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REPRESENTATION OF THE LIE GROUP OF AUTOMORPHISMS OF AN ALMOST r-PARACONTACT RIEMANNIAN MANIFOLD OF P-SASAKIAN TYPE

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ABSTRACT. In this paper the examples of Lie groups and Lie algebras on an almost r-paracontact Riemannian manifold M_n of P-Sasakian type are presented and their maximum dimensions are determined. It is proven that the set of all automorphisms of M_n is a Lie group, and the example of an almost r-paracontact Riemannian manifold of P-Sasakian type with the group of automorphisms of the maximum dimension is constructed. Finally, some representation of the group of automorphisms of an almost r-paracontact Riemannian manifold of P-Sasakian type is presented.

1. In this section we recall the definitions of almost r-paracontact Riemannian manifolds [3] and some types of these manifolds [4] together with some of their properties.

Definition 1.1. Let M_n be an n-dimensional Riemannian manifold with a positive definite metric g. If on M_n there exist: a tensor field ϕ of type (1,1), r vector fields $\xi_1, \xi_2, \dots, \xi_r$ $(r < n)$, r 1-forms $\eta^1, \eta^2, \dots, \eta^r$ such that

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \qquad \alpha, \beta \in (r) = 1, 2, ..., r \tag{1.1}$$

$$\phi^2 = Id - \eta^\alpha \otimes \xi_\alpha \tag{1.2}$$

$$\eta^\alpha(X) = g(X, \xi_\alpha), \qquad \alpha \in (r) \tag{1.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y) \tag{1.4}$$

for vector fields X, Y on M_n with $a^\alpha b_\alpha \overset{\text{def}}{=} \sum_\alpha a^\alpha b_\alpha$. Then $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$

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is said to be an *almost r-paracontact Riemannian structure* on M_n, and M_n is an *almost r-paracontact Riemannian manifold*.

Remark 1.1. From (1.1) through (1.4) we also have

$$\phi(\xi_\alpha) = 0, \qquad \alpha \in (r) \tag{1.5}$$

$$\eta^\alpha \circ \phi = 0, \qquad \alpha \in (r) \tag{1.6}$$

$$\Phi(X,Y) \stackrel{\text{def}}{=} g(\phi X, Y) = g(X, \phi Y) \tag{1.7}$$

From (1.7) we get

$$\nabla_Z \Phi(X,Y) = g((\nabla_Z \phi)X, Y) \tag{1.8}$$

Definition 1.2. If M_n is an almost r-paracontact manifold with the structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$, then Σ is said to be normal if an almost product structure F defined on $M_n \times R^r$ by $F(X, f^\alpha \frac{d}{dt^\alpha}) = (\phi X + f^\alpha \xi_\alpha, \eta^\alpha(X)\frac{d}{dt^\alpha})$ is integrable, i.e., its Nijenhuis tensor field N_F vanishes.

Theorem 1.1. *An almost r-paracontact structure Σ on M_n is normal if and only if $N(X,Y) = N_\phi(X,Y) - 2d\eta^a(X,Y)\xi_\alpha = 0$, where N_ϕ is the Nijenhuis tensor field for ϕ.*

Definition 1.3. An almost r-paracontact Riemannian manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is said to be of *P-Sasakian type* if

$$\Phi(X,Y) = (\nabla_Y \eta^\alpha)(X) \qquad \text{for all } \alpha \in (r) \tag{1.9}$$

$$\begin{aligned}(\nabla_Z \Phi)(X,Y) = &- \sum_\alpha \eta^\alpha(X) g(\phi Y, \phi Z) - \\ &- \sum_\alpha \eta^\alpha(Y) g(\phi X, \phi Z)\end{aligned} \tag{1.10}$$

for all vector fields X, Y, Z on M_n.

Remark 1.2. The conditions (1.9) and (1.10) are equivalent respectively to:

$$\phi X = \nabla_X \xi_\alpha \qquad \text{for all } \alpha \in (r) \tag{1.11}$$

$$\begin{aligned}(\nabla_Y \phi)(X) = &- \sum_\beta \eta^\beta(X)[Y - \eta^\alpha(Y)\xi_\alpha] \\ &- [g(X,Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y)] \sum_\beta \xi_\beta\end{aligned} \tag{1.12}$$

Corollary 1.1. *On a manifold M_n an almost r-paracontact Riemannian structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ of P-Sasakian type is normal and all η^α are closed.*

Remark 1.3 [5]. The Ricci tensor Ric satisfies

$$\mathrm{Ric}(X,\xi_\alpha) = -(n-r)\sum_\beta \eta^\beta(X), \quad \alpha \in (r) \tag{1.13}$$

Remark 1.4. On an almost r-paracontact Riemannian manifold M_n with the structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ the tensor ϕ has constant eigenvalues $1, -1$, and 0. Let p, q, and r be their multiplicities, respectively, with $p+q+r = n = \dim M_n$. M_n is said to be of type (p,q). If T_xM_n is the tangent space to M_n at a point $x \in M_n$, then M_n has an orthonormal frame $\{e_1,\ldots,e_p,e_{p+1},\ldots,e_{p+\lambda},e_{p+q+1} = \xi_1,\ldots,e_n = \xi_r\}$ with $\phi(e_a) = e_a$, $\phi(e_{p+\lambda}) = -e_{p+\lambda}$, $\phi(e_{p+q+\alpha}) = 0$, $a = 1,\ldots,p$, $\lambda = 1,\ldots,q$, $\alpha = 1,\ldots,r$. This frame is called an adapted frame.

Remark 1.5. The indices run always through the following ranges $A,B,C,\dots \in (n)$, $a,b,c,\dots \in (p)$, $\lambda,\mu,\nu,\dots \in (q)$, $\alpha,\beta,\gamma,\dots \in (r)$, $i,j,k,\dots \in (p+q)$, with $\lambda^* = p+\lambda$ and $\alpha^* = p+q+\alpha$.

2. In this section the Lie algebra of infinitesimal automorphisms of an almost r-paracontact Riemannian manifold of P-Sasakian type is defined and its maximunm dimension is found.

Definition 2.1. A vector field X on (M_n,Σ) is an infinitesimal automorphism of M_n if the following conditions are satisfied

$$\mathcal{L}_X\phi = 0, \quad \mathcal{L}_X\eta_\alpha = 0, \quad \mathcal{L}_X\xi_\alpha = 0, \quad \mathcal{L}_Xg = 0, \quad \alpha \in (r) \tag{2.1}$$

where $\mathcal{L}_X$ is the Lie derivative with respect to a vector field X.

Remark 2.1. An infinitesimal automorphism of M_n is an infinitesimal isometry.

Proposition 2.1. *The set L of all infinitesimal automorphisms of M_n is a Lie algebra.*

Now we shall study the maximum dimension of L. The equations (2.1), in local coordinates, may be written as follows

$$\mathcal{L}_X\phi^A_B = \nabla_C\phi^A_BX^C - \nabla_CX^A\phi^C_B + \nabla_BX^C\phi^A_C = 0 \tag{2.2}$$

$$\mathcal{L}_X\xi^A_\alpha = \nabla_B\xi^A_\alpha X^B - \nabla_BX^A\xi^B_\alpha = 0 \tag{2.3}$$

$$\mathcal{L}_X\eta^\alpha_A = \nabla_C\eta^\alpha_BX^C + \nabla_BX^C\eta^\alpha_C = 0 \tag{2.4}$$

$$\mathcal{L}_Xg_{AB} = \nabla_AX^Cg_{CB} + \nabla_BX^Cg_{CA} = 0 \tag{2.5}$$

The maximum dimension of L is equal to the number of independent algebraic solutions X^A and ∇_BX^C of (2.2) – (2.5) at a point P of M_n. Let $\{e_a, e_{\lambda^*}, e_{\alpha^*}\}$ be an adapted frame at a point P of M_n. Then the structure tensors of Σ are given by

$$\begin{gathered} g_{AB} = \delta_{AB} \\ \phi^A_B = \begin{cases} \delta^a_b & \text{if } A = a,\ B = b \\ -\delta^\lambda_\mu & \text{if } A = \lambda^*,\ B = \mu^* \\ 0 & \text{otherwise} \end{cases} \\ \xi^A_\alpha = \delta^A_{\alpha^*}, \quad \eta^\alpha_B = \delta^{\alpha^*}_B, \quad \alpha \in (r) \end{gathered} \tag{2.6}$$

Consider the solutions of (2.2) – (2.5) at P such that $X(P) = 0$. Then the formulas (2.3) – (2.5) are reduced to

$$\nabla_{\alpha^*} X^A(P) = 0 \tag{2.7}$$

$$\nabla_B X^{\alpha^*}(P) = 0 \tag{2.8}$$

$$\nabla_A X^B(P) + \nabla_B X^A(P) = 0 \tag{2.9}$$

respectively. From (2.2), if $A = a, B = \lambda^*$, and if $A = \mu^*, B = b$, we get

$$\nabla_{\lambda^*} X^a(P) = 0, \qquad \nabla_b X^{\mu^*}(P) = 0 \tag{2.10}$$

Hence, we conclude that the matrix $[\nabla_A X^B(P)]$ has the form

$$\begin{bmatrix} A_p & 0 & 0 \\ 0 & B_q & 0 \\ 0 & 0 & 0_r \end{bmatrix} \tag{2.11}$$

where A_p, B_q are skew symmetric matrices of orders p and q respectively, and 0_r is the zero matrix of order r. Therefore, the number of independent matrices of the form (2.11) at a point P is $\frac{1}{2}p(p-1) + \frac{1}{2}q(q-1)$. On the other hand, the number of non-vanishing vector fields at P is at most n. Thus we have

Theorem 2.1. *$\frac{1}{2}p(p-1) + \frac{1}{2}q(q-1) + n$, with $p + q + r = n = \dim M_n$ is the maximum dimension of the Lie algebra L of all infinitesimal automorphisms of an almost r-paracontact Riemannian manifold M_n of P-Sasakian type.*

3. In this section the maximum dimension of the group of isometries of an almost r-paracontact Riemannian manifold of P-Sasakian type is determined.

Suppose that an almost r-paracontact Riemannian manifold M_n of P-Sasakian type with the structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha \in (r)}$ is compact. From (1.11), for any ξ_α we have $\operatorname{div} \xi_\alpha = \operatorname{trace}\{X \to \nabla_X \xi_\alpha\} = \sum_{i=1}^{n} g(\nabla_{e_i} \xi_\alpha, e_i) = \sum_{i=1}^{n} g(\phi e_i, e_i)$. Hence,

$$\operatorname{div} \xi_\alpha = p - q, \quad \alpha \in (r) \tag{3.1}$$

By Green's theorem [10] we get $0 = \int_{M_n} \operatorname{div} \xi_\alpha \, dM_n = p - q$. We obtain

Proposition 3.1. *If an almost r-paracontact Riemannian manifold M_n of P-Sasakian type is compact, then $p = q = \frac{1}{2}(n-r)$ and $n - r$ is even.*

Now, from Theorem 2.1 we get

Theorem 3.1. *$\frac{1}{4}(n-r)(n-r-2) + n$ is the maximum dimension of the Lie algebra L of all infinitesimal automorphisms of an almost r-paracontact, compact, Riemannian manifold M_n of P-Sasakian type.*

Now, suppose that X is an infinitesimal isometry, i.e., X satisfies (2.5). Denote by

$$D^\alpha = \eta^\alpha_A X^A, \quad \alpha \in (r) \tag{3.2}$$

Making use of (1.7), (1.9), and (1.10) we get

$$\begin{aligned} g^{BA}\nabla_B\nabla_A D^\alpha = -(n-r)\sum_\beta D^\beta + 2\phi^B_A\nabla_B X^A \\ + g^{CB}\eta^\alpha_A\nabla_C\nabla_B X^A, \end{aligned} \qquad \alpha \in (r) \tag{3.3}$$

From (2.5) we get

$$\nabla_A X^A = 0, \qquad \xi^A_\alpha \nabla_A X^C g_{CB} + \xi^A_\alpha \nabla_B X^C g_{CA} = 0 \tag{3.4}$$

Differentiating covariantly $(3.4)_2$ and making use of (1.11) and the fact that Φ is symmetric we get

$$2\phi^A_B\nabla_A X^B + \xi^A_\alpha\nabla_B\nabla_A X^B + g^{BC}\eta^\alpha_A\nabla_B\nabla_C X^A = 0, \quad \alpha \in (r) \tag{3.5}$$

and from the Ricci identity, $(3.4)_1$ and (1.13) we get

$$\xi^A_\alpha\nabla_B\nabla_A X^B = -(n-r)\sum_\beta D^\beta, \quad \alpha \in (r) \tag{3.6}$$

Hence, from (3.3) we have $g^{BA}\nabla_A\nabla_B D^\alpha = 0, \quad \alpha \in (r)$. Thus, by Green's theorem [10] we have

$$D^\alpha = \eta^\alpha_A X^A = \text{constant}, \quad \alpha \in (r) \tag{3.7}$$

Differentiating covariantly (3.7) and making use of (1.9) we get

$$\mathcal{L}_X \eta^\alpha = 0, \quad \alpha \in (r) \tag{3.8}$$

From (3.8) and (2.5) we have

$$\mathcal{L}_X \xi_\alpha = 0, \quad \alpha \in (r) \tag{3.9}$$

For any vector field Y we have the identity [10]

$$\mathcal{L}_X\nabla_B Y^A = \nabla_B\mathcal{L}_X Y^A + (\mathcal{L}_X\Gamma^A_{BC})Y^C \tag{3.10}$$

Making use (1.11) we get for $Y^A = \xi^A_\alpha$

$$\mathcal{L}_X\phi^A_B = \nabla_B(\mathcal{L}_X\xi^A_\alpha) + (\mathcal{L}_X\Gamma^A_{BC})\xi^C_\alpha, \quad \alpha \in (r) \tag{3.11}$$

From (2.5) we have $\mathcal{L}_X\Gamma^A_{BC} = 0$, so from (3.9) and (3.11) we get

$$\mathcal{L}_X\phi = 0 \tag{3.12}$$

Thus we obtain

Theorem 3.2. *On a compact almost r-paracontact Riemannian manifold of P-Sasakian type every infinitesimal isometry is an infintesimal automorphism.*

We have

Proposition 3.2 [6]. *Let M be a complete Riemannian manifold, then the Lie algebra I of all infinitesimal isometries of M is isomorphic to the Lie algebra $i(M)$ of the group of isometries $I(M)$ of M.*

Combining Thoorems 3.1, 3.2 and Proposition 3.2 we obtain

Theorem 3.3. *$\frac{1}{4}(n-r)(n-r-2)+n$ is the maximum dimension of the group of isometries of a complete, compact, almost r-paracontact Riemannian manifold M_n of P-Sasakian type.*

Corollary 3.1. *The maximum dimension of the group of isometries of a complete, compact, P-Sasakian manifold M_n is $\frac{1}{4}(n^2+3)$.*

4. In this section it is shown that the automorphism group of an almost r-paracontact Riemannian manifold of P-Sasakian type is a Lie group and the example of the manifold with the maximum dimensional automorphism group is presented.

Definition 4.1. Let M_n with $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ be an almost r-paracontact Riemannian manifold of P-Sasakian type. A diffeomorphism f of M_n onto itself is an automorphism of M_n if it preserves all structure tensors, i.e.,

$$\phi \circ f_* = f_* \circ \phi, \quad f_*(\xi_\alpha) = \xi_\alpha, \quad f^*\eta^\alpha = \eta^\alpha, \quad f^*g = g, \quad \alpha \in (r) \tag{4.1}$$

where f_* is the differential of f and f^* is the dual map of f_*.

Proposition 4.1. *The set of all automorphisms of an almost r-paracontact Riemannian manifold M_n of P-Sasakian type constitutes a group which is called the automorphism group of M_n and is denoted by $A(M_n)$.*

Theorem 4.1. *The automorphism group $A(M_n)$ of an almost r-paracontact Riemannian manifold M_n of P-Sasakian type is a Lie group.*

Proof. According to (4.1) the group $A(M_n)$ is a subgroup of the isometry group $I(M_n)$ of the manifold M_n with the same metric tensor g. The group $A(M_n)$ is closed in $I(M_n)$, so it is a Lie group.

Remark 4.1. For any given infinitesimal automorphism X of an almost r-paracontact Riemannian manifold M_n of P-Sasakian type $\exp(tX)$ $(-\infty < t < \infty)$ is an automorphism of M_n. Also, any automorphism f of M_n can be generated by an infinitesimal automorphism X.

Taking this into consideration we get

Proposition 4.2. *The Lie algebra of all infinitesiaml automorphisms of an almost r-paracontact Riemannian manifold M_n of P-Sasakian type is isomorphic to the Lie algebra of the group $A(M_n)$.*

Combining Proposition 4.2 and Theorem 2.1 we obtain

Theorem 4.2. *$\frac{1}{2}p(p+1)+\frac{1}{2}q(q+1)+r$, with $p+q+r=n$, is the maximum dimension of the automorphism group $A(M_n)$ of an almost r-paracontact Riemannian manifold M_n of P-Sasakian type.*

Now we have

Theorem 4.3. *R^n with the structure $\Sigma=(\phi,\xi_\alpha,\eta^\alpha,g)_{\alpha\in(r)}$ given, in local coordinates $(x^1,\dots,x^n)$, by*

$$\phi_B^A=\begin{cases}\delta_b^a & \text{if } A=a,\ B=b\\ -\delta_\mu^\lambda & \text{if } A=\lambda^*,\ B=\mu^*\\ 0 & \text{otherwise}\end{cases}\tag{4.2 }$$

$$\xi_\alpha^A=\delta_{\alpha^*}^A,\quad \alpha\in(r)\tag{4.3}$$

$$\eta_B^\alpha=\delta_B^{\alpha^*}\quad \alpha\in(r)\tag{4.4}$$

$$g_{AB}=\begin{cases}g_{ab} &= e^{2(x^{p+q+1}+,\dots,+x^n)}\delta_{ab}\\ g_{\lambda^*\mu^*} &= e^{-2(x^{p+q+1}+,\dots,+x^n)}\delta_{\lambda\mu}\\ g_{\alpha^*\beta^*} &= \delta_{\alpha\beta}\\ 0 & \text{all others}\end{cases}\tag{4.5}$$

is the almost r-paracontact Riemannian manifold of P-Sasakian type whose the automorphism group $A(R^n)$ has the maximum dimension $\frac{1}{2}p(p+1)+\frac{1}{2}q(q+1)+r$.

Proof. In [4] it is shown that (R^n,Σ) is an almost r-paracontact Riemannian manifold of P-Sasakian type. Now, to find the dimension of $A(R^n)$ it suffices to determine the dimension of the Lie algebra L of all infinitesimal automorphisms of R^n. To this end, we have to solve the system (2.2) through (2.5) for a vector field V. Making use of (4.5) we calculate

$$\Gamma_{BC}^A=\begin{cases}-e^{2(x^{p+q+1}+,\dots,+x^n)} & \text{if } B=C=\alpha^*, A=\beta^*\\ e^{-2(x^{p+q+1}+,\dots,+x^n)} & \text{if } B=C=\lambda^*, A=\beta^*\\ \delta_{\alpha\beta} & \text{if } A=a, B=b, C=\beta^*\\ -\delta_{\lambda\mu} & \text{if } A=\lambda^*, B=\mu^*, C=\beta^*\\ 0 & \text{otherwise}\end{cases}\tag{4.6}$$

From (2.3), (2.4), and (2.2) we get $\partial_B\xi_\beta^A V^B-\partial_B V^A\xi_\beta^B=0$, $\partial_C\eta_B^\beta V^C+\partial_B V^C\eta_C^\beta=0$, and $\partial_C\phi_B^A V^C-\partial_C V^A\phi_B^C+\partial_B V^C\phi_C^A=0$, respectively, and after having used (4.3), (4.4), and (4.2) we obtain

$$\partial_{\beta^*}V^A=0,\qquad \partial_B V^{\beta^*}=0,\qquad \partial_{\lambda^*}V^a=0,\qquad \partial_b V^{\mu^*}=0\tag{4.7}$$

Putting $A=a,B=b$ in (2.5) and also putting $A=\lambda^*,B=\mu^*$ in (2.5), and making use of (4.5) and (4.6) give us

$$2\delta_{ab}\sum_{\beta^*}V^{\beta^*}+\partial_a V^b+\partial_b V^a=0\tag{4.8}$$

$$-2\delta_{\lambda\mu}\sum_{\beta^*}V^{\beta^*}+\partial_{\lambda^*}V^{\mu^*}+\partial_{\mu^*}V^{\lambda^*}=0\tag{4.9}$$

If we put any other set of values for A and B in (2.5) we get no new relation. Putting $a = b$ and $a \neq b$ in (4.8), and also $\lambda = \mu$ and $\lambda \neq \mu$ in (4.9) yield

$$\partial_a V^a + \sum_{\beta^*} V^{\beta^*} = 0, \quad \partial_a V^b + \partial_b V^a = 0 \tag{4.10}$$

$$\partial_{\lambda^*} V^{\lambda^*} - \sum_{\beta^*} V^{\beta^*} = 0, \quad \partial_{\lambda^*} V^{\mu^*} + \partial_{\mu^*} V^{\lambda^*} = 0 \tag{4.11}$$

From $(4.7)_1$ we obtain $V^{\beta^*} = c^{\beta^*}$, where c^{β^*} is a constant for any β. Let

$$\sum_{\beta^*} V^{\beta^*} = \sum_{\beta^*} c^{\beta^*} = c \tag{4.12}$$

Because of $(4.7)_1$ and $(4.7)_3$, V^a depends only on $x^1, \ldots, x^p$, so from $(4.10)_1$ we get

$$V^a = -cx^a + h^a(x^1, \ldots, \bar{x}^a, \ldots, x^p) \tag{4.13}$$

where h^a are differentiable functions and $\bar{x}^a$ means that a-th variable is omitted. Substituting (4.13) into $(4.10)_2$ gives

$$\partial_a h^b(x^1, \ldots, \bar{x}^b, \ldots, x^p) + \partial_b h^a(x^1, \ldots, \bar{x}^a, \ldots, x^p) = 0 \tag{4.14}$$

From this we get $\partial_b(\partial_b h^a) = 0$, hence $\partial_b h^a = c^a_b$ and

$$h^a(x^1, \ldots, x^p) = c^a_b x^b + c^a \tag{4.15}$$

where c^a_b and c^a are constants satisfying (because of (4.14))

$$c^a_b + c^b_a = 0 \tag{4.16}$$

Thus, from (4.13) we have

$$V^a = -cx^a + c^a_b x^b + c^a \tag{4.17}$$

Because of $(4.7)_1$ and $(4.7)_4$ V^{λ^*} depends only on $x^{p+1}, \ldots, x^{p+q}$, so from $(4.11)_1$ we get

$$V^{\lambda^*} = cx^{\lambda^*} + h^{\lambda^*}(x^{p+1}, \ldots, \bar{x}^{p+\lambda}, \ldots, x^{p+q}) \tag{4.18}$$

Substituting (4.18) into $(4.11)_2$ we obtain

$$\partial_{\lambda^*} h^{\mu^*} + \partial_{\mu^*} h^{\lambda^*} = 0 \tag{4.19}$$

Hence $\partial_{\lambda^*}(\partial_{\lambda^*} h^{\mu^*}) = 0$, and

$$h^{\lambda^*}(x^{p+1}, \ldots, x^{p+q}) = c^{\lambda^*}_{\mu^*} x^{\mu^*} + c^{\lambda^*} \tag{4.20}$$

where $c^{\lambda^*}_{\mu^*}$ and c^{λ^*} are constants satisfying (because of (4.19))

$$c^{\lambda^*}_{\mu^*} + c^{\mu^*}_{\lambda^*} = 0 \tag{4.21}$$

Thus, from (4.18) we have

$$V^{\lambda^*} = cx^{\lambda^*} + c^{\lambda^*}_{\mu^*} x^{\mu^*} + c^{\lambda^*} \tag{4.22}$$

Now, making use of (4.17), (4.22), (4.12), (4.16), and (4.21), any infinitesimal automorphism V on our manifold is of the form

$$\begin{aligned} V &= (-cx^a + c^a_b x^b + c^a)\frac{\partial}{\partial x^a} + (cx^{\lambda^*} + c^{\lambda^*}_{\mu^*} x^{\mu^*} + c^{\lambda^*})\frac{\partial}{\partial x^{\lambda^*}} \\ &+ c^{\alpha^*}\frac{\partial}{\partial x^{\alpha^*}} = c^a \frac{\partial}{\partial x^a} + c^{\lambda^*}\frac{\partial}{\partial x^{\lambda^*}} + \sum_{\lambda<\mu} c^{\mu^*}_{\lambda^*}(x^{\lambda^*}\frac{\partial}{\partial x^{\mu^*}} - x^{\mu^*}\frac{\partial}{\partial x^{\lambda^*}}) \\ &+ \sum_{a<b} c^b_a (x^a \frac{\partial}{\partial x^b} - x^b \frac{\partial}{\partial x^a}) + c^{\alpha^*}(\frac{\partial}{\partial x^{\alpha^*}} - x^a \frac{\partial}{\partial x^a} + x^{\lambda^*}\frac{\partial}{\partial x^{\lambda^*}}) \end{aligned}$$

Hence

$$V = c^a E_a + c^{\lambda^*} E_{\lambda^*} + c^{\alpha^*} E_{\alpha^*} + \sum_{a<b} c^b_a E^a_b + \sum_{\lambda^*<\mu^*} c^{\mu^*}_{\lambda^*} E^{\lambda^*}_{\mu^*} \tag{4.23}$$

where

$$\begin{aligned} E_a &= \frac{\partial}{\partial x^a}, \quad E_{\lambda^*} = \frac{\partial}{\partial x^{\lambda^*}}, \quad E_{\alpha^*} = \frac{\partial}{\partial x^{\alpha^*}} - x^a \frac{\partial}{\partial x^a} + x^{\lambda^*}\frac{\partial}{\partial x^{\lambda^*}}, \\ E^a_b &= x^a \frac{\partial}{\partial x^b} - x^b \frac{\partial}{\partial x^a} (a<b), \ E^{\lambda^*}_{\mu^*} = x^{\lambda^*}\frac{\partial}{\partial x^{\mu^*}} - x^{\mu^*}\frac{\partial}{\partial x^{\lambda^*}} (\lambda < \mu) \end{aligned}$$

The set of vector fields $\{E_a, E_{\lambda^*}, E_{\alpha^*}, E^a_b, E^{\lambda^*}_{\mu^*}, a<b, \lambda^*<\mu^*\}$ is a basis for the Lie algebra L of all infinitesimal automorphisms of the almost r-paracontact Riemannian manifold of P-Sasakian type R^n with the structure Σ given by (4.2) – (4.5). This set consists of $p+q+r+\frac{1}{2}(p^2-p)+\frac{1}{2}(q^2-q) = \frac{1}{2}p(p+1)+\frac{1}{2}q(q+1)+r$ vectors and this completes the proof.

5. In this section some representation of the Lie group of automorphisms of an almost r-paracontact Riemannian manifold of P-Sasakian type is presented.

Let $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)$ be an almost r-paracontact structure of P-Sasakian type on a Riemannian manifold M_n and let $A(M_n)$ be the group of automorphisms of M_n. Let $V = R^n \oplus R^p_q$ be a vector space, where R^p_q is a vector space generated by the set of $n \times n$-matrices $\{e^2_1, e^3_1, e^3_2, e^4_1, \ldots, e^p_1, \ldots, e^p_{p-1}, e^{p+1}_p, e^{p+2}_p, e^{p+2}_{p+1}, \ldots, e^{n-r}_{n-r-1}\}$ with $e^k_l = [a^i_j]$, where $a^i_j = 1$ if $i=k, j=l$ and $a^i_j = 0$ otherwise. Let Aut$V$ be the group of automorphisms of V. Then $\delta : A(M_n) \to \text{Aut}V, f \mapsto \delta(f) = id_V$ is a representation of the Lie group $A(M_n)$ of authomorphisms of M_n.

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