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$SO(n)$–INVARIANT EINSTEIN METRICS ON STIEFEL MANIFOLDS

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Abstract. A Stiefel manifold $V_k\mathbf{R}^n$ is the set of orthonormal k-frames in $\mathbf{R}^n$, and it is diffeomorphic to the homogeneous space $SO(n)/SO(n-k)$. We study $SO(n)$-invariant Einstein metrics on this space by using the variational approach. We determine when the standard metric on $SO(n)/SO(n-k)$ is Einstein, and we give an explicit solution for the space $V_2\mathbf{R}^n$.

1. Introduction

An Einstein metric is a Riemannian metric g with the property $Ric(g) = cg$. In this paper we are interested in $SO(n)$-invariant Einstein metrics on Stiefel manifolds $V_k\mathbf{R}^n$ of orthonormal k-frames in $\mathbf{R}^n$. These are diffeomorphic to the homogeneous space $SO(n)/SO(n-k)$. The existence of $SO(n)$-invariant Einstein metrics on Stiefel manifolds is due to A. Sagle [Sa] and G. Jensen [Jen]. The first author reduced the problem to an algebraic system of equations, and the second used the method of Riemannian submersions. A special case is the unit tangent bundle $T_1S^n = V_2\mathbf{R}^n$ of S^n, which follows from [Ko] (see also [Be, 9.77]). Here we are interested in explicit solutions for the Einstein equation on the Stiefel manifolds $V_2\mathbf{R}^n$.

First we examine when the standard metric on $V_k\mathbf{R}^n$ is Einstein. We use representation theory and a result from [Wa-Zi1] to show that this happens if and only if $V_k\mathbf{R}^n = S^n$ or $SO(n)$. Next we study a certain family of $SO(n)$-invariant metrics on $V_k\mathbf{R}^n$ and obtain a unique solution for the Einstein equation on $V_2\mathbf{R}^n$ for that particular family. An alternative approach to these results can be found in [A].

2. Preliminaries

Let $M = G/H$ be a compact, simply connected homogeneous space, where G is a semisimple, compact and connected Lie group, and H is a closed subgroup of

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G. Let $\mathbf{g}$ and $\mathbf{h}$ denote the Lie algebras of G and H respectively. We denote by $\mathrm{Ad}_G : G \to \mathrm{Aut}(\mathbf{g})$ (respectively $\mathrm{ad}_{\mathbf{g}} : \mathbf{g} \to \mathrm{End}(\mathbf{g})$) the adjoint representation of G (respectively $\mathbf{g}$).

Since G is semisimple and compact the Killing form $B(X,Y) = \mathrm{tr}\,\mathrm{ad}(X)\,\mathrm{ad}(Y)$ of $\mathbf{g}$ is nondegenerate and negative definite on $\mathbf{g}$, thus giving rise to an orthogonal decomposition of $\mathbf{g}$ as the direct sum $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$. This sum is $\mathrm{Ad}(H)$-invariant, i.e. $[\mathbf{m}, \mathbf{h}] \subset \mathbf{m}$. The tangent space T_oM of M at the identity coset $o = eH$ can be identified with $\mathbf{m}$ by the map

$$X \mapsto X^*(o) = \frac{\mathrm{d}}{\mathrm{d}\,t}(\exp tX \cdot o)\bigg|_{t=0}, \qquad X \in \mathbf{m}.$$

The restriction of $-B$ onto the complement $\mathbf{m}$ induces a G-invariant Riemannian metric g_B on M which is called the *standard metric* on M.

We recall the isotropy representation χ of H in T_oM. For any $\gamma \in G$, let L_γ denote the diffeomorphism of G/H given by left translation. The *isotropy representation* χ of H on T_oM is defined by $\chi(h) = (\mathrm{d}\,L_h)_0$. Using the identification of $\mathbf{m}$ with T_oM, this representation is identified with the adjoint representation of H on $\mathbf{m}$.

3. The standard metric on Stiefel manifolds

In this section we find the Stiefel manifolds $V_k\mathbf{R}^n = SO(n)/SO(n-k)$, $k \geq 1$ for which the standard homogeneous metric g_B is Einstein. For a homogeneous space G/H let $\chi = \chi_o \oplus \chi_1 \oplus \cdots \oplus \chi_s$ be the decomposition of the isotropy representation of H in $\mathbf{m}$ as a direct sum of a trivial representation χ_o and $\mathrm{Ad}(H)$-irreducible representations χ_i $(i = 1, \ldots, s)$. Let $\mathbf{m} = \mathbf{m}_o \oplus \mathbf{m}_1 \oplus \cdots \oplus \mathbf{m}_s$ be the corresponding decomposition of $\mathbf{m}$. According to [Wa-Zi1, Corollary 1.3] if the metric g_B is Einstein, then either H is trivial or $\mathbf{m}_o = 0$.

Theorem 1. *The standard metric on $V_k\mathbf{R}^n$ $(k \geq 1)$ is Einstein if and only if $V_k\mathbf{R}^n = S^n$ or $SO(n)$.*

Proof. We compute the isotropy representation for the Stiefel manifolds $V_k\mathbf{R}^n = SO(n)/SO(n-k)$, $k \geq 1$. Let λ_n denote the standard representation of $SO(n)$ (given by the natural action of $SO(n)$ on $\mathbf{R}^n$). If $\wedge^2\lambda_n$ denote the second exterior power of λ_n, then $\mathrm{Ad}_{SO(n)} = \wedge^2\lambda_n$. The isotropy representation χ of $SO(n-k)$ is characterized by the property $\mathrm{Ad}_{SO(n)}\big|_{SO(n-k)} = \mathrm{Ad}_{SO(n-k)} \oplus \chi$. We compute

$$\mathrm{Ad}_{SO(n)}\big|_{SO(n-k)} = \wedge^2\lambda_n\big|_{SO(n-k)} = \wedge^2(\lambda_{n-k} \oplus k) = \wedge^2\lambda_{n-k} \oplus \wedge^2 k \oplus (\lambda_{n-k} \otimes k),$$

where k denotes the trivial k-dimensional representation. The subrepresentation $\wedge^2\lambda_{n-k}$ is $\mathrm{Ad}_{SO(n-k)}$, thus $\chi = \wedge^2 k \oplus \lambda_{n-k} \oplus \cdots \oplus \lambda_{n-k}$ (k summands). If we further decompose $\wedge^2 k$ into a sum of $\binom{k}{2}$ 1-dimensional subrepresentations, we obtain

$$\chi = 1 \oplus \cdots \oplus 1 \oplus \lambda_{n-k} \oplus \cdots \oplus \lambda_{n-k}. \tag{1}$$

Hence, in our case, if g_B is Einstein, then either H is trivial (i.e. $V_k\mathbf{R}^n = SO(n)$) or $\mathbf{m}_o = 0$ in which case $k = 1$ hence $\chi = \lambda_{n-1}$ (i.e. $V_k\mathbf{R}^n = SO(n)/SO(n-1) = S^n$).

4. Homogeneous Einstein metrics on $V_2\mathbf{R}^n$

In this section we are interested in $SO(n)$-invariant Einstein metrics on the Stiefel manifolds $V_2\mathbf{R}^n = SO(n)/SO(n-2)$. We will use the variational approach to Einstein metrics.

Let $M = G/H$ be a homogeneous space and

$$\mathbf{m} = \mathbf{m}_1 \oplus \cdots \oplus \mathbf{m}_s \tag{2}$$

the decomposition of $\mathbf{m}$ into $\mathrm{Ad}(H)$-invariant subspaces. We consider the family of $\mathrm{Ad}(H)$-invariant *diagonal* metrics

$$\langle\ ,\ \rangle = x_1\ Q|_{\mathbf{m}_1} \oplus \cdots \oplus x_s\ Q|_{\mathbf{m}_s}, \qquad x_i > 0 \tag{3}$$

where Q is some multiple of the Killing form of G. If the $\mathbf{m}_i$'s are pairwise inequivalent representations, then any G-invariant metric on M is determined by $\langle\ ,\ \rangle$. Otherwise, by fixing the decomposition (2) of $\mathbf{m}$ there may exist G-invariant metrics on M not of this form. We will see later on that we face this problem with the Stiefel manifolds. For the following we use the notation from [Wa-Zi2].

Let $S(g)$ denote the scalar curvature of a Riemannian metric g on M. According to [Ber], [Hil] the Einstein metrics on M are the critical points of the total scalar curvature functional $T(g) = \int_M S(g) d\,\mathrm{vol}_g$ on the space of Riemannian metrics of volume one. Now, let $\mathcal{M}_G$ denote the set of all G-invariant metrics of volume one on M, and note that on $\mathcal{M}_G$, $T(g) = S(g)$. Then the critical points of $T|_{\mathcal{M}_G}$ are precisely the G-invariant Einstein metrics of volume one [Be, Theorem 4.23].

Let $\{e_\alpha\}$ be a Q-orthonormal basis adapted to the decomposition (2) of $\mathbf{m}$, i.e. $e_\alpha \in \mathbf{m}_i$ for some i and $\alpha < \beta$ if $i < j$. We set

$$C_{ij}^k = \sum_{\substack{e_\alpha \in \mathbf{m}_i \\ e_\beta \in \mathbf{m}_j \\ e_\gamma \in \mathbf{m}_k}} Q([e_\alpha, e_\beta], e_\gamma)^2. \tag{4}$$

Notice that C_{ij}^k is independent of the Q-orthonormal bases chosen for $\mathbf{m}_i$, $\mathbf{m}_j$ and $\mathbf{m}_k$, but it depends on the choice of the decomposition of $\mathbf{m}$. Also, C_{ij}^k is symmetric in all three indices. Let $\langle\ ,\ \rangle$ be the $\mathrm{Ad}(H)$-invariant metric having the form (3) with respect to a fixed decomposition of $\mathbf{m}$. Then $\{e_\alpha/\sqrt{x_i} : e_\alpha \in \mathbf{m}_i\}$ is a $\langle\ ,\ \rangle$- orthonormal basis of $\mathbf{m}$. According to [Wa-Zi2, equation (1.3)] the scalar curvature of $\langle\ ,\ \rangle$ is given by

$$S = \frac{1}{2}\sum_{i=1}^{s} \frac{d_i b_i}{x_i} - \frac{1}{4}\sum_{i,j,k} C_{ij}^k \frac{x_k}{x_i x_j} \tag{5}$$

where $d_i = \dim \mathbf{m}_i$ and $B|_{\mathbf{m}_i} = -b_i\ Q|_{\mathbf{m}_i}$.

We now come to the Stiefel manifolds. As shown in section 3, the isotropy representation of $V_k\mathbf{R}^n$ is decomposed as $\chi = 1 \oplus \cdots \oplus 1 \oplus \lambda_{n-k} \oplus \cdots \oplus \lambda_{n-k}$. We observe that the first $\frac{k(k-1)}{2}$ summands are equivalent to each other and the same is true for the remaining k summands.

We restrict ourselves to $V_2\mathbf{R}^n$ and we look for $SO(n)$-invariant Einstein metrics of the form (3). The isotropy representation is decomposed as $\chi = 1\oplus\lambda_{n-2}\oplus\lambda_{n-2}$, hence these metrics depend on three parameters $x_1,\ x_2,\ x_3,\ x_i>0$. Let $Q(X,Y) = -\frac{B(X,Y)}{2(n-2)} = -\frac{\operatorname{tr} XY}{2}$ and let E_{ij} denote the $n\times n$ matrix with 1 at the (ij)-entry and 0 elsewhere. Then the set $\mathcal{B} = \{e_{ij} = E_{ij} - E_{ji} : 1\leq i\leq k,\ 1\leq i<j\leq n\}$ constitutes a Q-orthonormal basis of $\mathbf{m}$ (containing $\frac{k(2n-k-1)}{2} = \dim V_k\mathbf{R}^n$ elements).

We adapt the basis $\mathcal{B}$ to the decomposition $\mathbf{m} = \mathbf{m}_1\oplus\mathbf{m}_2\oplus\mathbf{m}_3$, so that $\mathbf{m}_1 = \operatorname{span}\{e_{12}\}$, $\mathbf{m}_2 = \operatorname{span}\{e_{1j} : 3\leq j\leq n\}$, $\mathbf{m}_3 = \operatorname{span}\{e_{2j} : 3\leq j\leq n\}$. The scalar curvature is given by (5) and in our case $b_1 = b_2 = b_3 = 2(n-2)$ and $d_1 = 1,\ d_2 = d_3 = n-2$. For the computation of the numbers in (4) we need the following lemma.

Lemma. *The multiplication table of the elements in $\mathcal{B}$ is given as follows:*

$$[e_{ij}, e_{kl}] = \begin{cases} 0 & i\neq k,l \quad j\neq k,l \\ -e_{ik} & i\neq k,l \quad j=l \\ e_{il} & i\neq k,l \quad j=k \\ -e_{jl} & i=k, \quad j\neq k,l \\ e_{jk} & i=l, \quad j\neq k,l. \end{cases}$$

Proof. The computation is straightforward.

By using this lemma we find that $C_{12}^3 = n-2$ and its rearrangements are the only non-zero triples. The scalar curvature is given by

$$S = (n-2)\left[\frac{1}{x_1} + \frac{n-2}{x_2} + \frac{n-2}{x_3} - \frac{1}{2}(\frac{x_1}{x_2x_3} + \frac{x_2}{x_1x_3} + \frac{x_3}{x_1x_2})\right]$$

and the volume condition is $x_1x_2^{n-2}x_3^{n-2} = 1$. Let $\tilde{S} = S - \lambda(x_1x_2^{n-2}x_3^{n-2})$. Then

$$\begin{aligned} \frac{\partial\tilde{S}}{\partial x_1} &= \frac{2(n-2)}{x_1^2} + (n-2)(\frac{x_1}{x_2x_3} - \frac{x_2}{x_1^2x_3} - \frac{x_3}{x_1^2x_2}) - \mu x_2^{n-2}x_3^{n-2} \\ \frac{\partial\tilde{S}}{\partial x_2} &= \frac{2(n-2)}{x_2^2} - \frac{x_1}{x_2^2x_3} + \frac{x_2}{x_1x_3} - \frac{x_3}{x_1x_2^2} - \mu x_1x_2^{n-3}x_3^{n-2} \\ \frac{\partial\tilde{S}}{\partial x_3} &= \frac{2(n-2)}{x_3^2} - \frac{x_1}{x_2x_3^2} - \frac{x_2}{x_1x_3^2} + \frac{x_3}{x_1x_2} - \mu x_1x_2^{n-2}x_3^{n-3}. \end{aligned}$$

Setting the above equations equal to zero and by using the volume condition we obtain the following system of equations:

$$\begin{aligned} 2(n-2)x_2x_3 + (n-2)x_1^2 - (n-2)x_2^2 - (n-2)x_3^2 &= 2(n-2)x_1x_3 - x_1^2 + x_2^2 - x_3^2 \\ &= 2(n-2)x_1x_2 - x_1^2 - x_2^2 + x_3^2. \end{aligned}$$

Thus we have obtained:

Proposition. *The Einstein equation for $V_2\mathbf{R}^n$ reduces to the following system of algebraic equations:*

$$
\begin{aligned}
2(n-2)x_3(x_1-x_2) &= (n-1)x_1^2-(n-1)x_2^2-(n-3)x_3^2\\
(n-2)x_1(x_2-x_3) &= x_2^2-x_3^2
\end{aligned}
$$

This system of equations has only one positive solution, namely $x_2 = x_3 = \frac{n-1}{2(n-2)}x_1$. One can also check that as long as the Q-orthonormal basis $\mathcal{B}$ of $\mathbf{m}$ is adapted to the decomposition $\mathbf{m} = \mathbf{m}_1 \oplus \mathbf{m}_2 \oplus \mathbf{m}_3$ so that $\mathbf{m}_1 = \operatorname{span}\{e_{12}\}$, $\mathbf{m}_2 = \operatorname{span}\{e_{ij} : 1 \leq i,j \leq n, j \neq 2\}$, and $\mathbf{m}_3 = \operatorname{span}\{e_{kl} : e_{kl} \neq e_{ij}\}$, then the Einstein equation for $V_2\mathbf{R}^n$ reduces to the same algebraic system given in the previous proposition. Thus we have obtained

Theorem. *The Stiefel manifolds $V_2\mathbf{R}^n = SO(n)/SO(n-2)$ admit (up to scale) at least one diagonal $SO(n)$-invariant Einstein metric, explicitly given as $x_2 = x_3 = \frac{n-1}{2(n-2)}x_1$. If the Q-othonormal basis $\mathcal{B}$ is adapted to the decomposition (2) the way described above, then this metric is unique.*

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