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ON THE GENERAL TRACE DECOMPOSITION PROBLEM

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ABSTRACT. Another formulation of Krupka's trace decomposition theorem together with "constructive" proof is presented. Moreover the trace decomposition problem is generalized to the case of tensor spaces with affine structures.

1. Introduction

Let E be a real n-dimensional vector space, E^p_q the vector space of tensors of type (p,q) on E. We suppose that a fixed basis of E is chosen, which determines a unique basis of E^p_q. Any tensor $A \in E^p_q$ having components $A^{i_1\cdots i_p}_{j_1\cdots j_q}$ with respect to this basis will be denoted, as usual, by $A \equiv \left(A^{i_1\cdots i_p}_{j_1\cdots j_q}\right)$.

A tensor $A \in E^p_q$ will be called *traceless* if its traces are all zero. By the *trace decomposition problem* we mean the problem of existence and uniqueness of a decomposition of a tensor in which one term is traceless while the remaining terms are linear combinations of the Kronecker's δ-tensor. The following theorem is proved in [2] (Krupka's trace decomposition theorem):

Let p, q, n be positive integers, $p \le q$, and $A \equiv \left(A^{i_1\cdots i_p}_{j_1\cdots j_q}\right) \in E^p_q$. Then there exist a system $B \equiv \left(B^{i_1\cdots i_p}_{j_1\cdots j_q}\right)$ and systems $B^{(r)}_{(s)} \equiv \left(B^{(r)i_1\cdots i_{p-1}}_{(s)j_1\cdots j_{q-1}}\right)$, where $1 \le r \le p$, $1 \le s \le q$, such that

$$A^{i_1 i_2\cdots i_p}_{j_1 j_2\cdots j_q} = B^{i_1 i_2\cdots i_p}_{j_1 j_2\cdots j_q} + \sum_{r=1}^{p}\sum_{s=1}^{q} \delta^{i_r}_{j_s} B^{(r)\,i_1\cdots i_{r-1} i_{r+1}\cdots i_p}_{(s)\,j_1\cdots j_{s-1} j_{s+1}\cdots j_q}$$

where δ^h_i is the Kronecker δ and

$$\forall r = \overline{1,p}; \quad \forall s = \overline{1,q}: \quad B^{\cdots i_{r-1}\alpha\, i_{r+1}\cdots i_p}_{\cdots j_{s-1}\alpha\, j_{s+1}\cdots j_q} = 0.$$

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The system B is unique. If $p+q\le n+1$, the systems $B^{(r)}_{(s)}$ are also unique.

This paper is motivated by Weyl's theory [3] and by Krupka's results [2].

2. Another form of D. Krupka's trace decomposition theorem

We introduce another form of the trace decomposition theorem, whose presented proof has a constructive character and may be applied for determining the decomposition of a concrete tensor.

Theorem 1. *Let $A^{i_1i_2\cdots i_p}_{j_1j_2\cdots j_q}$ be a tensor of type (p,q). If $n+1\ge p+q$, then there exists a unique decomposition of A of the form*

$$A^{i_1i_2\cdots i_p}_{j_1j_2\cdots j_q}=B^{i_1i_2\cdots i_p}_{j_1j_2\cdots j_q}+\sum_{t=1}^{\min\{p,q\}}\sum_{\otimes}\delta^{i_{\rho_1}i_{\rho_2}\cdots i_{\rho_t}}_{j_{\sigma_1}j_{\sigma_2}\cdots j_{\sigma_t}}\,B^{\{\rho_1\rho_2\cdots\rho_t\}\cdots}_{\{\sigma_1\sigma_2\cdots\sigma_t\}\cdots}\,, \tag{1}$$

where

$$\otimes=\begin{cases}\forall\rho_1,\rho_2,\cdots,\rho_t\in\overline{1,p} & (\rho_1<\rho_2<\cdots<\rho_t)\\ \forall\sigma_1,\sigma_2,\cdots,\sigma_t\in\overline{1,q} & (\sigma_i \text{ are mutually different})\end{cases} \tag{2}$$

and $\delta^{i_1i_2\cdots i_t}_{j_1j_2\cdots j_t}=\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\cdots\delta^{i_t}_{j_t}$, such that the tensors B, $B^{\{\rho_1\rho_2\cdots\rho_t\}}_{\{\sigma_1\sigma_2\cdots\sigma_t\}}$ are traceless.

Proof. Under assumption $p\le q$ formula (1) may be rewritten in a more simple form:

$$A^{i_1i_2\cdots i_p}_{j_1j_2\cdots j_q}=B^{i_1i_2\cdots i_p}_{j_1j_2\cdots j_q}+\sum_{t=1}^{p}\sum_{\otimes}\delta^{i_{\rho_1}i_{\rho_2}\cdots i_{\rho_t}}_{j_{\sigma_1}j_{\sigma_2}\cdots j_{\sigma_t}}\,B^{\{\rho_1\rho_2\cdots\rho_t\}\cdots}_{\{\sigma_1\sigma_2\cdots\sigma_t\}\cdots} \tag{3}$$

where the indices $\rho_1,\rho_2,\cdots,\rho_t,\sigma_1,\sigma_2,\cdots,\sigma_t$ fulfill again the conditions (2). For any $t=1,\cdots,p$, put

$$W_1\equiv n+p+q-2;\quad W_t\equiv nW_{t-1}+p+q-2t\ \text{ if }\ t\ge 2. \tag{4}$$

It may be proved without troubles that for $n+1\ge p+q$ and for any $t=1,\cdots,p$ the inequality

$$n^t>W_t-n^t\qquad\text{for}\quad n+1\ge p+q. \tag{5}$$

is true.

a) First let us assume that the tensor A admits a decomposition of the form (3) with traceless tensor B and $B^{\{\rho_1\rho_2\cdots\rho_t\}}_{\{\sigma_1\sigma_2\cdots\sigma_t\}}$. Contracting (3) with the tensor $\delta^{j_1j_2\cdots j_p}_{i_1i_2\cdots i_p}$ we obtain

$$n^pB^{\{1,2,\cdots,p\}}_{\{\sigma_1,\sigma_2,\cdots,\sigma_p\}\cdots}+\sum B^{\{1,2,\cdots,p\}}_{\{\tau_1,\tau_2,\cdots,\tau_p\}\cdots}=T_{\cdots\alpha_1\cdots\alpha_p\cdots}^{\ \ \ \alpha_1\cdots\alpha_p} \tag{6}$$

The right sides of (6) are equal to the contractions of the tensor A with respect to all p contravariant indices, the sum $\sum$ has at most W_p-n^p summands $B^{\{1,2,\cdots,p\}}_{\{\tau_1,\tau_2,\cdots,\tau_p\}}$ where $\{\tau_1,\tau_2,\cdots,\tau_p\}\ne\{\sigma_1,\sigma_2,\cdots,\sigma_p\}$ with pairwise different indices $\tau_i\in\overline{1,q}$.

The system (6) is formed of (inhomogeneous) linear equations for the unknowns

$$B^{\{1,2,\cdots,p\}}_{\{\sigma_1,\sigma_2,\cdots,\sigma_p\}\,i_{p+1}\cdots i_q}\qquad(\sigma_1,\sigma_2,\cdots,\sigma_p\in\overline{1,q}). \tag{7}$$

If $n^p > W_p - n^p$ (this condition is fulfilled in the case $n+1 \geq p+q$) the system (6) has a unique solution (7). By this way, for any tensor A all tensors (7) are uniquely determined.

b) We can find in a similar way the tensors $B^{\{\rho_1\rho_2\cdots\rho_{p-1}\}h}_{\{\sigma_1\sigma_2\cdots\sigma_{p-1}\}\cdots}$ introduced as follows.

Supposing that the tensors

$$B \quad \text{and} \quad B^{\{\rho_1\rho_2\cdots\rho_t\}\cdots}_{\{\sigma_1\sigma_2\cdots\sigma_t\}\cdots} \quad (t \leq p-2)$$

are tracelles and using contraction of (3) with $\delta^{j_{\sigma_1}j_{\sigma_2}\cdots j_{\sigma_{p-1}}}_{i_{\rho_1}i_{\rho_2}\cdots i_{\rho_{p-1}}}$ we get a system of equations

$$n^{p-1}B^{\{\rho_1\rho_2\cdots\rho_{p-1}\}h}_{\{\sigma_1\sigma_2\cdots\sigma_{p-1}\}\cdots} + \sum B^{\{\varepsilon_1\varepsilon_2\cdots\varepsilon_{p-1}\}h}_{\{\tau_1\tau_2\cdots\tau_{p-1}\}\cdots} = \cdots . \tag{8}$$

The right sides of (8) may be calculated by means of the tensor A, the sum $\sum B^{\{\varepsilon_1\varepsilon_2\cdots\varepsilon_{p-1}\}h}_{\{\tau_1\tau_2\cdots\tau_{p-1}\}\cdots}$ has at $W_{p-1} - n^{p-1}$ summands

$$B^{\{\varepsilon_1\varepsilon_2\cdots\varepsilon_{p-1}\}h}_{\{\tau_1\tau_2\cdots\tau_{p-1}\}\cdots} \quad \left(\neq B^{\{\rho_1\rho_2\cdots\rho_{p-1}\}h}_{\{\sigma_1\sigma_2\cdots\sigma_{p-1}\}\cdots}\right).$$

Finally, (8) is a system of linear equations in the unknowns

$$B^{\{\rho_1\rho_2\cdots\rho_{p-1}\}h}_{\{\sigma_1\sigma_2\cdots\sigma_{p-1}\}j_p\cdots j_q} \qquad (\rho_i \in \overline{1,p}, \quad \sigma_i \in \overline{1,q}). \tag{9}$$

having for $n^{p-1} > W_{p-1} - n^{p-1}$ exactly one solution (the condition $n^{p-1} > W_{p-1} - n^{p-1}$ is fulfilled if $n+1 \geq p+q$).

Now we verify the tracelessness of tensors (9). The contractions of formulas (8) with tensors $\delta^{i_\sigma}_h$ $(\sigma \neq \sigma_1, \sigma_2, \cdots, \sigma_{p-1})$ annihilate the right sides of (8) because as to part a) of this proof. Thus we get a system

$$n^{p-1}B^{\{\rho_1\rho_2\cdots\rho_{p-1}\}\alpha}_{\{\sigma_1\sigma_2\cdots\sigma_{p-1}\}\cdots\alpha\cdots} + \sum B^{\{\varepsilon_1\varepsilon_2\cdots\varepsilon_{p-1}\}\alpha}_{\{\tau_1\tau_2\cdots\tau_{p-1}\}\cdots\alpha\cdots} = 0$$

having in the case $n+1 \geq p+q$ the trivial solution only.

c) In a similar way we find successively the tensors

$$B^{\{\rho_1\rho_2\cdots\rho_{p-2}\}\cdots}_{\{\sigma_1\sigma_2\cdots\sigma_{p-2}\}\cdots}, \quad B^{\{\rho_1\rho_2\cdots\rho_{p-3}\}\cdots}_{\{\sigma_1\sigma_2\cdots\sigma_{p-3}\}\cdots}, \quad \cdots \quad , \quad B^{\{\rho_1\}\cdots}_{\{\sigma_1\}\cdots}, \quad B^{\cdots}_{\cdots}$$

and we prove their tracelessness.

Now, the proof can be completed as follows. Let the tensors $B^{\{\rho_1\rho_2\cdots\rho_t\}\cdots}_{\{\sigma_1\sigma_2\cdots\sigma_t\}\cdots}$ $(t = 2, \cdots, p)$ be traceless and determined by the process described above. It remains to find the tensors $B^{\{\rho_1\}\cdots}_{\{\sigma_1\}\cdots}$ and $B^{\cdots}_{\cdots}$.

Supposing that the tensor B is traceless, we obtain by contraction of formulas (3) with tensors $\delta^{j_{\sigma_1}}_{i_{\rho_1}}$ a system of linear (inhomogeneous) equations

$$nB^{\{\rho_1\}\cdots}_{\{\sigma_1\}\cdots} + \sum_{\substack{\varepsilon=\overline{1,p}\ \setminus\ \{\rho_1\} \\ \tau=\overline{1,q}\ \setminus\ \{\sigma_1\}}} B^{\{\varepsilon\}\cdots}_{\{\tau\}\cdots} = \cdots \tag{10}$$

in the unknown tensor

$$B^{\{\rho\}\cdots}_{\{\sigma\}\cdots} \qquad (\rho = \overline{1,p}, \quad \sigma = \overline{1,q}). \tag{11}$$

The right sides of (10) are determined by the tensor A. If $n > p+q-2$ there exist exactly one solution $B^{\{\rho\}\cdots}_{\{\sigma\}\cdots}$ $(\rho = \overline{1,p}, \quad \sigma = \overline{1,q})$ of the system (10).

The contractions (10) give the system of homogeneous linear equations

$$nB^{\{\rho_1\}\cdots\alpha\cdots}_{\{\sigma_1\}\cdots\alpha\cdots} + \sum_{\substack{\varepsilon=\overline{1,p}\setminus\{\rho_1\}\\ \tau=\overline{1,q}\setminus\{\sigma_1\}}} B^{\{\varepsilon\}\cdots\alpha\cdots}_{\{\tau\}\cdots\alpha\cdots} = 0,$$

having the trivial solution $B^{\{\rho\}\cdots\alpha\cdots}_{\{\sigma\}\cdots\alpha\cdots} = 0$ $(\rho = \overline{1,p}$, $\sigma = \overline{1,q})$ only. Therefore the tensors $B^{\{\rho\}\cdots}_{\{\sigma\}\cdots}$ are traceless.

The tensor B may be calculated from formulas (3) when all tensors $B^{\{\rho_1\rho_2\cdots\rho_t\}\cdots}_{\{\sigma_1\sigma_2\cdots\sigma_t\}\cdots}$ are determined. Combining the contractions of (3) with tensors $\delta^{j_{\sigma_1}}_{i_{\rho_1}}$ and previous calculations, we are able to show that the tensor B is traceless too, which finishes the proof.

3. Generalization of trace decomposition problem

In the next text, let F^h_i denote an arbitrary affinor (thus $F^h_i \in E^1_1$). A generalization of the trace decomposition problem for tensors may be formulated in the following way:

Let a tensor $A \in E^p_q$ be given. Does there exist a decomposition of the form

$$\begin{aligned} A^{i_1i_2\cdots i_p}_{j_1j_2\cdots j_q} &= B^{i_1i_2\cdots i_p}_{j_1j_2\cdots j_q} + \\ &+ \sum_{r=1}^{p}\sum_{s=1}^{q}\left\{\delta^{i_r}_{j_s} B^{(r)\,i_1\cdots i_{r-1}i_{r+1}\cdots i_p}_{(s)\,j_1\cdots j_{s-1}j_{s+1}\cdots j_q} + F^{i_r}_{j_s} C^{(r)\,i_1\cdots i_{r-1}i_{r+1}\cdots i_p}_{(s)\,j_1\cdots j_{s-1}j_{s+1}\cdots j_q}\right\} \end{aligned} \tag{12}$$

and fulfilling

$$\forall r = \overline{1,p}; \quad \forall s = \overline{1,q}: \quad B^{\cdots i_{r-1}\alpha\, i_{r+1}\cdots i_p}_{\cdots j_{s-1}\alpha\, j_{s+1}\cdots j_q} = F^\alpha_\beta B^{\cdots i_{r-1}\beta\, i_{r+1}\cdots i_p}_{\cdots j_{s-1}\alpha\, j_{s+1}\cdots j_q} = 0$$

where $B^{(r)}_{(s)}, C^{(r)}_{(s)}$ are some tensors of the type $(p-1, q-1)$?

The decomposition (12) will be called the *F-traceless decomposition*. The F-traceless decomposition need not exist in general case.

Lemma 1. *Let an affinor F satisfy the condition*

$$\Delta \equiv nF^\alpha_\beta F^\beta_\alpha - (F^\alpha_\alpha)^2 \neq 0. \tag{13}$$

Then for any tensor $A \in E^1_1$ there exists its F-traceless decomposition.

Proof. The condition (13) implies that the F-traceless decomposition of an arbitrary affinor A^h_i has the form

$$A^h_i = B^h_i + B\,\delta^h_i + C\,F^h_i, \tag{14}$$

where

$$B \equiv (A^{\gamma}_{\gamma}\ F^{\alpha}_{\beta} F^{\beta}_{\alpha} - F^{\gamma}_{\gamma}\ A^{\alpha}_{\beta} F^{\beta}_{\alpha})/\Delta, \quad C \equiv (nA^{\alpha}_{\beta} A^{\beta}_{\alpha} - F^{\gamma}_{\gamma}\ A^{\alpha}_{\alpha})/\Delta.$$

The tensor B^h_i satisfies $B^{\gamma}_{\gamma} = F^{\alpha}_{\beta} B^{\beta}_{\alpha} = 0$.

The following lemma may be proved by analysing the formula (14).

Lemma 2. *a) Let the affinor F fulfil the condition $F^{\gamma}_{\gamma} = F^{\alpha}_{\beta} F^{\beta}_{\alpha} = 0$. Then the tensor A^h_i has no F-traceless decomposition if and only if $A^{\alpha}_{\beta} F^{\beta}_{\alpha} \neq 0$.*

b) Let an affinor F satisfy the condition $\Delta = 0$, $F^{\gamma}_{\gamma} \neq 0$. Then the tensor A has no F-traceless decomposition if and only if $nA^{\alpha}_{\beta} F^{\beta}_{\alpha} \neq F^{\gamma}_{\gamma}\ A^{\alpha}_{\alpha}$.

It may be proved that for any nilpotent affinor F there exists a tensor having no F-traceless decomposition.

Our further aim is to find a type of affinors allowing results analogous ones described by the trace decomposition theorem.

4. F-decomposition problem for e-structures

Let an affinor F be an e-structure fulfilling the relation

$$F^h_{\alpha} F^{\alpha}_i = e\delta^h_i, \tag{15}$$

where $e = \pm 1$.

The e-structures of this type exist in Kählerian spaces, Hermitian spaces, hyperbolical Kählerian spaces and in some other ones which demonstration the actuality of the F-decomposition problem in these spaces. Similar results have been applied to holomorphically projective mappings of Kählerian spaces [1].

The following theorem is true:

Theorem 2. *Let A be a tensor of type (p, q). If $n > 2(p+q)$, then there exists a unique decomposition of A in the form*

$$A^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} = B^{i_1 i_2 \cdots i_p}_{j_1 j_2 \cdots j_q} + \sum_{t=1}^{\min\{p,q\}} \sum_{\otimes} \overset{*}{Q}{}^{i_{\rho_1} i_{\rho_2} \cdots i_{\rho_t}}_{j_{\sigma_1} j_{\sigma_2} \cdots j_{\sigma_t}} \overset{\diamondsuit}{B}{}^{\cdots}_{\cdots} \tag{16}$$

where

$$\otimes = \begin{cases} \rho_1, \rho_2, & \cdots, & \rho_t & \in \overline{1,p} & (\rho_1 < \rho_2 < \cdots < \rho_t), \\ \sigma_1, \sigma_2, & \cdots, & \sigma_t & \in \overline{1,q} & (\sigma_i \textit{ are different to each other}), \\ \tau_1, \tau_2, & \cdots, & \tau_t & \in 0, 1; & \end{cases}$$

$$* = \{\tau_1, \tau_2, \cdots, \tau_t\}; \qquad \diamondsuit = \left\{ \begin{array}{l} \rho_1, \rho_2, \cdots, \rho_t \\ \sigma_1, \sigma_2, \cdots, \sigma_t \\ \tau_1, \tau_2, \cdots, \tau_t \end{array} \right\}$$

The tensors B and $\overset{\diamondsuit}{B}$ are F-traceless decomposable, and

$$\overset{*}{Q}{}^{i_{\rho_1} i_{\rho_2} \cdots i_{\rho_t}}_{j_{\sigma_1} j_{\sigma_2} \cdots j_{\sigma_t}} \equiv \overset{\{\tau_1\}}{Q}{}^{i_{\rho_1}}_{j_{\sigma_1}} \overset{\{\tau_2\}}{Q}{}^{i_{\rho_2}}_{j_{\sigma_2}} \cdots \overset{\{\tau_t\}}{Q}{}^{i_{\rho_t}}_{j_{\sigma_t}}; \qquad \overset{\{0\}}{Q}{}^{i}_{j} = \delta^i_j; \quad \overset{\{1\}}{Q}{}^{i}_{j} = F^i_j$$

(F^h_i is the structure fulfilling (15)).

The proof is analogous to that of Theorem 1, the formulas (16) need to be contracted with the Kronecker delta as well as with the affinor F.

5. F-traceless decomposition of Riemannian and Ricci tensors in Kählerian spaces

If we apply this decomposition to the *Ricci* and *Riemannian* tensors in Kählerian spaces (equipped with a structure tensor F_i^h) we get the following formulas:

$$R_i^h = E_i^h + \frac{R}{n}\delta_i^h$$

where $E_i^h = R_i^h - \frac{R}{n}\delta_i^h$ is the *Einstein's tensor, R is the scalar curvature in K_n,*

$$R_{ijk}^h = P_{ijk}^h + \frac{1}{n+2}\left(\delta_{[k}^h R_{j]i} + F_{[k}^h F_{j]}^\alpha R_{\alpha i} + 2F_i^h F_j^\alpha R_{\alpha k}\right)$$

where P_{ijk}^h is tensor of *holomorphically-projective curvature*, and

$$R^{hi}{}_{jk} = B^{hi}{}_{jk} +$$

$$+\frac{1}{n-2}\left(\delta_{[k}^h E_{j]}^i - \delta_{[k}^i E_{j]}^h + F_{[k}^h F_{j]}^\alpha E_\alpha^i - F_{[k}^i F_{j]}^\alpha E_\alpha^h\right) + \frac{R}{n^2}\left(\delta_{[k}^h \delta_{j]}^i + F_{[k}^i F_{j]}^h\right),$$

$$R^h{}_{ij}{}^k = B^h{}_{ij}{}^k -$$

$$-\frac{1}{n}\left(\delta_j^h E_i^k + \delta_i^k E_j^h + F_j^h F_i^\alpha E_\alpha^k + F_i^k F_j^\alpha E_\alpha^h\right) - \frac{R}{n^2}\left(\delta_j^h \delta_i^k + F_j^h F_i^k\right),$$

where B is a F-traceless tensor.

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