DIFFERENTIAL GEOMETRY AND APPLICATIONS
Proc. Conf., Aug. 28 – Sept. 1, 1995, Brno, Czech Republic
Masaryk University, Brno 1996, 239–244

SOME SITUATIONS WHEN THE TOPOLOGICAL φ-CATEGORY IS INFINITE

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Abstract. The topological φ-category $\varphi_{top}(M,N)$ of a pair of manifolds (M,N) is introduced and studied in the authors' paper [3]. The main result of this present paper gives sufficient conditions in terms of fundamental groups of the compact manifolds M^m, N^n $(m \geq n \geq 2)$, in order that $\varphi_{top}(M,N) = +\infty$. Some applications and examples are given.

1. Preliminaries

Let M^m, N^n be two topological manifolds and let $f : M \to N$ be a continuous map. The point $x_0 \in M$ is called *topological regular point* of f if f has a local representation $f_{\varphi\psi} = \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ at x_0 such that for any $z = (z^1, \dots z^m) \in \varphi(U) \subseteq R^m$

$$f_{\varphi\psi}(z) = \begin{cases} (z^1, \dots, z^m, \underbrace{0, \dots, 0}_{n-m}) & \text{if} \quad m \leq n \\ (z^1, \dots, z^n) & \text{if} \quad m \geq n \end{cases} \tag{1}$$

There are different definitions of regular points in the case of real functions which are defined on a topological manifold. We mention here the papers of **N.H. Kuiper** [15], **W. Smith** [17], **F. Wille** [19] and **B. Doubrovine**, a.o.[12, pp 186-187].

If the point x_0 is not topological regular, it is called *topological critical point* of the map f. This notion was introduced and studied in our paper **D. Andrica, C. Pintea** [3]. An interesting problem is to compare our definition with those of critical points obtained from the above mentioned papers, but this is not the intention of the present paper.

Let us denote by $C_{top}(f)$ the set of all such above defined points and let us call it the *topological critical set* of f.

When M^m, N^n are differentiable manifolds and $f : M \to N$ is a C^1-differentiable mapping, recall that the critical set of f is given by

$$C(f) = \{x \in M \,:\, \text{rank}_x f < \min(m,n)\}$$

1991 *Mathematics Subject Classification.* 58E05, 58C27.
Key words and phrases. Topological critical points, topological φ-category, φ-category.
This paper is in final form and no version of it will be submitted for publication elsewhere.

where $\text{rank}_x f = \dim_R \text{Im}(Tf)_x$, $(Tf)_x : T_x M \to T_{f(x)} N$ being the tangent map of f at x. According to the well-known rank's theorem (see for instance **Th. Bröker, L. Lander** [6, pp. 2-5]) it follows the relation:

$$C_{top}(f) \subseteq C(f) \tag{2}$$

Generally, the equality in (2) not holds (see Proposition 1.3 and Example in [3]).

For a continuous map $f : M \to N$, where M, N are topological manifolds, let us consider $\mu_{top}(f) = |C_{top}(f)|$, the cardinal number of topological critical set of f. Let us denote $\varphi(M, N) = \min\{\mu_{top}(f) \; : \; f \in C(M, N)\}$ the *topological φ-category* of the pair (M, N).

If M^m, N^n are smooth differentiable manifolds we define the *$\varphi-category$* of the pair (M, N) by $\varphi(M, N) = \min\{\mu(f) \; : \; f \in C^\infty(M, N)\}$, where $\mu(f)$ denotes the cardinal number of $C(f)$. Note that when $N = R$, $\varphi(M, R)$ is even the φ-category of M (or the functional category of M), it is simply denoted $\varphi(M)$ and it was investigated by **F. Takens** [18] (see also **D. Andrica** [1], [2]).

$\varphi(M, N)$ points out a class of mappings $f \in C^\infty(M, N)$ having the critical set $C(f)$ minimal. Let us remark that in a series of papers **P.T. Church**, [8], [9] and **P.T. Church, J. G. Timourian**, [10], [11] using results from dimension theory, obtain informations concerning mappings $f \in C^\infty(M, N)$ with "small" critical set.

Using inclusion (2), it follows that if M, N are smooth differentible manifolds and $f \in C^\infty(M, N)$ the relation $\mu_{top} \leq \mu(f)$ holds and consequently one obtains the inequlity

$$\varphi_{top}(M, N) \leq \varphi(M, N) \tag{3}$$

2. The main result

The principal result of our paper [3] is the following

Theorem 2.1. *Let M^m be a compact topological manifold and let k be an integer number with $m \geq k \geq 2$. Then*

$$\varphi_{top}(M, R^k) = +\infty \tag{4}$$

Taking into account the inequality (3) one obtains (see [3, Corollary 3.2]) that if M^m is a compact differentiable manifold and if k is an integer with $m \geq k \geq 2$, then the relation

$$\varphi(M, R^k) = +\infty \tag{5}$$

is satisfied.

The main purpose of this paper is to obtain sufficient conditions for the manifolds M, N such that $\varphi(M, N) = +\infty$, i.e. in the differential case $\varphi(M, N) = +\infty$, via the inequality (2). Let us mention that in the first author's paper [2] are obtained such results for $\varphi(M, N)$, but the conditions are given in terms of Stiefel-Whitney characteristic classes.

We begin with two useful results in the proof of main theorem. Let us consider the topological manifolds M, N

Lemma 2.2. *If there exists a local homeomorphism $\tilde{N} \to N$, then $\varphi_{top}(M, N) \leq \varphi_{top}(M, \tilde{N})$*

Proof. Let $p : \tilde{N} \to N$ be the asserted local homeomorphism. If $f \in C(M, \tilde{N})$ consider $g \in C(M, N)$ so that $g = p \circ f$. It is not difficult to note that $C_{top}(f) = C_{top}(g)$ (see also [3, Proposition 1.4 (ii)]), that is $\mu_{top}(f) = \mu_{top}(g) \geq \varphi_{top}(M, N)$. Because the inequality holds for any map $f \in C(M, N)$, one obtains the desired relation. □

Lemma 2.3. *Let $p : \tilde{N} \to N$ be the universal covering of manifold N. If $Hom(\pi_1(M), \pi_1(N)) = 0$, then the relation*

$$\varphi_{top}(M, N) = \varphi_{top}(M, \tilde{N}) \tag{6}$$

is satisfied.

Proof. For the begining let us recall that the manifold N has an universal covering $p : \tilde{N} \to N$, and $\tilde{N}$ is also a manifold (see **M.J. Greenberg, J.R. Harper** [14, Corollary 6.8]).

Taking into account the result of Lemma 2.2 it follows that it is sufficient to prove the inequality $\varphi_{top}(M, \tilde{N}) \leq \varphi_{top}(M, N)$. Let us consider $f \in C(M, N)$. Then $f_* : \pi_1(M) \to \pi_1(N)$ is the null homomorphism, thus the lifting crterion (see **M.J. Greenberg, J.R. Harper** [14, Theorem 6.1] or **C. Godbillon** [13, Théoreème 3.2]) is verified. One obtains the continuous map $\tilde{f} : M \to \tilde{N}$ such that $p \circ \tilde{f} = f$. Because the projection p is a local homeomorphism it implies the relation $C_{top}(f) = C_{top}(\tilde{f})$ (see [3, Proposition 1.4 (ii)]), i.e. $\mu_{top}(f) = \mu_{top}(\tilde{f}) \geq \varphi_{top}(M, \tilde{N})$, and the desired inequality $\varphi_{top}(M, N) \geq \varphi_{top}(M, \tilde{N})$ is proved. □

Our main result is contained in the following theorem.

Theorem 2.4. *Let M^m, N^n be two connected compact manifolds with $m \geq n \geq 2$. Suppose that:*

(i) $\tilde{N}$ is homeomorphic with R^n, where $p : \tilde{N} \to N$ is the universal covering of N.

(ii) $Hom(\pi_1(M), \pi_1(N)) = 0$

Then the following relation holds

$$\varphi_{top}(M, N) = +\infty \tag{7}$$

Proof. Let $h : \tilde{N} \to R^n$ be the asserted homeomorphism and let us put $H : (M, \tilde{N}) \to (M, R^n)$, $H = (1_M, h)$. For the continuous map $f \in C(M, \tilde{N})$ consider the map $f' \in C(M, R^n)$ such that the following diagram is commutative

$$\begin{array}{ccc} M & \xrightarrow{f} & \tilde{N} \\ {\scriptstyle 1_M}\downarrow & & \downarrow{\scriptstyle \mathrm{h}} \\ M & \xrightarrow{f'} & R^n \end{array}$$

that is $f' = h \circ f$. One obtains $C_{top}(f') = C_{top}(f)$ (see [3, Proposition 1.4 (ii)]) so $\mu_{top}(\Lambda(f)) = \mu_{top}(f)$, where $\Lambda : C(M, \tilde{N}) \to C(M, R^n)$ is defined by $\Lambda(f) = f'$. The map Λ is bijective and Λ^{-1} is given by $\Lambda^{-1}(f') = h^{-1} \circ f'$. So we have succesively $\varphi_{top}(M, \tilde{N}) = \min\{\mu_{top}(f) : f \in C(M, \tilde{N})\} = \min\{\mu_{top}(\Lambda(M)) : f \in C(M, \tilde{N})\} = \min\{\mu_{top}(f') : f' \in C(M, R^n\} = \varphi_{top}(M, R^n)$.

On the other hand from Lemma 2.3 one obtains $\varphi_{top}(M, N) = \varphi_{top}(M, \tilde{N}) = \varphi_{top}(M, R^n)$. Using the result contained in Theorem 2.1 we get the relation (7). □

Remark For any topological manifold N the total space $\tilde{N}$ of its universal covering $p : \tilde{N} \to N$ is simply connected (see **C. Godbillon** [13, Théorème 5.3 and Théorème 4.8]) but it is not neccesary that $\tilde{N}$ to be homeomorphic with R^n. For example, we have $\pi_1(S^3) = 0$, $\pi_1(S^2) = 0$ so the condition (ii) in Theorem 2.4 is satisfied. The universal covering of S^2 is $1_{S^2} : S^2 \to S^2$, i.e. the hypothesis (i) is not verified. On the other hand $\varphi(S^3, S^2) = 0$, because of the well-known Hopf fibration $S^3 \to S^2$.

It is an interesting problem to characterize the manifolds N whose universal space $\tilde{N}$ satisfies condition (i) in Theorem 2.4.

Corollary 2.5. *Suppose that the hypothesis (i) in Theorem 2.4 is satisfied. If $\pi_1(M)$ is a torsion group (i.e. $Tor\pi_1(M) = \pi_1(M)$) and $\pi_1(N)$ is without torsion (i.e. $Tor\pi_1(N) = 0$), then the relation (7) holds.*

Corollary 2.6. *Suppose that the hypothesis (i) in Theorem 2.4 is satisfied. If M is simply-connected, then the relation (7) holds.*

For the proof of the above corollaries it is sufficient to observe that the condition $Hom(\pi_1(M), \pi_1(N)) = 0$ is automatically verified.

3. Applications

Proposition 3.1. *Let M^m be a compact connected topological manifold which is simply-connected. If the finite group G acts properly discontinuous on M, then*

$$\varphi_{top}(M/G, T^n) = +\infty \tag{8}$$

where $T^n = S^1 \times S^1 \times \cdots \times S^1$ (n times) is the n-dimensional torus and $m \geq n \geq 2$.

Proof. Because G acts properly discontinuous on M it follows that $p : M \to M/G$ is a covering space of M/G, where p is the canonical map. Taking into account that M is simply-connected one obtains that $p : M \to M/G$ is even the universal covering of M/G (see **C. Godbillon** [13, Théorème 4.8]) and $\pi_1(M/G) \cong G$ (see **G.E. Bredon** [5, Corollary 7.3]).

On the other hand the universal covering of the torus T^n is $R^n \to R^n/Z^n = T^n$, i.e. the condition (i) from Theorem 2.4 is verified. The fundamental group of T^n is $\pi_1(T^n) = Z \times \cdots \times Z$ (n times), therefore $\pi_1(T^n)$ is without torsion.

We have $Hom(\pi_1(M/G), \pi_1(T^n)) = 0$ since $\pi_1(M/G)$ is a finite group. The desired conclusion is obtain from Theorem 2.4. □

If $M_j^{m_j}, j = \overline{1,k}$, are connected and the finite group G_j acts properly discontinuous on M_j, $j = \overline{1,k}$, with an analogous argument one obtains

$$\varphi_{top}(M_1/G_1 \times \cdots \times M_k/G_k, T^n) = +\infty \tag{9}$$

where $m_1 + m_2 + \cdots + m_k \geq n \geq 2$.

Examples 1) Consider $S^{2m-1} = \{z \in C^m : ||z|| = 1\}$. Let $\varepsilon = e^{2\pi i/p}$ be a primitive p^{th} rooth of unity and let $q_1, \dots, q_m$ be integers relatively prime to p. Consider the multiplicative group $Z_p = \{1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{p-1}\}$ and let us consider its action on S^{2m-1} given by $\varepsilon \cdot z = (\varepsilon^{q_1} z_1, \dots, \varepsilon^{q_m} z_m)$, where $z = (z_1, \dots, z_m)$. The above action is properly discontinuous since it is a free action of a finite group. The orbit space S^{2m-1}/Z_p denoted by $L(p; q_1, \dots, q_m)$ is called a "lens space" and it is a (2m-1)-dimensional real manifold. Applying the result of Proposition 2.7 we have

$$\varphi_{top}(L(p; q_1, \dots, q_m), T^n) = +\infty \tag{10}$$

where $2m - 1 \geq n \geq 2$.

2) From the relation (9) one obtains for the real projective spaces $P^{m_i}(R)$, $i = \overline{1, k}$, that if $m_1 + \cdots + m_k \geq n \geq 2$, then

$$\varphi_{top}(P^{m_1}(R) \times \cdots \times P^{m_k}(R), T^n) = +\infty \tag{11}$$

Theorem 3.2. *Let M^m be a compact connected topological manifold and let N^n be a compact Riemannian manifold with negative curvature. If $\pi_1(M)$ is commutative torsion group and $m \geq n \geq 2$, then $\varphi_{top}(M, N) = +\infty$.*

Proof. Using the Hadamard theorem (see **M.P. doCarmo** [7, Theorem 3.1, pp 149]) it follows that the universal covering space $\tilde{N}$ of N is diffeomorphic with R^n. Let us show the relation $Hom(\pi_1(M), \pi_1(N)) = 0$. If $F : \pi_1(M) \to \pi_1(N)$ is a homomorphism such that $F \neq 0$, one obtains immediately that ImF is a nontrivial commutative torsion subgroup of $\pi_1(N)$. But, taking into account the theorem of Preissman [16] (see also **M.P. doCarmo** [7, Theorem 3.2, pp 260]) we have that any nontrivial commutative subgroup of $\pi_1(N)$ is infinite cyclic, which is a contradiction with the fact that ImF is a torsion group. Therefore $Hom(\pi_1(M), \pi_1(N)) = 0$, i.e. the conditions of Theorem 2.4 are satisfied and one can apply it to obtain the desired conclusion. □

Corollary 3.3. *Let G be a compact Lie group and let N^n be a compact Riemannian manifold with negative curvature. If $\pi_1(G)$ is a torsion group and $dim\, G \geq n \geq 2$, then $\varphi_{top}(G, N) = +\infty$.*

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