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SOME APPLICATIONS OF THE EXTRINSIC DISTANCE TO IMMERSED AND EMBEDDED SUBMANIFOLDS

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§ 0. Introduction

We survey here some recent results about embedded or immersed submanifolds of complex or real space forms obtained by F.J. Carreras, F. Giménez and the author (cfr [CGM1,2], [GM2] and [Mi]). The common thread of these results is in their proofs: A basic tool used in all of them is a computation of some laplacians (in the submanifold) of some well chosen functions of the extrinsic distance to a point of the ambient space.

§ 1. Notation

Let $\psi : M \longrightarrow \overline{M}$ be an isometric immersion of a compact Riemannian manifold M, of dimension m, into an n-dimensional Riemannian manifold $\overline{M}$.

Let $o \in \overline{M}$ and let $r : \overline{M} \longrightarrow I\!\!R$ be the function defined by

$$r(p) = d(o,p) \text{ for every } p \in \overline{M},\ d = \text{ distance in } \overline{M}.$$

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We shall also denote by r the composition $r \equiv r \circ \psi : M \longrightarrow \mathbb{R}$. It is called the *extrinsic distance* to o in M.

∂_r and $\partial_r^\top$ will denote the gradients of r in $\overline{M}$ and in M respectively. Let us observe that $\partial_r^\top(m)$ is the image by ψ_*^{-1} of the component of $\partial_r(\psi(m))$ tangent to $\psi_*(T_m M)$, for every $m \in M$.

If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is any function, $f(r)$ will denote the composition $f \circ r : M \longrightarrow \mathbb{R}$. Then the notation $f(r)$ reminds us that $f(r)$ is an extrinsic radial function on M, i.e., that it depends only on the extrinsic distance to o.

By $\mathrm{cut}(o)$ we shall denote the set of cut points of o in $\overline{M}$.

If R denotes the curvature tensor of $\overline{M}$, $R(r)$ will be the $(1,1)$-tensor on $\overline{M} - \mathrm{cut}(o)$ defined by

$$R(r)A = R(\partial_r, A)\partial_r \text{ for every vector } A \text{ on } \overline{M} - \mathrm{cut}(o).$$

It is called radial curvature operator ([GW]). Observe that $R(r)\partial_r = 0$.

By $R_\lambda(r)$ we shall denote the radial curvature operator of a (real or complex) space form $\mathbb{K}^n(\lambda)$.

We denote by $S(r)$ the $(1,1)$-tensor on $\overline{M}$ defined by

$$S(r)(A) = -\overline{\nabla}_A \partial_r \text{ for every } A \text{ tangent to } \overline{M},$$

Let us observe that $S(r)\partial_r = 0$, and $S(r)$ restricted to the vectors tangent to the geodesic sphere $\partial B_r(o)$ of $\overline{M}$ of centre o and radius r is the Weingarten map of this sphere, and $\mathrm{tr}\, S(r)$ is $(n-1)$ times the mean curvature of this sphere.

When $\overline{M} = \mathbb{K}^n(\lambda)$, then $S(r)$ will be denoted by $S_\lambda(r)$.

We shall define the "transplanted" operator $\overline{R}_\lambda(r)$ of $R_\lambda(r)$ in $\overline{M}$ in the following way: let $\phi : T_{p'}\mathbb{K}^n(\lambda) \longrightarrow T_p\overline{M}$ be an (holomorphic if $\overline{M}$ and $\mathbb{K}^n(\lambda)$ are Kähler) isometry, then

$$\overline{R}_\lambda(r)A(q) = \overline{\tau}_t \circ \phi \circ \tau_t^{-1} \circ R_\lambda(r) \circ \tau_t \circ \phi^{-1}(\overline{\tau}_t^{-1}A), \qquad t = \mathrm{distance}(o,q) \text{ in } \overline{M},$$

where $\overline{\tau}_t$ is the parallel transport along the minimizing geodesic γ of $\overline{M}$ from p to q and τ_t is the parallel transport along the minimizing geodesic $exp_{p'} s\phi^{-1}(\gamma'(0))$ of $\mathbb{K}^n(\lambda)$ from p' to $exp_{p'} t\phi^{-1}(\gamma'(0))$.

The "transplanted" operator $\overline{S}_\lambda(r)$ is defined analogously.

We shall denote also by $R(r)$, $\overline{R}_\lambda(r)$, $S(r)$ and $\overline{S}_\lambda(r)$ the quadratic forms associated to these operators and the metric of $\overline{M}$.

Given two quadratic forms, A and B, we shall say that $A \geq B$ if $A - B$ is positive semidefinite.

Given a real number λ, we shall denote by $\mathrm{s}_\lambda(t)$ the solution of

$$\mathrm{s}'' + \lambda\,\mathrm{s} = 0, \qquad \mathrm{s}(0) = 0, \qquad \mathrm{s}'(0) = 1,$$

and let

$$\mathrm{c}_\lambda = \mathrm{s}_\lambda', \quad \mathrm{ta}_\lambda = \lambda\frac{\mathrm{s}_\lambda}{\mathrm{c}_\lambda}, \quad \mathrm{co}_\lambda = \frac{\mathrm{c}_\lambda}{\mathrm{s}_\lambda}.$$

§ 2. Basic formula

Let $\overline{\nabla}$ and $\overline{\Delta}$ denote the covariant derivative and the Laplacian respectively on $\overline{M}$. Let us denote by ∇ and Δ the corresponding operators on M.

From the definition of $S(r)$ it follows

$$\overline{\nabla}^2 r(A,B) = -\langle S(r)A, B\rangle \text{ for every } A, B \text{ on } \overline{M} - \mathrm{cut}(o). \tag{2.1}$$

In the local computations below, we shall denote by the same letter a local vector field X on M and its image $\psi_* X$.

If α is the second fundamental form of the immersion $\psi : M \longrightarrow \overline{M}$, then an easy computation shows that

$$\nabla^2 r(X,Y) = \overline{\nabla}^2 r(X,Y) + \langle \alpha(X,Y), \partial_r\rangle \tag{2.2}$$

for every X, Y tangent to M.

If $\{e_i\}_{i=1}^m$ is a local orthonormal frame of vector fields tangent to M and H denotes the mean curvature vector of the immersion $\psi : M \longrightarrow \overline{M}$, from (2.1) and (2.2), it follows that

$$\Delta r = \sum_{i=1}^m \langle S(r)(e_i), e_i\rangle - m\langle H, \partial_r\rangle. \tag{2.3}$$

If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is any C^2 function, it follows from (2.3) that

$$\Delta f(r) = -f''(r)|\partial_r^\top|^2 + f'\{\sum_{i=1}^m \langle S(r)(e_i), e_i\rangle - m\langle H, \partial_r\rangle\}. \tag{2.4}$$

In the next sections we shall give applications of the formula (2.4) to some problems. Our emphasis will be more on computations than on arguments. For more details, see the references.

§ 3. Immersibility into a geodesic ball

Riemannian case.

Here $\mathbb{K}^n(\lambda)$ will denote the simply connected real space form of dimension n and constant sectional curvature λ, and $B_\rho^{\lambda,n}$ will be a geodesic ball of radius ρ in this space. One has

$$R_\lambda(r)A = \lambda A, \quad S_\lambda(r)A = -\mathrm{co}_\lambda(r)A \quad \text{for every } A \text{ orthogonal to } \partial_r.$$

Moreover

3.1. Lemma. *If $R(r) \leq \overline{R}_\lambda(r)$ in $B_t(o)$ for some $t \leq c(o)$ if $\lambda \leq 0$, and $t \leq \min\{c(o), \frac{\pi}{\sqrt{\lambda}}\}$ if $\lambda > 0$, then*

$$S(r) \leq \overline{S}_\lambda(r) \text{ for every } q \in B_t(o). \tag{3.1.1}$$

If the equality in (3.1.1) holds in $\partial B_\rho(o)$, for some $\rho < t$, then there exists an isometry between $B_\rho(o)$ and $B_\rho^{\lambda,n}$ which takes $\partial B_\rho(p)$ onto $\partial B_\rho^{\lambda,n}$.

Then, under the general hypothesis $R(r) \leq \overline{R}_\lambda(r)$, Lemma 3.1 says that

$$\sum_{i=1}^m \langle S(r)e_i, e_i\rangle \leq \sum_{i=1}^m \langle \overline{S}_\lambda(r)(e_i - \langle e_i, \partial_r\rangle\partial_r), e_i\rangle = -m\ \mathrm{co}_\lambda(r) + \mathrm{co}_\lambda(r)|\partial_r^\top|^2.$$

By substitution of this inequality in (2.4), we get

$$\Delta f(r) \leq (-f''(r) + f'(r)\,\mathrm{co}_\lambda(r))\,|\partial_r^\top|^2 - mf'(r)\,(\mathrm{co}_\lambda(r) + \langle H, \partial_r\rangle).$$

Now we have to choose a function f which is the best suited for each situation. In this case, it is that which makes the coefficient of $|\partial_r^\top|^2$ vanish, for which f must satisfy $f'(r) = \mathrm{s}_\lambda(r)$. With this choice we get

$$\Delta f(r) \leq -m(\mathrm{c}_\lambda(r) + \langle \mathrm{s}_\lambda(r)\, H, \partial_r\rangle).$$

If $|H| \leq \mathrm{co}_\lambda(\rho)$ (with $\rho \leq \frac{\pi}{\sqrt{\lambda}}$ if $\lambda > 0$) and $\psi(M) \subset B_\rho(o)$,

then

$$\begin{aligned}\Delta f(r) &\leq -m\ \mathrm{c}_\lambda(r) + m\,\mathrm{s}_\lambda/r)|H| \\ &\leq -m\ \mathrm{c}_\lambda(r) + m\frac{\mathrm{s}_\lambda(r)}{\mathrm{s}_\lambda(\rho)}\,\mathrm{c}_\lambda(\rho) \leq -m\,\mathrm{c}_\lambda(r) + m\frac{\mathrm{s}_\lambda(r)}{\mathrm{s}_\lambda(r)}\,\mathrm{c}_\lambda(r) = 0.\end{aligned}$$

From Hopf principle, this inequality implies that $\Delta f(r) = 0$. Then the inequalities in the above formula must all be equalities and we have:

$r|_M = \rho$, $S(r) = \mathrm{s}_\lambda(r) Id$ on the vectors tangent to M, $\partial_r = N$ on M, $|H| = \mathrm{co}_\lambda(\rho)$ and $-\langle H, \partial_r\rangle = \mathrm{co}_\lambda(r)$. Then $H = -\,\mathrm{co}_\lambda(\rho)\partial_r$, and $\psi : M \longrightarrow \partial B_\rho(o)$ is minimal.

If $m = n-1$, $\psi(M) = \partial B_\rho(o)$, ψ is an embedding and $B_\rho(o)$ is isometric to $B_\rho^{\lambda,n}$.

This is essentially the union of Markvorsen Theorem ([Ma]) and a compact version of a Theorem by Jorge and Xavier ([JX]).

Kähler case.

Here $\mathbb{K}^n(\lambda)$ will denote the complex space form of real dimension $2n$ and constant holomorphic sectional curvature 4λ, and $B_\rho^{\lambda,n}$ will be a geodesic ball of radius ρ in this space. Now

$$\begin{aligned}&R_\lambda(r)J\partial_r = 4\lambda J\partial_r, \quad R_\lambda(r)X = \lambda X,\\ &S_\lambda(r)J\partial_r = -\,\mathrm{co}_{4\lambda}(r)J\partial_r \quad \text{and} \quad S_\lambda(r)X = -co_\lambda(r)X\end{aligned}$$

for every X orthogonal to ∂_r and $J\partial_r$.

We have

3.2. Lemma. *If $R(r) \leq \overline{R}_\lambda(r)$ in $B_t(o)$ for some $t \leq c(o)$ if $\lambda \leq 0$, and $t \leq \min\{c(o), \frac{\pi}{2\sqrt{\lambda}}\}$ if $\lambda > 0$, then*

$$S(r) \leq \overline{S}_\lambda(r) \text{ for every } q \in B_t(o). \tag{2.1.1}$$

Moreover, if the equality in (2.1.1) holds in $\partial B_\rho(o)$, for some $\rho < t$, then there exists a holomorphic isometry between $B_\rho(o)$ and $B_\rho^{\lambda,n}$ which takes $\partial B_\rho(o)$ onto $\partial B_\rho^{\lambda,n}$.

Computing as above, we get

$$\begin{aligned}\Delta f(r) \leq &(-f''(r) + f'(r)\,\mathrm{co}_\lambda(r))\,|\partial_r^\top|^2 \\ &+ f'(r)\,\mathrm{ta}_\lambda(r)\,|(J\partial_r)^\top|^2 - mf'(r)\,(\mathrm{co}_\lambda(r) + \langle H, \partial_r\rangle).\end{aligned}$$

The result analogous to that in the riemannian case will be

3.3. Theorem ([CGM2]). *Let us suppose that $R(r) \leq \overline{R}_\lambda(r)$ in $B_\rho(p)$, $\psi(M) \subset B_\rho(p)$, and*

$$m|H| \leq (m-1)\,\mathrm{co}_\lambda(\rho) + co_{4\lambda}(\rho).$$

When $\lambda \leq 0$, let us assume, also, that $J\mathcal{N}M \subset TM$. Then $\psi(M) \subset \partial B_\rho(p)$, $m\,H = -\{(m-1)\,\mathrm{co}_\lambda(\rho) + co_{4\lambda}(\rho)\}\partial_r$, and $J\partial_r$ is tangent to $\psi(M)$ at each point.

If $m = 2n-1$, then $B_\rho(p)$ is isometric to a geodesic ball of radius ρ in $\mathbb{K}^n(\lambda)$, M is isometric to $\partial B_\rho^{\lambda,n}$ and ψ is an embedding.

Now, the proof requires to distinguish between the cases $\lambda > 0$ and $\lambda < 0$. The reason of that is the second term in the above expression of $\Delta f(r)$, which does not appear in he Riemannian case.

For $\lambda > 0$ we are, as above, interested in the vanishing of the term containing $|\partial_r^\top|^2$ and in bounding $|(J\partial_r)^\top|$ by 1. Arguments similar to those in the previous case give us the conclusion of Theorem 3.3.

For $\lambda < 0$ we add the condition $J\mathcal{N}M \subset TM$ (which is always satisfied if $m = 2n-1$). Now it is convenient to work with $|\partial_r^\top|^2$ and $|(J\partial_r)^\top|^2$ together. For it we need that f satisfies the equation

$$-f'' + f'\,\mathrm{co}_\lambda = f'\,\mathrm{ta}_\lambda,$$

and it is enough to take f with $f' = \mathrm{s}_{4\lambda}$. Now, computing, and bounding $|\Pi_{\mathcal{N}M \oplus J\mathcal{N}M}\partial_r|$ by 1, one gets the same result that with $\lambda > 0$.

Let us denote by $\Pi : \partial B_\rho^{\lambda,n} \longrightarrow \mathbb{C}P^{n-1}(1/\mathrm{s}_\lambda^2(\rho))$ the Hopf fibration on the geodesic sphere $\partial B_\rho^{\lambda,n}$. Two interesting corollaries (which have no Riemannian analogs) are

3.4. Corollary ([CGM2]). *Let M be of dimension $2n-2$. Let us suppose that $\overline{M} = \mathbb{K}^n(\lambda)$, $\psi(M) \subset B_\rho^{\lambda,n}$ and $(2n-2)|H| \leq (2n-3)\,\mathrm{co}_\lambda(\rho) + \mathrm{co}_{4\lambda}(\rho)$. When $\lambda \leq 0$, we assume also that $J\mathcal{N}M \subset TM$. Then there is a compact riemannian manifold G of dimension $2n-3$, a riemannian submersion $\pi : M \longrightarrow G$, and a minimal immersion $\varphi : G \longrightarrow \mathbb{C}P^{n-1}(1/\mathrm{s}_\lambda^2(\rho))$ such that $\Pi \circ \psi = \varphi \circ \pi$. Moreover, $\psi(M) = \Pi^{-1}(\varphi(G))$.*

3.5. Corollary ([CGM2]). *Let M and ψ be as in Corollary 3.3, but with $n=2$. Then $\psi : M \longrightarrow \mathbb{K}^2(\lambda)$ is an isometric immersion of a flat torus in $\mathbb{K}^2(\lambda)$, such that $\psi(M)$ is an embedded torus and also a tubular hypersurface of radius $(\pi/4)\,\mathrm{s}_\lambda(\rho)$ around an integral curve of $J\partial_r$ in $\partial B_\rho^{\lambda,2}$.*

A natural question to be posed: Apart from the technical reason given by the formula for $\Delta f(r)$, why are different (for $m < 2n-1$)) the hypotheses when $\lambda < 0$ and when $\lambda > 0$?.

Let us observe that

$$m\ \mathrm{co}_\lambda(\rho) < (m-1)\ \mathrm{co}_\lambda(\rho) + co_{4\lambda}(\rho) \text{ if } \lambda < 0$$

and

$$m\ \mathrm{co}_\lambda(\rho) > (m-1)\ \mathrm{co}_\lambda(\rho) + co_{4\lambda}(\rho) \text{ if } \lambda > 0.$$

That is, the bound for $m|H|$ is the lowest of the possible sums of principal curvatures of $\partial B_\rho^{\lambda,n}$ if $\lambda > 0$, but not if $\lambda < 0$. This explains why, to get the same strong conclusion, we have been obliged to add conditions.

The lowest of the possible sums of principal curvatures of $\partial B_\rho^{\lambda,n}$ for $\lambda < 0$ is $m\ \mathrm{co}_\lambda(\rho)$. If we take it as an upper bound of $|H|$ and choose f as in the Riemannian case, we get

$$\Delta(-\frac{1}{\lambda}\,\mathrm{c}_\lambda) \leq \lambda\frac{\mathrm{s}_\lambda^2}{\mathrm{c}_\lambda}|(J\partial_r)^\top|^2 - m\,\mathrm{c}_\lambda - m\langle H,\partial_r\rangle\,\mathrm{s}_\lambda \leq -m\ \mathrm{c}_\lambda + m\,|H|\,\mathrm{s}_\lambda \leq 0.$$

Applying again Hopf principle and using known facts about the Hopf fibration on $\partial B_\rho^{\lambda,n}$ one gets

3.6. Theorem ([CGM2]). *If $\psi : M \longrightarrow \mathbb{C}H^n(\lambda)$, $\lambda < 0$, is an isometric immersion such that $\psi(M) \subset B_\rho(p)$ and $|H| \leq \mathrm{co}_\lambda(\rho)$, then $1 \leq m \leq n-1$, $|H| = \mathrm{co}_\lambda(\rho)$ and there exists a minimal totally real isometric immersion $\varphi : M \longrightarrow \mathbb{C}P^{n-1}(1/\,\mathrm{s}_\lambda^2(\rho))$ such that ψ is a lifting of φ by Π.*

§ 4. Bounds for the first Dirichlet eigenvalue of an extrinsic ball of a complex submanifold.

In this section $\psi : M \longrightarrow \overline{M}$ will be a Kähler embedding of a Kähler manifold M of real dimension $2q$ into a Kähler manifold $\overline{M}$ of real dimension $2n$ satisfying $R(r) \leq R_\lambda(r)$. Then $H = 0$, $|J\partial_r| = |\partial_r|$, and the formula for the laplacian becomes

$$\Delta f(r) \leq (-f''(r) + f'(r)(\mathrm{co}_\lambda(r) + \lambda\frac{\mathrm{s}_\lambda}{\mathrm{c}_\lambda}))|\partial_r^\top|^2 - 2qf'(r)\,\mathrm{co}_\lambda(r).$$

We take $f(r)$ positive and such that the coefficient of $|\partial_r^\top|$ vanishes.

Now, we follow a technique of [CLY]. We consider the eigenfunction ϕ associated to the first eigenvalue $\mu_1(B_\rho^{\lambda,q})$ of the Dirichlet eigenvalue problem on $B_\rho^{\lambda,q}$, which is a radial function. We define ϕ over M by $\phi(m) = \phi(r(m))$, and we change the variable r by f and consider ϕ as a function of f. Denoting by $\cdot$ and $'$ the derivatives respect to f and r respectively, one gets

$$\Delta\phi = -\phi^{\cdot\cdot} {f'}^2 |\partial_r^\top|^2 + \phi^\cdot \Delta f.$$

If $\phi^{\cdot\cdot} \geq 0$ and $\phi^\cdot \leq 0$, then one concludes

$$\Delta\phi \geq -\phi^{\cdot\cdot}(f(r)){f'}^2(r) + 2qf'(r)\,\mathrm{co}_\lambda(r)\phi^\cdot = \mu_1(B_\rho^{\lambda,q})\phi.$$

Now, by using a classical Lemma of Barta for lower bounds of the first eigenvalue one gets

4.1. Theorem ([CGM1]). *Let us assume that $R(t) \leq R_\lambda(t)$. If $cut(o) \cap B_\rho(o) =$ (and $\rho < \frac{\pi}{2\sqrt{\lambda}}$ if $\lambda > 0$), then*

$$\mu_1(B_\rho(o) \cap M) \geq \lambda_1(B_\rho^{\lambda,q}). \tag{4.1.1}$$

Moreover, if the equality holds in (4.1.1), then $B_\rho(o) \cap M$ is holomorphically isometric to a geodesic ball of radius ρ in $I\!K^q(\lambda)$.

Two main facts used in the proof of the above Theorem are the inequalities $\phi^\cdot \leq 0$ and $\phi^{\cdot\cdot} \geq 0$. The first is an obvious consequence of the properties of the functions ϕ and f. The study of the second inequality requires a more elaborated argument.

§ 5. Looking for a Kähler version of an Alexandrov Theorem.

A well known theorem by A. D. Alexandrov says that a connected compact embedded hypersurface of constant mean curvature in the Euclidean space or in the real hyperbolic space must be a geodesic sphere of the ambient space. It seems natural to conjecture a Kähler version. Using the notation of section 3, Kähler case, this conjecture will say that *a connected compact embedded real hypersurface of constant mean curvature in $I\!K^n(\lambda)$ (or in some geodesic ball if $\lambda > 0$) must be a geodesic sphere.* We have proved a weaker result. To state it we need to introduce some notation:

If L is the Weingarten map of M associated to a unit normal vector field N, then the *JN- normal curvature k_{JN} of M is defined by $k_{JN} = \langle L\,JN\,,\,JN\rangle$ and the JN-mean curvature of M is defined by $H_{JN} = (1/(2n-2))(\mathrm{tr}\,L - k_{JN})$.*

We say that a real hypersurface M is a *Hopf hypersurface* if JN is an eigenvector of the Weingarten map L.

5.1. Theorem ([Mi]). *Let M be a connected compact embedded Hopf hypersurface of $I\!K^n(\lambda)$ with constant mean curvature and $k_{JN} \geq k_\lambda$, where k_λ is the value of the JN-normal curvature of a geodesic sphere of $I\!K^n(\lambda)$ having a JN-mean curvature equal to that of M. If $\lambda > 0$, we suppose also that it is contained in a geodesic ball of radius $R < \pi/(2\sqrt{\lambda})$. Then M is a geodesic sphere of $I\!K^n(\lambda)$.*

To prove Theorem 5.1 we have used a special Laplacian that we introduce now:

For a $(1,1)$-tensor φ on M, the φ-divergence $\operatorname{div}_\varphi$ of a vector field X on M is defined as the function $\operatorname{div}_\varphi X$ given by

$$(\operatorname{div}_\varphi X)(p) = -\operatorname{tr}(Y \mapsto \varphi(\nabla_Y X))(p) = -\sum_{i=1}^{2n-1} < \varphi(\nabla_{e_i} X), e_i > (p).$$

One gets a nice $\operatorname{div}_\varphi$-theorem similar to the well known divergence theorem when $\operatorname{div}_\varphi = 0$. In fact, let us denote by div the usual divergence operator in M, and by ω the volume form of M, then

5.2. Proposition. *Given a vector field X on M,*

$$-(\operatorname{div}_\varphi X)\omega = d\iota_{\varphi(X)}\omega$$

if and only if $(\operatorname{div} \varphi)X = 0$.

The φ-Laplacian Δ_φ is defined by

$$\Delta_\varphi f = \operatorname{div}_\varphi \operatorname{grad} f$$

for every C^2 function f on M.

Now, let us denote by $\{JN\}^\perp$ the orthogonal complement in TM of the space $< \{JN\} >$ generated by JN. Let η and ζ be, respectively, the $(1,1)$-tensors on M defined by the projections from TM onto $\{JN\}^\perp$ and $< \{JN\} >$ respectively, that is

$$\zeta(X) = \langle X, JN\rangle JN \quad \text{and} \quad \eta = \mathbf{I}\mathrm{d} - \zeta, \quad \text{for every} \quad X \in TP.$$

The following elementary proposition is crucial.

Proposition ([Mi]). *M is a Hopf hypersurface of $I\!\!K^n(\lambda)$ if and only if* $\operatorname{div}\eta = 0$ *or, equivalently,* $\operatorname{div}\zeta = 0$.

Then for a Hopf hypersurface M, computations like those in preceeding sections give rise to the formula

$$\Delta_\eta f(r) = \{-f''(r) + f'(2\,\mathrm{co}_\lambda(r) - \mathrm{co}_{4\lambda}(r))\}|\partial_r^J|^2 \\ - (2n-2)f'(r)(\mathrm{co}_\lambda(r) + H_{JN}\langle N, \partial_r\rangle),$$

where $\partial_r^J = \eta(\partial_r^\top)$, and the convenient choice of $f(r)$ is $f'(r) = \mathrm{ta}_\lambda(r)$, which makes the coefficient of $|\partial_r^J|^2$ to vanish.

The combination of this expression with one for $\overline{\Delta} f(r)$ and a formula for the volume in [GM1], in a similar way as S. Montiel and A. Ros do in [MR] complete the proof of Theorem 5.1.

§ 6. More about immersibility into a geodesic ball.

There are many theorems about nonimmersibility of compact riemannian manifolds with sectional curvature bounded from above inside a metric ball of a simply connected space form. Recently, S. Deshmukh and M.A. Al-Gwaiz ([DA]) have proved a theorem, which deals with Ricci curvatures instead of sectional curvatures. This theorem works for immersions into a euclidean space of even dimension. They used the following well known integral formula

$$\int_M \{\rho(X,X) + \frac{1}{2}|\mathcal{L}_X g|^2 - |\nabla X|^2 - |\operatorname{div} X|^2\} dM = 0.$$

We saw that a good choice of the vector field X in the above formula and a combination of it with formula (2.4) (now taking $f' = \mathrm{s}_{4\lambda}$) allow to extend their theorem (with slightly weaker hypothesis) to immersions in every real space form $\mathbb{K}^n(\lambda)$ of any dimension. In particular, we have

6.1. Theorem ([GM2]). *Let M be a compact $(n-1)$-dimensional Riemannian manifold. Let us define the tangent vector field u on M by $u(m) = \mathrm{s}_\lambda(r(m))\partial_r^\top(m)$. Let us suppose that its Ricci curvature ρ and scalar curvature τ satisfy*

$$\int_M \{\rho(u,u) + \tau|u|^2\} dM \geq (n+1)(n-2)\lambda \int_M |u|^2 dM$$

(and $\tau \geq 0$ if $\lambda < 0$) and

$$\int_M \tau dM \leq (n-1)(n-2)\operatorname{vol}(M)\, \mathrm{s}_\lambda^{-2}(R).$$

If there is an isometric immersion ψ of M into $\mathbb{K}^n(\lambda)$ contained in $B_R(o)$, then $\psi(M)$ is the boundary of this geodesic ball.

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