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HOMOGENEOUS RIEMANNIAN MANIFOLDS WITH EINSTEIN METRICS

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Abstract. Homogeneous Einstein manifolds $(M = G/H, \rho)$ which have some restrictions either on the algebraic structure of G/H, or on the dimension of G/H, or on the class of G — invariant Riemannian metrics of G/H are studied in this paper.

In this paper we'll be discussed some questions from the theory of Einstein manifolds, that is Riemannian manifolds (M, ρ) satisfying the Einstein equation $\mathrm{Ricc}(\rho) = C \cdot \rho$, where $C \in R$, ρ is the Riemannian metric tensor, Ricc is the Ricci tensor of (M, ρ).

In general case the problem of the classification of all Einstein manifolds is very difficult, it is also difficult in homogeneous case $(M = G/H)$. Therefore different authors assume that homogeneous Riemannian manifold $(G/H, \rho)$ has some restrictions either on the algebraic structure of G/H, or on the dimension of $M = G/H$ and others.

The main purpose of this paper is to present some theorems of such sort.

Theorem 1 [Ro12]. *Let G/H be a compact simply-connected homogeneous space which admit a homogeneous Riemannian metric of positive sectional curvature. If $(G/H, \rho)$ is a homogeneous Einstein manifold, then it is isometric, up to a homothety,to one of the manifolds in Table 1.1.*

We note that compact simply connected homogeneous spaces admitting homogeneous Riemannian metric of positive sectional curvature have been classified by M.Berger [Ber], N.Wallach [Wal] and B.Bergery [Berg]. There are compact rank one symmetric spaces (CROSS) and also some finite list of manifolds.

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Table 1.1

N	G/H	Isotropy representation	Einstein metrics
1.	CROSS	Irreducible	Standard [Wol]
2. 3.	$\frac{SU(n+1)}{SU(n)} = S^{2n+1}$ $\frac{U(n+1)}{U(n)} = S^{2n+1}$	$p = p_o \oplus p_1$ $p = p_o \oplus p_1$	Metric with constant sectional curvature [Zil]
4. 5. 6.	$\frac{Sp(n+1)}{Sp(n)} = S^{4n+3}$ $\frac{Sp(n+1)\times Sp(1)}{Sp(n)\times Sp(1)} = S^{4n+3}$ $\frac{Sp(n+1)\times U(1)}{Sp(n)\times U(1)} = S^{4n+3}$	$p = p_o \oplus p_1$ $p = p_1 \oplus p_2$ $p = p_o \oplus p_1 \oplus p_2$	a. Metric with constant sectional curvature [Zil] b. Metric of the distance sphere S^{4n+3} in $P^{n+1}H$ [Bo-Ka]
7.	$\frac{Spin(9)}{Spin(7)} = S^{15}$	$p = p_1 \oplus p_2$	a. Metric with constant sectional curvature [Zil] b. Metric of the distance sphere S^{15} in CaP^2 [Bo-Ka]
8.	$\frac{SU(3)}{i_{k,l}(S^1)}$	$p = p_o \oplus p_1 \oplus p_2 \oplus p_3$	Two invariant Einstein metrics [Ko-Vl], [Wan]
9.	$\frac{Sp(n+1)}{Sp(n)\times U(1)} = CP^{2n+1}$	$p = p_1 \oplus p_2$	Two invariant Einstein metrics [Zil]
10.	$\frac{SU(3)}{T_{max}}$	$p = p_1 \oplus p_2 \oplus p_3$	Two invariant Einstein metrics [Ro2],[Ro9]
11.	$\frac{Sp(3)}{Sp(1)\times Sp(1)\times Sp(1)}$	$p = p_1 \oplus p_2 \oplus p_3$	Two invariant Einstein metrics [Ro2],[Ro9]

Table 1.1 (continuation)

12.	$\frac{F_4}{Spin(8)}$	$p = p_1 \oplus p_2 \oplus p_3$	Two invariant Einstein metrics [Ro2],[Ro9]
13.	$\frac{SU(5)}{Sp(2)\times S^1}$	$p = p_1 \oplus p_2$	Two invariant Einstein metrics [Ro4],[Ro9]

It is known that any homogeneous Einstein manifold M^n of dimension $n \leq 4$ is a Riemannian symmetric space. On the other hand, in the case $n \geq 5$ the problem of the complete classification of homogeneous Einstein manifolds still remains open [Bes].

Theorem 2 ([Ro 1], [Ro 5], [Ro 10], [Ro 11]). *Let G be a compact, connected, simply connected and semisimple Lie group which acts almost effectively on the five-dimensional simply connected homogeneous space $M^5 = G/H$, where H is a connected closed subgroup of G. If $(G/H, \rho)$ is a homogeneous Einstein manifold, then it is isometric, up to a homothety, to one of the manifolds in Table 2.1.*

Table 2.1

N	M^5	Invariant Einstein metrics
1.	$\frac{SU(3)}{i(SU(2))} = S^5$, where i is the standard embedding	Normal homogeneous Riemannian metric with constant sectional curvature
2.	$\frac{SU(3)}{SO(3)}$, an irreducible symmetric space of type AI	Symmetric
3.	$\frac{SU(2)\times SU(2)}{SU(2)} \times \frac{SU(2)}{S^1} = S^3 \times S^2$	Unique invariant Einstein metric

Table 2.1 (continuation)

4.	$\frac{SU(2)\times SU(2)}{i(S^1)} = M_i^5$	For any nondiagonal embedding there exists exactly one invariant Einstein metric, and for a diagonal embedding at least one invariant Einstein metric

We note that this theorem allows us to give an answer on a problem from A.Besse book [Bes] for different embeddings.

Problem. *Given a compact 1 – connected homogeneous space $M = G/H$ is the set of G-invariant Einstein metrics on M finite (up to homotheties)?*

Indeed, every space M_i^5 is diffeomorphic to the Riemannian product $S^3 \times S^2$ and our family of homogeneous Einstein metrics on $S^3 \times S^2$ is non-homothetic and non-finite.

We also note that D.V.Alekseevskii, Isabel Dotti Miatello, and Sergio Ferrara have obtained a classification of homogeneous Einstein 5-manifolds as well as a number of new results about Einstein manifolds of positive Ricci curvature.

In dimension 6, we have obtained a classification of compact simply connected homogeneous Einstein manifolds. There are symmetric spaces, some spaces which are diffeomorphic to the symmetric spaces, and also the space $SU(3)/T_{max}$ with two invariant Einstein metrics (see Theorem 1.).

Now we can suppouse, without the loss of generality, that G is a compact semisimple Lie group and H is a compact subgroup of G.

Definition. *A G – homogeneous Riemannian manifold $(G/H, \rho_B)$ (with compact semisimple G and compact H) is called standard homogeneous Riemannian manifold if its metric ρ_B is obtained from the mines Killing's form B of the Lie algebra g under the natural projection $\pi : G \mapsto G/H$.*

Let us consider the reductive decomposotion $g = h\perp_B p$ for G/H. Then the Ricci curvature of $(G/H, \rho_B)$ is given by the formula [Wa-Zi]:

$$ricc(X) = \frac{1}{4} + \frac{1}{2} \cdot B(C_{\chi,B}(X), X),$$

where χ is the isotropy representation of H in $gl(p)$, and $C_{\chi,B}$ is the Casimir operator of the isotropy representation χ .

Finally, from this formula one may obtain a formula for the calculation of the Ricci curvature of standard Riemannian manifolds with the help of the marked Dynkin's diagrams. Using this algorithm we proved the theorems:

Theorem 3 ([Ro 8], [Ro 12]). *Let $(G/H, \rho_B)$ be a 1-connected compact standard homogeneous Riemannian manifold, G is a compact connected semisimple Lie group acting effectively on $M = G/H$, H is a closed simple subgroup of G. Then $(G/H, \rho_B)$ is Einstein manifold if and only if the Lie algebras (g, h) of the Lie groups (G, H) are contained in the following list:*

(1) *$(g, h) = (h \oplus ... \oplus h, \triangle h)$, where $\triangle h$ is the diagonal inclusion;*
(2) *g is a simple Lie algebra and the pair (g, h) is one from the Einstein pairs of Cartan - Manturov - Wolf - Wang - Ziller.*

We note that the list of pairs of Cartan - Manturov - Wolf - Wang - Ziller is finite and containes some symmetric pairs of Cartan, compact non-symmetric strongly isotropy irreducible pairs of Manturov - Wolf, and compact non-strongly isotropy irreducible pairs of Wang - Ziller corresponding to the standard homogeneous Einstein manifolds with simple transitive group G [Bes].

Theorem 4 [Ro 12]. *Let $(G/H, \rho_B)$ be a compact 1-connected special reductive standard homogeneous Einstein manifold, G a connected 1-connected semisimple Lie group acting effectively on $M = G/H$, and H a closed subgroup of G whose Lie algebra h is a regular subalgebra of Lie algebra g. Then (M, ρ_B) is isometric to the direct Riemannian product of irreducible (in sence of de'Rham) standard Einstein manifolds $(M, \rho_B) = (M_1, \rho_{B_1}) \times ... \times (M_r, \rho_{B_r})$, where $M_i = G_i/H_i$, and also:*

(1) *Einstein constants $C_1, ..., C_r$ of $M_1, ..., M_r$ coincide;*
(2) *g_i are simple algebras;*
(3) *$rkg_i = rkh_i$;*
(4) *the pair (g_i, h_i) is one from the Einstein pairs of Manturov - Wolf - Wang - Ziller satisfying the condition (3).*

Moreover, with the help of this algorithm we have constructed new examples of homogeneous Einstein manifolds. It is interesting that these examples correspond to the solutions of some Diophantine equations.

We consider the embedding

$$H = K \times L \subset (K \times ... \times K) \times L \subset K \times ... \times K \times G_o = G$$

where the first embedding is of the form $diag \times id$ (K is taken t times) and the second is of the form $id \times ... \times id \times \pi$, where $\pi : K \times L \subset G_o$ is some embedding; G_o, K, L are compact connected simple Lie groups.

Theorem 5 ([Ro 8], [Ro 12]). *Let $(g_o, k \oplus l)$ be a compact irreducible symmetric or compact nonsymmetric strictly isotropically - irreducible pair. Then the space*

$(G/H, \rho_B)$ will be an Einstein manifold if and only if the Lie algebras g_o, k, l appear in the list presented in Table 5.1, and also in the first two cases the embeddings

$$\pi : so(n) \oplus so(m) \subset so(n+m),$$
$$\pi : sp(n) \oplus sp(m) \subset sp(n+m)$$

must be the standard embeddings whereas in the last two they must be nonstandard embeddings.

We also note that in the orthogonal and symplectic cases we have the following nontrivial decisions of Einstein equations correspondingly:

(SO) $\quad (n, m, t) = (t^2 - 4t + 6, t - 2, t) \quad (t \in N)$

(SP) $\quad (n, m, t) = (2s^2 - 1, s, 2s) \quad (s \in N)$

Table 5.1

g_0	k	l	Einstein equations
$so(n+m)$	$so(n)$	$so(m)$	$n^2 + (t-5)n + 6 - 2t$ $= m[m + (n-2)(t-1)]$
$sp(n+m)$	$sp(n)$	$sp(m)$	$2n^2 + (5-t)n + 3 - t$ $= 2m[(t-1)(n+1) + m]$
$so(4n)$	$sp(n)$	$sp(1)$	The metric is Einstein when $t = 11$ and $n = 8$
e_6	g_2	$su(3)$	The metric is Einstein when $t = 2$

Theorem 6 ([Ro 8], [Ro 12]). *Let*

$$H = SO(k) \times SO(n) \times SO(m) \subset$$
$$\subset SO(k) \times [SO(n) \times SO(n)] \times SO(m) \subset SO(k+n) \times SO(n+m) = G$$

where the first embedding is of the form $id \times diag \times id$ and the second is of the form $[(\rho_k \otimes id) \oplus (id \otimes \rho_n)] \times [(\rho_n \otimes id) \oplus (id \otimes \rho_m)]$. Then $(G/H, \rho_B)$ is Einstein manifold if and only if $k = m$, $n = 2m + 1$.

These examples show that the case of a semisimple isotropy group is less trivial and leads to new Einstein manifolds.

Definition. *Periodic space with multiplication of order k is a manifold M with a multiplication $\mu : M \times M \to M$, $(x, y) \mapsto x \cdot y$ such that the mappings $s_x : M \to M$, $y \mapsto s_x(y) = x \cdot y$ satisfy the following axioms:*

(1) $s_x(x) = x \quad \forall x \in M;$

(2) $s_x \circ s_y = s_z \circ s_x$, *where* $z = s_x(y)$ $\forall x, y, z \in M$;
(3) *for every point* $x \in M$ *the symmetry* s_x *is a diffeomorphism and endomorphism* $id - (ds_x)_x$ *of the tangent space* $T_x M$ *is non-degenerated;*
(4) *there exists a number* $k \in Z^+$ *such that for all* $x, y \in M$ *we have* $(s_x)^k(y) = y$ *and* k *is a minimal number with this property.*

Theorem 7 ([Ro 3], [Ro 7]). *Let* $(G/H, \rho_B)$ *be a simply connected compact standard periodic Riemannian manifold,* G *a compact connected and simply connected semisimple Lie group acting effectively on* $M = G/H$, *and* H *a closed subgroup of* G. *Then* $(G/H, \rho_B)$ *is an Einstein manifold if and only if it is either isometric to a compact symmetric space or isometric to the direct product of compact standard periodic Riemannian manifolds:* $(G/H, \rho_B) = (G_1/H_1, \rho_{B_1}) \times ... \times (G_n/H_n, \rho_{B_n})$, *each of which has the property that the Lie algebras* (g_i, h_i), $i \in \{1, ..., n\}$, *of the Lie groups* (G_i, H_i) *occur in Table 7.1, and the Einstein constants* $C_1, ..., C_n$ *of these spaces coincide.*

Table 7.1. Application. t_{max} is the maximal Abelian subalgebra of the corresponding simple algebra g_i.

$No.$	g_i	h_i	C_i
1	$g_o \oplus ... \oplus g_o$ n *times*	$\triangle g_o$ g_o *simple*	$1/4 + 1/(2n)$
2	$su(nk)$ $2 \leq k, 3 \leq n$	$n \cdot su(k) \oplus R^{n-1}$	$1/4 + 1/(2n)$
3	$sp(3n-1)$ $1 \leq n$	$sp(n) \oplus R \oplus su(2n-1)$	$5/12$
4	$so(3n+2)$ $3 \leq n$	$so(n) \oplus R \oplus su(n+1)$	$5/12$
5	e_6	$so(8) \oplus R^2$	$5/12$
6	$su(n)$ $3 \leq n$	t_{max}	$1/4 + 1/(2n)$
7	$so(2n)$ $3 \leq n$	t_{max}	$1/4 + 1/(4n-4)$
8	e_6	t_{max}	$7/24$
9	e_7	t_{max}	$5/18$
10	e_8	t_{max}	$4/15$
11	$sp(2)$	$su(2)$	$9/20$

Table 7.1. (continuation)

$No.$	g_i	h_i	C_i
12	g_2	$su(3)$	$5/12$
13	f_4	$su(3) \oplus su(3)$	$5/12$
14	e_6	$3su(3)$	$5/12$
15	e_7	$su(6) \oplus su(3)$	$5/12$
16	e_8	$su(9)$	$5/12$
17	e_8	$e_6 \oplus su(3)$	$5/12$
18	e_8	$2su(5)$	$7/20$

Further we shall establish a connection between standard homogeneous Einstein manifolds and Einstein triple algebras. Before this:

Definition. *Lie triple algebra is an anticommutative algebra* $(A, \circ)$ *over field* F *with trilinear operation* $A \times A \times A \ni (X, Y, Z) \mapsto [X, Y, Z] \in A$ *and axioms:*

(1) $[X, X, Y] = 0$;
(2) $\sigma\{[X, Y, Z] - (X \circ Y) \circ Z\} = 0$;
(3) $\sigma\{[X \circ Y, Z, U]\} = 0$;
(4) $[X, Y, U \circ V] = [X, Y, U] \circ V + U \circ [X, Y, V]$;
(5) $[U, V, [X, Y, Z]] = [[U, V, X], Y, Z] + [X, [U, V, Y], Z] + [X, Y, [U, V, Z]]$,

where σ *denotes cyclic sum over* $X, Y, Z \in A$.

Let G/H be a reductive homogeneous space with the reductive decomposition $g = h \oplus m$. Let us define two operations on m by the rule:

$$X * Y = [X, Y]_m, \quad I(X, Y, Z) = [[Y, X]_h, Z]$$

for $X, Y, Z \in m$.

It is well-known that the triple $(m, *, I)$ is a Lie triple algebra and its called Lie triple algebra (L.t.a.) of the reductive homogeneous space G/H.

Theorem 8 ([Ro 6], [Ro 12]). *Let* $(G/H, \rho_B)$ *be a compact irreducible standard homogeneous Riemannian manifold. Then* $(G/H, \rho_B)$ *is an Einstein manifold if and only if the Lie triple algebra* $(m, *, I)$ *of* G/H *satisfies the identity:*

$$X * \sum_i I(X, X_i, X_i) = 0 \quad \forall X \in m,$$

where $\{X_i\}$ *is a* B*-orthonormal basis of* m.

We note that in non-irreducible case we can apply the theorem of Costant-de'Rham and reduce this case to the irreducible case.

Definition. *A finite-dimensional L.t.a. $(A, *, I)$ over R with $*$-invariant scalar product Q is called Einstein triple algebra if for every Q-orthonormal basis $\{X_i\}$ of A and each element $X \in A$ the following identity*

$$X * \sum_i I(X, X_i, X_i) = 0$$

is true.

So we have connections between standard homogeneous Einstein manifolds, marked Dynkin's diagrams and Einstein triple algebras.

Examples of Einstein triple algebras.

(1) If $(G/H, \rho_B)$ is a compact irreducible symmetric space with $H \neq e$ then for the decomposition $g = h \perp_B m$ one has $m * m = 0$ and therefore $(m, *, I)$ is an Einstein triple algebra.
(2) Let (G, ρ_B) be a compact simple Lie group with biinvariant Riemannian metric. Then $(g, [.,.]) = (m, *)$ and $I \equiv 0$. Hence $(g, [.,.], I \equiv 0)$ is an Einstein triple algebra and (G, ρ_B) is an Einstein manifold.
(3) The Lie triple algebras of the spaces Manturov-Wolf-Wang-Ziller give us another example of Einstein triple algebras.
(4) The space of Ledger-Obata $(G_o \times ... \times G_o / \triangle G_o, \rho_B)$ (with G_o compact simple and ρ_B standard) give us Lie triple algebra which is Einstein triple algebra too.
(5) Above examples of standard homogeneous Einstein manifolds give also examples of Einstein triple algebras.

We shall present a classification of the reductive homogeneous Riemannian manifolds corresponding to the special simple Lie triple algebras which were considered at first by A.Sagle [Sag]. From this classification it follows that such spaces are, in general, standard homogeneous Einstein manifolds.

Definition. *A finite-dimensional Lie triple algebra $(A, \circ, [.,.,.])_R$ is triple S-algebra if the identity*

$$[X, Y, Z] = \alpha(X \circ Y) \circ Z + \beta(Y \circ Z) \circ X + \gamma(Z \circ X) \circ Y$$

is true for all $X, Y, Z \in A$ and some constants $\alpha, \beta, \gamma \in R$.

A.Sagle has been proved that for simple algebra $(A, \circ)$ we have the following possibilities only:

(L) $\{\alpha, \beta, \gamma\} = \{1, 0, 0\}$ and $(A, \circ)$ is a simple Lie algebra;
(M) $\{\alpha, \beta, \gamma\} = \{-1, 1, 1\}$ and $(A, \circ)$ is a simple Malcev algebra;
(S) $\{\alpha, \beta, \gamma\} = \{1/2, 1/4, 1/4\}$ and $(A, \circ)$ is a simple algebra satisfying the identity $J(X, Y, Z) \circ W = J(W, X, Y \circ Z) + J(W, Y, Z \circ X) + J(W, Z, X \circ Y)$

where $J(X, Y, Z)$ is the Jacobian.

We have studied homogeneous spaces G/H with Lie triple algebras $(m, *, I)$ of S-type and have proved the following

Theorem 9 [Ro 12]. *Let $M = G/H$ be a reductive homogeneous space whose Lie triple algebra is an algebra of types (L) or (M). Then*

(1) *there exists only one, up to a homothety, G-invariant Riemannian metric on M (canonical Riemannian metric);*
(2) *in case (L) the space (M, ρ) is isotropy irreducible and locally symmetric;*
(3) *in case (M) the universal covering (M_o, ρ_o) of (M, ρ) is isometric to the space $Spin(7)/G_2$.*

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