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SOME CONNECTION CLASS ON A HYPERBOLIC KAEHLERIAN SPACE

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ABSTRACT. We consider a special semi-symmetric connection on a hyperbolic Kaehlerian space. We find conditions for such a connection to be of zero curvature; then, we find conditions for the curvature tensor of such a connection to be of special form.

1. Introduction

A hyperbolic Kaehlerian space is an $n(=2m)$-dimensional pseudo-Riemannian space with metric tensor (g_{ij}) and non-degenerate structure tensor $(F_j{}^i)$ satisfying

$$F_i^j F_j^k = \delta_i^k \tag{1}$$

$$F_{ik} = g_{ka} F_i^a = -F_{ki} \tag{2}$$

$$\overset{\circ}{\nabla}_k F_j^i = 0 \tag{3}$$

for the Levi-Civita connection. Formally, a hyperbolic Kaehlerian space looks very similar to a Kaehlerian space (we even call a Kaehlerian space an *elliptic* Kaehlerian space), but geometrically it is different.

We call a connection on a hyperbolic Kaehlerian space a semi-symmetric connection of the second kind if its torsion tensor is of the form

$$T_{jk}^i = \delta_j^i p_k - \delta_k^i p_j + F_j^i q_k - F_k^i q_j - 2F_{kj} q^i \tag{4}$$

for some vector fields p, q, if it is a metric connection i. e. if there holds

$$\nabla_k g_{ij} = 0 \tag{5}$$

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and if it is an F-connection, that is

$$\nabla_k F_j^i = 0. \tag{6}$$

It is easy to construct coefficients of such a connection with given components of torsion tensor. For such a connection,

$$\Gamma_{ki}^a = \overset{\circ}{\Gamma}{}_{ki}^a + \delta_k^a p_i - g_{ki} p^a + F_k^a q_i - F_{ki} q^a. \tag{7}$$

The connection with coefficient (1. 7) is authomatically a metric connection and it is an F-connection if and only if

$$q_i = F_i^a p_a. \tag{8}$$

The connection with coefficients (1. 7) is called a special semi-symmetric connection if $p_i = \frac{\partial p}{\partial x^i}$.

Yano and Imai [3] constructed a connection satisfying conditions (1. 4)-(1. 6) on a Kaehlerian space. Its coefficients had the form (1. 7), but it satisfied $q_i = -F_i^a p_a$. They proved that, in a Kaehlerian space, there does not exist such a connection of zero curvature. They investigated some special forms of curvature tensor of this connection and covariant derivatives of the torsion tensor.

We are going to prove here that on a hyperbolic Kaehlerian space there exists a special semi-symmetric connection of zero curvature (it is a geometric difference) and also to do the rest calculations which Yano and Imai have done for Kaehlerian spaces.

2. The form of curvature tensor

The Riemann curvature tensor of the special semi-symmetric metric connection on a hyperbolic Kaehlerian space has the form

$$\begin{aligned} R_{kjih} = {} & K_{kjih} - g_{kh} p_{ji} + g_{jh} p_{ki} - g_{ji} p_{kh} + g_{ki} p_{jh} \\ & - F_{kh} q_{ji} + F_{jh} q_{ki} - F_{ji} q_{kh} + F_{ki} q_{jh} + F_{kj} \beta_{hi} \end{aligned} \tag{9}$$

where

$$p_{ji} = \overset{\circ}{\nabla}_j p_i - p_j p_i - q_j q_i + \frac{1}{2} g_{ji} p_a p^a \tag{10}$$

$$q_{ji} = \overset{\circ}{\nabla}_j q_i - p_j q_i - q_j p_i - \frac{1}{2} F_{ji} p_a p^a \tag{11}$$

$$\beta_{hi} = 2(p_h q_i - q_h p_i). \tag{12}$$

It is easy to notice that

$$q_{ji} = p_{ja} F_i^a.$$

It is obvious that the tensor (p_{ji}) is symmetric and the tensor (β_{hi}) is skew-symmetric.

3. The case of vanishing curvature

If the considered connection on a hyperbolic Kaehlerian space is of zero curvature, then the Riemann-Christoffel tensor of the basic connection gets the form

$$K_{kjih} = g_{kh}p_{ji} - g_{jh}p_{ki} + g_{ji}p_{kh} - g_{ki}p_{jh} \tag{13}$$

$$+F_{kh}q_{ji} - F_{jh}q_{ki} + F_{ji}q_{kh} - F_{ki}q_{jh} - F_{kj}\beta_{hi}.$$

It is obvious that K_{kjih} is skew-symmetric in indices i,h and k,j. For K_{kjih} there should also hold

$$K_{kjih} = K_{ihkj}. \tag{14}$$

According to (3. 1), we have

$$F_{kh}(q_{ji} + q_{ij}) - F_{jh}(q_{ki} + q_{ik}) + F_{ji}(q_{kh} + q_{hk}) \tag{15}$$

$$-F_{ki}(q_{jh} + q_{hj}) - F_{kj}\beta_{hi} + F_{ih}\beta_{jk}.$$

Transvecting (3. 3) by F^{hk}, we obtain

$$(n-2)(q_{ji} + q_{ij}) = 0,$$

and, consequently,

$$q_{ji} = -q_{ij}. \tag{16}$$

Besides, by (3. 3) there holds

$$F_{kj}\beta_{hi} - F_{ih}\beta_{jk} = 0.$$

Taking into account

$$\beta_{jk}F^{jk} = 4p_a p^a$$

and transvecting the equation above by F^{jk}, we obtain

$$\beta_{si} = \frac{4}{n}p_a p^a F_{is},$$

that is

$$p_s q_i - q_s p_i = \frac{2}{n}p_a p^a F_{is}. \tag{17}$$

Transvecting (3. 5) by p^s, we obtain

$$p_a p^a q_i = \frac{2}{n}p_a p^a q_i.$$

as $n \geq 4$, then

$$p_a p^a = 0, \tag{18}$$

$$p_s q_i = q_s p_i \tag{19}$$

$$p_s p_j = q_s q_j. \tag{20}$$

In such a case, the gradient vector field (p_i) is an isotropic one.

We have just proved

Theorem 1. *If a special semi-symmetric connection (1. 7) on a hyperbolic Kaehlerian space is of zero curvature, then the vector field (p_i) is isotropic and the Riemann-Christoffel tensor of the Levi-Civita connection is of the form*

$$K_{kjih} = g_{kh}p_{ji} - g_{jh}p_{ki} + g_{ji}p_{kh} - g_{ki}p_{jh} + F_{kh}q_{ji} - F_{jh}q_{ki} + F_{ji}q_{kh} - F_{ki}q_{jh},$$

where

$$p_{ji} = \overset{\circ}{\nabla}_j\, p_i - 2p_jp_i,$$
$$q_{ji} = \overset{\circ}{\nabla}_j\, q_i - 2p_jq_i.$$

Suppose, now, that

$$\nabla_k T^h_{ji} = b_k T^h_{ji} + \tilde{b}_k (F^h_a T^a_{ji} - F_{ji}p^h) \tag{21}$$

for some 1-forms (b_k) and $(\tilde{b}_k)$. Then we can obtain

$$\nabla_k p_j = b_k p_j + \tilde{b}_k q_j.$$

But,

$$\nabla_k p_j = p_{kj} + \frac{1}{2}p_a p^a g_{kj},$$

and, as the vector field (p_a) is isotropic,

$$\nabla_k p_j = p_{kj}. \tag{22}$$

Contracting the component of the curvature tensor by g^{kh}, we obtain

$$K_{ji} = np_{ji} + g_{ji}\, \overset{\circ}{\nabla}_a\, p^a$$

and

$$K = 2n\, \overset{\circ}{\nabla}_a\, p^a,$$

as $p^a_a = \overset{\circ}{\nabla}_a\, p^a$, by the fact of isotropy of (p_a). Then

$$p_{ji} = \frac{1}{n}(K_{ji} - \overset{\circ}{\nabla}_a\, p^a g_{ji}) = \frac{1}{n}(K_{ji} - (b_a p^a + \tilde{b}_a q^a)g_{ji}) \tag{23}$$

$$q_{ji} = \frac{1}{n}(K_{ja}F^a_i + (b_a p^a + \tilde{b}_a q^a)F_{ji}) \tag{24}$$

and the curvature tensor of the Levi-Civita connection takes the form

$$\begin{aligned} K_{kjih} = \frac{1}{n}[&g_{kh}K_{ji} - g_{jh}K_{ki} + g_{ji}K_{kh} - g_{ki}K_{jh} \\ &+F_{kh}S_{ji} - F_{jh}S_{ki} + F_{ji}S_{kh} - F_{ki}S_{jh} \\ &-2(b_a p^a + \tilde{b}_a q^a)(g_{kh}g_{ji} - g_{jh}g_{ki} - F_{kh}F_{ji} + F_{jh}F_{ki})] \end{aligned} \tag{25}$$

where S_{ji} stands for $K_{ja}F_i^a$.

There arises the question of existence of such a construction. In the hyperbolic Kaehlerian space, for an isotropic vector field there holds

$$p_i = u_i + v_i, \tag{26}$$

in the adapted coordinate system, where $u = u_i l^i$, l^i being $m = \frac{n}{2}$ eigenvectors for the structure and eigenvalue 1 and $v = v_j \lambda^j$, λ^j being $m = \frac{n}{2}$ eigenvectors and eigenvalue -1. l^i, λ^j form an **adapted** basis of the hyperbolic Kaehlerian space. Then $p_\alpha = u_\alpha$ $(\alpha = 1, \ldots, m)$ and $p_{\hat\alpha} = v_{\hat\alpha}$ $(\hat\alpha = \alpha + m = m+1, \ldots, n)$ as $u_{\hat\alpha} = v_\alpha = 0$. Then

$$q_i = u_i - v_i \tag{27}$$

by the obvious reason. u and v are mutually orthogonal because p and q are mutually orthogonal. Then

$$p_{\alpha\beta} = \overset{\circ}{\nabla}_\alpha u_\beta - 2u_\alpha u_\beta = b_\alpha u_\beta + \tilde{b}_\alpha u_\beta$$

$$q_{\alpha\beta} = \overset{\circ}{\nabla}_\alpha v_\beta - 2u_\alpha v_\beta = b_\alpha v_\beta + \tilde{b}_\alpha v_\beta$$

and

$$b_\alpha p_\beta + \tilde{b}_\alpha q_\beta = 0$$

$$p_{\hat\alpha\hat\beta} = b_{\hat\alpha} v_{\hat\beta} - \tilde{b}_{\hat\alpha} v_{\hat\beta}$$

$$q_{\hat\alpha\hat\beta} = -b_{\hat\alpha} v_{\hat\beta} + \tilde{b}_{\hat\alpha} v_{\hat\beta}$$

$$b_{\hat\alpha} p_{\hat\beta} + \tilde{b}_{\hat\alpha} q_{\hat\beta} = 0.$$

In the similar way, we can show that

$$b_\alpha p_{\hat\beta} + \tilde{b}_\alpha q_{\hat\beta} = 0$$

and, consequntly, there holds

$$b_i p_j + \tilde{b}_i q_j = 0 \tag{28}$$

in any coordinate system. We proved

Theorem 2. *A hyperbolic Kaehlerian space admits a special semi-symmetric connection of zero curvature satisfying (3. 9) for an isotropic vector field p_i if and only if (3. 16) holds. Then, by (3. 10), $K_{kjih} = 0$ and such a space is flat.*

4. The case $R_{kjih} = \alpha_{kj} F_{ih}$

If the curvature tensor of a special semi-symmetric metric connection is of the form $\alpha_{kj}F_{ih}$ for some skew-symmetric tensor α_{kj}, then the Riemann-Christoffel tensor of the Levi-Civita connection is of the form

$$K_{kjih} = g_{kh}p_{ji} - g_{jh}p_{ki} + g_{ji}p_{kh} - g_{ki}p_{jh} \tag{29}$$

$$+F_{kh}q_{ji} - F_{jh}q_{ki} + F_{ji}q_{kh} - F_{ki}q_{jh} - F_{kj}\beta_{hi} - \alpha_{kj}F_{ki}$$

Then, as there holds

$$K_{kjih} = K_{ihkj},$$

we obtain

$$F_{kh}(q_{ij} + q_{ji}) - F_{jh}(q_{ki} + q_{ik}) + F_{ji}(q_{kh} + q_{hk}) \tag{30}$$

$$-F_{ki}(q_{jh} + q_{hj}) - F_{kj}(\beta_{hi} - \alpha_{hi}) + F_{ih}(\beta_{jk} - \alpha_{jk}) = 0.$$

Transvecting (4. 2) by F^{hk}, we obtain

$$(n-2)(q_{ij} + q_{ji}) = 0,$$

and, consequently,

$$q_{ij} + q_{ji} = 0. \tag{31}$$

By (4. 3),we can see from (4. 2) that

$$F_{kj}(\alpha_{hi} - \beta_{hi}) - F_{ih}(\alpha_{jk} - \beta_{jk}) = 0.$$

Transvecting the equation above by F^{jk}, we obtain

$$\alpha_{hi} = \beta_{hi} - \frac{1}{n}(\alpha - 4p_a p^a)F_{hi}, \tag{32}$$

where α stands for $\alpha_{jk}F^{jk}$. By the first Bianchi identity, there holds

$$F_{kh}(2q_{ji} + \alpha_{ji}) - F_{jh}(2q_{ki} + \alpha_{ki}) + F_{ih}(2q_{kj} + \alpha_{kj})$$

$$+F_{ji}(2q_{kh} + \beta_{kh}) - F_{ki}(2q_{jh} + \beta_{jh}) + F_{kj}(2q_{ih} + \beta_{ih}) = 0.$$

If we transvect the equation above by F^{kh}, then

$$(n-4)(2q_{ji} + \alpha_{ji}) = 2(p_a^a + 2p_a p^a)F_{ji}.$$

As

$$p_a^a = p_{ai}g^{ia} = \overset{\circ}{\nabla}_a\, p^a + \frac{n}{2}p_a p^a,$$

then

$$\alpha_{ji} = -2q_{ji} + \frac{2}{n-4}(\overset{\circ}{\nabla}_a\, p^a + \frac{n+4}{2}p_a p^a)F_{ji}. \tag{33}$$

After transvection by F^{ji}, we obtain

$$\alpha = \alpha_{ji}F^{ji} = -\frac{2}{n-4}[2(n-2)\, \overset{\circ}{\nabla}_a\, p^a + n^2 p_a p^a]$$

and, consequently

$$\alpha - 4p_a p^a = -\frac{4(n-2)}{n-4}(\overset{\circ}{\nabla}_a p^a + \frac{n+4}{2}p_a p^a).$$

Then

$$\alpha_{ji} = \beta_{ji} + \frac{4(n-2)}{n(n-4)}(\overset{\circ}{\nabla}_a p^a + \frac{n+4}{2}p_a p^a)F_{ji}. \tag{34}$$

$$\beta_{ji} = -2q_{ji} - \frac{2}{n}(\overset{\circ}{\nabla}_a p^a + \frac{n+4}{2}p_a p^a)F_{ji}. \tag{35}$$

We can substitute (4. 7) and (4. 5) into the first Bianchi identity. Then

$$\begin{aligned} 0 &= (\frac{2}{n-4} - \frac{2}{n})(\overset{\circ}{\nabla}_a p^a + \frac{n+4}{2}p_a p^a)(F_{kh}F_{ji} - F_{jh}F_{ki} + F_{ih}F_{kj}) \\ &= \frac{8}{n(n-4)}(\overset{\circ}{\nabla}_a p^a + \frac{n+4}{2}p_a p^a)(F_{kh}F_{ji} - F_{jh}F_{ki} + F_{ih}F_{kj}). \end{aligned}$$

As

$$F_{kh}F_{ji} - F_{jh}F_{ki} + F_{ih}F_{kj} \neq 0,$$

then $\overset{\circ}{\nabla}_a p^a + \frac{n+4}{2}p_a p^a = 0$, and

$$\alpha_{ji} = \beta_{ji} = -2q_{ji}. \tag{36}$$

Then, according to (4. 1),

$$\begin{aligned} K_{kjih} = g_{kh}p_{ji} - g_{jh}p_{ki} + g_{ji}p_{kh} - g_{ki}p_{jh} + F_{kh}q_{ji} - F_{jh}q_{ki} \\ + F_{ji}q_{kh} - F_{ki}q_{jh} + 2F_{kj}q_{hi} + 2F_{hi}q_{kj}. \end{aligned}$$

After transvection by g^{kh}, we get

$$K_{ji} = (n+4)p_{ji} + g_{ji}p_a^a, \tag{37}$$

As

$$\overset{\circ}{\nabla}_a p^a = -\frac{n+4}{2}p_a p^a,$$

then

$$\begin{aligned} p_a^a &= -2p_a p^a, \\ K &= -4(n+2)p_a p^a, \\ p_a p^a &= -\frac{K}{4(n+2)}, \\ p_a^a &= \frac{K}{2(n+2)}, \end{aligned}$$

$$p_{ji} = \frac{1}{n+4}[K_{ji} - \frac{K}{2(n+2)}g_{ji}], \tag{38}$$

$$q_{ji} = \frac{1}{n+4}[K_{ja}F_i^a + \frac{K}{2(n+2)}F_{ji}]. \tag{39}$$

If we substitute (4. 10) and (4. 11) into (4. 1), we obtain that there holds:

Theorem 3. *If the curvature tensor of the special semi-symmetric connection on a hyperbolic Kaehlerian space admits the form $R_{kjih} = \alpha_{kj} F_{ih}$, then the HB-tensor [2] of the hyperbolic Kaehlerian space vanishes.*

Suppose that the covariant derivative of the torsion tensor vanishes

$$\nabla_k T_{ij}^h = 0. \tag{40}$$

Then $\nabla_k T_{hj}^h = 0$. As $T_{hj}^h = np_j$, then there holds

$$\nabla_k p_j = 0. \tag{41}$$

According to (1. 7)

$$\nabla_k p_j = \overset{\circ}{\nabla}_k \, p_j + g_{kj} p_a p^a - p_k p_j - q_k q_j = 0. \tag{42}$$

Then

$$p_{kj} = -\frac{1}{2} p_a p^a g_{kj}. \tag{43}$$

Comparing (4. 15) to (4. 10), we obtain

$$K_{ji} = K \frac{n+8}{8(n+2)} g_{ji}$$

and

$$K \frac{(n+4)(n-4)}{8(n+2)} = 0,$$

and, consequently,

$$K = 0.$$

We have proved

Corollary 1. *If the curvature tensor of the special semi-symmetric connection on a hyperbolic Kaehlerian space admits the form $R_{kjih} = \alpha_{kj} F_{ih}$, then the condition (4. 12) is satisfied if and only if the curvature scalar of the space vanishes and the vector (p_i) is isotropic.*

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