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LORENTZIAN MANIFOLD WITH A GROUP OF CONFORMAL TRANSFORMATIONS THAT HAS A NORMAL ONE-PARAMETER HOMOTHETY SUBGROUP

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The following result was proved in [1] :

Theorem. (C.Barbance). *Let (M, g) be a complete homogeneous Lorentzian manifold and G be a group of conformal transformations, which acts on M. Suppose, that for some $p \in G$ a stable subgroup G_p contains a connected homothety subgroup H, which is a normal subgroup in G. Then G is a homothety group.*

This result was proved by the author [2] in a more general case. We declined the assumptions of homogeneity for (M, g) and of connectedness for H and proved that (M, g) is a plain-wave space; the actions of G and H on M were described. Then we found the necessary and sufficient condition of homogeneity of such manifold and finally proved the theorem using the global technique. Instead of completeness assumption for M, some restrictions on its metric were used; only the conformally flat manifolds from the very limited class does not satisfy this restrictions.

Now we decline the assumption, that homothety group Φ has a stable point, but we consider only the case, when Φ is a one-parameter group.

Lorentzian manifolds with one-parameter group of homotheties Φ were examined by D.Alekseevskii [3]. Two cases were considered:

i) Φ has a stable point;

ii) Φ acts without stable point.

In the second case the assumption was made, that M is a causal manifold. The remaining possibility, when M is not causal, was examined by the author of the present paper [4]. Manifold (M, g) was proved to be the plane-wave space and either (M, g), or its two-sheet covering is topologically equivalent to the cylinder.

Theorem. *Let (M, g) be a distinguishing space-time, and G be a group of conformal transformations, that acts transitively on M. Suppose, that G contains a normal one-parameter homothety subgroup Φ , which acts without stable points*

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and Φ has at least one non-isotropic orbit. Then there is a function $f : M \to R^+$, such that for the manifold (M, h) , $h = f \cdot g$, the group Φ is an isometry group and G is a homothety group.

The following example shows that the theorem is not true, when each orbit of the group Φ is isotropic.

Let N be an arbitrary Riemannian manifold, which has a group H of conformal transformations and L be the two-dimensional Minkowski space with coordinates (u, v) and the metric $ds^2 = dudv$. Let $(y^2, \dots, y^{n-1})$ be local coordinates defined on a neighborhood V on N. Let $M = N \times L$ be supplied by the metric h of direct product. Define the action of H on (M, h). In the local coordinates on the neighborhood $\overline{V} = V \times L$ an arbitrary transformation $\psi \in H$ acts as follows:

$$\psi(u, y^2, \dots, y^{n-1}, v) = (f^2(y)u, \psi(y^2, \dots, y^{n-1}), v),$$

where $f(y) = f(y^2, \dots, y^{n-1})$ - is the homothety ratio of the transformation $\psi : N \longrightarrow N$ at the point p with the coordinates $(y^2, \dots, y^{n-1})$. It easy to verify, that ψ is a conformal transformation of the manifold M with the same homothety ratio. The manifold M admits a one-parameter isometry group $\Phi = \{\varphi_t\}$, that acts by the formula

$$\varphi_t(u, y^2, \dots, y^{n-1}, v) = (u, \varphi_t(y^2, \dots, y^{n-1}), v + t).$$

It is easy to verify that Φ commutes with H. Let $G = \Phi \times H$ be the direct product of transformation groups. Let us introduce another metric $g = e^v h$ on M. Then Φ becomes the homothety group. Manifold (M, g) with the group G satisfies the conditions of the theorem, except the only one: each orbit of the group Φ is isotropic. And we see that G is not a homothety group.

Proof of the theorem. According to our assumptions (M, g) is a causal Lorentzian manifold and (M, g) has a normal one-parameter homothety subgroup, that acts without stable points. Then, according to D.Alekseevskii [3]

1. M is diffeomorphic to the direct product $M = R \times N$, where N is a manifold;
2. Φ is a one-parameter group of translations of the line R

$$\varphi_t : (u, y) \to (u + t, y) \ , \qquad (u, y) \in R \times N \ ;$$

3. the metric g can be written in the form $g = e^u h$, where h is Φ – invariant metric on M:

$$h = h_{00}(y)\, du^2 + h_{0i}(y)\, du\, dy^i + h_{ij}\, dy^i\, dy^j \ , \qquad y = 1, \dots, n-1.$$

Here $(y^1, \dots, y^{n-1})$ is a local coordinate system on the manifold N.

Thus Φ is an isometry group for the manifold (M, h) , and G is a conformal transformation group for (M, h), since two metrics h and g are conformally equivalent. Let π_1 and π_2 be the projections of the fiber bundle $R \times N$ on the first and the second factors respectively. Denote the fiber $\{u\} \times N$ of the fiber bundle

$R \times N$ by N_u , and denote the fiber $R \times \{\pi_2(p)\}$, which passes through $p \in M$, by R_p . It is obvious, that an arbitrary transformation $\{\varphi_t\} \in \Phi$ maps each fiber R_p on itself and it maps each fiber N_u isometrically on N_{u+t}.

Let p be an arbitrary point in M , and ψ be an arbitrary element from G. Each conformal transformation either preserves the time orientation of a space-time, or inverse it. We consider, that ψ preserves the time orientation. The opposite case could be considered in the similar way. Denote the dilatation ratio of the transformation ψ at the point p by $f_\psi(p)$. Let q be an arbitrary point from R_p. Then there is such $t_1 \in R$, that $q = \varphi_{t_1}(p)$. Since Φ is normal subgroup in G, we have

$$\psi(\varphi_{t_1}(p)) = \varphi_{t_2}(\psi(p)) \tag{1}$$

for some $t_2 \in R$. This implies that $\psi(q) \in R_{\psi(q)}$. Moreover, equality (1) implies, that the dilatation ratios of the transformation ψ at the points p and q are equal. It means, that the function f_ψ is constant along each fiber R_p.

Thus, ψ maps each fiber R_p on the fiber $R_{\psi(p)}$ and moreover the transformation $\psi|_{R_p} : R_p \to R_{\psi(p)}$ is homothetic.

We say that a function f is not constant on the open set V, if each set $\{s|f(s) = f_0 = \mathrm{const}\}$ is nowhere dense in V.

Suppose, that f_ψ is not constant along the fiber N_0 in a neighborhood $V \subset N_0$ of a point $p \in N_0 \subset M$ and the fiber R_p is not isotropic. Then we can choose the neighborhood V so tight, that for each point $q \in V$ the fiber R_q is isotropic as well. Denote by $\overline{V} = \bigcup_{q \in V} R_q$ the tubular neighborhood of the point p.

Let $a = f_\psi(p)$, $p' = \psi(p)$. Choose an arbitrary point $q \in V$ so, that $b = f_\psi(q) \neq a$. Let for the definiteness be $b > a$. The transformation ψ induces mappings of the fiber R_p on the fiber $R_{p'}$ and of the fiber R_q on the fiber $R_{q'}$, $q' = \psi(q)$. Since each of the fibers is isometric to the straight line R, these mappings are linear, i.e. $\psi|_{R_p} : R_p \longrightarrow R_{p'}$, $\psi|_{R_q} : R_q \longrightarrow R_{q'}$ act respectively by the formulas

$$u' = au + c\ , \qquad u' = bu + d\ , \qquad a, b, c, d \in R\ .$$

Choose an arbitrary value $u_0 \in R$. Let $u_1 = au_0 + c$, $u_2 = bu_0 + d$. Then

$$u_2 - u_1 = (b - a)u_0 + (d - c)\ .$$

We see, that the value $w = u_2 - u_1$ increases infinitely as u_0 increases, and it can get any preassigned value depending on the choice of u_0.

Let us consider the transformation $\psi_{u_0} = \psi^{-1} \circ \phi^{-1}_{u_1 - c} \circ \psi \circ \phi_{u_0}$. Then $\psi_{u_0}(p) = p$, and $\psi_{u_0}(q) = q_0 \in N_w \cap R_q$. It means, that depending on the choice of u_0, the point q can be relocated in any predefined position on the fiber R_q under the action of ψ_{u_0} and the point p can be remained immovable. Moreover, the transformation ψ_{u_0} is isometric and it preserves the time orientation independently on the fact, whether ψ preserves the orientation or not.

Denote as usual the chronological future of the point p by $I^+(p)$, the causal future of the point p by $J^+(p)$. The set $I^+(p)$ is open, and p is its limit point. The set $F_a = \{s \in M | f_\psi(s) = a\}$ is nowhere dense in the neighborhood V according to its choice, and F_a is nowhere dense in $\overline{V}$ because f_ψ is constant along the fibers R_s. Therefore p is a limit point of the set $W = (I^+(p) \cap \overline{V}) \setminus F_a$.

Let us choose a sequence of points $\{q_n\}$ from W, that converges to p. As it was demonstrated above, under the action of some isometric transformation, which remains the point p fixed, each of the points could be removed in any predefined position of its fiber. Thus each of the fibers R_{q_n} is contained entirely in $I^+(p)$. It means that the fiber R_p is contained in closure of the set $I^+(p)$, and so contingent of the set $J^+(p)$ includes the vectors $(\partial/\partial u)_p$ and $(-\partial/\partial u)_p$. Thus it does not coincide with the cone $K^+(p) \subset T_pM$, which consists of all future-directed nonspacelike vectors. So, the space-time (M, h) (and (M, g) as well) is not distinguishing (see the Lemma below). It contradicts to our assumption. Therefore f_ψ is constant in the neighborhood V of the point p. So f_ψ is constant on N_0, and that is why it is constant on the entire M. □

If we waive the assumption of transitivity for G, we can say the following. Manifold M can be represented as a union $M_1 \cup M_2 \cup \dots \cup M_n \cup \dots$ of G–invariant sets, where each M_i is either an open manifold, on which G acts as a homothety group, or a manifold with a boundary, on which all the orbits of the group Φ are isotropic.

The following Lemma can be derived from the results of article [5].

Lemma. *For a space-time (M, g) the following conditions are equivalent:*
i) (M, g) is distinguishing;
ii) $cont_p J^+(p) = K^+(p)$ and $cont_p J^-(p) = K^-(p)$ for any $p \in M$.

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