DIFFERENTIAL GEOMETRY AND APPLICATIONS
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CONNECTION–FREE DIFFERENTIAL GEOMETRY

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Abstract. Both vector 1-forms (tensor fields of type (1,1)) and Lie brackets map (tangent) vectors to the same. The former do so in a manner that is linear over the ring of functions; the latter are bilinear, but only over the constants. By composing one Lie bracket and m vector 1-forms (we consider $2 \leq m \leq 4$) in all possible ways many maps of vector fields are obtained, bilinear over the constants.
We study linear combinations of these maps with some special properties:
1. Bilinearity over the functions. This yields the F-N bracket as a differential invariant of pairs of vector 1-forms, as well as an additional differential invariant J of triples of vector 1-forms; both are tensors of type (1,2).
2. Linearity over the functions in one variable, while the dependence on the other variable extends to an action on all tensor fields, as a derivation of type (0,1).
We give a complete analysis of all linear relations between these differential invariants. Most derive from a known relation between F-N brackets, and from the Leibniz rule for derivations. There is, however, an additional relation for quadruples of vector 1-forms that is independent of these.
The commutator of two derivations of type (0,1) gives rise to another, of type (0,2). Its "index" (i.e., its action on functions) is expressed in terms of F-N brackets and the J-invariant.
Several of the calculations are made simple by studying an appropriate graph. Others are performed with the use of a computer.

1. Introduction. The most fundamental property of a differential n-manifold is the ever-presence of local diffeomorphisms; in fact, these are the cement that holds together its building blocks: chunks of $\mathbb{R}^n$. Bundles over manifolds, with the property that local diffeomorphisms can be lifted (in a factorial way), are rightly called natural. These include tangent and co-tangent bundles, tensor bundles constructed from these, frame bundles, connection bundles, jet bundles of sections in these, and so on. (See Kolář, Michor and Slovák **[7]**, where the foundations of differential geometry are presented from this point of view.) The structures that differential geometers usually study are sections in natural bundles.

Families of diffeomorphisms, i.e., flows, generate tangent vector fields, and conversely. The lifting $\bar{u}$ of a vector field u to a natural bundle is tangent to the lifted

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flow in the bundle. If s is a section in the bundle, and $s_*(u)$ is tangent to the section and covers u, then $\mathcal{L}_u s = s_*(u) - \bar{u}$ is the Lie derivative; it is tangent to the fibers, and is a section in another natural bundle. When s is a section in a natural *vector* bundle, $\mathcal{L}_u s$ is usually interpreted as a section in that same bundle. $\mathcal{L}_u s$ measures the rate of change of s from the point of view of an observer who travels with the flow.

The Lie derivative is a natural operator: it commutes with (liftings of) diffeomorphisms. As a result, it also "commutes" with other natural operators. For example, it satisfies a "Leibniz Rule" for (tensor) products and other multilinear operations.

This paper is a study of vector 1-forms, so called because they are differential forms of degree 1 whose values are tangent vectors. They act as linear transformations in each tangent space. The composition of two V1F is again a V1F. As a result, they are ideal objects of study in connection with the Lie bracket of (tangent) vector fields. If h, k are V1F and u, v vector fields, it is natural to study expressions such as $[hu, kv], k[u, hv]$, etc. Here are some results.

(a) Given V1F h, k, there is a vector 2-form $[[h, k]]$ defined as a sum of eight expressions of the above type. Following **[7]**, we call it the F-N bracket. (It actually predates (see **[8]**) the work by Frölicher and Nijenhuis **[2]**.)

(b) Given two V1F h, k and a vector field u, the operator $\mathcal{D}_u^{h,k}$ **[9]** acts as a derivation on tensor fields of all types. Like the Lie derivative, it preserves types, satisfies a Leibniz rule, etc., but in contrast to the Lie derivative $\mathcal{L}_u$, its dependence on u is point-wise, i.e., it does not depend on higher order jets of u, more like a covariant derivative ∇_u. Its geometric significance requires further study. – When h, k commute, $\mathcal{D}_u^{h,k}$ is an "inner" derivation.

(c) Given three V1F h, k, ℓ, one can form $\mathcal{D}_u^{h,k}\ell$; then $(u, v) \mapsto (\mathcal{D}_u^{h,k}\ell)v$ is a tensor of type (1,2). Contrary to reasonable expectations, it is not expressible in terms of the F-N bracket. Instead, a study of Lie brackets like $h[ku, \ell v]$, $[ku, h\ell v]$, etc. shows that, in addition to vector 2-forms such as $h[[k, \ell]]$, $[[\ell, kh]]$ that can be formed from the F-N bracket, there is "one" additional differential invariant that is independent of these. Thus arises $J_1(h, k, \ell)$, a sum of twelve Lie brackets; other, sometimes more convenient to handle, expressions (denoted $J_2, \ldots, J_4$) can be defined by adding suitable F-N brackets. In particular, $\mathcal{D}_u^{h,k}\ell$ is expressible in terms of J_1 and a few F-N bracket terms.

(d) Given two derivations, $\mathcal{D}_u^{h,k}$ and $\mathcal{D}_v^{\ell,m}$, their commutator is again a derivation. It depends linearly on four V1F. Its index (the part that acts on scalar functions) is a tensor of type (1,2). This raises the question as to whether it is expressible in terms of J and F-N brackets – or are there additional differential invariants of quadruples of V1F?

(e) To answer (d), we consider all composites that can be formed from one Lie bracket, four V1F and two vector fields, linear in each of them. We show that all linear combinations of these that are tensors of type (1,2) are expressible in terms of J and the F-N bracket.

(f) Although there is just one differential invariant of pairs of V1F, namely the F-N bracket, when it is applied to triples of V1F, such as $[[hk, \ell]]$, $k[[\ell, h]]$, etc., there is "one" relation between them. (More precisely, there are three, each obtained from another by permuting h, k, ℓ, and these form a basis for all.) When considering quadruples of V1F, there are additional relations. First of all, the Leibniz rule for $\mathcal{D}_u^{h,k}$ applied to products of V1F, provides relations between F-N brackets and J_1. However, there is one brand-new relation, to complete a basis for all. These relations are a complete set of identities one uses in calculations, without the need to refer back to the original definitions.

An initial part of the material in this paper is similar to that of **[10]**; however, here we maintain a symmetry in the ubiquitous vector fields u, v. This gives rise to stronger assertions and an approach that is more amenable to calculations by computer.

Portions of this material have been presented at the International Congress of Mathematicians (Kyoto, 1990) and at the A.M.S. Summer Institute on Mathematical Aspects of Classical Field Theory (Seattle, 1991), but no part of it has appeared in the Proceedings of these events.

2. Notation. We use the following symbols.

F - ring of real functions on an n-manifold;
f, g - functions $\in F$;
u, v, w - tangent vector fields; $u \cdot f$ - directional derivative;
$\mathcal{V}$ - F-module of tangent vector fields;
df - differential of f, $df \cdot u = u \cdot f$;
h, k, ℓ, m - vector 1-forms (**V1F**);
$\mathcal{VF}$ - F-module of V1F;
hu - action of h on u; F-linear map $\mathcal{V} \to \mathcal{V}$;
$[u, v]$ - Lie bracket of vector fields,

$$[u, v] \cdot f = u \cdot (v \cdot f) - v \cdot (u \cdot f);$$

$\mathcal{L}_u$ - Lie derivative of any tensor field, e.g.

$$\mathcal{L}_u f = u \cdot f, \qquad \mathcal{L}_u v = [u, v];$$

i_h - inner derivation acting on any tensor field, e.g.

$$i_h f = 0, \quad i_h u = hu, \quad i_h k = hk - kh.$$

All maps smooth ($= C^\infty$).

3. Linearity over the function ring. If B is a vector bundle, then ΓB, the set of sections in B, is a module over F. If B_1, B_2 are vector bundles over the same base space, and a map $\phi : B_1 \to B_2$ is fiber-preserving, linear on each fiber and covers the identity map of the base space, then ϕ determines an F-linear map $\Phi : \Gamma B_1 \to \Gamma B_2$, by $\Phi(s)(x) = \phi(s(x))$. The converse is also true, and we use it repeatedly.

An F-linear map $\Gamma B_1 \to \Gamma B_2$ induces a map $B_1 \to B_2$ that sends each fiber of B_1 linearly to the fiber of B_2 over the same base point.

If T^p_q denotes the bundle of tensors of type (p, q) (so $\mathcal{V} = \Gamma T^1$ and $\mathcal{VF} = \Gamma T^1_1$), then $\mathcal{VF} = \mathrm{Hom}_F(\mathcal{V}, \mathcal{V})$ says that V1F are linear maps of tangent vectors to tangent vectors, both in the point-wise sense (linear over the constants) and as sections (linear over the functions). Thus, $h(f \cdot u) = f \cdot hu$, and any map h from vector fields to vector fields, linear over F, with this property, is a V1F.

In what follows, many maps can be seen to be $\mathbb{R}$-linear by plain inspection; the question at hand is their F-linearity.

4. Lie and F-N brackets. The Lie bracket $[u, v]$ of vector fields u, v is linear over the constants in each of u, v, but not over the functions, and is therefore not in T^1_2. In fact, a simple calculation shows

$$[fu, gv] = fg[u, v] + f(u \cdot g)v - g(v \cdot f)u. \tag{4.1}$$

This property will be used repeatedly, we therefore re-write (4.1) as

$$[fu, gv] - fg[u, v] = (v \otimes fdg) \cdot u - (u \otimes gdf) \cdot v = Vu - Uv, \tag{4.2}$$

where $U = u \otimes gdf$ and $V = v \otimes fdg$ measure the failure of $[u, v]$ to be F-linear in u and v, respectively. Observe that U and V are V1F. The terms on the right in (4.2) are the *correction terms* for $[u, v]$.

If h is a V1F, we may calculate the correction terms for $[hu, v]$. We find

$$\begin{aligned}[hfu, gv] - fg[hu, v] &= [fhu, gv] - fg[hu, v] \\ &= (v \otimes fdg) \cdot hu - ((hu) \otimes gdf) \cdot v = Vhu - hUv.\end{aligned}$$

In the same way, we may take two V1F, h and k, and place them in all (legal) ways inside and outside $[u, v]$. There are 12 ways of doing so. In each case we calculate the correction terms. We find

(4.3)

bracket	corr. terms	bracket	corr. terms
$hk[u, v]$	$hkVu - hkUv$	$kh[u, v]$	$khVu - khUv$
$h[ku, v]$	$hVku - hkUv$	$k[hu, v]$	$kVhu - khUv$
$h[u, kv]$	$khVu - hUkv$	$k[u, hv]$	$khVu - kUhv$
$[hu, kv]$	$kVhu - hUkv$	$[ku, hv]$	$hVku - kUhv$
$[hku, v]$	$Vhku - hkUv$	$[khu, v]$	$Vkhu - khUv$
$[u, hkv]$	$hkVu - uhkv$	$[u, khv]$	$khVu - Ukhv$

Since many of the correction terms appear more than once, we investigate if it is possible to find linear combinations of the brackets whose correction terms vanish. Each bracket has two correction terms, with opposite signs, so one may consider the correction terms vertices of an oriented graph (or, a 1-chain complex) whose edges are the brackets.

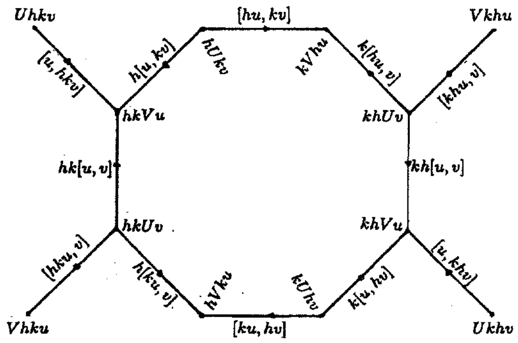

Fig. 1. The graph G_2

In this way any linear combinations of the latter with canceling correction terms will appear as a cycle. We find

THEOREM 1. [8] *In a manifold of dimension > 1 the 12 bracket terms (4.3) with generic V1F h, k are linearly independent, and so are their distinct correction terms. All linear combinations of bracket terms whose correction terms cancel are multiples of*

$$\begin{aligned}[][h,k]](u,v) &= [hu,kv] - k[hu,v] + kh[u,v] - k[u,hv] \\ &\quad + [ku,hv] - h[ku,v] + hk[u,v] - h[u,kv]\end{aligned} \tag{4.4}$$

Furthermore, $[[h,k]](u,v)$ is symmetric in h,k, and alternates in u,v; $[[h,k]]$ is a vector 2-form.

Proof. The linear independence is shown in the same way as in the proofs of Lemmas 1 and 2 in **[10]**. The graph G_2 shows the rest. See also section 8. ■

Collecting pairs of successive terms in (4.3) we find an expression that exhibits F-linearity in v:

$$[[h,k]](u,v) = (\mathcal{L}_{hu}k - k\mathcal{L}_u h + \mathcal{L}_{ku}h - h\mathcal{L}_u k)v.$$

Applications: (1) (Almost-) Complex structures, multifoliate structures, deformation theory, Dolbeault cohomology, CR structures, (almost-) quaternian structures, f-structures, etc.

(2) Parallel developments in deformations of (associative and Lie) algebras, homomorphisms, modules, etc. have been stimulated by the formalism of geometric deformation theory.

(3) The original problem **[8]** leading to the bracket was this: given a V1F h with distinct real eigenvalues, when are subsets of these eigenvector fields tangent to surfaces of the appropriate dimension? If $e_1, \dots, e_n$ are eigenvector fields, and $\lambda_1, \dots, \lambda_n$ the respective eigenvalues, then the answer can be found in the formula

$$\tfrac{1}{2}[[h,h]](e_i,e_j) = (h-\lambda_i I)(h-\lambda_j I)[e_i,e_j] + (e_i\cdot\lambda_j)(\lambda_i-\lambda_j)e_j - (e_j\cdot\lambda_i)(\lambda_j-\lambda_i)e_i.$$

The much more elegant integrability condition

$$h^2[[h,h]](u,v) + [[h,h]](hu,hv) - h[[h,h]](hu,v) - h[[h,h]](u,hv) = 0$$

is due to Haantjes **[3]**. More general situations are dealt with by Kobayashi **[5,6]**.

5. Derivations. Definition. An $\mathbb{R}$-linear, type-preserving mapping of tensors to tensors is a **derivation** (of type (0,0)) if it obeys Leibniz' rule for $\otimes$, and commutes with contractions.

(Folk) **THEOREM. (a)** *A derivation is uniquely determined by its action on F and $\mathcal{V}$; conversely, every map D of functions and vector fields to the same, linear over $\mathbb{R}$, that satisfies*

$$D(fg) = (Df)g + fDg, \qquad D(fu) = (Df)u + f(Du)$$

uniquely extends to a derivation of the tensor algebra.

(b) *A derivation D is uniquely decomposable:*

$$D = \mathcal{L}_u + i_h. \tag{5.1}$$

Definition. A derivation of type (0,1) is an F-linear map of $\mathcal{V}$ into the derivations of type (0,0). Typical notation: $u \mapsto D_u$.

Example: ∇ (covariant derivative); image ∇_u.

Lemma. *Let $D : \mathcal{V}\times\mathcal{V} \to \mathcal{V}$ be $\mathbb{R}$-bilinear, and let there be a V1F m such that*

$$D(fu, gv) - fgD(u,v) = Vmu,$$

then there is an $\mathbb{R}$-linear map $M : \mathcal{V} \to \mathcal{V}\mathcal{F}$ such that

$$M(fu) = fM(u) + mu \otimes df$$

and the formula

$$D_u = \mathcal{L}_{mu} + i_{M(u)}$$

defines a derivation of type $(0,1)$ with $D_u v = D(u,v)$.

THEOREM 2. [9] *The formula*

$$\mathcal{D}_u^{h,k} = \mathcal{L}_{(hk-kh)u} + i_{\mathcal{L}_{ku}h - k\mathcal{L}_u h} \tag{5.2}$$

defines a derivation of type $(0,1)$.

Proof. The path on the lower right half of Fig. 1 (including the "hairs" at the ends) has the correction terms $Vhku - Vkhu$, so satisfies the conditions of the preceding lemma, with $m = hk - kh$. The edges of this path spell out

$$\begin{aligned} D(u,v) &= -[khu,v] + kh[u,v] - k[u,hv] + [ku,hv] - h[ku,v] + [hku,v] \\ &= \mathcal{L}_{(hk-kh)u}v - (k\mathcal{L}_u h)v + (\mathcal{L}_{ku}h)v. \end{aligned}$$

Thus, $D(u,v) = D_u v$.■

We calculate the actions of $\mathcal{D}_u^{h,k}$ on a tangent vector v and a 1-form ω, and express them in such a way as to exhibit by simple inspection the F-linearity in u.

$$\begin{aligned} \mathcal{D}_u^{h,k}v &= -[v,(hk-kh)u] - [hv,ku] + h[v,ku] + k[hv,u] - kh[v,u] \\ &= -[v,hku] + h[v,ku] - kh[v,u] + [v,khu] - [hv,ku] + k[hv,u] \\ &= -(\mathcal{L}_v h)ku + (\mathcal{L}_v kh)u - (\mathcal{L}_{hv}k)u. \end{aligned} \tag{5.3}$$

For the calculation of $\mathcal{L}_u\omega$ we use the formulas

$$\mathcal{L}_u\omega = (d\omega)u + d(\omega u) \quad \text{and} \quad \omega[u,v] = -d\omega(u,v) + u\cdot\omega v - v\cdot\omega u,$$

with various substitutions for ω, u, v involving h, k.

$$\begin{aligned} (\mathcal{D}_u^{h,k}\omega)v &= (\mathcal{L}_{(hk-kh)u}\omega)v - \omega(\mathcal{L}_{ku}h - k\mathcal{L}_u h)v \\ &= d\omega((hk-kh)u,v) + d(\omega(hk-kh)u)v \\ &\quad - \omega([ku,hv] - h[ku,v]) + \omega(k[u,hv] - kh[u,v]) \\ &= d\omega((hk-kh)u,v) + v\cdot(\omega hku) - v\cdot(\omega khu) \\ &\quad + d\omega(ku,hv) - (ku)\cdot\omega hv + (hv)\cdot(\omega ku) \\ &\quad - d(\omega h)(ku,v) + (ku)\cdot(\omega hv) - v\cdot(\omega hku) \\ &\quad - d(\omega k)(u,hv) + u\cdot(\omega khv) - (hv)\cdot(\omega ku) \\ &\quad + d(\omega kh)(u,v) - u\cdot(\omega khv) + v\cdot(\omega khu). \end{aligned}$$

All the "$\cdot$" terms cancel, and the "d" terms give, upon the deletion of v

$$\mathcal{D}_u^{h,k}\omega = d\omega(hk-kh)u + d\omega(ku)h - d(\omega h)(ku) - d(\omega k)uh + d(\omega kh)u, \tag{5.4}$$

in a somewhat compressed notation.

In the presence of a connection, $\mathcal{D}_u^{h,k}$ can be expressed in terms of a covariant derivative ∇ and the "reverse" $\bar\nabla$ (with the opposite torsion), so that $[u,v] = \nabla_u v - \bar\nabla_v u$.

$$\begin{aligned} \mathcal{D}_u^{h,k}v &= [(hk-kh)u,v] + [ku,hv] - h[ku,v] - k[u,hv] + kh[u,v] \\ &= \nabla_{(hk-kh)u}v - \bar\nabla_v((hk-kh)u) \\ &\quad + \nabla_{ku}(hv) - \bar\nabla_{hv}(ku) \quad - h\nabla_{ku}v + h\bar\nabla_v(ku) \\ &\quad - k\nabla_u(hv) + k\bar\nabla_{hv}u \quad + kh\nabla_u v - kh\bar\nabla_v u \\ &= \nabla_{(hk-kh)u}v - (\bar\nabla_v h)ku + (\bar\nabla_v kh)u + (\nabla_{ku}h)v - (\bar\nabla_{hv}k)u - k(\nabla_u h)v. \end{aligned}$$

Thus,

$$\mathcal{D}_u^{h,k} = \nabla_{(hk-kh)u} + i_{\nabla_{ku}h - k\nabla_u h - (\bar{\nabla}_{h*}k)u - (\bar{\nabla}_* kh)u - (\bar{\nabla}_* h)ku}. \tag{5.5}$$

(Note that, for example, $(\bar{\nabla}_* h)u : w \mapsto (\bar{\nabla}_w h)u$ is a V1F.)

6. Triples of V1F. The formulas at the conclusion of the previous section have demonstrated that the action of $\mathcal{D}_u^{h,k}$ on a tangent vector field can be expressed in a form that clearly exhibits its F-linearity in u and uses the operation typical for tangent vector fields, namely the Lie derivative. Similarly, the action of $\mathcal{D}_u^{h,k}$ on a 1-form can, as shown, be expressed in that form using exterior derivatives. And, finally, in the presence of a connection the action of $\mathcal{D}_u^{h,k}$ can be in a compatible form: exhibiting F-linearity in u, using covariant derivatives.

The action of $\mathcal{D}_u^{h,k}$ on a V1F, ℓ is, by definition, given by (5.2). In analogy with the above, one would expect to be able to express this in terms of F-N brackets, in a form that displays simultaneously the F-linearity in u and v. Persistent attempts at this have failed. Therefore, it seems reasonable to back to basics, find an expression in terms of the elementary Lie brackets, and proceed from there.

$$\begin{aligned}
(\mathcal{D}_u^{h,k}\ell)v &= (\mathcal{L}_{(hk-kh)u}\ell + (\mathcal{L}_{ku}h - k\mathcal{L}_u h)\ell - \ell(\mathcal{L}_{ku}h - k\mathcal{L}_u h))v \\
&= [(hk-kh)u, \ell v] - \ell[(hk-kh)u, v] + [ku, h\ell v] - h[ku, \ell v] \\
&\quad + k[u, h\ell v] + kh[u, \ell v] - \ell[ku, hv] + \ell h[ku, v] + \ell k[u, hv] - \ell kh[u, v] \\
&= [hku, \ell v] - \ell[hku, v] + \ell h[ku, v] - \ell[ku, hv] + \ell k[u, hv] - \ell kh[u, v] \\
&\quad + \ell[khu, v] - [khu, \ell v] + kh[u, \ell v] - k[u, h\ell v] + [ku, h\ell v] - h[ku, \ell v].
\end{aligned} \tag{6.1}$$

The last member simply re-arranges the terms from the previous member; the order chosen is such that of the two correction terms of each bracket, one cancels against one of the preceding bracket, and one against one of the next bracket. That is, the correction terms are, in order,

$$(-hkU\ell v + \ell V hku) - (\ell V hku - \ell hkUv) + (-\ell hkUv + \ell hV ku) - \ldots - (h\ell V ku - hkU\ell v).$$

Each successive pair of terms in (6.1) combines to a Lie derivative of a (product of) V1F. Combining the first and second term, etc., reproduces the second member in (6.1), which shows the F-linearity in v. Interchanging the entries in each Lie bracket and then combining instead the second and third terms, etc., produces

$$(\mathcal{D}_u^{h,k}\ell)v = (\ell(\mathcal{L}_v h)k + \ell\mathcal{L}_{hv}k + \ell\mathcal{L}_v(kh) + \mathcal{L}_{\ell v}(kh) + \mathcal{L}h\ell vk + (\mathcal{L}_{\ell v}h)k)u, \tag{6.2}$$

which exhibits the F-linearity in u.

7. The graph $\mathbf{G_3}$. The edges of this graph are all brackets of the form $\alpha[\beta u, \gamma v]$, where α, β, γ are (possibly empty) products of some of h, k, ℓ, such that $\alpha\beta\gamma$ is a product of h, k, ℓ in some order. The vertices of G_3 are the totality of the endpoints of the edges, which are the correction terms, so they are $\alpha\gamma V\beta u - \alpha\beta U\gamma v$.

There are 3 places to insert h into $?[?u, ?v]$; for each of these there are 4 ways to insert k; then there are 5 ways to insert ℓ, so G_3 has 60 edges. Similarly, there are 2 ways to insert h into $?U?v$, 3 to insert k, and 4 to insert ℓ. The same number of insertions into $?V?u$ are possible, so G_3 has 48 (potential) vertices (all of which occur). Fig. 2 shows G_3.

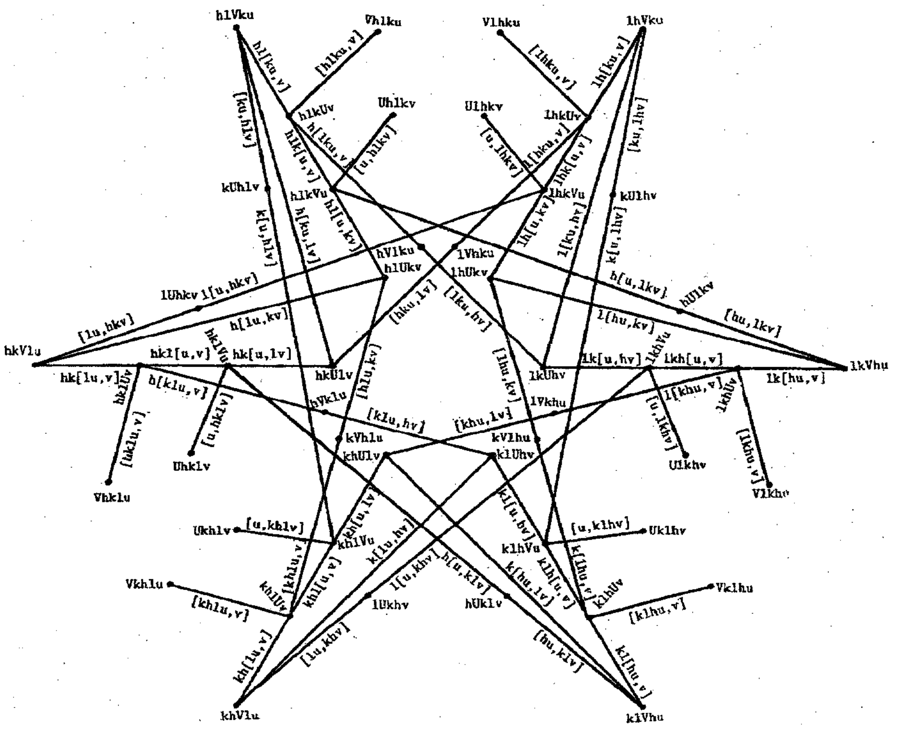

Fig 2. The graph G_3

G_3 has many symmetries. There are three axes, reflections in which correspond to interchanges of two of h, k, ℓ. A rotation of 120° corresponds to a cyclic permutation of h, k, ℓ. Finally, an inversion with respect to the center of symmetry (or better: a reversal if G_3 is seen as a perspective drawing of a graph on a cylinder) corresponds to an interchange of u and v.

Every cycle of G_3 corresponds to a map $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$ which is bilinear over F, so gives rise to tensor of type (1,2). A well-known theorem in graph theory states that the number of independent cycles equals #components -#vertices +#edges, in this case 1-48+60=13. Many of these are well known, for example

$$h[[k,\ell]](u,v), \quad [[h,k]](\ell u,v), \quad [[h,k]](u,\ell v), \quad [[hk,\ell]](u,v).$$

In fact, upon permuting h, k, ℓ, these are all that can be produced from the F-N bracket. Due to the symmetry of the F-N bracket in its V1F entries, the first three of these give rise to 3 cycles each; the last to 6, for a total of 15. Clearly, these cycles are not all linearly independent, since the total number of independent ones is 13. The four cycles listed above,

$$\begin{aligned} h[[k,\ell]](u,v) &= h[ku,\ell v] - h\ell[ku,v] + h\ell k[u,v] - h\ell[u,kv] \\ &\quad + h[\ell u,kv] - hk[\ell u,v] + hk\ell[u,v] - hk[u,\ell v], \\ [[h,k]](\ell u,v) &= [h\ell u,kv] - k[h\ell u,v] + kh[\ell u,v] - k[\ell u,hv] \\ &\quad + [k\ell u,hv] - h[k\ell u,v] + hk[\ell u,v] - h[\ell u,kv], \\ [[h,k]](u,\ell v) &= [hu,k\ell v] - k[hu,\ell v] + kh[u,\ell v] - k[u,h\ell v] \\ &\quad + [ku,h\ell v] - h[ku,\ell v] + hk[u,\ell v] - h[u,k\ell v], \\ [[hk,\ell]](u,v) &= [hku,\ell v] - \ell[hku,v] + \ell hk[u,v] - \ell[u,hkv] \\ &\quad + [\ell u,hkv] - hk[\ell u,v] + hk\ell[u,v] - hk[u,\ell v], \end{aligned} \tag{7.1}$$

can easily be located in G_3. Readers who wish to follow the discussion below, should be prepared to draw all 15 cycles , with colored pencil, in one or more copies(!) of G_3.

The construction described below is a visual version of the row reduction of a matrix. Its purpose is to find a basis for the cycles in G_3 which contains as many of the 15 known cycles as possible. The following steps are executed as indicated.

(0) The starting graph is G_3. The starting list of cycles consists of the 15 known. The list of basic cycles starts out as empty.

(1) Remove "hairs" from the graph, repeatedly if necessary, until none are left. These are the vertices of degree 1, and their incident edges. (They do not contribute to any cycles.)

(2) Locate one of the cycles from the list, let it be C, and choose one of its edges, say E.

(3) Every cycle other than C on the list that does not have E as an edge is left unchanged, and remains on the list.

(4) Every cycle, not C, say C', on the list, that has E as an edge,is modified by adding to it such a multiple of C that edge E in the resulting cycle has coefficient 0. This new cycle replaces C' on the list, unless it is the null cycle, in which case it is deleted.

(5) Cycle C is removed from the list of cycles, and is added to the list of basic cycles.

(6) If the list of cycles is non-empty, return to (1); otherwise stop.

The list of cycles is now empty, and the list of basic cycles contains a set that belong to a basis. In fact, these, plus any basis for the cycles of the remaining graph form a basis for the cycles of G_3.

The actual cycles that are on the list of basic cycles depends on the order in which they were chosen from the list. However, two things are always the same.

(A) On three occasions the construction in step (3) produces a null-cycle. These 3 cycles are (linear combinations of) the null-cycle

$$\begin{aligned} &h[[k,\ell]](u,v)+k[[h,\ell]](u,v)+[[h,k]](\ell u,v)+[[h,k]](u,\ell v)\\ &\qquad -[[h\ell,k]](u,v)-[[h,k\ell]](u,v)=0, \end{aligned} \tag{7.2}$$

and the two obtained from it by permuting h,k,ℓ. (Note the symmetry in h,k.)

(B) Upon termination, a graph with just one cycle is left. Modulo the cycles in the basic list, this cycle is

$$\begin{aligned} J_1(h,k,\ell)(u,v) &= [hku,\ell v]-\ell[hku,v]+\ell hk[u,v]-\ell h[u,kv]+[\ell hu,kv]-k[\ell hu,v]\\ &\quad +k\ell h[u,v]-k\ell[u,hv]+[k\ell u,hv]-h[k\ell u,v]+hk\ell[u,v]-hk[u,\ell v] \end{aligned} \tag{7.3}$$

The analysis is similar to that in **[10]**, where it is illustrated with pictures.

Again, J_1 can be expressed in terms of Lie derivatives, exhibiting F-linearity in u or in v, but not both simultaneously:

$$\begin{aligned} J_1(h,k,\ell)(u,v) &= (\mathcal{L}_{hku}\ell-\ell h\mathcal{L}_u k+\mathcal{L}_{\ell hu}k-k\ell\mathcal{L}_u h+\mathcal{L}_{k\ell u}h-hk\mathcal{L}_u\ell)v\\ &= (\ell\mathcal{L}_v(hk)-\mathcal{L}_{kv}(\ell h)+k\mathcal{L}_v(\ell h)-\mathcal{L}_{hv}(k\ell)+h\mathcal{L}_v(k\ell)-\mathcal{L}_{\ell v}(hk))u. \end{aligned} \tag{7.4}$$

THEOREM 3. *Among the 60 compositions of three V1F h,k,ℓ, with one Lie bracket in a base manifold of dimension > 2, there are 13 independent linear combinations whose correction terms cancel, yielding a tensor of type (1,2) (i.e., cycles in G_3). These are generated by the 15 cycles (7.1) and their h,k,ℓ-permutes, and (7.3). The formula (7.2) and its h,k,ℓ-permutes form a basis for all the linear relations between the F-N brackets listed in(7.1) and their h,k,ℓ-permutes.*

Proof. The 48 distinct correction terms are linearly independent provided the dimension of the underlying manifold exceeds 2, see section 8. Any linear combination with canceling correction terms is therefore a cycle in G_3. These have been analyzed above.

THEOREM 4. *The following holds:*

$$\begin{aligned} (\mathcal{D}_u^{h,k}\ell)v &= J_1(h,k,\ell)(u,v)-\ell[[h,k]](u,v)+[[h,k]](u,\ell v)\\ &\qquad -[[k\ell,h]](u,v)+[[k,\ell]](hu,v). \end{aligned} \tag{7.5}$$

Proof. By direct calculation (i.e., expressing all terms as Lie brackets), or by drawing all terms into a copy of G_3.∎

Remark. Readers who have drawn the cycles (7.1), (7.3) and (6.1) into G_3 will have observed

(1) the cycles (7.1) have winding number 0 with respect to the center of symmetry of G_3;

(2) the cycles (7.4) and (6.1) have winding number -1 with respect to the center.

These observations alone already show that (7.5) must contain a J-term.

8. The case of two dimensions. Consider m generic square $n \times n$ real matrices $A_1, \dots, A_m$, and all $m!$ products $A_{i_1}, \dots, A_{i_m}$, where $(i_1, \dots, i_m)$ is any permutation of $(1, \dots, m)$. According to a theorem by Amitsur and Levitski **[1]**, these products are linearly independent when $m < 2n$. When $m = 2n$, however, there is exactly one linear relation between them,

$$\sum_{\pi \in S_{2n}} \operatorname{sgn}(\pi) A_{\pi(1)} \cdots A_{\pi(2n)} = 0.$$

This implies that the correction terms (vertices) in G_2 are linearly independent for base manifolds of dimension > 1 (in dimension 1, a V1F is just a scalar, and all V1F commute). Similarly, in G_3 every correction term consists of a product of 4 V1F (acting on a generic vector), so these are linearly independent for base manifolds of dimension > 2. When the dimension is 2, however, we have

$$\sum_{\operatorname{perm}(h,k,\ell)} \operatorname{sgn}(\operatorname{perm})(hk\ell U - hkU\ell + hUk\ell - Uhk\ell)v = 0,$$

and a similar formula with U replaced by V, and v by u. These two relations provide the cancellation of correction terms of two additional expressions. The first is

$$\begin{aligned} J_0(h,k,\ell)(u,v) &= \sum_{\operatorname{perm}(h,k,\ell)} \operatorname{sgn}(\operatorname{perm})([u, hk\ell v] - h[u, k\ell v] + hk[u, \ell v] - hk\ell[u,v]) \\ &= \sum_{\operatorname{perm}(h,k,\ell)} \operatorname{sgn}(\operatorname{perm})(\mathcal{L}_u(hk\ell) - h(\mathcal{L}_u k)\ell)v; \end{aligned} \tag{8.2}$$

the second is the same, except u and v interchanged: $J_0(h,k,\ell)(v,u)$.

THEOREM 5. *In a 2-dimensional base manifold there are 15 linearly independent compositions of generic V1F h, k, ℓ with a Lie bracket: the 13 listed in Theorem 3 , plus the additional $J_0(h,k,\ell)(u,v)$ of (8.2), and $J_0(h,k,\ell)(v,u)$.*

The proof is left to the reader. See also **[10]**.

9. Properties of $\mathbf{J_1}$. The only symmetry property of $J_1(h,k,\ell)(u,v)$ is its invariance under cyclic permutations of h, k, ℓ. Since $J_1(h,k,\ell)(u,v) + J_1(k,h,\ell)(u,v)$ is a cycle in G_3 with winding number 0, it is expressible in terms of F-N brackets. Anti-symmetry in h, k, ℓ is obviously found in

$$\begin{aligned} J_2(h,k,\ell)(u,v) &= J_1(h,k,\ell)(u,v) - J_1(k,h,\ell)(u,v) \\ &= 2J_1(h,k,\ell)(u,v) - \sum_{\operatorname{cycl}(h,k,\ell)} ([[h,k]](\ell u, v) + h[[k,\ell]](u,v)). \end{aligned} \tag{9.1}$$

Also, $\mathcal{D}_u^{h,k}$ has no symmetry properties with respect to h, k. Clearly, $\mathcal{D}_u^{h,k} + \mathcal{D}_u^{k,h}$ has no $\mathcal{L}$-term and no J-term, but $\tilde{\mathcal{D}}_u^{h,k} = \mathcal{D}_u^{h,k} - \mathcal{D}_u^{k,h}$ is antisymmetric in h, k; we find

$$\begin{aligned}(\tilde{\mathcal{D}}_u^{h,k}\ell)v = J_2(h,k,\ell)(u,v) - [[k\ell, h]](u,v) + [h\ell, k]](u,v)\\ + [[k,\ell]](hu,v) - [h,\ell]](ku,v).\end{aligned} \tag{9.2}$$

It satisfies the interesting identity (use (7.2))

$$\sum_{\mathrm{cycl}(h,k,\ell)} \tilde{\mathcal{D}}_u^{h,k}\ell = 3J_2(h,k,\ell)(u,v).$$

The expression $J_1(k,h,\ell)(u,v) - J_1(k,h,\ell)(v,u)$ has winding number 0 with respect to the center, so is expressible in terms of F-N brackets, while, obviously,

$$\begin{aligned}J_3(h,k,\ell)(u,v) &= J_1(h,k,\ell)(u,v) + J_1(h,k,\ell)(v,u)\\ &= 2J_1(h,k,\ell)(u,v) - \sum_{\mathrm{cycl}(h,k,\ell)} [[hk,\ell]](u,v)\end{aligned} \tag{9.3}$$

is symmetric in u, v. It is *not* a vector 2-form!

Finally, one can form a $J_4(h,k,l)(u,v)$ that reflects both (anti-)symmetrizations.

$$\begin{aligned}J_4(h,k,\ell)(u,v) &= J_2(h,k,\ell)(u,v) + J_2(h,k,\ell)(v,u)\\ &= J_3(h,k,\ell)(u,v) - J_3(k,h,\ell)(u,v)\\ &= 2J_2(h,k,\ell)(u,v) - \sum_{\mathrm{perm}(h,k,\ell)} \mathrm{sgn(perm)}[[hk,\ell]](u,v)\\ &= 4J_1(h,k,\ell)(u,v) - \sum_{\mathrm{cycl}(h,k,\ell)} \{[[h,k]](\ell u,v) + [[kh,\ell]](u,v)\}\end{aligned} \tag{9.4}$$

The "essential" symmetry of $J_1(h,k,\ell)(u,v)$ in u, v implies that there is no "Jacobi" (or "Bianchi") identity involving J_1.

THEOREM 6. *Let u, v, w, h, K be, respectively, three vector fields, a V1F and a tensor field of type (1,2), all generic. Consider all the terms obtained from the expressions*

$$?K(?x, ?[?y, ?z]), \quad ?K(?[?x, ?y], ?z) \quad \textit{and} \quad ?[?x, ?K(?y, ?z)]$$

by replacing (x, y, z) by all permutations of (u, v, w) and placing h in any one place denoted ?, deleting the other ?. Then there is, up to a constant factor, precisely one linear combination of these terms which is F-linear in u, v, w, namely the vector 3-form $[h, \hat{K}]$, which depends only on the antisymmetric part $\hat{K}$ of K: $\hat{K}(u,v) = K(u,v) - K(v,u)$, and which can be defined by

$$[h, \hat{K}]u = \mathcal{L}_{hu}\hat{K} - h\mathcal{L}_u\hat{K} - [[h, \hat{K}u]] - \hat{K}\,\bar{\wedge}\mathcal{L}_u h, \tag{9.5}$$

where

$$(K \barwedge h)(u,v) = K(hu,v) + K(u,hv). \tag{9.6}$$

A proof can be found by drawing the graph whose edges are the various terms, and the edges the correction terms. (Some edges have 3 endpoints!) It has just one cycle. Details are left to the reader. Another proof is given in section 12. The formula (9.5) (as does (7.2))expresses a form of the Jacobi identity in a graded Lie algebra of all vector forms, in which $[u,v],[u,h],[h,k]$ denote, respectively, the Lie bracket, $\mathcal{L}_u h$ and $[[h,k]]$. For details, see **[2]** or **[7]**. A historical survey of graded Lie algebras is found in **[4]**.

10. The graph $\mathbf{G_4}$ for manifolds of dimension >2. The edges of G_4 are all meaningful terms that can be formed from one Lie bracket, four V1F h,k,ℓ,m and two vector fields u,v, such as $h[\ell u, mkv], [hk\ell mu, v]$, etc. There are a total of $3\cdot 4\cdot 5\cdot 6 = 360$ of these. The correction terms for $\alpha[\beta u,\gamma v]$ are $-\alpha U\gamma v + \alpha\gamma V\beta u$; there are a total of $2\times 2\cdot 3\cdot 4\cdot 5 = 240$ of these. The number of independent cycles is therefore #components#vertices+#edges = 121. This graph is difficult to deal with pictorially.

Every cycle in G_4 represents an expression which is F-linear in u,v, is therefore a differential invariant of h,k,ℓ,m, and is a tensor of type (1,2).

Many of these tensors are immediately identifiable; for example

$$h[[k,\ell m]](u,v), \quad [[\ell,km]](hu,v), \quad J_2(h,k,\ell)(mu,v), \quad hJ_2(h,\ell,m)(u,v),$$

etc. The number of expressions $\alpha[[\beta,\gamma]](\delta u,\varepsilon v)$ can be counted as follows. Since at least one of the V1F must appear in β and γ each, and since $[[\beta,\gamma]] = [[\gamma,\beta]]$, there are $\binom{4}{2} = 6$ ways to determine the first factors of β and γ. The remaining two V1F can then be placed in $5\cdot 6 = 30$ ways, giving a total of 180 F-N bracket expressions.

The number of J-terms is similarly determined. Since the three entries in $J_2(?,?,?)$ are permutable modulo sign, there are $\binom{4}{3}$ ways to place the first three V1F. The last one can then be placed in 6 ways, for a total of 24.

Thus, 180+24=204 cycles in G_4 are known, but at most 121 of them can be independent.

THEOREM 7. *The space of cycles in G_4 is 121-dimensional. The 180 cycles that are sums of F-N brackets span a 114-dimensional subspace. The 24 cycles that are sums of J_2-terms span a 7-dimensional subspace; together, the 204 cycles span the entire 121-dimensional space.*

Theorem 7 can be summarized as: *There are no new differential invariants formed from a Lie bracket and four V1F.*

A proof of Theorem 7 is outlined in section 12.

Theorem 7 states that there are 66 independent relations between the F-N brackets, and 17 additional relations that involve J_2 as well. The former all are consequences of (7.2), as we shall show. To state this precisely, let, for any two V1F h, k, $B(h, k)$ denote a map of $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$; such that $B(h, k)(u, v)$ has the same symmetry properties as $[[h, k]](u, v)$. Define (see (7.2) and (9.6))

$$R(h, k, \ell) = hB(k, \ell) + kB(h, \ell) + B(h, k) \bar{\wedge} \ell - B(h, k\ell) - B(k, h\ell). \tag{10.1}$$

The number of expressions obtainable from (10.1) by substitutions of (products of) h, k, ℓ and m for h, k, ℓ in $\alpha R(\beta, \gamma, \delta)(\varepsilon u, \zeta v)$, so that each entry "?" in $R(?, ?, ?)$ is replaced by at least one V1F, is easily seen to be $3 \times \binom{4}{3} \times 6 = 72$. Each of these, when $B(\beta, \gamma) = [[\beta, \gamma]]$, is a relation between cycles in G_4. However, there are 6 "relations between these relations".

THEOREM 8. *Let $B(h, k)$ be any map $\mathcal{V} \times \mathcal{V} \to \mathcal{V}$, symmetric in h, k, and R defined by (10.1). Then the following identity holds.*

$$\begin{aligned} &R(h, k, \ell) \bar{\wedge} m - R(h, k, m) \bar{\wedge} \ell - R(h, k, \ell m) + R(h, k, m\ell) \\ &- hR(k, \ell, m) - kR(h, \ell, m) + hR(k, m, \ell) + kR(h, m, \ell) \\ &+ R(h\ell, k, m) - R(hm, k, \ell) + R(k\ell, h, m) - R(km, h, \ell) = 0. \end{aligned} \tag{10.2}$$

In view of the symmetry in h, k and the antisymmetry in ℓ, m permutations of h, k, ℓ, m, yield 6 identities; these are linearly independent.

Proof. By direct verification, using $(\ldots \bar{\wedge}\ell) \bar{\wedge} m - (\ldots \bar{\wedge} m) \bar{\wedge} \ell = \ldots \bar{\wedge}(\ell m - m\ell)$. Independence is immediate: the 6 identities do not have a single term in common.■

As to the relations involving J_2 as well as F-N brackets, a number can be obtained from the Leibniz rule for $\tilde{\mathcal{D}}_u^{h,k}$ when acting on products of V1F

$$(\tilde{\mathcal{D}}_u^{h,k}(\ell m))v = (\tilde{\mathcal{D}}_u^{h,k}\ell)mv + \ell(\tilde{\mathcal{D}}_u^{h,k} m)v \tag{10.3}$$

and applying (9.2) to each of the terms. This yields

$$\begin{aligned} &J_2(h, k, \ell m)(u, v) - \ell J_2(h, k, m)(u, v) - J_2(h, k, \ell)(u, mv) \\ &- [[k\ell m, h]](u, v) + [h\ell m, k]](u, v) - [[k, \ell m]](hu, v) + [h, \ell m]](ku, v) \\ &+ \ell[[km, h]](u, v) - \ell[[hm, k]](u, v) + \ell[[k, m]](hu, v) - \ell\ [h, m]](ku, v) \\ &+ [[kl, h]](u, mv) - [[h\ell, k]](u, mv) + [[k, \ell]](hu, mv) - [[h, \ell]](ku, mv) = 0. \end{aligned} \tag{10.4}$$

Under permutations of h, k, ℓ, m, respecting the antisymmetry in h, k, there are 12 relations here.

THEOREM 9. *The relation (10.4) and its h, k, ℓ, m permutes, added to 66 independent relations between V1F in G_4 span a subspace of dimension 78 in the 83-dimensional space of all relations between cycles produced by F-N brackets and J_2.*

The proof is given in section 12.

Leibniz' rule yields more relations, obtained by interchanging u and v in (10.3) and then using (see (9.4))

$$J_2(h,k,\ell)(v,u) = J_2(h,k,\ell)(u,v) - \sum_{\mathrm{perm}(h,k,\ell)} \mathrm{sgn}(\mathrm{perm})[[kh,\ell]](u,v) \tag{10.5}$$

and the antisymmentry $[[\ ,\ ]](v,u) = -[[\ ,\ ]](u,v)$, to put u and v back in their places. Thus 12 more relations are obtained, one of which is

$$\begin{aligned} &J_2(h,k,\ell m)(u,v) - [[hk,\ell m]](u,v) - [[\ell mh,k]](u,v) + [[kh,\ell m]](u,v) \\ &\qquad + [[\ell mk, h[[(u,v) + [[k,\ell m]](u,hv) - [[h,\ell m]](u,kv) \\ &-\ell J_2(h,k,m)(u,v) + \ell[[hk,m]](u,v) + \ell[[mh,k]](u,v) - \ell[[kh,m]](u,v) \\ &\qquad - \ell[[mk,h]](u,v) - \ell[[k,m]](u,hv) + \ell[[h,m]](u,kv) \\ &-J_2(h,k,\ell)(mu,v) + [[hk,\ell]](mu,v) + [[\ell h\, k](mu,v) - [[kh,\ell]](mu,v) \\ &\qquad - [[\ell k,h]](mu,v) - [k,\ell]](mu,hv) + [h,\ell]](mu,kv) = 0. \end{aligned} \tag{10.6}$$

THEOREM 10. *The relations (10.4), (10.6) and permutes, added to the relations obtained from (7.2) by substitutions span an 82-dimensional subspace of the 83-dimensional space of all relations between cycles generated in G_4 by F-N brackets and J_2.*

This theorem is proved in section 12.

The 12 latest relations (10.6), added to the previous ones, yield only 4 new ones. The 8 "relations between relations" that cause the reduction are easily found. Subtraction of (10.6) from (10.4) yields a formula for $J_2(h,k,\ell)(mu,v) - J_2(h,k,\ell)(u,mv)$ in terms of F-N brackets. Three such formulas occur among the 12 permutes of (10.4) and (10.6); the above expression arises as the difference of permutes in which the first term is $J_2(h,k,\ell m)$, $J_2(h,km,\ell)$ and $J_2(hm,k,\ell)$, respectively. These three, by subtraction, give rise to 2 identities involving only F-N brackets – and by Theorem 10 these are not new ones.

The final identity is

$$\begin{aligned} &J_2(k,\ell,m)(u,hv) - J_2(h,\ell,m)(u,kv) + J_2(h,k,m)(u,\ell v) - J_2(h,k,\ell)(u,mv) \\ &-hJ_2(k,\ell,m)(u,v) + kJ_2(h,\ell,m)(u,v) - \ell J_2(h,k,m)(u,v) + mJ_2(h,k,\ell)(u,v) \\ &+[[hk-kh,\ell m - m\ell]] - [[h\ell - \ell h, km - mk]] + [[hm - mh, k\ell - \ell k]] = 0. \end{aligned} \tag{10.7}$$

THEOREM 11. *The full 83-dimensional space of relations between cycles in G_4 generated by F-N brackets and J_2 with substitutions of four V1F is spanned by the relations obtained from (7.2) by substitutions, augmented by the Leibniz rule for $\tilde{\mathcal{D}}$ and (10.7).*

A proof is given in section 12.

In popular terms, Theorem 11 states that for calculations involving F-N brackets and J_2 (or J_1, J_3 or J_4) and no more than 4 V1F the only identities one needs to avoid returning to the original definitions are the symmetry properties of the F-N bracket and J_2, the identity (7.2), the Leibniz rule for $\tilde{\mathcal{D}}$ (as incorporated in (10.4) and (10.6)), and (10.7).

11. Commutators. Let D and D' be derivations of type (0,1), so D_u and D'_v depend F-linearly on u and v, respectively. Then $D_u D'_v - D'_v D_u$ is a derivation, but is not F-linear in u and v. However,

$$[D, D']_{u,v} = D_u D'_v - D'_v D_u - D'_{D_u v} + D_{D'_v u} \tag{11.1}$$

is a derivation, and is F-linear in u and v.

Lemma. *If D and D' are derivations of type(0,1), then the commutator defined by (11.1) is a derivation of type (0,2) (i.e., is F-linear in u and v) and satisfies a Jacobi identity.*

Lemma. *If D^* is a derivation of type (0,2), then there are a unique tensor field t of type (1,2), $t : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$, and an $\mathbb{R}$-bilinear function $T : \mathcal{V} \times \mathcal{V} \to \mathcal{V}\mathcal{F}$ satisfying*

$$T(fu, gv) - fgT(u, v) = t(u, v) \otimes d(fg)$$

such that

$$[D, D']_{u,v} = \mathcal{L}_{t(u,v)} + i_{T(u,v)}.$$

As a consequence of the above lemmas, there is a tensor field t of type (1,2) such that

$$[\mathcal{D}^{h,k}, \mathcal{D}^{\ell,m}]_{u,v} f = t(u, v) \cdot f. \tag{11.2}$$

As a consequence of Theorem 7, $t(u, v)$ is expressible in terms of F-N brackets and J_1 operators, applied to combinations of h, k, ℓ, m.

THEOREM 12. *The tensor field t in (11.2) is given by*

$$2t(u, v) = (\mathcal{D}^{h,k}_u (\ell m - m\ell))v - (\mathcal{D}^{\ell,m}_v (hk - kh))u + A, \tag{11.3}$$

where $A = A_1 + A_2$;

$$\begin{aligned}
A_1 = {} & k[u, h\ell mv] - kh\ell[u, mv] + kh[\ell u, mv] - khm[\ell u, v] + khm\ell[u, v] - k[u, hm\ell v] \\
& + [ku, hm\ell v] - h[ku, m\ell v] + hk[u, m\ell v] - hkm\ell[u, v] + hkm[\ell u, v] - hk[\ell u, mv] \\
& + hk\ell[u, mv] - hk[u, \ell mv] + h[ku, \ell mv] - [ku, h\ell mv],
\end{aligned} \tag{11.4}$$

while A_2 is obtained from A_1 by the substitution $(h,k,\ell,m,u,v)\to(\ell,m,h,k,v,u)$ and a reversal of signs. A_1 and A_2 each represent a cycle in G_4 (the terms in (11.4) are listed in cyclic order), and are expressible in terms of F-N brackets:

$$\begin{aligned} A_1 = {} & [[h,k]](u,m\ell v) - [[h,km]](u,\ell v) - [[h\ell,k]](u,mv) \\ & + [[h\ell,km]](u,v) + h[[k,\ell]](u,mv) - h[[km,\ell]](u,v) \\ & + k[[h,m]](u,\ell v) - k[[h\ell,m]](u,v) + kh[[\ell,m]](u,v), \end{aligned} \tag{11.5}$$

while A_2 is obtained from (11.5) by the above substitution and sign change.

Proof (first part) (sketch).

$$\begin{aligned} & \mathcal{D}_u^{h,k} D_v^{\ell,m} f - \mathcal{D}_v^{\ell,m}\mathcal{D}_u^{\ell,m} f - \mathcal{D}^{\ell,m}_{\mathcal{D}_u^{h,k}v} f + \mathcal{D}^{h,k}_{\mathcal{D}_v^{\ell,m}u} f \\ = & \mathcal{D}_u^{h,k}(\mathcal{L}_{(\ell m - m\ell)v} f) - \mathcal{D}_v^{\ell,m}(\mathcal{L}_{(hk-kh)u} f) - \mathcal{L}_{(\ell m - m\ell)\mathcal{D}_u^{h,k}v} f + \mathcal{L}_{(hk-kh)\mathcal{D}_v^{\ell,m}u} f \\ = & [\mathcal{L}_{(hk-kh)u}, \mathcal{L}_{(\ell m - m\ell)v}] f - \mathcal{L}_{(\ell m - m\ell)\mathcal{D}_u^{h,k}v} f + \mathcal{L}_{(hk-kh)\mathcal{D}_v^{\ell,m}u} f. \end{aligned}$$

It follows that

$$t(u,v) = [(hk-kh)u, (\ell m - m\ell)v] - (\ell m - m\ell)\mathcal{D}_u^{h,k} v + (hk-kh)\mathcal{D}_v^{\ell,m} u.$$

The right side is expressed in terms of Lie brackets, using (5.3). The expressions in terms of Lie brackets of $(\mathcal{D}_u^{h,k}(\ell m - m\ell))v$, $(\mathcal{D}_v^{\ell,m}(hk-kh))u$ and $A = 2t(u,v) - (\mathcal{D}_u^{h,k}(\ell m - m\ell))v + (\mathcal{D}_v^{\ell,m}(hk-kh))u$ are calculated (by computer) from (5.3) by a sequence of substitutions and linear combinations.

Next, the correction terms for the terms in A were calculated (by hand), from which the cycles represented by A_1 and A_2 were identified. Finally, A_1 and A_2 were expressed in terms of F-N brackets; this was again done by computer; see section 12.

12. Proofs of Theorems 6, 7, 9, 10, 11 and 12.

All the proofs described in this section consist of the determination of the rank of matrices, by Gaussian reduction. These calculations were done by computer.

Proof of Theorem 6. The edges and correction terms are of the types

edge	endpoints
$\alpha K(\beta x, \gamma[\delta y, \varepsilon z])$	$\alpha K(\beta x, \gamma\varepsilon Z\delta y) - \alpha K(\beta x, \gamma\delta Y\varepsilon z)$
$\alpha K(\beta\gamma x, \delta y], \varepsilon z)$	$\alpha K(\beta\delta Y\gamma x, \varepsilon z) - \alpha K(\beta\delta Y\gamma x, \varepsilon z)$
$\alpha[\beta x, \gamma K(\delta y, \varepsilon z)]$	$\alpha\gamma K(\delta Y\beta x, \varepsilon z) + \alpha\gamma K(\delta y, \varepsilon Z\beta x) - \alpha\beta X\gamma K(\delta y, \varepsilon z)$

Here, x, y, z stand for any permutation of u, v, w; X, Y, Z are the corresponding corrections. Any one of $\alpha, \ldots, \varepsilon$ can be equal to h; the others are blanks.

Due to the antisymmetry of the Lie bracket, the number of edges is $5\cdot 3 + 5\cdot 3 + 5\cdot 6 = 60$. The number of distinct vertices is $3\cdot 4 + 3\cdot 4 + 6\cdot 4 = 48$. A split matrix $[A|I]$ is set up, where I is an identity matrix. The rows of the combined matrix, and the columns of I are labeled with the edges; the columns of A with the vertices. The entries of A are the incidence numbers of the vertices and edges. A Gaussian row reduction of the entire matrix is performed, with all pivots chosen in A. The resulting matrix is denoted $[B|C]$. B contains one row of zeros; the corresponding row of C gives the coefficients of the one linear combination of edges whose correction terms vanish. The terms are precisely those of the right side of (9.5) when fully written out in terms of Lie brackets.

Proof of Theorem 7. A matrix A is set up, whose rows are labeled with the F-N brackets; its columns with the edges. The entries of A are the coefficients of the latter in each of the former. A Gaussian row reduction is performed, which shows that the reduced matrix has 66 rows of zeros, so the rank of A is 114. Next, A is augmented with more rows, one for each J_2-term. Another Gaussian row reduction is performed. This time there are 66+17=83 rows of zeros. The rank of the augmented matrix is therefore 180+24-66-17=121. ∎

Next, the 204-row matrix A is augmented with an identity matrix I of size 204, yielding a matrix $[A|I]$. Its row reduction, with all pivots in A (so, the same reduction as before) yields a matrix $[A', B]$, where A' had 83 rows of zeros. Let C denote the matrix consisting of these 83 rows of B. The entries in C give the coefficients of a basis for all dependencies between the F-N brackets and the J_2's. C is used in subsequent calculations as a "check" matrix.

Proof of Theorem 9. A matrix A is set up, whose columns are labeled with the F-N brackets and J_2's. The first 72 rows are filled with the coefficients of the permutes of (10.4) (among which there are 6 dependencies) the following 12 with the permutes of (10.4). A Gaussian row reduction shows the occurrence of 6 dependencies in the total matrix, so its rank is 78.

Proof of Theorem 10. The matrix A in the previous proof is further enlarged by adding 12 new rows, the coefficients of the permutes of (10.6). The Gaussian row reduction now shows 14 dependencies, so the rank is 82.

Proof of Theorem 11. The enlarged matrix A of the previous proof is further enlarged with the coefficients of (10.7). Gaussian row reduction shows 14 dependencies. The rank of the matrix is 83. The validity of (10.7) was ascertained by the procedure described in the paragraph following the proof of Theorem 12.

Proof of Theorem 12 (continuation). A matrix B is set up, whose rows are labeled by the 180 F-N brackets. Two additional rows are labeled A_1 and A_2. The columns of B are labeled with the edges, i.e. the Lie brackets that define the F-N brackets, and the coefficients are entered. For the last two rows, the coefficients of the Lie brackets representing A_1 and A_2 are entered. Next, B is expanded to $[B|I]$ by the addition of an identity matrix. Its columns are labeled in the same way as the rows of A. A Gaussian reduction of $[B|I]$ is performed, with pivots limited to the rows and columns of the F-N brackets. The last two rows of the reduced matrix contain only zeros in the B-columns, and the coefficients in (11.5) are in the last part of the second-last row. The last row, similarly, contains a set of coefficients of A_2 in terms of F-N brackets. The coefficients are not unique, because of relations that follow from (7.2). The "best" expression obtained, with the smallest number of terms (which depend on the pivoting strategy) has been used in (11.5).

In the proofs of Theorems 9, 10 and 11 an extra safety check against errors was performed. Prior to the matrix reductions there, the matrix denoted A in each of them is examined to make sure all its rows are indeed relations between F-N brackets and J_2's. (A single incorrect sign could cause problems!) To this end, a matrix $\begin{bmatrix} C \\ A \end{bmatrix}$ is formed, where C is the "check" matrix, see the proof of Theorem 7. A Gaussian row reduction, with pivots limited to the rows in C, is performed. The resulting matrix should have all zeros in the other rows, for there to be no errors.

All calculations described in this section were performed on a '486 computer, using Turbo Pascal v5.0, with v5.5 compiler. All matrix entries were limited to integers in the range -32768..32767. To guard against overflow, the largest absolute value occurring in each Gaussian reduction was reported as a part of its output. In each case, this value was well below 100.

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