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AREA AND LENGTH FOR ROULETTES VIA CURVATURE

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Abstract. Formula for the area bounded by a roulette and centroid of curvatures are given.

Introduction

A roulette $\mathcal{R}(c,l,P)$ is a curve traced out by a point P in fixed position with respect to a rolling curve c which rolls without slipping along a fixed base curve l. Area bounded by the roulette and base curve, area $\mathcal{R}(c,l,P)$ in some cases is minimal if tracing point is barycenter B of curve c (which is) pondered by a difference of curvature measures $\kappa_c - \kappa_l$. Moreover,

$$\operatorname{area}\mathcal{R}(c,l,P) = \operatorname{area}\mathcal{R}(c,l,B) + \text{area sector}(BP, \kappa_c^T - \kappa_l^T) + \operatorname{area}\square PC_1BC_2,$$

where the sector is circular with radius BP and the angle of rotation of curve c, i. e., difference of total curvatures $\kappa_c^T - \kappa_l^T$; also $c = \widehat{C_1C_2}$. If the rolling curve is closed, then the last addend vanishes and points from the circle with center B trace equal areas. By this formula the areas for cycloid, cardioid, epi and hypocycloid, trochoid etc. simply achieve. Given formula follows from

$$\operatorname{area}\mathcal{R}(c,l,P) = \text{area cone}Pc + \frac{1}{2}\int_{c\ni X} PX^2\, d(\kappa_c - \kappa_l),$$

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the conePc is bounded by c and straight segments PC_1 and C_2P. For an ellipse e and straight line L obtains area$\mathcal{R}(e, L, P) = \pi(a^2 + b^2 + OP^2)$, a and b are half-axes, and O is center of the ellipse, cf. E. Ehrhart [1]. For symmetric closed curves $\kappa_c = -\kappa_l$, the cardioid—cycloid proportion holds: area$\mathcal{R}(c, l, P) +$ area$\,c =$ 2area$\mathcal{R}(c, L, P)$. The area between two roulettes traced by points P and Q is

$$\text{area}\,\mathcal{R}(c, l, PQ) = \frac{1}{2}(\kappa_c^T - \kappa_l^T)(BP^2 - BQ^2).$$

Finally,

$$\text{length}\,\mathcal{R}(c, l, P) = \int_{c \ni X} PX\, d(\kappa_c - \kappa_l).$$

Definitions

Let $c = \overset{\frown}{C_1C_2}$ and $l = \overset{\frown}{L_1L_2}$ be oriented piecewise regular curves with parametric equations $r(t) = (x(t), y(t))$, $a \le t \le b$, and $\rho(t) = (u(t), v(t))$, $c \le t \le d$, respectively. Tangent vector $\dot{r}(t) = (\dot{x}t), \dot{y}(t))$ at regular point $X(t)$ is non-zero. A parametric representation is regular if vector function r is of class C^1 and $\dot{r}(t) \neq 0$ for all $a \le t \le b$. Tangent vectors can be translated so that they have the same origin. The total curvatre κ_c^T is difference $\theta_1 - \theta_0$ in the values of inclination θ of the tangent to the curve at the end points of the curve. For closed curve c ($C_1 = C_2$) it holds $\kappa_c^T = 2z\pi$, z is an integer. Natural parameter is length of arc $s = \int |\dot{r}(t)|\, dt$; then tangent vectors have length 1. If c is smooth in some neighborhood of a point X, then curvature of the curve c at the point X is

$$\kappa(X) = \lim \frac{\Delta\alpha}{\Delta s} = \frac{d\alpha}{ds}, \qquad \Delta s \to 0, \qquad Y \to X,$$

$\Delta\alpha$ is angle between tangent vectors at X and Y, Δs is length of arc $\overset{\frown}{XY}$ (limes of quotient of tangent inclination and increase of curve length). Classical formula for curvature is

$$\kappa(X) = \frac{\dot{x}(t)\ddot{y}(t) - \ddot{x}(t)\dot{y}(t)}{(\dot{x}(t)^2 + \dot{y}(t)^2)^{3/2}}.$$

At singular point S the point curvature is $\kappa(S) = \measuredangle(\dot{r}_-(S), \dot{r}_+(S))$, so that $-\pi \le \kappa(S) \le \pi$. We take $\kappa(S) = \pm\pi$ if $\lim \angle(\dot{r}_-(S), \overset{\frown}{SX}) = \pm\pi$, $X \to S$, $x \in \overset{\frown}{S_2}$. If the whole arc $\overset{\frown}{SC_2}$ is straight segment opposite to $\dot{r}_-(s)$, we can choose either π or $-\pi$. Mention that curvature κ as function of length of arc is natural equation of curve. If two curves have equal curvatures as functions of length, then they can coincide by motion.

Main results

Definition 1. *Curvature measure κ on curve c is defined on regular segments by the curvature functional $\kappa(t)$, and at singular points the point-measure is equal to point-curvature.*

Definition 2. *Center B of curvature of curve c is barycenter of curve c which is by the curvature measure κ pondered.*

Hence, $\kappa_c^T = \int_c d\kappa$. Alternative terms for B are barycenter (centroid) of curvature or κ-center of curve. Obviously, if c is axially symmetric curve, then B belongs to the axis of symmetry; also $B \in \operatorname{conv} c$. $\kappa_c^- \kappa_l^T = \rho$ is the angle of rotation of curve c.

Let $\mathcal{R}(c, l, P) = \overset{\frown}{P_1P_2}$, and let $l = \overset{\frown}{L_1L_2}$ where L_1 and L_2 are first and last contact point respectively. The area $\mathcal{R}(c, l, P)$ is oriented area $L_1L_2P_2P_1$ bounded by the base curve $l = \overset{\frown}{L_1L_2}$, the roulette $\overset{\frown}{P_2P_1}$, and straight segments L_2P_2 and P_1L_1.

Theorem 1. *Let B be a barycenter of curve c by difference of curvature measures $\kappa_c - \kappa_l$ pondered, and let κ^T be total curvature. Then, area bounded by a roulette and base curve l is*

$$\begin{aligned} \text{area}\,\mathcal{R}(c, l, P) &= \text{area cone}\,Pc + \frac{1}{2}\int_c PX^2\, d(\kappa_c - \kappa_l), \qquad X \in c \\ &= \text{area}\,\mathcal{R}(c, l, B) + \frac{\kappa_c^T - \kappa_l^T}{2} BP^2 + \text{area}\,\square PC_1BC_2. \end{aligned}$$

For closed rolling curve area cone Pc = area c and area $\square PC_1BC_2 = 0$.

For closed rolling curve c points from the circle with center B trace roulettes which bound equal areas.

The equivalence between two given formulas from theorem 1. follows from

$$\int_c PX^2\, d\kappa_c = \int_c (PB^2 + BX^2)\, d\kappa_c = \kappa_c^T PB^2 + \int_c BX^2\, d\kappa_c.$$

This theorem is used in solving the problem [1] proposed by E. Ehrhart, Université de Strasbourg: "The curve traced out by a fixed point of a closed convex curve as that curve rolls without slipping along a second curve will be called a "roulette". Let S be the area of one arc of a roulette traced out by an ellipse of area s rolling on a straight line. Prove or disprove that $S \geq 3s$, with equality only if the ellipse is a circle."

Let $x = a\cos t$, $y = b\sin t$, $0 \leq t \leq 2\pi$ be parametric equations of an ellipse, L be a straight line, and let $BP = h$. Then

$$\begin{aligned} \text{area}\,\mathcal{R}(e, L, P) &= \text{area}\,\mathcal{R}(e, L, B) + \frac{1}{2}2\pi BP^2 \\ &= \text{area}\,e + \frac{1}{2}\int_0^{2\pi} \frac{a^2\cos^2 t + b^2\sin^2 t}{a^2\sin^2 t + b^2\cos^2 t} \begin{vmatrix} -a\sin t & b\cos t \\ -a\cos t & -b\sin t \end{vmatrix} dt + \pi h^2 \\ &= \pi ab + 2ab\left[\frac{a^2+b^2}{ab}\arctan\frac{a}{b}\tan t - t\right]\Bigg|_0^{\pi/2} + \pi h^2 \\ &= \pi(a^2 + b^2 + h^2). \end{aligned}$$

For cycloid, trochoid, epicycloid, hypocicloid, cardioid, and evolvent of circle, the areas are simply obtained.

The area under the cycloid cusp is $3\pi r^2$, and the area bounded by a cardioid is $6\pi r^2$. This relation holds in more general case.

Corollary. Cardioid—cycloid proportion. *Let c and l be closed symmetric curves $\kappa_c(t) = -\kappa_l(t)$, $a \le t \le b$, and L be a straight line, then*

$$\operatorname{area}\mathcal{R}(c, l, P) + \operatorname{area} l = 2\operatorname{area}\mathcal{R}(c, L, P).$$

From corollary and solution of the problem one obtains the area bounded by an epiellipse.

Theorem 2. *Let $\operatorname{area}\mathcal{R}(c, l, PQ)$ be the area traced by a segment PQ in fixed position with respect to the rolling curve c, i. e., the area between two roulettes traced by points P and Q which ends are connected by straight segments. Then*

$$\operatorname{area}\mathcal{R}(c, l, PQ) = \frac{1}{2}(\kappa_c^T - \kappa_l^T)(BP^2 - BQ^2).$$

Especially,

$$\operatorname{area}\mathcal{R}(c, l, BP) = \frac{1}{2}(\kappa_c^T - \kappa_l^T)BP^2.$$

Note that

$$\operatorname{area}\mathcal{R}(c, l, P) = \operatorname{area}\mathcal{R}(c, l, B) + \operatorname{area}\mathcal{R}(c, l, BP) + \operatorname{area}\square PC_1BC_2.$$

References

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