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# ON HIGHER ORDER CONTACT MANIFOLDS

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Let $M$ be a smooth $m$-dimensional manifold and $C^{k,n}M$, $n \leq m$, the manifold of contact elements of order $k$ and dimension $n$ over M [3]. Given an $n$-submanifold $S \subset M$, denote by $C^k_x S$ the contact element of order $k$, of $S$ at $x \in S$ and by $\pi^k_0$ the canonical projection over $M$. Let $\tilde{C}^{1,n}(C^{k,n}M)$ be the manifold of all contact elements of order 1 and dimension $n$, of $n$-submanifolds $\mathcal{S} \subset C^{k,n}M$, such that the natural projection $\pi_0^{1,k}$ restricted to $\mathcal{S}$ over M, has maximal rank [4].

The manifold $\tilde{C}^{1,n}(C^{k,n}M)$ is embedded in $C^{1,n}(C^{k,n}M)$, and the natural immersion,
$i^{1,k} : C^{k+1,n}M \to \tilde{C}^{1,n}(C^{k,n}M)$, defined by $i^{1,k}(C^{k+1}_x S) = C^1_{C^k_x S} C^k S$, is an embedding; where $C^k S \subset C^{k,n}M$ denotes the $n$-submanifold in $C^{k,n}M$, defined by the immersion $x \in S \subset M \mapsto C^k_x S \in C^{k,n}M$.

We prove that, there exists an involution $\tilde{\alpha}$, i.e. an automorphism with $\tilde{\alpha} \circ \tilde{\alpha}$ equal to the identity,

$$\tilde{\alpha} : \tilde{C}^{1,n}(C^{k,n})M \longrightarrow \tilde{C}^{1,n}(C^{k,n})M,$$

such that, $i^{1,k}(C^{k+1,n}M) \subset \tilde{C}^{1,n}(C^{k,n}M)$ is the set of the fixed points of $\tilde{\alpha}$.

This result generalizes the following: Let $J^k_0(\Re^n, M)$, be the $k$-jet bundle of maps from $\Re^n$ into $M$, with source at $0 \in \Re^n$ [1]. There is a canonical involution, $\alpha$ in $J^1_0(\Re^n, J^k_0(\Re^n, M))$ such that, $J^{k+1}_0(\Re^n, M)$ can be immersed in $J^1_0(\Re^n, J^k_0(\Re^n, M))$ as the invariant set of $\alpha$ ([2], [5]).

Using the properties of the fixed point set of $\tilde{\alpha}$, we we will prove the following:

1)The $n$-planes $P$ in a submanifold $C^{1,n}W$, with rank $(\pi^k_o)_*|P = n$, contained in $C^{k+1,n}M$, are fixed by $\tilde{\alpha}$.

2)If a submanifold $W \subset C^{k+1}M$, has at each point an $n$-plane fixed by $\tilde{\alpha}$, then $W$ generates an involutive $n$-distribution on $C^{k,n}M$. Moreover, given $X^k \in W$, there exists a submanifold $S \subset M$, with $\pi^k_o(X^k) = x \in S$ such that: $C^k_x = X^k$ and $C^k S$ is a solution of the distribution defined by $W$, through $X^k$.

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This paper is in final form and no version of it will be submitted for publication elsewhere.

## 1. General Results.

Let $M$ be a smooth manifold of dimension $m$. Let $\tilde{J}_0^k(\Re^n, M)$ be the tangent bundle of $n^k$- velocities of maximal rank with $n < m$, and $\rho_0^k$ the canonical projection [1]. Two jets $X_1, X_2 \in \tilde{J}_0^k(\Re^n, M)$, which project onto the same point $x \in M$, are equivalent if there is a jet $Y \in \tilde{J}_0^k(\Re^n, \Re^n)$, such that $X_2 = X_1 \circ Y$. An equivalence class of jets, is a contact element of order $k$ and dimension $n$ at $x \in M$, or a contact $n^k$-element at $x$. Denote by $[j_0^k f]$ the equivalence class of a $k$-jet $j_0^k f$, where $f : \Re^n \to M$ is a map with maximal rank.

Let $S \subset M$ be an embedded $n$-dimensional submanifold of $M$ and $k$ a positive integer. Then $S$ determines a contact $n^k$-element, denoted by $C_x^k S = [j_0^k f]$, where $f$ is a local parametrization of $S$, with $f(0) = x$. The set $C^{k,n}M$ of all contact $n^k$-elements over $M$, has a structure of smooth manifold, and is fibered over $M$ with the canonical projection $\pi_0^k(C_x^k S) = x$. If $X^k \in C^{k,n}M$ we denote by $X^s$ the projection $\pi_s^k(X^k) \in C^{s,n}M$ [3].

Given $X^k = [j_o^k f]$, we consider a fibered neighborhood $(V, U, \rho)$, with $U \subset \Re^n$ and $f(U)$ a transversal section to this fibering. A coordinate neighborhood of $X^k$ can be identified with the $k$-jets of sections of the bundle $(V, U, \rho)$ denoted by $J^k(V, U, \rho)$ or $J^k V$ [4].

**Proposition 1.** *Let $M$ be a smooth manifold and $k$ a positive integer. Then there exists a canonical immersion of $C^{k+1}M$ in $C^{1,n}(C^{k,n}M)$.*

**Proof**: Let $X = C_{x_0}^{k+1} S$ and consider the map,

$$c^k : S \longrightarrow C^{k,n}M, \quad x \in S \longmapsto C_x^k S.$$

This map is an immersion and $c^k(S)$, denoted by $C^k S$, is an immersed submanifold of dimension $n$ in $C^{k,n}M$.

Now, let

$$i^{1,k} : C^{k+1,n}M \longrightarrow C^{1,n}(C^{k,n}M),$$

be defined by $i^{1,k}(X^{k+1}) = C_{X^k}^1 C^k S$.

This map is well defined: given two submanifolds $S_1$, $S_2$ of dimension $n$ in M, with $C_{x_0}^{k+1} S_1 = C_{x_0}^{k+1} S_2$, we have that $C^k S_1$ and $C^k S_2$ are equivalent up to order 1 in $X^k$, hence their tangent planes at $X^k$ coincide, consequently, $C_{X^k}^1 C^k S_1 = C_{X^k}^1 C^k S_2$. It can be shown using coordinates that $i^{1,k}$ is an immersion [3],[6]. $\sharp$

Let $\tilde{C}^{1,n}(C^{k,n}M)$ be the set of all contact $n^k$- elements of $n$-submanifolds $S \subset C^{k,n}M$, such that the projection $\pi_0^k|_S$ has rank $n$. We will show that this set is an immersed submanifold in $C^{1,n}(C^{k,n}M)$. We will use the notation given in [2].

**Proposition 2.** *The set $\tilde{C}^{1,n}(C^{k,n}M) \subset C^{1,n}(C^{k,n}M)$ is an immersed submanifold.*

**Proof.** We consider the case $k = 1$, $n = 2$, for simplicity, the same procedure can be used for $k > 1$, $n > 2$.

Let $X^{1,1} = C_{X^1}^1 f(U)$, where

$$f : U \subset \Re^2 \longrightarrow C^{1,2}M, \quad f(0) = X^1, \quad rank(\pi_0^1|_{f(U)}) = 2,$$

then, there exists a fibration $(V, U, \rho)$ of a neighborhood $V$ at $x = (\pi_0^1 \circ f)(0)$, such that $(\pi_0^1 \circ f)(U)$ is a transversal section of this bundle. Indeed: let $V$ be a local neighborhood of $x \in M$ and $\phi = (x^i)$, a coordinate systems defined in $V$ such that: $x^i \circ \pi_0^1 \circ f = t^i$, $1 \leq i \leq 2$, are the canonical coordinates of $\Re^2$. Let $p_2$ be the natural projection, of $\Re^m$ onto its first 2-coordinates, and $\rho = p_2 \circ \pi_0^1 \circ f$, then $(V, U, \rho)$ with $U = \rho(V)$, is a fiber bundle, and $f \circ \pi_0^1$ is a transversal section.

Let $\pi_{-1}^1 = \pi_0^1 \circ \rho$ and $\mathcal{V}^1$ defined by:
$\mathcal{V}^1 = \{C_u^1 \sigma(U),\ \sigma : U \to V$, section of $(V, U, \rho)$, $u \in U\}$,
then we have the fibered chart $(\mathcal{V}^1, U, \pi_{-1}^1)$, at $X^1 = \pi_1^{1,1}(X^{1,1})$.

Let $\mathcal{V}^{1,1}$ be the set of all contact elements of order 1 and dimension 2, transversal to this fibering, i.e.

$$\mathcal{V}^{1,1} = \{C_{g(u)}^1 ug(U), \quad g : U \to \mathcal{V}^1, \quad \textit{section} \quad \textit{of} \quad (\mathcal{V}^1, U, \pi_{-1}^1), \quad u \in U\}.$$

Then, the contacts elements in $\mathcal{V}^{1,1}$ can be identified with the 1-jets of sections $J^1\mathcal{V}^1$, of the bundle $(\mathcal{V}^1, U, \pi_{-1}^1)$ [4]. This identification gives the coordinates in the neighborhood $\mathcal{V}^{1,1}$ at $X^{1,1}$.

If $V$ has fibered coordinates $(x^1, x^2, x^3(x^1, x^2), \cdots, x^m(x^1, x^2))$, then $\mathcal{V}^1$ has coordinates $(x^i, y^j, z^j)$, $1 \leq i \leq m$, $3 \leq j \leq m$.

For $\sigma : U \to V$ section, we have:

$$\begin{aligned} y^j(C_{\sigma(u)}^1 \sigma(U)) &= \frac{\partial}{\partial x^1}(x^j \circ \sigma), \\ z^j(C_{\sigma(u)}^1 \sigma(U)) &= \frac{\partial}{\partial x^2}(x^j \circ \sigma). \end{aligned} \tag{1.1}$$

Thus, $\mathcal{V}^{1,1}$ has coordinates,

$$(x^i, y^j, z^j, x^{j+m}, x^{j+2m}, y^{j+m}, y^{j+2m}, z^{j+m}, z^{j+2m}).$$

For $Z^{1,1} = C_{g(u)}^1(g(U))$, with $g : U \to \mathcal{V}^1$ a section, we have:

$$\begin{aligned} x^{j+m}(Z^{1,1}) &= \frac{\partial g^j}{\partial x^1}(0), & x^{j+2m}(Z^{1,1}) &= \frac{\partial g^j}{\partial x^2}(0), \\ y^{j+m}(Z^{1,1}) &= \frac{\partial g^{j+m}}{\partial x^1}(0), & y^{j+2m}(Z^{1,1}) &= \frac{\partial g^{j+m}}{\partial x^2}(0), \\ z^{j+m}(Z^{1,1}) &= \frac{\partial g^{j+2m}}{\partial x^1}(0), & z^{j+2m}(Z^{1,1}) &= \frac{\partial g^{j+2m}}{\partial x^2}(0), \end{aligned} \tag{1.2}$$

where $x^j \circ \pi_0^1 \circ g = g^j$, $\quad y^j \circ g = g^{j+m}$, $\quad z^j \circ g = g^{j+2m}$. $\sharp$

**Proposition 3.** *The immersion $i^{1,1}(C^{2,2}M)$ in $\tilde{C}^{1,2}(C^{1,2}M)$, is given in coordinates by,*

$$x^{j+m} = y^j, \qquad y^{j+2m} = z^{j+m}, \qquad x^{j+2m} = z^j. \tag{1.3}$$

**Proof.** Let $X = C_x^2 S$, with $S$ a transversal section of the fibered chart $(V, U, \rho)$ at $x$ defined in Proposition 2.,

If $S$ is parametrized by $(x^1, x^2) \longmapsto (x^1, x^2, \sigma^j(x^1, x^2))$, then $C_x^2 S$ is given by,

$$(x^1, x^2, \sigma^j, \frac{\partial}{\partial x^1}\sigma^j, \frac{\partial}{\partial x^2}\sigma^j, \frac{\partial}{\partial x^1}(\frac{\partial}{\partial x^1}\sigma^j), \frac{\partial}{\partial x^1}(\frac{\partial}{\partial x^2}\sigma^j), \frac{\partial}{\partial x^2}(\frac{\partial}{\partial x^2}\sigma^j)) =$$

$(x^1, x^2, x^j \circ \pi_o^1 \sigma, y^j \circ \sigma, z^j \circ \sigma, y^{j+m} \circ \sigma, y^{j+2m} \circ \sigma, z^{j+2m} \circ \sigma),$
and we have,

$$x^{j+m} = y^j, \qquad y^{j+2m} = z^{j+m}, \qquad x^{j+2m} = z^j.$$

$\sharp$

Given an immersed submanifold $W \subset C^{2,2}M$ and $X \in W$, denote by $\mathcal{F}_X$ the fiber over $\rho_1^2(X)$ of $C^{2,2}M$, which contains $X$. The following proposition gives a local characterization of the submanifold $C^{1,2}W$, for $W$ transverse to the fiber $\mathcal{F}_X$.

**Proposition 4.** *Let $W \subset C^{2,2}M$ be an immersed submanifold, $X \in W$. Suppose $W$ is transverse to the fiber $\mathcal{F}_X$, with $dimW + dim\mathcal{F}_X = dimC^{2,2}M$. Then, there exist functions $F_{11}^j$, $F_{12}^j$, $F_{22}^j$, defined in a neighborhood of $\rho_1^2(X)$ such that $C^{1,2}W$ can be expressed as:*

$$y^{j+m} = F_{11}^j(x^i, y^j, z^j), \quad y^{j+2m} = F_{12}^j(x^i, y^j, z^j), \quad z^{j+2m} = F_{22}^j(x^i, y^j, z^j).$$

**Proof:** Let $W \subset C^{2,2}M$ be an immersed submanifold, $X^2 \in W$ and consider the coordinate neighborhood,

$$(\mathcal{V}^2, (x^i, y^j, z^j, y^{j+m}, y^{j+2m}, z^{j+2m})),$$

about $X^2$, defined as in Proposition 3.

Let $F_1, \cdots, F_s$ be differentiable functions defined in a neighborhood of $X^2$, (also written $\mathcal{V}^2$), such that:

$$\mathcal{V}^2 \cap W = \{X \in \mathcal{V}^2 : \quad F_1(X) = \cdots = F_s(X) = 0\}.$$

If $\eta \in T_X C^{2,2}M$ and is expressed in coordenates as:

$$\eta = \eta^i \frac{\partial}{\partial x^i} + p_1^j \frac{\partial}{\partial y^j} + p_2^j \frac{\partial}{\partial z^j} + p_{11}^j \frac{\partial}{\partial y^{j+m}} + p_{12}^j \frac{\partial}{\partial y^{j+2m}} + p_{22}^j \frac{\partial}{\partial z^{j+2m}},$$

then, $\eta \in T_X W \Leftrightarrow \eta(F_r) = o, \qquad 1 \le r \le s,$

$$\Leftrightarrow \frac{\partial F_r}{\partial x^i}\eta^i + \frac{\partial F_r}{\partial y^j}p_1^j + \frac{\partial F_r}{\partial z^j}p_2^j + \frac{\partial F_r}{\partial y^{j+m}}p_{11}^j + \frac{\partial F_r}{\partial y^{j+2m}}p_{12}^j + \frac{\partial F_r}{\partial z^{j+2m}}p_{22}^j = 0.$$

The coordinates of the fiber $\mathcal{F}_X$ are: $(y^{j+m}, y^{j+2m}, z^{j+2m})$, then we have:

the submanifold $W$ is transversal to the fiber $\mathcal{F}_X$,

$\Leftrightarrow T_X W \cap T_X \mathcal{F}_X = 0 \Leftrightarrow \{v \in T_X \mathcal{F}_X, \quad v(F_r) = 0 \Rightarrow v = 0\}$

$\Leftrightarrow$ is the unique solution of the system,

$$\begin{pmatrix} \frac{\partial F_1}{\partial y^{j+m}} & \frac{\partial F_1}{\partial y^{j+2m}} & \frac{\partial F_1}{\partial z^{j+2m}} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \frac{\partial F_s}{\partial y^{j+m}} & \frac{\partial F_s}{\partial y^{j+2m}} & \frac{\partial F_s}{\partial z^{j+2m}} \end{pmatrix} \begin{pmatrix} p^j_{11} \\ p^j_{12} \\ p^j_{22} \end{pmatrix},$$

is trivial. Then, the matrix which defines the system has maximal rank $d = \dim \mathcal{F}_X$.

By the implicit Function theorem, there exists an open neighborhood $\mathcal{U}$ of $X^2$ and functions $F^j_{11}$, $F^j_{12}$, $F^j_{22}$, defined on a neighborhood of $\rho^2_1(X^2)$ such that:

$$y^{j+m} = F^j_{11}(x^i, y^j, z^j), \quad y^{j+2m} = F^j_{12}(x^i, y^j, z^j), \quad z^{j+2m} = F^j_{22}(x^i, y^j, z^j). \quad \sharp$$

## 2. The involution on the contact manifold.

The following Proposition shows the existence of an involution on the manifold $\tilde{C}^{1,n}(C^{k,n}M)$ such that the invariant set of this involution is $C^{k+1,n}M$.

For simplicity, we shall only study the case $k = 1$, $n = 2$; the other cases can be shown anologously, using coordinates.

**Proposition 5.** *There exists an involution*

$$\tilde{\alpha} : \tilde{C}^{1,2}(C^{1,2}M) \longrightarrow \tilde{C}^{1,2}(C^{1,2}M),$$

*such that the inmersion* $i^{1,1}(C^{2,2}M) \subset \tilde{C}^{1,2}C^{1,2}M$ *is the set of fixed points of* $\tilde{\alpha}$.

**Proof.** Let $X^{1,1} = C^1_{X^1_0}S$, where rank $(\pi^1_0|_S) = 2$, then there exists a fibration $(\mathcal{V}^1, U, \pi)$ such that $S$ is transversal, and a coordinate neighborhood,

$$\mathcal{V}^{1,1} = \{C^1_{g(u)}g(U), \quad g : U \to \mathcal{V}^1, \textit{section} \quad \textit{of} \quad (\mathcal{V}^1, U, \pi^1_{-1}, \quad u \in U\},$$

with coordinates,

$$(x^i, y^j, z^j, x^{j+m}, x^{j+2m}, y^{j+m}, y^{j+2m}, z^{j+m}, z^{j+2m}),$$

defined in (1.1) and (1.2).

Now, the contact element $X^{1,1}$ can be represented by the tangent space $T_{X^1_0}S$, which is generated by,

$$L_1 = \frac{\partial}{\partial x^1} + x^{m+j}\frac{\partial}{\partial x^j} + y^{m+j}\frac{\partial}{\partial y^j} + z^{m+j}\frac{\partial}{\partial z^j};$$

$$L_2 = \frac{\partial}{\partial x^2} + x^{2m+j}\frac{\partial}{\partial x^j} + y^{2m+j}\frac{\partial}{\partial y^j} + z^{2m+j}\frac{\partial}{\partial z^j}. \tag{2.1}$$

Let $Y^1_o$ be defined by,

$$(x^i(X^1_0), y^j(X^1_0), z^j(X^1_0), x^{j+m}(X^1_0), x^{j+2m}(X^1_0), y^{j+m}(X^1_0)),$$

and $Y^{1,1}$ the contact element defined by the tangent space at $Y_0^1$ generated by,

$$\tilde{L}_1 = \frac{\partial}{\partial x^1} + y^j \frac{\partial}{\partial x^j} + z^j \frac{\partial}{\partial y^j} + y^{m+j} \frac{\partial}{\partial z^j};$$

$$\tilde{L}_2 = \frac{\partial}{\partial x^2} + z^{m+j} \frac{\partial}{\partial x^j} + y^{2m+j} \frac{\partial}{\partial y^j} + z^{2m+j} \frac{\partial}{\partial z^j}. \tag{2.1'}$$

We define $\tilde{\alpha}(X^{1,1}) = Y^{1,1}$.

The map $\tilde{\alpha}$ is well defined, because if $S_1, S_2 \subset C^{1,2}M$ are 2-dimensional submersions with $X_0^1 \in S_1 \cap S_2$ and $C^1_{X^1_0} S_1 = C^1_{X^1_0} S_2$, then $T_{X^1_0} S_1 = T_{X^1_0} S_2$.

Now, both of the projections, $\pi_0^1$ and $\pi_0^{1,1}$ restricted to the tangent spaces associated to the contact elements $Y_0^1$ and $Y_0^{1,1}$, respectively, have maximal rank.

Then, $\tilde{\alpha}$, is given using coordinates as follows:

$$\begin{aligned}&\tilde{\alpha}(x^1, x^2, x^j, y^j, z^j, x^{j+m}, y^{j+m}, z^{j+m}, x^{j+2m}, y^{j+2m}, z^{j+2m})\\ &\quad = (x^1, x^2, x^j, x^{j+m}, x^{j+2m}, y^j, z^j, y^{j+m}, z^{j+m}, y^{j+2m}, z^{j+2m}).\end{aligned} \tag{2.2}$$

It is clear that $\tilde{\alpha} \circ \tilde{\alpha}$ is the identity.

Let $X^{1,1}$ be a contact element fixed by $\tilde{\alpha}$, then,
$\pi_1^{1,1}(X^{1,1}) = \pi_1^{1,1}(Y^{1,1})$, i. e.,
$y^j(X^{1,1}) = x^{j+m}(X^{1,1}), \quad z^j(X^{1,1}) = x^{j+2j}(X^{1,1})$.
Now, $\tilde{\alpha}(X^{1,1}) = X^{1,1}$ if and only if,
$X_0^1 = Y_0^1$, and $Z^{j+m}(X^{1,1}) = x^{j+2m}(X^{1,1})$.
Then contact element fixed by $\tilde{\alpha}$ is given as,

$$x^{j+m} = y^j, \qquad y^{j+2m} = z^{j+m}, \qquad x^{j+2m} = z^j.$$

Using (1.3) we have that the fixed point set of $\tilde{\alpha}$ is $i^{1,1}(C^{2,2}M)$. $\sharp$

**Corollary 1.** *Let $W$ be an immersed submanifold transverse to $\mathcal{F}_X$, with $X \in W$. Consider the involution $\tilde{\alpha}$ of $\tilde{C}^{1,2}C^{2,2}M$ and let $Y \in \tilde{C}_X^{1,n}W$ such that $\tilde{\alpha}(Y) = Y$ then,*

$$\frac{\partial F_{11}^j(X)}{\partial x^2} = \frac{\partial F_{12}^j(X)}{\partial x^1}, \quad \frac{\partial F_{22}^j(X)}{\partial x^1} = \frac{\partial F_{12}^j(X)}{\partial x^2}.$$

**Proof:** Using Propositions 3 and 4. $\sharp$

**Proposition 6.** *Let $(M, \Re^2, \rho)$ be a fibered manifold and $W \subset C^{1,2}M$ a submanifold. If $\pi_0^1|_W$ is an immersion and for any $X \in W$ there is an element*

*$\tilde{X} \in C^{1,2}_X W$ which is invariant by $\tilde{\alpha}$, then $W$ generates an involutive 2- distribution on $M$.*

**Proof.** Let $W = \sigma(M)$, with $\sigma : M \longrightarrow C^{1,2}M$ be an immersion defined by,
$\sigma(x^1, x^2, x^j(x^1, x^2)) =$

$$(x^1, x^2, x^j(x^1, x^2), F^j_1(x^1, x^2, x^j(x^1, x^2)), F^j_2(x^1, x^2, x^j(x^1, x^2))),$$

and $\tilde{X} \in C^{1,2}_X W$, a contact element, which is invariant by $\tilde{\alpha}$, expressed as,

$$\tilde{X} = (x^1, x^2, x^j(x^1, x^2), F^j_1, F^j_2, \frac{\partial x^j}{\partial x^1}, \frac{\partial x^j}{\partial x^2}, \frac{\partial F^j_1}{\partial x^1}, \frac{\partial F^j_1}{\partial x^2}, \frac{\partial F^j_2}{\partial x^1}, \frac{\partial F^j_2}{\partial x^2}).$$

Consider the distribution on $M$ generared by the vector fields:

$$L_1 = \frac{\partial}{\partial x^1} + F^j_1 \frac{\partial}{\partial x^j}; \qquad L_2 = \frac{\partial}{\partial x^2} + F^j_2 \frac{\partial}{\partial x^j}.$$

The Lie bracket is given by,
$[L_1, L_2](X) =$

$$(\frac{\partial}{\partial x^1} F^j_2 + \sum_{r=3}^{m} F^r_1 \frac{\partial}{\partial x^r} F^j_2)(X) - (\frac{\partial}{\partial x^2} F^j_1 + \sum_{r=3}^{m} F^r_2 \frac{\partial}{\partial x^r} F^j_1)(X).$$

By Proposition 3, we have,

$$F^j_1(X) = \frac{\partial x^j}{\partial x^1}(X), \qquad F^j_2(X) = \frac{\partial x^j}{\partial x^2}, \qquad \frac{\partial F^j_1}{\partial x^2}(X) = \frac{\partial F^j_2}{\partial x^1}(X).$$

Using the chain rule and substituting the last expression in the bracket, we obtain $[L_1, L_2] = 0$, and the distribution is involutive. $\sharp$

Now, Let $W \subset C^{2,2}M$ be an immersed submanifold such that $\pi^2_1(W) \subset C^{1,2}M$ is an immersion. Let $\sigma = (\pi^2_1)^{-1}$. We can define a differentiable distribution $\mathcal{D}$ in $C^{1,2}M$ as follows:

$$X \in C^{1,2}M \longmapsto i^{1,1}(\sigma(X)) \in C^{1,2}(C^{1,2}M).$$

Clearly $\mathcal{D}$ is a 2-distribution because $i^{1,1}(\sigma(X))$ is identified with the corresponding 2-plane at $X$. Moreover, $\sigma$ is an immersion and hence $\mathcal{D}$ is differentiable.

**Proposition 7.** *Let $W \subset C^{2,2}M$ be an immersed submanifold such that:*
*i) $\pi^2_1 : W \longrightarrow C^{1,2}M$ is a local summersion in a neighborhood of $X_0 \in W$,*
*ii) given $X \in W$ there exits $Y \in T_X W$ invariant by the involution $\tilde{\alpha}$*
*then, the distribution $\mathcal{D}$ defined by $W$ is involutive and there exits a submanifold $S \subset M$, through $x_0 = \rho^2_0(X_0)$, such that $C^1 S$ is the integral submanifold of $\mathcal{D}$.*

**Proof.** Let $X^2 \in W$ and $\mathcal{V}^2 \subset C^{2,2}$ the open neighborhood at $X_0$ with coordinates as above. Let $\mathcal{U}$ be the projection of $\mathcal{V}^2$ by $\pi^2_1$, and $\sigma = (\pi^2_1)^{-1}|_{\mathcal{U}}$. The map $\sigma$ can by expressed in coordinates as:

$$\sigma(x^i, y^j, z^j) = (x^1, x^2, x^j, y^j, z^j, F^j_{11}, F^j_{12}, F^j_{22}).$$

Given $X \in W$, then the tangent space $T_X W$ is generated by the fields,

$$L_1 = \frac{d}{dt}\sigma(t, x^2, x^j, y^j, z^j) = \frac{\partial}{\partial x^1} + y^j \frac{\partial}{\partial x^j} + F^j_{11}\frac{\partial}{\partial y^j} + F^j_{12}\frac{\partial}{\partial z^j},$$

$$L_2 = \frac{d}{dt}\sigma(x^1, t, x^j, y^j, z^j) = \frac{\partial}{\partial x^2} + z^j \frac{\partial}{\partial x^2} + z^j \frac{\partial}{\partial x^j} + F^j_{12}\frac{\partial}{\partial y^j} + F_{22}\frac{\partial}{\partial z^j}$$

Now,

$[L_1, L_2] = (L_1(F^j_{12}) - L_2(F^j_{11}))\frac{\partial}{\partial y^j} + (L_1(F^j_{22}) - L_2(F^j_{21}))\frac{\partial}{\partial z^j}$

$= (\frac{\partial F^j_{12}}{\partial x^1} - \frac{\partial F^j_{11}}{\partial x^2})\frac{\partial}{\partial y^j} + (\frac{\partial F^j_{22}}{\partial x^1} - \frac{\partial F^j_{12}}{\partial x^2})\frac{\partial}{\partial x^2})$.

Using the corollary 2, we have $[L_1, L_2] = 0$ and the distribution is involutive.

Let $\Sigma$ be an integral submanifold of $\mathcal{D}$ through $X^2$. Since $\sigma(\mathcal{U})$ can be fibered over $U \subset \Re^2$, there exits a parametrization such that $\Sigma$ is expressed as:

$$\Sigma = (x^1, x^2, x^j, f^j(x^1, x^1), g^j(x^1, x^2), h^j(x^1, x^2)).$$

Now, $\Sigma \subset C^{1,2}M$, then $g^j = \frac{\partial f^j}{\partial x^1}$; $\quad h^j = \frac{\partial f^2}{\partial x^2}$.

Moreover, $C^{1,2}\Sigma \subset C^{2,2}M$, in consequence, given the submanifold $S \subset M$, defined by $S = (x^1, x^2, f^j)$ we have $C^{1,2}S$ is a solution of $\mathcal{D}$ through $X_0$, then $C^{1,2}S = \Sigma$. $\sharp$

## References

[1] C. Ehresmann, Les prolongements d'un espace fibré diffeérentiable, C. R. Acad. Sci. Paris 240 (1955), 1755-1757.

[2] F. Etayo, On jets of surfaces, Collect. Math. 42, 2(1991), 171-176.

[3] G. Jensen, Higer order of submanifolds on homogeneous spaces, Lectures notes in Math. Vol. 610, Springer, Berlin (1977).

[4] I. Kolář, On the invariant method in differential geometry of submanifolds, Czechoslovak Math. Journal, 27 (102) 1977,Praha.

[5] A Morimoto, Prolongation of Geometric Structures, Math. Inst. Nagoya, Nagoya Univ., 1969.

[6] Y. Villarroel, Superficies Singulares en una cuadrica. Acta Científica Venezolana. Vol 37, 6, p. 625-631. (1987), Venezuela.

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