformally–Kählerian space K_n $(n > 10)$ admit a geodesic mapping. Let us contract the integrability condition (6) of the equations (5) with $F_{j'}^{j}F_{k'}^{k}$. After omitting primes, from the obtained relation we shall subtract (6). On the basis of (3) one gets

$$\begin{aligned} & a_{[k|h}\bar{\varphi}_{j]i} + a_{[k|i}\bar{\varphi}_{j]h} + \bar{a}_{[k|h}\varphi_{j]i} + \bar{a}_{[k|i}\varphi_{j]h} + \\ & + g_{[k|h}\bar{\Phi}_{j]i} + g_{[k|i}\bar{\Phi}_{j]h} + F_{[k|h}\Phi_{j]i} + F_{[k|i}\Phi_{j]h} = 0 \end{aligned} \tag{11}$$

where

$$\Phi_{ji} \equiv \lambda_{i\alpha}F_j^\alpha - \varphi_{j\alpha}a_i^\alpha; \quad \bar{\Phi}_{ji} \equiv \Phi_{\alpha i}F_j^\alpha; \quad \bar{a}_{ji} \equiv a_{\alpha i}F_j^\alpha.$$

A detailed analysis of the conditions (11) for $n > 10$ leads us to a conclusion that either conditions (10a) or one of conditions

$$(a) \quad \varphi_{ij} = \alpha_1 g_{ij} + \alpha_2 F_{ij} + \alpha_3 a_{ij} + \alpha_4 \bar{a}_{ij},$$

$$\text{or} \tag{12}$$

$$(b) \quad \Phi_{ij} = \beta_1 g_{ij} + \beta_2 F_{ij} + \beta_3 a_{ij} + \beta_4 \bar{a}_{ij},$$

are satisfied, where α_1, ...,β_4 are invariants.

If $(\alpha_3)^2 + (\alpha_4)^2 \neq 0$ then from (12a) condition (10b) follows. In the opposite case (12a) turns into $\varphi_{ij} = \alpha_1 g_{ij} + \alpha_2 F_{ij}$, and from (12b) one obtains

$$\lambda_{i,j} = \mu g_{ij} + B a_{ij} + \alpha_5 F_{ij} + \alpha_6 \bar{a}_{ij}, \tag{13}$$

where μ,B,α_5,α_6 are invariants. Alternating (13) we obtained

$$2\alpha_5 F_{ij} + \alpha_6\, \bar{a}_{[ij]} = 0.$$

Differentiating covariantly the latter, we see after some analysis that either conditions (10a) are satisfied, or $\alpha_5 = \alpha_6$. Then formula (13) takes the form of (9), and, consequently, K_n is a $V_n(B)$-space which completes the proof.

4. Conformally-Kählerian spaces, admitting nontrivial geodesic mappings

It is easy to show that the space V_n with a metric form

$$ds^2 = e(dx^1)^2 + f(x^1) d\tilde{s}^2, \tag{14}$$

where $e = \pm 1$, f $(\neq 0)$ is a function of the argument x^1 and $d\tilde{s}^2$ is a metric of a Sasaki space, is a conformally-Kählerian space with concircular vector field.

In [9] it is shown, that (14) for $f \equiv const\,(x^1)^2$ determines the metric of a Kählerian space.

The direct analysis of the basic equations of the geodesic mappings makes it possible to prove

Theorem 3. *The conformally-Kählerian space K_n with the metric (14) admits a geodesic mapping onto the conformally-Kählerian space $\bar{K}_n$ with the metric*

$$ds^2 = e\alpha(1+\beta f)^{-2}(dx^1)^2 + \alpha f(1+\beta f)^{-1} d\tilde{s}^2,$$

where α, β are constants, satisfying the conditions $\alpha \neq 0$, $1 + \beta f \neq 0$. When $\beta f' \not\equiv 0$, then the mapping is nontrivial one.

It is not difficult to show that if $f = \cos^2 \alpha x^1$, (or $sh^2 \alpha x^1$), $\alpha \neq 0$, then conformally-Kählerian space with the metric (14) admits nontrivial projective transformations.

One should note that the spaces with metrics as in (14) are a particular case of $V_n(B)$–spaces.

5. Holomorphically-projective mappings of a conformally-Kählerian space onto an almost Hermitian space

We recall the concept of holomorphically-projective mapping of a conformally-Kählerian space $K_n(g_{ij}, F_i^h)$ onto an almost Hermitian space $\bar{H}_n(\bar{g}_{ij}, \bar{F}_i^h)$.

A curve $L : x^h = x^h(t)$ in K_n will be called an analytically planar one, if its tangential vector $\lambda^h \equiv dx^h/dt$ under a parallel motion is always complanar to the tangential vector λ^h and the conjugate vector $\lambda^\alpha F_\alpha^h$. Similarly we introduce this notion in $\bar{H}_n$.

A diffeomorphism $f: K_n \to \bar{H}_n$ will be called a *holomorphically-projective mapping*, if with respect to f, all the analytically planar curves in the Kählerian space are transformed into analytically planar curves in $\bar{H}_n$.

Taking into account the modified results reported in [11], one can show easily that, in a general "common" coordinate system,

$$\bar{\Gamma}^h_{ij} = \Gamma^h_{ij} + \delta^h_{(i}\psi_{j)} + F^h_{(i}\Theta_{j)} \quad \textit{and} \quad \bar{F}^h_i = \pm F^h_i,$$

where ψ_i, Θ_i are vectors, Γ^h_{ij}, $\bar{\Gamma}^h_{ij}$ are the Christoffel symbols of K_n and $\bar{H}_n$.

These conditions are equivalent to the equations

$$\bar{g}_{ij,k} = 2\psi_k \bar{g}_{ij} + \psi_{(i}\bar{g}_{j)k} + \Theta_{(i}\tilde{F}_{j)k} \tag{15}$$

where $\tilde{F}_{ji} \equiv \bar{g}_{j\alpha} F^\alpha_i$.

Differentiate covariantly the relation $\bar{g}_{\alpha(i}\bar{F}^\alpha_{j)} = 0$, which holds in $\bar{H}_n$. After the substitution from (2) and (15) we symmetrize the obtained expression. In the result we get

$$\xi_i g_{jk} + \xi_j g_{ki} + \xi_k g_{ij} + \bar{\xi}_i \bar{g}_{jk} + \bar{\xi}_j \bar{g}_{ki} + \bar{\xi}_k \bar{g}_{ij} = 0,$$

where

$$\xi_i \equiv -\bar{g}_{i\alpha}\varphi^\alpha; \quad \bar{\xi}_i \equiv \psi_\alpha F^\alpha_i + \Theta_i + \varphi_i.$$

Hence it follows, that either $\xi_k = \bar{\xi}_k = 0$, or K_n is Kählerian space. The latter case leads to the homothetic mappings $K_n \to \bar{H}_n$ that we consider as trivial. In result, we have proved

Theorem 4. *Conformally-Kählerian spaces K_n, different from Kählerian ones, do does not admit nontrivial holomorphically-projective mappings onto almost Hermitian spaces.*

References

1. D.V. Becklemishev, *Of differential geometry of spaces with almost complex structure*, Itogi Nauki, VINITI, Moscow (1963), 165-212 (in Russian).
2. A. Gray, L.M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. di Mat. Pura ed Appl. **4** (1980), 35-58.
3. S. Ianus, M. Visinescu, *Kaluza-Klein theory with skalar fields and generalized Hopf manifolds*, Class. Quantum Grav. **4**, N. 3 (1987), 1317-1325.
4. A.V. Karmasina, I.N. Kurbatova, *On some questions of geodesic mappings of almost Hermitian spaces*, Dep. in UkrNIINTI, N458 (1990), (in Russian).
5. V.F. Kirichenko, *Local conformally Kählerian manifolds with constant holomorphically sectional curvatures*, Mat. Sbornik **182**, N. 3 (1991), 354-363 (in Russian).
6. V.B. Koncevaya, *Differential geometry of local conformally Kählerian manifolds*, Autoref., Moscow (1991), (in Russian).
7. J. Mikeš, *On geodesic mappings of the Ricci-2-symmetric Riemannian spaces*, Mat. Zametki **28**, N. 2 (1980), 313-317 (in Russian).
8. J. Mikeš, *On equidistant Kählerian spaces*, Mat. Zametki **38**, N. 4 (1985), 627-633 (in Russian).
9. J. Mikeš, *On the Sasakian and equidistant Kählerian space*, Dokl. AN SSSR **291**, N. 1 (1986), 33-36 (in Russian).
10. J. Mikeš, *On global concircular vector fields on compact Riemannian spaces*, Dep. in UkrNIINTI, N. 615 (1988), (in Russian).
11. J. Mikeš, N.S. Sinyukov, *Quasigeodesic mappings of affine connected spaces*, Izv. VUZOV, Math. **N. 1** (1983), 55-61 (in Russian).
12. N.S. Sinyukov, *Geodesic mapping Riemannian spaces*, Nauka, Moscow (1979), 256 (in Russian).

13. W.Y. Westlake, *Hermitian spaces in geodesic correspondence*, Proc. Amer. Math. Soc. **5**, N. 2 (1954), 301-303.
14. K. Yano, *Differential geometry on complex and almost complex spaces*, Oxford Pergamon Press (1965), 326.
15. K. Yano, T. Nagano, *Some theorems on projective and conformal transformations*, Koninkl. Nederl. Akad. Wet. **A60**, N. 4 (1957), 451-458.

Želko Radulovič and Josef Mikeš, Odessa university, Petra Velikogo 2, Odessa, Ukraine

Received 8 October 1992
Revised 30 January 1993