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ON THE GEOMETRY OF SOME 4–DIMENSIONAL EINSTEIN MANIFOLDS

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Abstract. Using a skew-symmetric curvature operator a classification of some 4-dimensional Einstein manifolds has been done in [1]. In the present paper we study one of the classes of these Einstein manifolds and answer the question about the global constancy of the functions, characterizing them.

§1 Introduction.

Let (M, g) be an n-dimensional Riemannian manifold and R be the curvature tensor. In [1] we have considered a linear curvature operator $\kappa_{X,Y}$, acting in the tangent space M_p at a point $p \in M$, defined by

$$\kappa_{X,Y} = R(X,Y)u, \tag{1}$$

where X, Y is an arbitrary othonormal pair of vectors in M_p and $u \in M_p$. The curvature operator $\kappa_{X,Y}$ is a skew-symmetric one and it leaves the 2-dimensional subspace $E^2(p; X, Y)$, spaned by X, Y, invariant. That is why we have introduced the curvature operator

$$\kappa_{E^2} = \kappa_{X,Y} \tag{2}$$

with respect to any $E^2 \subset M_p$ at any point $p \in M$. The operator κ_{E^2} has a characteristical equation which in the case $n = 4$ has the form

$$\lambda^4 + J_2.\lambda^2 + J_4 = 0,$$

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where

$$J_2 = J_2(p; E^2) = \sum_{1 \le i < j \le 4} [R(X, Y, e_i, e_j)]^2 , \tag{3}$$

$$\begin{aligned} J_4 = J_4(p; E^2) = [\ & R(X, Y, e_1, e_2).R(X, Y, e_3, e_4) \\ & + R(X, Y, e_2, e_3).R(X, Y, e_1, e_4) + R(X, Y, e_3, e_1).R(X, Y, e_2, e_4)\ ]^2 . \end{aligned} \tag{4}$$

If the manifold is of constant sectional curvature c, then the coefficients in characteristical equation of the operator κ_{E^2} are constants $J_2 = c^2$ and $J_4 = 0$. We study the manifolds, for which the second characteristical coefficient of the operator κ_{E^2} is zero, i.e. the manifolds with the property

$$J_4(p; E^2) = 0. \tag{5}$$

A classification of such kind of manifolds has been done in [1]. Using the curvature operator κ_{E^2} we have proved ([1], [2]), that a 4-dimensional Einstein manifold (M, g) has the property (5) iff it is of one of the following two types:

1) (M, g) is of constant sectional curvature;

2) The curvature tensor R satisfies the conditions:

$$\left\{ \begin{array}{l} R_{1213} = R_{1214} = R_{1314} = 0 \\ R_{1223} = R_{1224} = 0,\ \ R_{1323} \neq 0 \\ R_{1234} = R_{1423} = R_{1324} = 0 \\ K_{13} + K_{23} = 0,\ \ K_{13} \neq 0 \\ K_{12}^2 = K_{13}^2 + R_{1323}^2 \\ (K_{12} \text{ is a global constant}), \end{array} \right. \tag{Γ}$$

where $K_{ij} = R_{ijij}$ is the sectional curvature of the plane $(e_i \wedge e_j)$.

The manifolds satisfying (Γ) are characterized by the functions $K_{12}(p; E^2)$, which is a global constant of the manifold, $K_{13}(p; E^2)$ and $R_{1323}(p; E^2)$. There is an orthogonal group of transformations in the tangent space M_p, keeping the curvature conditions (Γ) invariant. We have found these transformations and also determined the changing of the functions $K_{13}(p; E^2)$ and $R_{1323}(p; E^2)$ by them [3].

§2. A representation of the curvature tensor.

Let (M, g) be a 4-dimensional nonflat Einstein manifold satisfying the curvature conditions (Γ) and e_1, e_2, e_3, e_4 be an orthonormal base of M_p. Then, the following relations

$$\begin{aligned} & R_{ijjk} + R_{illk} = 0,\ \ i, j, k, l - \text{different}, \\ & K_{12} = K_{34},\ \ K_{13} = K_{24},\ \ K_{23} = K_{14} \end{aligned} \tag{6}$$

also hold.

Let x, y, z, u be arbitrary tangent vectors. Then,

$$x=\sum_{i=1}^{4} x^i e_i,\ y=\sum_{j=1}^{4} y^j e_j,\ z=\sum_{k=1}^{4} z^k e_k,\ u=\sum_{l=1}^{4} u^l e_l,$$

and

$$R(x,y,z,u)=R\,(\sum_{i=1}^{4} x^i e_i, \sum_{j=1}^{4} y^j e_j, \sum_{k=1}^{4} z^k e_k, \sum_{l=1}^{4} u^l e_l). \tag{7}$$

Because of (Γ), only the components K_{12}, K_{13} and R_{1323} of the sum (7) are nonzero. Let us notice

$$\begin{aligned} a_{ij} &= x^i y^j - x^j y^i \\ b_{ij} &= z^i u^j - z^j u^i. \end{aligned} \tag{8}$$

Then, taking into account (6), the coefficient of K_{12} in (7) can be written in the form

$$a_{12}b_{12}+a_{34}b_{34}. \tag{9}$$

Since $K_{13}=K_{24}, K_{23}=K_{14}$ and $K_{13}+K_{23}=0$, the coefficient of K_{13} is

$$a_{13}b_{13}+a_{24}b_{24}-a_{23}b_{23}-a_{14}b_{14}. \tag{10}$$

Analogously, for the coefficients of R_{1323} we obtain

$$a_{13}b_{23}+a_{23}b_{13}-a_{14}b_{24}-a_{24}b_{14}. \tag{11}$$

Now, using that

$$x^i=g(x,e_i),\ y^j=g(y,e_j),\ z^k=g(z,e_k),\ u^l=g(u,e_l),$$

we can present the coefficients (9),(10),(11) in the following way:

$$\begin{aligned} &\alpha_{12}(x,y,z,u)\\ &=[g(x,e_1)g(y,e_2)-g(x,e_2)g(y,e_1)][g(z,e_1)g(u,e_2)-g(z,e_2)g(u,e_1)]\\ &+[g(x,e_3)g(y,e_4)-g(x,e_4)g(y,e_3)][g(z,e_3)g(u,e_4)-g(z,e_4)g(u,e_3)], \end{aligned}$$

$$\begin{aligned} &\alpha_{13}(x,y,z,u)\\ &=[g(x,e_1)g(y,e_3)-g(x,e_3)g(y,e_1)][g(z,e_1)g(u,e_3)-g(z,e_3)g(u,e_1)]\\ &+[g(x,e_2)g(y,e_4)-g(x,e_4)g(y,e_2)][g(z,e_2)g(u,e_4)-g(z,e_4)g(u,e_3)]\\ &-[g(x,e_2)g(y,e_3)-g(x,e_3)g(y,e_2)][g(z,e_2)g(u,e_3)-g(z,e_3)g(u,e_2)]\\ &-[g(x,e_1)g(y,e_4)-g(x,e_4)g(y,e_1)][g(z,e_1)g(u,e_4)-g(z,e_4)g(u,e_1)], \end{aligned}$$

$$\begin{aligned}
&\alpha_{1323}(x,y,z,u)\\
&=[g(x,e_1)g(y,e_3)-g(x,e_3)g(y,e_1)][g(z,e_2)g(u,e_3)-g(z,e_3)g(u,e_2)]\\
&+[g(x,e_2)g(y,e_3)-g(x,e_3)g(y,e_2)][g(z,e_1)g(u,e_3)-g(z,e_3)g(u,e_1)]\\
&-[g(x,e_1)g(y,e_4)-g(x,e_4)g(y,e_1)][g(z,e_2)g(u,e_4)-g(z,e_4)g(u,e_2)]\\
&-[g(x,e_2)g(y,e_4)-g(x,e_4)g(y,e_2)][g(z,e_1)g(u,e_4)-g(z,e_4)g(u,e_1)]
\end{aligned}$$

and the expression (7) takes the form:

(12)

$$R(x,y,z,u)=\alpha_{12}(x,y,z,u).K_{12}+\alpha_{13}(x,y,z,u).K_{13}+\alpha_{1323}(x,y,z,u).R_{1323}.$$

The tensors $R, \alpha_{12}, \alpha_{13}$ and α_{1323} are of type (0,4). For the corresponding tensors of type (1,3) we get

$$R(z,u,y)=\alpha_{12}(z,u,y).K_{12}+\alpha_{13}(z,u,y).K_{13}+\alpha_{1323}(z,u,y).R_{1323}, \tag{13}$$

where

$$\begin{aligned}
&\alpha_{12}(z,u,y)\\
&=[g(z,e_1)g(u,e_2)-g(z,e_2)g(u,e_1)][g(y,e_2)e_1-g(y,e_1)e_2]\\
&+[g(z,e_3)g(u,e_4)-g(z,e_4)g(u,e_3)][g(y,e_4)e_3-g(y,e_3)e_4],\\
&\alpha_{13}(z,u,y)\\
&=[g(z,e_1)g(u,e_3)-g(z,e_3)g(u,e_1)][g(y,e_3)e_1-g(y,e_1)e_3]\\
&+[g(z,e_2)g(u,e_4)-g(z,e_4)g(u,e_2)][g(y,e_4)e_2-g(z,e_2)e_4]\\
&-[g(z,e_2)g(u,e_3)-g(z,e_3)g(u,e_2)][g(y,e_3)e_2-g(y,e_2)e_3]\\
&-[g(z,e_1)g(u,e_4)-g(z,e_4)g(u,e_1)][g(y,e_4)e_1-g(y,e_1)e_4],\\
&\alpha_{1323}(z,u,y)\\
&=[g(z,e_2)g(u,e_3)-g(z,e_3)g(u,e_2)][g(y,e_3)e_1-g(y,e_1)e_3]\\
&+[g(z,e_1)g(u,e_3)-g(z,e_3)g(u,e_1)][g(y,e_3)e_2-g(y,e_2)e_3]\\
&-[g(z,e_2)g(u,e_4)-g(z,e_4)g(u,e_2)][g(y,e_4)e_1-g(y,e_1)e_4]\\
&-[g(z,e_1)g(u,e_4)-g(z,e_4)g(u,e_1)][g(y,e_4)e_2-g(y,e_2)e_4]
\end{aligned}$$

So, we have represented the curvature tensor only by the metric tensor of the manifold. Moreover, this representation of R is with respect to an arbitrary orthonormal base of M_p.

§3. Global constancy of the characterizing functions.

According to the curvature conditions (Γ) the functions $K_{12}(p;E^2)$, $K_{13}(p;E^2)$ and $R_{1323}(p;E^2)$ have a special role for the characterizing the manifold. From (6) and (Γ) we have $K_{13}+K_{14}=0$ and then for the Ricci curvature with respect to e_1, we obtain:

$$\rho(e_1,e_1)=\sum_{i=2}^{4}K_{1i}=K_{12}. \tag{14}$$

(M, g) is an Einstein manifold and it follows from (14) and the well known theorem of Herglotz that K_{12} is a global constant of the manifold. It is interesting to answer the question whether the other two characteizing functions $K_{13}(p; E^2)$ and $K_{1323}(p; E^2)$ are global constants too? For this purpose, let us differentiate the representation (13), considering now x, y, z, u and e_1, e_2, e_3, e_4 as smooth vector fields around a point $p \in M$.

$$
\begin{aligned}
&(\nabla_x R)(z,u,y)\\
&= \nabla_x[\alpha_{12}(z,u,y).K_{12} + \alpha_{13}(z,u,y).K_{13} + \alpha_{1323}(z,u,y).R_{1323}]\\
&- [\alpha_{12}(\nabla_x z,u,y).K_{12} + \alpha_{13}(\nabla_x z,u,y).K_{13} + \alpha_{1323}(\nabla_x z,u,y).R_{1323}]\\
&- [\alpha_{12}(z,\nabla_x u,y).K_{12} + \alpha_{13}(z,\nabla_x u,y).K_{13} + \alpha_{1323}(z,\nabla_x u,y).R_{1323}]\\
&- [\alpha_{12}(z,u,\nabla_x y).K_{12} + \alpha_{13}(z,u,\nabla_x y).K_{13} + \alpha_{1323}(z,u,\nabla_x y).R_{1323}].
\end{aligned}
\tag{15}
$$

From the well known second Bianchi identity

$$\sum_{(x,z,u)} (\nabla_x R)(z,u,y) = 0$$

and (15), we get:

$$
\begin{aligned}
\sum_{(x,z,u)} \{ & \, x(K_{12}).\alpha_{12}(z,u,y) + K_{12}.\nabla_x(\alpha_{12}(z,u,y))\\
&+ x(K_{13}).\alpha_{13}(z,u,y) + K_{13}.\nabla_x(\alpha_{13}(z,u,y))\\
&+ x(R_{1323}).\alpha_{1323}(z,u,y) + R_{1323}.\nabla_x(\alpha_{1323}(z,u,y))\\
&- K_{12}[\alpha_{12}(\nabla_x z,u,y) + \alpha_{12}(z,\nabla_x u,y) + \alpha_{12}(z,u,\nabla_x y)]\\
&- K_{13}[\alpha_{13}(\nabla_x z,u,y) + \alpha_{13}(z,\nabla_x u,y) + \alpha_{13}(z,u,\nabla_x y)]\\
&- R_{1323}[\alpha_{1323}(\nabla_x z,u,y) + \alpha_{1323}(z,\nabla_x u,y) + \alpha_{1323}(z,u,\nabla_x y)] \, \} = 0.
\end{aligned}
\tag{16}
$$

We may not write (16) in detail (which takes more than 20 pages) because we can choose x, y, z, u to be some of e_1, e_2, e_3, e_4 in a proper way, so that many of the addends in (16) are zero. Let us choose $x = e_4, y = e_1, z = e_1, u = e_3$. Using

$$
\begin{aligned}
&g(\nabla_{e_i} e_j, e_j) = 0\\
&g(\nabla_{e_i} e_j, e_k) = -g(\nabla_{e_i} e_k, e_j),
\end{aligned}
\tag{17}
$$

we get

$$
\begin{aligned}
&e_4(K_{13}).e_3 + K_{13}.\nabla_{e_4} e_3\\
&+ K_{12}\{ [g(\nabla_{e_4} e_2, e_3).e_2 - 2g(\nabla_{e_1} e_1, e_4)].e_3 + g(\nabla_{e_1} e_1, e_3).e_4 \}\\
&+ K_{13}\{ [2g(\nabla_{e_1} e_4, e_3) + g(\nabla_{e_4} e_1, e_3)].e_1 + 2g(\nabla_{e_1} e_1, e_4).e_3\\
&\qquad + [g(\nabla_{e_1} e_1, e_3) - g(\nabla_{e_4} e_4, e_3)].e_4 \}\\
&+ R_{1323}\{ [2g(\nabla_{e_1} e_4, e_3) + g(\nabla_{e_4} e_1, e_3)].e_2 - 2[g(\nabla_{e_1} e_4, e_2)\\
&\qquad + g(\nabla_{e_4} e_1, e_2)].e_3 - g(\nabla_{e_1} e_3, e_2).e_4 \} = 0.
\end{aligned}
$$

The vectors e_1, e_2, e_3, e_4 are linearly independent. Having in mind

$$\nabla_{e_4} e_3 = g(\nabla_{e_4} e_3, e_1).e_1 + g(\nabla_{e_4} e_3, e_2).e_2 + g(\nabla_{e_4} e_3, e_4).e_4 \tag{19}$$

and substituting it into (18), it is easily seen that

$$e_4(K_{13}) = \lambda_1.K_{12} + \lambda_2.K_{13} + \lambda_3.R_{1323}, \tag{20}$$

where

$$\begin{aligned} \lambda_1 &= \ 2g(\nabla_{e_1} e_1, e_4) \\ \lambda_2 &= -2g(\nabla_{e_1} e_1, e_4) \\ \lambda_3 &= \ 2[g(\nabla_{e_1} e_4, e_2) + g(\nabla_{e_4} e_1, e_2)]. \end{aligned} \tag{21}$$

The coefficients $\lambda_i, i = 1, 2, 3$ in (21) do not depend on the appropriate choice of the vectors (x, y, z, u). If for (x, y, z, u), inspite of (e_4, e_1, e_1, e_3) we take (e_4, e_2, e_2, e_3), i.e. changing $1 \to 2$, then the new coefficients are obtained from (21) by changing $1 \to 2$, too.

In order to express $e_4(R_{1323})$ we replace (x, y, z, u) by (e_1, e_4, e_2, e_3). Then from (16) and (17) we get

$$\begin{aligned} & e_4(R_{1323}).e_3 + R_{1323}.\nabla_{e_4} e_3 \\ & + K_{12} \{ \ g(\nabla_{e_4} e_3, e_2).e_1 - [g(\nabla_{e_1} e_2, e_4) + g(\nabla_{e_2} e_1, e_4)].e_3 + g(\nabla_{e_2} e_1, e_3).e_4 \ \} \\ & + K_{13} \{ \ g(\nabla_{e_4} e_2, e_3).e_1 + [g(\nabla_{e_1} e_3, e_4) - g(\nabla_{e_1} e_4, e_3)].e_2 + [g(\nabla_{e_1} e_4, e_2) \\ & - g(\nabla_{e_4} e_2, e_1) + g(\nabla_{e_4} e_1, e_2) - g(\nabla_{e_2} e_4, e_1)].e_3 - g(\nabla_{e_2} e_3, e_1).e_4 \ \} \\ & + R_{1323} \{ \ [g(\nabla_{e_1} e_4, e_3) - g(\nabla_{e_1} e_3, e_4)].e_1 + [g(\nabla_{e_4} e_2, e_3).e_2 \\ & - [g(\nabla_{e_1} e_4, e_1) + g(\nabla_{e_2} e_4, e_2)].e_3 - [g(\nabla_{e_2} e_3, e_2) + g(\nabla_{e_4} e_3, e_4)].e_4 \ \} = 0. \end{aligned}$$

From (19),(20) and due to the linear independence of the vectors e_1, e_2, e_3, e_4 one can easily obtain

$$e_4(R_{1323}) = \mu_1.K_{12} + \mu_2.K_{13} + \mu_3.R_{1323}, \tag{23}$$

where

$$\begin{aligned} \mu_1 &= g(\nabla_{e_1} e_2, e_4) + g(\nabla_{e_2} e_1, e_4) \\ \mu_2 &= g(\nabla_{e_1} e_2, e_4) + g(\nabla_{e_2} e_4, e_1) - 2g(\nabla_{e_4} e_1, e_2) \\ \mu_3 &= -g(\nabla_{e_1} e_1, e_4) - g(\nabla_{e_2} e_2, e_4). \end{aligned} \tag{24}$$

The coefficients $\mu_i, i = 1, 2, 3$ in (24) do not depend on the appropriate choice of (x, y, z, u). Indeed, if inspite of (e_1, e_4, e_2, e_3), we take (e_4, e_2, e_1, e_3) or (e_4, e_1, e_2, e_3), the new coefficients coincide with these in (24). In this way, we obtained the system for the characterizing functions:

$$\begin{cases} e_4(K_{12}) = 0 \\ e_4(K_{13}) = \lambda_1.K_{12} + \lambda_2.K_{13} + \lambda_3.R_{1323} \\ e_4(R_{1323}) = \mu_1.K_{12} + \mu_2.K_{13} + \mu_3.R_{1323}, \end{cases} \tag{25}$$

where $\lambda_i, i = 1,2,3$ and $\mu_j, j = 1,2,3$ are given by (21) and (24), respectively.

Now let us differentiate the last equation of the curvature conditions (Γ):

$$K_{12}^2 = K_{13}^2 + R_{1323}^2. \tag{26}$$

We have

$$K_{12}.e_4(K_{12}) = K_{13}.e_4(K_{13}) + R_{1323}.e_4(R_{1323}). \tag{27}$$

Substituting (25) into (27), we obtain

$$K_{12}(\lambda_1.K_{13} + \mu_1.R_{1323}) = -[\lambda_2.K_{13}^2 + \mu_3.R_{1323}^2 + (\lambda_3 + \mu_2).K_{13}.R_{1323}]. \tag{28}$$

From (26) we can express K_{12}, replace it into (28) and square both sides of that equality. The result is a polynomial of the functions $K_{13}(p; E^2)$ and $R_{1323}(p; E^2)$, namely:

$$A_1.K_{13}^4 + A_2.K_{13}^3.R_{1323} + A_3.K_{13}^2.R_{1323}^2 + A_4.K_{13}.R_{1323}^3 + A_5.R_{1323}^4 = 0, \tag{29}$$

where

$$\begin{aligned} A_1 &= \lambda_1^2 - \lambda_2^2 \\ A_2 &= 2\lambda_1\mu_1 - 2\lambda_2(\lambda_3 + \mu_2) \\ A_3 &= \lambda_1^2 + \mu_1^2 - (\lambda_3 + \mu_2)^2 - 2\lambda_2\mu_3 \\ A_4 &= 2\lambda_1\mu_1 - 2\mu_3(\lambda_3 + \mu_2) \\ A_5 &= \mu_1^2 - \mu_3^2. \end{aligned} \tag{30}$$

From (21) and (24) we have

$$\begin{cases} \lambda_1 = -\lambda_2 \\ \lambda_3 + \mu_2 = -\mu_1, \end{cases} \tag{31}$$

and then $A_1 = A_2 = 0$. The equation (29) takes the form

$$R_{1323}^2(A_3.K_{13}^2 + A_4.K_{13}.R_{1323} + A_5.R_{1323}^2) = 0.$$

Because of (Γ), the polynomial of K_{13} and R_{1323} in the brackets of the last equation is identically equal to zero, which implicates that all of its coefficients are zero, i.e.

$$A_3 = A_4 = A_5 = 0. \tag{32}$$

The equations (30),(31) and (32) give us the system

$$\begin{aligned} &33.1. \quad \lambda_1^2 + 2\lambda_1\mu_3 = 0 \\ &33.2. \quad \lambda_1\mu_1 + \mu_1\mu_3 = 0 \\ &33.3 \quad \mu_1^2 - \mu_3^2 = 0 \end{aligned} \tag{33}$$

From (33.2) it follows that there are 3 possible cases:

1) $\mu_1 = 0, \lambda_1 \neq -\mu_3$. Then from (33.3) we get $\mu_3 = 0$ and from (33.1): $\lambda_1 = 0$, i.e. in this case we have $\lambda_1 = -\lambda_2 = 0, \mu_1 = \mu_3 = 0$. But $\lambda_1 \neq -\mu_3$ and thus, 1) is impossible;

2) $\mu_1 \neq 0, \lambda_1 = -\mu_3$. Then, from (33.1) we obtain $\mu_3 = 0$ and from (33.3) it follows $\mu_1 = 0$, which is impossible;

3) $\mu_1 = 0, \lambda_1 = -\mu_3$. Then from (33.3) and (33.1) it follows $\mu_3 = 0$ and $\lambda_1 = 0$, respectively. So, in this case we have

$$\lambda_1 = -\lambda_2 = 0,\ \mu_1 = \mu_3 = 0,\ \lambda_3 + \mu_2 = 0. \tag{34}$$

Then the system (25) can be written in the form

$$\begin{aligned} &e_4(K_{12}) = 0 \\ &e_4(K_{13}) = \lambda_3 R_{1323} \\ &e_4(R_{1323}) = -\lambda_3 . K_{13}. \end{aligned} \tag{35}$$

This means that in the general case K_{12} is the only global constant of the manifold. The functions $K_{13}(p; E^2)$ and $R_{1323}(p; E^2)$ are not global constants.

Let us now suppose that K_{13} and R_{1323} are global constants too, i.e.

$$e_4(K_{13}) = e_4(R_{1323}) = 0.$$

Then, from (35) we get $\lambda_3 = 0$ which is equivalent to the equation

$$g(\nabla_{e_1} e_4, e_2) = -g(\nabla_{e_4} e_1, e_2). \tag{36}$$

As we have already noticed, the coefficients $\lambda_i, i = 1, 2, 3$ do not depend on the appropriate choice of $(x, y, z, u) = (e_i, e_j, e_k, e_l)$ and from (36) we have

$$g(\nabla_{e_i} e_j, e_k) = -g(\nabla_{e_j} e_i, e_k)\ ,\ i, j, k = 1, 2, 3, 4. \tag{37}$$

According to (17) and (37) we can write

$$g(\nabla_{e_i} e_j, e_k) = g(\nabla_{e_j} e_k, e_i) = g(\nabla_{e_k} e_i, e_j). \tag{38}$$

Thus, we have proved the following

Theorem. *Let (M, g) be a 4-dimensional nonflat Einstein manifold satisfying the curvature conditions (Γ). Then, (M, g) is one of the following types:*

1) Only one of the characterizing functions, namely K_{12}, is a global constant of the manifold and the other two functions $K_{13}(p; E^2)$, $R_{1323}(p; E^2)$ are not global constants.

2) Three of the functions K_{12}, K_{13} and R_{1323} are global constants of the manifold and the metric tensor has the property (38).

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