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THE INTEGRAL OF MIXED SCALAR CURVATURE ALONG A LEAF OF FOLIATION

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Abstract. By using the method of matrix Riccati ODE, we study Riemannian manifolds with foliations (distributions), whose mixed Ricci or scalar curvature is nonnegative and the norms of integrability tensors are bounded above. We obtain the integral formula for mixed scalar curvature along the complete leaf, rigidity of totally geodesic distribution, splitting of foliation with "large" dimension.

1. Main results

For two complementary orthogonal distributions T_1 and T_2 on TM we define the *structural tensors* $B_1 : T_2 \times T_1 \to T_1$, $B_2 : T_1 \times T_2 \to T_2$ by formulae, where $P_i : TM \to T_i$ are orthoprojectors

$$B_1(y, x) := P_1(\nabla_x y), \quad B_2(x, y) := P_2(\nabla_y x). \tag{1}$$

Obviously, the equation $B_i = 0$ means that T_i is tangent to totally geodesic foliation. The equalities $B_1 = B_2 = 0$ mean by de Rham decomposition theorem that M is locally a product $L_1 \times L_2$, where $TL_1 = T_1$, $TL_2 = T_2$. Tensors (1) (as well as Gray's tensors O, T [Gra2] or O'Neill's tensors A, T [O'Ne]) contain the whole local information about distributions, for example, their *mean curvature vector fields* $H_1 \subset T_2$, $H_2 \subset T_1$ are given by formulae

$$(H_2, x) = -\text{trace } B_2(x, *), \; (H_1, y) = -\text{trace } B_1(y, *). \tag{2}$$

Firstly such so called *co-nullity operator* was studied by [Ros], [Gra1] for *nullity foliation* of curvature operator and by [O'NS] for *relative nullity foliation* on submanifold in space form (see [Fer], [Dom] for totally geodesic foliation).

The *second fundamental forms* $h_i : T_i \times T_i \to T_i^\perp$ and *integrability tensors* $A_i : T_i \times T_i \to T_i^\perp$ are defined by formulae

$$\begin{aligned} h_1(x, u) = P_2(\nabla_x u + \nabla_u x)/2, \quad A_1(x, u) = P_2(\nabla_x u - \nabla_u x)/2, \\ h_2(y, z) = P_1(\nabla_y z + \nabla_z y)/2, \quad A_2(y, z) = P_1(\nabla_y z - \nabla_z y)/2. \end{aligned} \tag{3}$$

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Note, that h_i is a second fundamental form at a point $p \in M$ of submanifold, consisting of geodesics in M, tangent to subspace $T_i(p)$, the equality $A_i = 0$ means that T_i is tangent to foliation.

Definition 1. A distribution (or foliation) T_i on a Riemannian manifold M is called *(totally) geodesic* or *minimal* if $h_i = 0$ or $H_i = 0$.

Geodesic distribution is characterized by the property: *if geodesic in M is tangent to T_i at one point, then it stays tangent to T_i at every point.*

Proof. Let $\gamma : I \to M$ be a T_i-*geodesic*, i.e. $\gamma' \subset T_i$ and $\nabla_{\gamma'}\gamma' = h_i(\gamma', \gamma') \perp T_i$. If $h_i = 0$, then $\nabla_{\gamma'}\gamma' = 0$, i.e. γ is geodesic in M. □

The symmetric and skew-symmetric with respect to $T_i^\perp$ components B_i^+, B_i^- of structural tensors may be considered as self-adjoint *Weingarten tensors* associated to h_i, and as equivalent terms of integrability tensors for T_i in view of identities

$$(B_1^+(y,x),u) = -(h_1(x,u),y), \quad (B_2^+(x,y),z) = -(h_2(y,z),x), \tag{4}$$

$$(B_1^-(y,x),u) = -(A_1(x,u),y), \quad (B_2^-(x,y),z) = -(A_2(y,z),x). \tag{5}$$

Lemma 1 [Rov3]. *The following differential equation is true (similar to Codazzi equation (2.4) in [Gra2])*

$$\begin{aligned}((\nabla_{x_1}B_2)(x_2,y),z) + (B_2(x_2,B_2(x_1,y),z) + ((\nabla_y B_1)(z,x_1),x_2) +\\ (B_1(z,B_1(y,x_1)),x_2) + (R(y,x_1)x_2,z) = 0.\end{aligned} \tag{6}$$

If $\gamma : I \to M$ is a unit speed T_1-geodesic, then the endomorphism field $B_2(t) = B_2(\gamma'(t), *)$ of T_2 along γ satisfies the Riccati ODE

$$B_2' + B_2^2 + F(t) = 0, \quad (t \in I), \tag{7}$$

where $(F(t)y,z) = (R(y,\gamma')\gamma',z) + ((\nabla_y B_1)(z,\gamma'),\gamma') + (B_1(z,B_1(y,\gamma')),\gamma')$. For totally geodesic foliation $F(t)$ coincides with *Jacobi operator* $R_{\gamma'}$, see [Fer], [Dom].

The *expansion* θ, *sheaf* tensor σ and *vorticity* tensor ω are calculated from B_2: $\theta = trace\, B_2^+$, $\sigma = B_2^+ - a\,\theta\, Id$, $\omega = B_2^-$, where $a = \frac{1}{\dim T_2}$. Note, that $\sigma^2 \geq 0$, but $\omega^2 \leq 0$, and $\sigma = 0$ exactly for conformal operator B_2. It is clear, that θ satisfies the following equation, which is also called the *Raychaudhuri equation* in Relativity theory:

$$\theta' + a\theta^2 + \{trace\, F(t) + trace\,(\sigma^2) + trace\,(\omega^2)\} = 0. \tag{8}$$

Let $\{x_i\} \subset T_1$, $\{y_j\} \subset T_2$ are local orthonormal bases. The *mixed Ricci and scalar curvature* are defined by

$$\mathrm{Ric}_1(x) = \sum_j K(x,y_j), \quad \mathrm{Ric}_2(y) = \sum_i K(x_i,y), \quad s_{mix} = \sum_{i,j}(R(y_j,x_i)x_i,y_j).$$

The method of matrix Riccati ODE (see Lemma 4 below) allows to prove

Proposition 1 [BW] ([Abe] *for* $K_{mix} \geq 0$**).** *Let* $\{L\}$ *be a totally geodesic foliation with complete leaves, integrable* $T_2 = TL^{\perp}$ *and* $Ric_1 \geq 0$. *Then foliation splits locally.*

The contraction of (6) with respect to i, j gives us equation

$$\operatorname{div}(H_1 + H_2) = s_{mix} + ||h_1||^2 + ||h_2||^2 - ||A_1||^2 - ||A_2||^2 - |H_1|^2 - |H_2|^2, \quad (9)$$

where we used $\operatorname{div}_1 H_2 = \operatorname{div} H_2 + |H_2|^2, \quad \operatorname{div}_2 H_1 = \operatorname{div} H_1 + |H_1|^2$,

$$\sum_{i,j}(B_2(x_i, B_2(x_i, y_j)), y_j) = ||B_2^+||^2 - ||B_2^-||^2 = ||h_2||^2 - ||A_2||^2, \quad (10)$$

$$\sum_{i,j}(B_1(y_j, B_1(y_j, x_i)), x_i) = ||B_1^+||^2 - ||B_1^-||^2 = ||h_1||^2 - ||A_1||^2. \quad (11)$$

Integration of (9) along compact manifold, or over compact leaf L_0 of foliation, in view of Green's Theorem, leads to *integral formulae* having many interesting global corollaries ([Wal], [Ran], [Roc])

$$\int_M \{s_{mix} + ||B_1^+||^2 + ||B_2^+||^2 - ||B_1^-||^2 - ||B_2^-||^2 - |H_1|^2 - |H_2|^2\} = 0, \quad (12)$$

$$\int_{L_0} \{s_{mix} + \operatorname{div}_2 H_1 + ||B_1^+||^2 + ||B_2^+||^2 - ||B_2^-||^2\} = 0. \quad (13)$$

In relation with Proposition 1 and Theorems 1, 2 we give

Proposition 2. *A compact minimal foliation* $\{L\}$ *on a Riemannian manifold* M *with integrable normal bundle and inequality* $s_{mix} \geq 0$ *splits locally:* $M = L \times L^{\perp}$. *(In particular, a minimal foliation* $\{L\}$ *with integrable normal bundle and inequality* $s_{mix} > 0$ *has no compact leaves [Wal]).*

Proof. From (13) in view $B_2^- = 0$ it follows $B_1^+ = B_2^+ = 0$, i.e. TL and $TL^{\perp}$ are tangent to totally geodesic foliations. By de Rham theorem foliation splits. □

The consequence $\rho(n) = 8b + 2^c$, where $n = (\text{odd})\, 2^{4b+c}$, $(b \geq 0,\ 0 \leq c \leq 3)$ is well-known in topology: $\rho(n) - 1$ is *number of continuous pointwise linearly independent vector fields on* $(n{-}1){-}$*sphere.* It is true inequality $\rho(n) \leq 2 \log_2 n + 2$. Let $|A_i|(e) = \sup\{|A_i(e,v)| : |v| = 1\}, \quad |A_i|(p) = \sup\{|A_i|(e) : |e| = 1\}$.

Theorem 1. *Let* M *be a complete Riemannian manifold equipped with two complementary orthogonal distributions* T_1, T_2, $(\dim T_1 = \nu,\ \dim T_2 = n)$, *and* T_1 *is totally geodesic. a) Then for every* $T_1{-}$*geodesic* $\gamma(t) \subset M$

$$\liminf_{s \to +\infty} \int_{-s}^{s} \left\{Ric_1(\gamma') - (\nu - 1)|A_1|^2(\gamma) - n(n-1)|A_2|^2(\gamma)\right\} dt \leq 0, \quad (14)$$

and equality holds for every γ iff T_2 is totally geodesic distribution, and skew-symmetric operators B_2, B_1 have "constant torsion":

$$B_2(x, B_2(x, y)) = -|A_2|^2(p)x^2 y, \quad B_1(y, B_1(y, x)) = -|A_1|^2(p)y^2 x. \tag{15}$$

b) If equality holds in (14) for every γ and additionally $\nu \geq \rho(n)$, then T_2 is also integrable and hence is tangent to Riemannian totally geodesic foliation.

Proof. a) Consider (6) along any unit speed T_1-geodesic $\gamma : \mathbb{R} \to M$

$$(\nabla_{\gamma'} B_2)(\gamma', y) + B_2(\gamma', B_2(\gamma', y)) + (B_1(*, B_1(y, \gamma')), \gamma') + P_2(R(y, \gamma')\gamma') = 0 \tag{16}$$

with use $((\nabla_y B_1)(*, \gamma'), \gamma') = 0$, because B_1 is skew-symmetric for totally geodesic T_1. The symmetric for T_2 component of (16) is

$$\begin{aligned}(\nabla_{\gamma'} B_2^+)(\gamma', y) + B_2^+(\gamma', B_2^+(\gamma', y)) + B_2^-(\gamma', B_2^-(\gamma', y)) +\\ (B_1(*, B_1(y, \gamma')), \gamma') + P_2(R(y, \gamma')\gamma') = 0.\end{aligned} \tag{17}$$

Consider the Raychaudhuri equation (8), for $\theta(t) = trace\, B_2^+(\gamma')$

$$\begin{aligned}&\theta' + a\theta^2 + \\ &\{Ric_1(\gamma') + trace\, \sigma^2(\gamma') - \sum_i |B_2^-(\gamma', y_i)|^2 - \sum_i |B_1(y_i, \gamma')|^2\} = 0.\end{aligned} \tag{18}$$

Note that for $z_1 = \gamma'$

$$\begin{aligned}\sum_i |B_1(y_i, \gamma')|^2 = \sum_j |A_1(\gamma', z_j)|^2 \leq (\nu - 1)|A_1|^2(\gamma'),\\ \sum_i |B_2^-(\gamma', y_i)|^2 \leq n(n-1)|A_2|^2(\gamma).\end{aligned} \tag{19}$$

From Lemma 4 below and (19) it follows (14) and for the equality case:
$\theta = \sigma(\gamma') = B_2^+(\gamma', *) = 0$ and $|B_2^-(\gamma', *)| = |A_2|(\gamma)$, $|B_1(*, \gamma')| = |A_1|(\gamma)$. Applying this to every T_1-geodesic we obtain for the equality case: $B_2^+ = 0$, i.e. T_2 is totally geodesic and (15) is true.

b) By point a), the distribution T_2 is totally geodesic and (15) is right. Assume the opposite, that T_2 in non-involutive somewhere, i.e. by (5) there exist point $p \in M$ and unit vectors $x_0 \in T_1(p)$, $y_0 \in T_2(p)$ with the property $B_2(x_0, y_0) \neq 0$. Therefore, $|A_2|(p) \neq 0$ and in view of (15) $B_2(x, y) \neq 0$ for each nonzero $x \in T_1(p)$, $y \in T_2(p)$. Since skew-symmetric linear operator $B_2(x, *)$, $(x \neq 0)$ has no eigenvectors, then following [Fer] we choose a basis $\{u_i\}$ for $T_1(p)$ and define ν continuous pointwise linearly independent vector fields $\{v_i\}$ on the unit sphere $S^{n-1} \subset T_2(p)$ by the rule: $v_i(y) = B_2(u_i, y)$. Thus $\nu < \rho(n)$ – a contradiction. □

Combine with the method [Gui], we obtain the inequality for integral of mixed scalar curvature along a leaf.

Theorem 2. *Let $\{L^\nu\}$ be a foliation on Riemannian manifold $M^{n+\nu}$ with complete minimal leaves, and quadratic form*

$$Ric_1(x) + trace\, B_2^-(x, B_2^-(x,*)) - trace\,(h_1(x, B_1^+(*,x)) + (\nabla_* h_1)(x,x))$$

on a unit tangent bundle SL_0 of every leaf L_0 has an integrable positive or negative part. Then

$$\int_{L_0} \{s_{mix} - n(n-1)|A_2|^2\}\, d\,vol \le 0, \tag{20}$$

and equality holds for every leaf iff $\{L\}$ is a Riemannian totally geodesic foliation. Moreover, if equality in (20) holds for all leaves and $\nu \ge \rho(n)$, then M is locally a product $L \times L^\perp$.

Remark 1. If $TL^\perp$ in Theorem 2 is integrable, then without inequality for dimension foliation splits. The certain Corollary generalizes Proposition 1. For compact leaf L_0 see Proposition 2.

Proof of Theorem 2. is based on some facts from ergodic theory, see 2.2. Consider arbitrary leaf L_0. By symmetric part of structural tensor (1) we define tensor U on SL_0, such that for every $x \in SL_0$: $U(x) = B_2^+(x,*)$ is a linear operator on $TL_0^\perp$. Then U satisfies the Riccati equation $U' + U^2 + \tilde R(x) = 0$, see (7), where $(\tilde R(x)y,z) = (R(y,x)x,z) + (B_2^-(x, B_2^-(x,y)),z) + (B_1^+(x, B_1^+(y,x)),z) + ((\nabla_y B_1^+)(z,x),x)$, $R(x) = R(*,x)x$ – the Jacobi operator on $TL_0^\perp$ and $B_2^-(x,*)$ – the vorticity tensor. Theorem will follow from the inequality

$$\int_{SL_0} \widetilde{\mathrm{Ric}}_1(x)\, d\mu \le -\int_{SL_0} trace\, U^2(x)\, d\mu, \tag{21}$$

where
$\widetilde{\mathrm{Ric}}_1(x) = \mathrm{Ric}_1(x) + trace\, B_2^-(x, B_2^-(x,*)) - trace\,(h_1(x, B_1^+(*,x)) + (\nabla_* h_1)(x,x))$.
To see this, by Fubini's theorem and Lemma 6 below, we obtain

$$\int_{L_0} \tilde s_{mix}\, d\,vol = \frac{1}{\omega_{n-1}} \int_{SL_0} \widetilde{\mathrm{Ric}}_1(x)\, d\mu \le -\int_{SL_0} trace\, U^2(x)\, d\mu \le 0,$$

where $\tilde s_{mix} := trace\, \widetilde{\mathrm{Ric}}_1 = s_{mix} - ||B_2^-||^2 + ||B_1^+||^2$ in view of $H_1 = 0$, $s_{mix} = trace\, \mathrm{Ric}_1$ are functions on L_0. Note that $||B_2^-||^2(p) \le n(n-1)|A_2|^2(p)$.
Furthermore, if $\int_{L_0} \tilde s_{mix}\, d\,vol = 0$, then U^2 is identically zero and so $\tilde R(x) = 0$ on SL_0 and $B_1 = 0$, i.e. L_0 is totally geodesic submanifold. Since L_0 is an arbitrary leaf, then distribution $TL^\perp$ and foliation $\{L\}$ are totally geodesic. Moreover, B_2 has constant torsion in sence (15). From the given inequality for dimension of L_0 and Theorem 1 b) it follows that $TL^\perp$ is also integrable. The de Rham decomposition Theorem completes the proof.

To prove (21) let $r_1^+ = (\widetilde{\mathrm{Ric}}_1)^+$ and $r_1^- = (\widetilde{\mathrm{Ric}}_1)^-$ – the positive and negative parts of $\widetilde{\mathrm{Ric}}_1$. If $\int_{SL_0} r_1^-\, d\mu = +\infty$, then $\int_{SL_0} \widetilde{\mathrm{Ric}}_1\, d\mu = -\infty$, because in this case we have $\int_{SL_0} r_1^+\, d\mu < \infty$. So (21) holds.

Hence we may suppose $\int_{SL_0} r_1^- \, d\mu < \infty$. Consider nonnegative integrable function $f \le trace\,(U^2) + r_1^+$ on SL_0. Thus it suffices to show that

$$\int_{SL_0} f \, d\mu \le \int_{SL_0} r_1^- \, d\mu. \tag{22}$$

To prove (22) we will verify that the integral of $f - r_1^-$ is nonpositive in each one of the sets D, C^+ and $C^- \setminus C^+$. Note that SL_0 is the disjoint union $SL_0 = D \cup C^+ \cup (C^- \setminus C^+)$. We integrate the Raychaudhuri equation for $u = trace\, U$, which corresponds to Riccati equation $U' + U^2 + \tilde{R}(x) = 0$, and obtain

$$u(s) - u(t) + \frac{1}{n-1}\int_t^s u^2(\lambda)\, d\lambda + \int_t^s \left\{ trace\,(U^2) + \widetilde{\mathrm{Ric}_1} \right\} d\lambda = 0,$$

i.e.

$$u(s) - u(t) + \frac{1}{n-1}\int_t^s u^2(\lambda)\, d\lambda + \int_t^s (f - r_1^-)(T_\lambda(v))\, d\lambda = 0 \tag{23}$$

for every $v \in SL_0$ and $s, t \in \mathbb{R}$. By Lemma 3 below $\int_t^s (f - r_1^-)(T_t(v))\, dt \le 0$ for almost all $v \in D$. Then by Lemma 5 below

$$\int_D (f - r_1^-)\, d\mu = \int_{\pi(D)} \int_{-\infty}^{\infty} (f - r_1^-)(T_t(v))\, dt\, d\tilde{\mu} \le 0.$$

In order to compute $\int_{C^+} (f - r_1^-) d\mu$, let $\delta > 0$ and $f_0 > 0$ be integrable on SL_0. Define $g = f - r_1^- - \delta f_0$. From (23) we have

$$\delta \int_0^s f_0(T_t(v))\, dt - u(0) + u(s) + \frac{1}{n-1}\int_0^s u^2(t)\, dt + \int_0^s g(T_t(v))\, dt = 0$$

for every $v \in SL_0$ and $s \in \mathbb{R}$. If $v \in C^+$, using Lemma 2 a) below we conclude that $\liminf\limits_{s \to +\infty} g(T_t(v)\, dt = -\infty$. Then by Theorem 3 below we obtain $\int_{C^+} (f - r_1^- - \delta f_0)\, d\mu = \int_{C^+} g\, d\mu$ and therefore $\int_{C^+} (f - r_1^-)\, d\mu \le 0$, by letting $\delta \to 0$. In order to compute $\int_{C^+ \setminus C^-} (f - r_1^-)\, d\mu$ we derive from (23) that

$$\delta \textstyle\int_{-s}^0 f_0(T_t(v)) dt + u(0) - u(s) + \frac{1}{n-1} \int_{-s}^0 u^2(t) dt + \int_{-s}^0 g(T_t(v)) dt \le 0$$

for every $v \in SL_0$ and $s \in \mathbb{R}$. Hence

$$\delta \textstyle\int_0^s f_0(T_{-t}(v)) dt + u(0) - u(-s) + \frac{1}{n-1} \int_0^s u^2(-t) dt + \int_0^s g(T_{-t}(v)) dt \le 0. \tag{24}$$

If $v \in C^-$, then $\liminf\limits_{s \to +\infty} g(T_{-t}(v))\, dt = -\infty$ by (31) and Lemma 2 a) below. Using Theorem 3 below we obtain $\int_{C^- \setminus C^+} (f - r_1^- - \delta f_0)\, d\mu = \int_{C^- \setminus C^+} g\, d\mu$. Hence letting $\delta \to 0$ leads to $\int_{C^- \setminus C^+} (f - r_1^-)\, d\mu \le 0$. $\square$

As from Theorem 1 so from Theorem 2 it follows

Corollary 1 [KT] *(rigidity). Let $\{L\}$ be a Riemannian foliation on a complete Riemannian manifold M. If for each geodesic γ orthogonal to leaves we have $Ric_2(\gamma') \ge (n-1)|A_2|^2(\gamma')$, then $\{L\}$ is totally geodesic.*

Corollary 2 [Rov2] *(splitting). Let M, $\{L\}$ be a foliated Riemannian manifold with complete totally geodesic leaves and $Ric_1(x) \ge n(n-1)|A_2|^2(p)$, $(x \in T_pM)$. If $\nu \ge \rho(n)$, then M is locally a product $L \times L^\perp$.*

2. Necessary facts

2.1 Ferus's scheme. Ferus [Fer] proved the most effective estimate $\dim L < \rho(\operatorname{codim} L)$ of dimension of totally geodesic foliation $\{L\}$ with complete leaves and $K_{mix} = const > 0$. The proof is based on the following scheme, which was used later by many geometers.

1. O'Neill and Steil [O'NS] studied relative nullity foliations on submanifolds in space forms and have used the following property of matrix Riccati equation $B' + B^2 + kE = 0$ with $K_{mix} = k = const$:

the eigen-vectors of solution $B(t)$ do not depend on t, and eigen-function $\lambda(t)$ satisfies the scalar Riccati equation $\lambda(t)' + \lambda^2(t) + k = 0$, and for $k > 0$ can't be continued on the whole real line.

2. The key for the existence of eigen-vector for structural tensor B of totally geodesic foliation $\{L\}$ is the following [Fer]:

in case $\dim L \geq \rho(codim\, L)$ for any point $m \in L$ there exist nonzero vectors $x \in T_m L,\ y \in T_m L^{\perp}$ and the real number λ with the property $B(x, y) = \lambda y$.

The inequality with dimensions in local Theorem [Fer] is exact, but is wrong without additional conditions for totally geodesic foliation with $K_{mix} > 0$ (in Riccati equation kE replaced by non-constant symmetric matrix $R(t) > 0$) [Rov1].

Lemma 2 [EK]. *Let $\int_0^{+\infty} f(t)dt$ and $\int_{-\infty}^0 f(t)dt$ converge conditionally and there exists a solution $\lambda(t)$ of the Riccati equation $\lambda' + a\lambda^2 + f = 0,\ (t \in \mathbb{R})$. Then a) $\lim\limits_{s\to+\infty} \lambda(t) = \lim\limits_{s\to-\infty} \lambda(t) = 0$,*
b) $\int_{-\infty}^{+\infty} f(t)dt = -a\int_{-\infty}^{+\infty} \lambda^2(t)dt \leq 0$ where equality holds iff $\lambda = f = 0$.

Lemma 3 [Gui]. *Let $\lambda : [0, +\infty) \to \mathbb{R}$ and $\mu : [-\infty, +\infty) \to \mathbb{R}$ be continuous functions and $a > 0$. Then a) $\limsup\limits_{s\to+\infty} \left(\lambda(s) + a\int_0^s \lambda^2(t)dt\right) \geq 0$ with equality iff $\lambda = 0$, b) $\limsup\limits_{s\to+\infty} \left(\mu(s) - \mu(-s) + a\int_{-s}^s \mu^2(t)dt\right) \geq 0$ with equality iff $\mu = 0$.*

Lemma 4. *Let $a = const > 0$, $R(t)$ – a symmetric continuous $(n \times n)-$matrix, and the Riccati equation $B' + aB^2 + R(t) = 0,\ (t \in \mathbb{R})$ admits a solution $B(t)$ with a symmetric $B(0)$. Then $\liminf\limits_{s\to+\infty} \int_{-s}^s \{trace\, R(t) + trace\,(\sigma^2)\}\, dt \leq 0$ and equality holds iff $trace\, B(t) = trace\, R(t) + trace\,(\sigma^2) \equiv 0$. Moreover, if $trace\, R(t) \geq 0$, then $B(t) = R(t) \equiv 0$.*

Remark 2. Without the symmetry for $B(0)$ Lemma 4 is wrong [Rov1]. Application of Lemma 4 for manifolds without conjugate points see in [EK].

Proof. Since $B^-(0) = 0$, then $B(t)$ is symmetric for all t [Abe]. Then from (8) assuming $f(t) = trace\, R(t) + trace\,(\sigma^2)$ and $\theta(t) = trace\, B(t)$ we have $\theta(s) - \theta(-s) + a\int_{-s}^s \theta^2(t)dt = -\int_{-s}^s f(t)dt$. Hence, by Lemma 3 b) $\limsup\limits_{s\to+\infty}\left(-\int_{-s}^s f(t)dt\right) = -\liminf\limits_{s\to+\infty} \int_{-s}^s f(t)dt \geq 0$. If equality holds, then $\theta = 0$, and hence $f = 0$ from (8). From $trace\, R(t) \geq 0$ it follows $trace\, R(t) = \sigma = 0$, i.e. $B = a\theta = 0$. Finally from given matrix Riccati equation we obtain $R = 0$. □

2.2. Ergodic theory. Let SL_0 – a unit tangent bundle of a leaf L_0, T_t – a geodesic flow on SL_0, μ – Liuville measure on the Borel sets of SL_0, which is $\sigma-$finite and invariant under T_t.

(Maximal Ergodic) Theorem 3 [Gui]. *Let f be an integrable function on SL_0 and let $D \subset SL_0$ be T_t-invariant Borel set. Define $E(f) = \{v \in D : \sup_{s>0} \int_0^s f(T_t(v))\,dt > 0\}$. Then $\int_{E(f)} f\,d\mu \geq 0$. If we set $T = T_1$, and if A is a $T-$invariant Borel set and $E[f] = \{v \in A :\ \sup_{n\geq 0} \sum_{j=0}^n f(T^j(v)) > 0\}$, then $\int_{E[f]} f\,d\mu \geq 0$.*

Let f_0 be an integrable function on SL_0. Then the Borel sets

$$D^+ = \{v \in SL_0 :\ \int_0^\infty f_0(T_t(v))\,dt < \infty\},\quad C^+ = SL_0 \setminus D^+ \tag{25}$$

are T_t-invariant; they are independent of f_0 in the following sence [Gui]: if f_1 is another integrable function, then the intersection of set $D_1^+ = \{v \in SL_0 : \int_0^\infty f_1(T_t(v))\,dt = \infty\}$ with D^+ has measure zero.

The components D^+ and C^+ are called, respectively, the *dissipative* and the *conservative parts* of the *E.Hopf decomposition* $SL_0 = D^+ \cup C^+$. Let $SL_0 = D^- \cup C^-$ be the decomposition of SL_0 associated to the *inverse geodesic flow* $\{T_{-t}\}$, $(t \in \mathbb{R})$, i.e. $D^- = \{v \in SL_0 :\ \int_{-\infty}^0 f_0(T_t(v))\,dt < \infty\}$, $C^- = SL_0 \setminus D^-$. If $D = D^+ \cap D^-$, then for every integrable function f on SL_0, the Lebesgue integral

$$\int_{-\infty}^\infty f(T_t(v))\,dt \tag{26}$$

exists for almost all $v \in D$. Also, if $g_0 > 0$ is integrable, then the $T-$invariant sets

$$D[T] = \{v \in SL_0 :\ \sum_{-\infty}^\infty g_0(T^j(v)) < \infty\},\quad C(T) = SL_0 \setminus D(T)$$

are independent of g_0, up to subset of measure zero. Moreover, if $g \geq 0$ is integrable, then $\sum_{-\infty}^\infty g(T^j(v)) < \infty$ for almost all $v \in D(T)$.

Next we consider a measure on the space of orbits. Let $\sim$ be the equivalence relation in SL_0 defined by $v \sim w \iff \exists j \in \mathbb{Z} :\ T^j(v) = w$. Let $\tilde{S}L_0$ be the space of orbits, and denote by $\pi : \tilde{S}L_0 \to SL_0$ the natural projection $\pi(v) = [v]$. Consider in $\tilde{S}L_0$ the $\sigma-$algebra $\tilde{\beta}$ induced by π. Then $\pi(E)$ is measurable for every Borel set E, because $\pi^{-1}(\pi(E)) = \bigcup_{-\infty}^\infty T^j(E)$.

Given a function f integrable on SL_0, then by (26) $x \in D \to \int_{-\infty}^\infty f(T_t(x))\,dt$ defines a measurable function on the Borel sets of D. Furthermore, it is constant in each orbit on D. Thus $[x] \in \pi(D) \to \int_{-\infty}^\infty f(T_t(x))\,dt$ defines a $\tilde{\mu}-$measurable function on $\pi(D)$.

Lemma 5 [Gui]. *If f is an integrable function on SL_0, then*

$$\int_D f\,d\mu = \int_{\pi(D)} \int_{-\infty}^\infty f(T_t(v))\,dt\,d\tilde{\mu}.$$

Lemma 6. *Let $S^{n-1} \subset \mathbb{R}^n$ be a unit sphere, ω_{n-1} – its volume, A – a quadratic form in $\mathbb{R}^n$. Then $\frac{1}{\omega_{n-1}} \int_{S^{n-1}} A(x)\,dx = trace\,A$.*

Proof. The function $f(x) = A(x) - (trace\,A)x^2$ is, obviously, harmonic in $\mathbb{R}^n$ and by the mean value Theorem the integral of $f(x)$ over S^{n-1} is zero. □

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