DIFFERENTIAL GEOMETRY AND APPLICATIONS
Proc. Conf., Aug. 28 – Sept. 1, 1995, Brno, Czech Republic
Masaryk University, Brno 1996, 73–79

THE NATURAL CLASSIFICATION OF REAL LIE ALGEBRAS

M. RAINER AND H.-J. SCHMIDT

ABSTRACT. We classify the n-dimensional real Lie algebras according to algebraic, geometric, and topological points of view. For the case that Lie algebra, Riemannian geometry, and general topology yield the same classification we call it the natural one. It turns out that for dimension $n \leq 3$ our classification is natural. For $n \geq 4$ partial results are given.

1. Introduction

The classification of real Lie algebras can be performed under several points of view; no generally accepted notation exists up to now. Here we consider three approaches which give suitable classification schemes. For the case that all approaches coincide we call it the natural classification.

2. Notation

The dimension is $n \geq 1$. Let $i, j, k = 1, \dots, n$ and $C := \{C^k_{ij}\}$ be a real tensor antisymmetric in i and j. With $m := \frac{n-1}{2} \cdot n^2$ one gets: C has m independent components. So, the set of all these tensors C can be identified with $I\!R^m$. Even for $n = 1$ this makes sense, because $I\!R^0$ is a one-point set. The Jacobi condition, $C^l_{[ij} C^m_{k]l} = 0$, is a continuous relation for these tensors C. We define

$$W^n = \{C | C \text{ satisfies the Jacobi condition}\}. \tag{2.1}$$

W^n is a closed subset of $I\!R^m$. It is a proper subset only if $n \geq 3$, because for $n \leq 2$ the Jacobi condition is satisfied identically. The set of real Lie algebras of dimension n will be denoted by K^n. Every Lie algebra can be characterized by its Lie product, i.e. the adjoint representation

$$[e_i, e_j] = C^k_{ij} e_k$$

1991 *Mathematics Subject Classification.* 53 B 20, 22 E 15.
Key words and phrases. Real Lie algebras, homogeneous Riemannian manifolds, general topology.
This paper is in final form and no version of it will be submitted for publication elsewhere.

w.r.t. basis elements e_i of the Lie algebra. So we have a surjective map

$$\pi : W^n \to K^n. \tag{2.2}$$

Two sets of structure constants C represent the same Lie algebra iff they go into each other by a regular basis change $e_i \to A_i^j e_j$ with $A \in GL(n)$, where $GL(n)$ is represented by the set of all regular real $n \times n$-matrices. So we can write

$$K^n = W^n / GL(n). \tag{2.3}$$

For $n = 3$, Lie algebras can be defined by

$$[e_1, e_2] = n_3 e_3 - h e_2,$$
$$[e_2, e_3] = n_1 e_1, \qquad [e_3, e_1] = n_2 e_2 + h e_3$$

via the quadrupels (h, e_1, e_2, e_3) as follows:

$$IV : (1, 0, 0, 1); VI_h : (h, 1, -1, 0); III = VI_1; VII_h : (h, 1, 1, 0).$$

3. First set of examples

For every $n \geq 1$ there exists one commutative Lie algebra, we denote it by $I^{(n)}$ (and if n is known from the context, simply by I). It corresponds to $C^i_{jk} = 0$. For $n = 1$ it is the only Lie algebra, $K^1 = \{I\}$.

A Lie algebra h is called trivial iff for all $x, y \in h$ the vector $[x, y]$ is a linear combination of x and y. For every $n \geq 2$ there exists exactly one non-commutative trivial Lie algebra. We denote it by $V^{(n)}$ (or V). This algebra, which is also called the pure vector type Lie algebra, corresponds to $C^i_{jk} = \delta^i_{[j} v_{k]}$ with any non-vanishing vector v_k. For $n = 2$ we find $K^2 = \{I, V\}$.

For every $n \geq 3$ there exists exactly one Lie algebra possessing some basis $\{e_i\}_{i=1 \dots n}$ such that $[e_1, e_2] = e_3$ is the only non-vanishing Lie product. We denote it by $II^{(n)}$ (or II). It represents a non-trivial Lie algebra. For $n \geq 3$ the set $\{I, II, V\}$ is a proper subset of K^n.

4. Algebraic classification scheme

For transformations $C \to \tilde{C}$, given by the formula

$$\tilde{C}^k_{ij} := A_i{}^a A_j{}^b A^{-1}{}^k{}_c C^c_{ab}, \tag{4.1}$$

we look for those elements $A \in GL(n)$ for which $\tilde{C} = C$. For any Lie algebra h and structure constants $C(h)$ of the adjoint representation, we have the set

$$K(h) := \{A \in GL(n) | \tilde{C}(h) = C(h)\}.$$

$K(h)$ is a closed subgroup of $GL(n)$, namely $K(h) = Aut(h)$, which is the automorphism group of h. It holds: $K(h) = GL(n)$ iff $h = I$.

Let $h \neq I$, then $K(h)$ is a proper closed subgroup of $GL(n)$. It is $GL(n) = SL(n) \otimes IR_+$, where $SL(n) := \{A \in GL(n) | \det A = \pm 1\}$ is its unimodular subgroup and $IR_+ := \{A \in GL(n) | A^i_j = \lambda\delta^i_j, \lambda > 0\}$ is the group of positive scalar multiplication operators.

For $h \neq I^{(n)}$, IR_+ is never a subgroup of $K(h)$. Hence, in this case $K(h)$ is isomorphic to a subgroup of $SL(n)$. For the canonical basis freedom $k := \dim K$, we obtain $k(I^{(n)}) = n^2$, $k(h) \leq n^2 - 1$ for $h \neq I^{(n)}$. Let us take all n-dimensional Lie algebras and look for the possible values of k. Denoting the number of different k-values by $i(n)$, it is $1 \leq i(n) \leq n^2$. Let these $i(n)$ values be numbered in a decreasing sequence as $k_1, \dots, k_{i(n)}$. Thus, $k_1 = n^2$, $n^2 - 1 \geq k_2 \dots > k_{i(n)} \geq 0$. We define

$$K^n[i] := \{h \in K^n | k(h) = k_{i+1}\}. \tag{4.2}$$

Then it is $K^n = \cup_{j=0}^{i(n)-1} K^n[j]$, a disjoint union of non-empty sets. [Remark: Using instead of $K^n[i]$ the simplified notion of $\tilde{K}^n[i] = \{h \in K^n | k(h) = i\}$ has the disadvantage that then most of all sets $\tilde{K}^n = \emptyset$.]

By definition it is clear that $K^n[0] = \{I^{(n)}\}$. It holds $i(1) = 1$, $i(2) = 2$ and $i(3) = 4$. For $n = 2$ it holds $K^2[1] = \{V^{(2)}\}$. For $n = 3$ it holds $K^3[1] = \{II, V\}$, $K^3[3] = \{VIII, IX\}$, where $IX = SO(3)$ and $VIII = SO(2,1)$ are the semi-simple Lie algebras. All remaining 3-dimensional Lie algebras together form the set $K^3[2]$. For completeness let us mention[1]: $k_1 = 9$, $k_2 = 6$, $k_3 = 4$, $k_4 = 3$.

5. Geometric classification scheme

Let h be a Lie algebra of dimension n and H the corresponding simply connected Lie group. Let g be a positive definite left–invariant metric on H. Let $V_n(h, g)$ be the underlying Riemannian manifold of the pair (H, g) (forgetting its Lie group structure). [Of course the group structure of H can be resurrected as some subgroup of the isometry group of $V_n(h, g)$.] There exists a $\frac{n(n+1)}{2}$-parameter set of positive definite left invariant metrics on H. We identify two metrics g and $\tilde{g}$ iff $V_n(h, g)$ is isometric to $V_n(h, \tilde{g})$. This identification reduces the $\frac{n(n+1)}{2}$-parameter set to a set of dimension $d(h) \leq \frac{n(n+1)}{2}$.

Examples: $d(h) = 0$ iff all left-invariant metrics are isometric to each other. It is clear that this implies flatness of V_n, and this happens iff h is the commutative Lie algebra. Therefore it holds [with l.h.s algebraically defined in sct. 4, r.h.s. geometrically defined in sct. 5]:

Theorem 1:
(1) *For all* $n \geq 1$ *one has* $K^n[0] = \{h \in K^n | d(h) = 0\}$.
(2) *For all* $n \leq 3$ *and for all* i *one has* $K^n[i] = \{h \in K^n | d(h) = i\}$.
(3) *For all* $n \geq 2$, $d(V^{(n)}) = 1$.
(4) *For all* $n \geq 3$, $d(II^{(n)}) = 1$.

Proof: (1) is trivial. (2) is lengthy but straightforward (we applied computer algebra). (3) follows from the fact that every positive definite left-invariant metric of $V^{(n)}$ leads to a space of constant negative curvature. (4) holds because of 2 with $II \in K^3[1]$, 1, and the fact that $II^{(n)}$, $n \geq 3$, is the direct Abelian extension of $II^{(3)}$, and hence for any g, there is a g_2 of $II^{(3)}$, such that $V_n(II^{(n)}, g) = V_3(II^{(3)}, g_2) \times V_3(I^{(n-3)}, g_1)$, where g_1 is the unique flat metric of $I^{(n-3)}$. q.e.d.

Remark: $d(h) = 1$ *if and only if* there exists a non-flat positive definite left-invariant metric on h *and* all such metrics are homothetically equivalent.

6. Topological classification scheme

In this section we propose two versions of a topological classification of the n-dimensional Lie algebras. Theorem 2 shall help to understand the underlying topology, Theorem 3 compares the first version of a topological classification with the algebraic classification of eq. (4.2). Theorem 4 does it with the second version.

Let us go back to (2.2), $\pi : W^n \to K^n$, which is a surjective map. $W^n \subset IR^m$ carries the Euclidean subspace topology. So it is natural to define a topology κ^n in K^n by the condition: κ^n is the finest topology which makes the function π continuous. It holds:

Theorem 2:
(1) *$A \subset K^n$ is closed in κ^n iff $\pi^{-1}(A)$ is closed in W^n.*
(2) *Given $h_i, h \in K^n$, then $h_i \to h$ in κ^n iff there exist*
$X_i, X \in W^n$ with $X_i \to X$ in W^n and $\pi(X_i) = h_i$, $\pi(X) = h$.

Proof: (1) is only a reformulation of the definition. (2) applies the usual construction of quotient topologies. q.e.d.

Remark: Theorem 2.2 can be formulated as follows: A sequence of Lie algebras converges, $h_i \to h$, if and only if for each element of the sequence there exists a basis such that the corresponding structure constants converge as real numbers.

It holds: (K^n, κ^n) is a Hausdorff space iff $n = 1$. (This is trivial for $n = 1$, because K^1 is a one-point set.) Let $n \geq 2$ in the following. Then it holds: K^n is the only open set containing $I^{(n)}$. Therefore, (K^n, κ^n) is a T_0 but not a T_1-space. Such spaces are not very intuitive, so we try the following: We write (K^n, κ^n) as disjoint union of compact Hausdorff subsets. If there are more than one possibilities we use a union which has the least number of subsets. To be precise, let us write: We use the symbol $K^n\{i\}$ to denote the elements of a minimal disjoint decomposition of K^n into compact Hausdorff subspaces. For $n < 4$ this represents a definition, for larger values n it is open yet whether the following requirements make such a decomposition unique. $K^n = \cup_{i=0}^{j(n)} K^n\{i\}$ be a disjoint union. The minimality condition implies $K^n\{i\} \neq \emptyset$ for all i. $j(n)$ be the smallest number such that this decomposition into compact Hausdorff sets exists. Let N_i be the number of connected components of the set $K^n\{i\}$. We set $N_{(0)} := \max\{N_i | 0 \leq i \leq j(n)\}$, and $N_{(k)} := \max(\{N_i | 0 \leq i \leq j(n)\} \setminus \{N_{(j)} | 0 \leq j \leq k-1\})$. If there is more than one possibility, then we select that possibilities where $N_{(0)}$ has its least possible value. If there are still more than one possibilities we take those with minimal value $N_{(1)}$, and so on, until there remains -hopefully- only one possibility.

Now we expect the representation to be unique up to a permutation of the sets. We fix the order by the requirement that $K^n\{i\}$ is a closed subset of $\cup_{k=i}^{j(n)} K^n\{k\}$. Then it holds

Theorem 3:
(1) *For all* $n \geq 1$, $K^n\{0\} = \{I^{(n)}\}$.
(2) *For* $n = 2$, $K^n\{1\} = \{V^{(2)}\}$, *i.e.* $K^2\{i\} = K^2[i]$ *for all* i.
(3) *For* $n = 3$, $K^3\{i\} = K^3[i]$ *for all* i. $K^3\{1\} = \{II, V\}$. $K^3\{2\}$ *has the topology of a closed interval, from* VI_0, *increasing* h *of* VI_h *to* III *at* $h = 1$, *and further to* $\lim_{h\to\infty} VI_h = IV = \lim_{h\to\infty} VII_h$, *and finally decreasing* h *of* VII_h, *to* VII_0. $K^3\{3\} = \{VIII, IX\}$.

Proof: (1) $\{I^{(n)}\}$ is the only Hausdorff subset of K^n containing $I^{(n)}$. So $\{I^{(n)}\} = K^n\{i\}$ for some i. But a set not containing $I^{(n)}$ cannot be closed in K^n, hence $i = 0$. (2) is trivial. (3) is not trivial; obviously one needs at least four subsets to have all of them Hausdorff ones. With exactly four, one has two different possibilities. One of them will be excluded because it contains the three-point set $\{IV, VIII, IX\}$. The remaining one contains only two connected sets and two two-point sets. So it is a unique decomposition. q.e.d

Now we define another topological classification: For $i \in IN_0$,

$$\tilde{K}^n\{i\} := \{h \in K^n | \{h\} \text{ is a closed subset of } S_i := K^n \setminus \cup_{j=0}^{i-1} \tilde{K}^n\{j\}\}.$$

The defining condition is sometimes (see [3]) also reformulated as "h is an atom w.r.t. S_i". As usual, the result of $\bigcup_{j=0}^{-1}$ is defined as empty set. So every $\tilde{K}^n\{i\}$ is a well-defined set (in principle, for each $i \in IN_0$). It holds

Theorem 4:
(1) *For all* $n \geq 1$, $\tilde{K}^n\{0\} = K^n\{0\}$.
(2) *For all* $n \leq 3$ *and all* i, $\tilde{K}^n\{i\} = K^n\{i\}$.
(3) *For all* $n \geq 3$, $\tilde{K}^n\{1\} = \{II^{(n)}, V^{(n)}\}$.

Proof: (1) is trivial, (2) uses Theorem 1.2, (3) uses the definition of atoms, (see below). q.e.d.

Definition: For every $n \geq 2$, the n-dimensional *atoms* are the elements of $\tilde{K}^n\{1\}$.

Another definition of an *atom* can be found in [1].

7. The idea of the proofs

The number $k(h)$ defined in sct. 4 can be found as follows (cf. (2.2), $\pi : W^n \to K^n$):

$$k(h) = n^2 - \dim \pi^{-1}(h). \tag{7.1}$$

Now we define a subset V^n of W^n of eq. (2.1) by

$$V^n := \{C \in W^n | \sum_{i,j,k=1}^{n} (C_{ij}^k)^2 = 1\}. \tag{7.2}$$

We get $V^n \otimes IR = W^n \setminus \{0\}$, where $C_{ij}^k \otimes \lambda = \lambda C_{ij}^k$ with the factor $\lambda \in IR_+$ corresponding to a scalar multiplication operator, $A_i^j = \lambda \delta_i^j$, in $GL(n)$. Therefore, we have the restriction of eq. (2.2):

$$\pi : V^n \to K^n \setminus \{I^{(n)}\} \tag{7.3}$$

is a surjective map. V^n is compact, hence $K^n \setminus \{I^{(n)}\}$ is compact, too. [The continuous image of a compact set is compact.] In connection with eq. (7.1) we get for $h \neq I$

$$\dim \pi^{-1}|_{V^n}(h) = n^2 - k(h) - 1. \tag{7.4}$$

The largest value k gives the subsets of V^n of smallest dimension. So they are the best candidates for closed subsets.

8. Second set of examples

For $n = 3$, Bianchi type IV is the only Lie algebra which has both atoms in its closure. All other non-commutative non-atoms converge to II but not to V, cf. Theorem 4.3.

For $n \geq 4$ we only know: every non-commutative non-atom converges at least to one of the two atoms.

Theorem 5: *Let $n \leq 3$ and g, h be n-dimensional Lie algebras. Then it holds $g \to h$ in κ^n if and only if to every positive definite left-invariant metric η in h there exists a sequence of left-invariant metrics in g which converge, as Riemannian manifolds, to the Riemannian manifold defined by (h, η).*

Note: There exists exactly one further decomposition of (K^3, κ^3) into four disjoint compact Hausdorff subsets. It has the shape $\{I\}$, $\{II\}$, $\{IV, VIII, IX\}$ plus a closed interval which is $K^3\{2\}$ with IV replaced by V. It was excluded because a 3-point set is contained.

By the way: The decomposition into Hausdorff compact subsets is not always possible. It holds: The open interval cannot be represented as countable infinite union of disjoint compact subsets.

9. Outlook

The present paper represents a continuation of the papers [1, 2, 3, 4]. It would be interesting to check which of these statements hold true for higher dimension and different signature.

The numeration of Lie algebras as used here was first introduced by Bianchi [5]; now it is the standard classification within relativity theory. For the general background the reader is referred to the following basic monographs: [6] for homogeneous structures, [7] for Lie algebras, [8] for homogeneous cosmological models and explanation of ref. [5], [9] for General topology.

Acknowledgements

The authors would like to thank for financial support by Deutsche Forschungsgemeinschaft (M.R.) and by the Wissenschaftler - Integrations - Programm (H.-J.S.).

References

[1] H.-J. Schmidt, *On Geroch's limit of space–times and a new topology in the space of Lie groups*, J. Math. Phys. **28** (1987), 1928.
[2] H.-J. Schmidt, *Addendum*, J. Math. Phys. **29** (1988), 1264.
[3] M. Rainer, *The topology of the space of real Lie algebras up to dimension* 4, *with applications to homogeneous cosmological models*, Thesis, University Potsdam (1994), Edition Wissenschaft Reihe Mathematik Bd. 3, Tectum Verlag Marburg 1995.
[4] M. Rainer, H.-J. Schmidt, *Inhomogeneous cosmological models with homogeneous inner hypersurface geometry*, Gen. Relat. Grav. **27** (1995), 1265–1293, and references cited there.
[5] L. Bianchi, *Sugli spazii a tre dimensioni che ammettono un gruppo continuo di movimenti* , Mem. della Soc. Italiana delle Scienze di Mat. Ser. 3a, **11** (1897), 267.
[6] F. Tricerri and L. Vanhecke, *Homogeneous structures on Riemannian manifolds*, Cambridge University Press 1983.
[7] J. Humphreys, *Introduction to Lie algebras and representation theory*, Springer New York 1972.
[8] L. Landau and E. Lifshitz, *Klassische Feldtheorie*, Akademieverlag Berlin 1992, esp. §116.
[9] W. Rinow, *Topologie*, Verlag d. Wissenschaften Berlin 1975.

Projektgruppe Kosmologie, Institut für Mathematik
Universität Potsdam, PF 601553
D-14415 Potsdam, Germany

E–mail addresses: mrainer@aip.de, hjschmi@rz.uni-potsdam.de