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INVARIANCE THEORY AND AFFINE DIFFERENTIAL GEOMETRY

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ABSTRACT. The main purpose of this paper is to study the characteristic classes of manifolds and submanifolds endowed with a complex affine connection D whose curvature tensor R and an almost complex structure J satisfy the condition $R(JX, JY) = -R(X,Y)$. These connections are naturally appeared in the context of pseudo-Riemannian manifolds with the metric of the signature (n, n) as well as in complex affine and projective differential geometry (see [1–5, 7–10], etc.). We show especially a complex equiaffine surface endowed with a complex affine connection D has the vanishing first Chern class c_1. We prove also the second Chern class c_2 and c_1^2 vanish for all complex surfaces endowed with a complex affine connection D. Some new and known examples of manifolds endowed with a complex affine connection D are given.

§1 Introduction

Let M be n-dimensional smooth manifold and T_pM its tangent space in an arbitrary point $p \in M$. Then we denote by $TM = \bigcup_{p\in M}(T_pM)$ the tangent bundle of M. ∇ is an affine symmetric connection or a torsion free affine connection if

$$\nabla_X Y - \nabla_Y X = [X, Y], \tag{1.1}$$

for arbitrary smooth vector fields $X, Y \in \mathcal{X}(M^n)$. A smooth manifold M^n endowed with an affine symmetric connection ∇ we denote by (M^n, ∇).

The *Riemann-Christoffel curvature tensor* R of an affine connection ∇ is the (1,3)-tensor field defined by

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, \tag{1.2}$$

and the *Ricci tensor* ρ is defined by

$$\rho(X,Y) = \mathrm{tr}(Z \to R(Z,X)Y), \tag{1.3}$$

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where $X, Y, Z \in \mathcal{X}(M^n)$.

We say (M^n, J) is an almost complex manifold if M^n can be endowed with a smooth (1,1)-tensor field J such that $J^2 = -I$, where I is a field of identity endomorphisms. If the Nijenhuis tensor field

$$N(X,Y) = [JX, JY] - [X,Y] - J[JX,Y] - J[X,JY] \tag{1.4}$$

vanishes, i.e. $N = 0$, then J is a complex structure and moreover M^n is a complex manifold whose transition functions are holomorphic mappings of the complex space $\mathbf{C}^n$. $N = 0$ is equivalent to the condition $\nabla J = 0$. (M^n, ∇, J) is a complex manifold, i.e. a manifold endowed with a symmetric affine connection ∇ and an almost complex structure J such that $\nabla J = 0$. We call also ∇ a complex connection if $\nabla J = 0$.

Let $\langle\cdot,\cdot\rangle$ is a metric defined on M^n. If

$$\langle JX, JY\rangle = \langle X, Y\rangle, \tag{1.5}$$

or

$$\langle JX, JY\rangle = -\langle X, Y\rangle, \tag{1.6}$$

for all $X, Y \in \mathcal{X}(M)$ then $\langle\cdot,\cdot\rangle$ is an almost Hermitian metric and an almost complex metric respectivelly. If (M^n, ∇, J) is a complex manifold then the corresponding metric is a Hermitian and a complex metric respectivelly. Let us remark that a complex metric is always a pseudo-Riemannian metric of signature (m, m), where $n = 2m$.

If $\{E_1, \cdots, E_n\}$ is an orthonormal local basis of M^n and $\{\theta^1, \ldots, \theta^n\}$ the dual basis of $\{E_1, \cdots, E_n\}$, then we call $\omega = \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n$ the *metric volume form* of M^n. The metric volume form is only locally defined and unique up to sign. If M^n is oriented, then ω can be defined on the whole of M^n.

If M^n is a *complex manifold* (in the sense that there exists a system of charts covering M^n such that the transition functions are holomorphic mappings of $\mathbf{C}^n$), then there exists a complex structure on M^n in a natural way. Indeed, suppose that $(z_1, z_2, \ldots, z_n)$ are complex coordinates on M^n. Then every vector field X can be written as

$$X = \sum_{a=1}^{n} X^a \frac{\partial}{\partial z_a} \quad \text{and we can define} \quad JX \quad \text{by } JX = iX = \sum_{a=1}^{n} iX^a \frac{\partial}{\partial z_a}. \tag{1.7}$$

A manifold with Hermitian metric and complex Levi Civita connection is called a *Kaehler manifold.* A complex manifold M^n with complex affine connection ∇ endowed with a parallel complex n-form ${}^c\omega$, is called a *complex equiaffine manifold* and $(J, \nabla, {}^c\omega)$ a *complex equiaffine structure.* We call ${}^c\omega$ a *complex volume form.* It is easy to check that on a complex equiaffine manifold there is exactly one parallel real volume form ω satisfying

$$|{}^c\omega(X_1, \ldots, X_n)|^2 = \omega(X_1, JX_1, \ldots X_n, JX_n), \tag{1.8}$$

for all $X_1, \ldots X_n \in \mathcal{X}(M^{2n})$.

Torsion free non-metric complex connections were studied by many authors (see for example [1–5, 7–10] etc).

Our paper is organized as follows. In the section §2 we define a connection D that we study throughout this paper and prove lemmas considering properties of its curvature tensor. We apply the classical Weyl invariance theory to prove Chern classes c_1^2, c_2 vanish on complex surfaces M endowed with the connection D. Moreover, if M is equiaffine then $c_1 = 0$. The section §4 is devoted to the construction of nonflat connection D on the torus T^4 and the Euclidean space $\mathbf{R}^4$. To see how are these connections appeared naturally in geometry we review also some known results of F. Dillen, L. Vrancken and L. Verstraelen as well as of A.P. Norden in §5-6.

§2 The curvature tensor of a complex affine connection

If M^n is a complex manifold with complex affine connection ∇ ($\nabla J = 0$) then it holds

$$R(X,Y) \circ J = J \circ R(X,Y). \tag{2.1}$$

Among complex affine connection we pay the attention especially on these that are considered in the following lemma.

Lemma 1.1: *If the Riemann-Christoffel curvature tensor of a complex affine connection D has the following property*

$$R(JX,Y) = R(X,JY) \tag{2.2}$$

then it has also the following property

$$R(JX,Y) = R(X,JY) = R(X,Y) \circ J = J \circ R(X,Y). \tag{2.3}$$

Proof: We use the first Bianchi identity, (1.9) and (1.10) to see

$$\begin{aligned} R(X,JY)Z &= -R(Z,X)JY - R(JY,Z)X \\ &= -JR(Z,X)Y - R(Y,JZ)X \\ &= -JR(Z,X)Y + R(X,Y)JZ + R(JZ,X)Y \\ &= JR(X,Y)Z + JR(Y,Z)X + R(X,Y)JZ + R(Z,JX)Y \\ &= 2JR(X,Y)Z + JR(Y,Z) - R(JX,Y)Z - R(Y,Z)JX \\ &= 2JR(X,Y)Z - R(X,JY)Z, \end{aligned} \tag{2.4}$$

and hence

$$R(X,JY)Z = JR(X,Y)Z. \tag{2.5}$$

This relation together with (2.1) and (2.2) prove our lemma. ■

Remark: If a complex affine connection ∇ satisfies the condition $\nabla_{JX}Y = J(\nabla_X Y)$ then it satisfies also the relations (2.3).

From now on we study only complex affine conections satisfying (2.3) and usually denote them by D.

Hence R is $\mathbf{C}$-trilinear, and we can define a complex Ricci tensor ${}^c\rho$ by

$$ {}^c\rho(X,Y) = \mathrm{tr}^c[Z \to R(Z,X)Y], \tag{2.6}$$

whereby tr^c denotes the trace of a $\mathbf{C}$-linear mapping over $\mathbf{C}$. Then it is easy to show that ${}^c\rho(X,Y) = (1/2)[Ric(X,Y) - iRic(JX,Y)]$, such that ${}^c\rho$ is symmetric if and only if ρ is symmetric. One can show that ${}^c\rho$ is symmetric if and only if around each point there exists a parallel holomorphic complex volume form, see [4].

It will be useful for the studying of the Chern characteristic classes to know symmetry properties of the Riemann-Christoffel curvature tensor in complex and real coordinate system. Consequently, let $\tilde{R}$ and R be components in complex and real coordinate systems for the curvature tensor of a complex affine connection D. If (E_i, JE_i) is a basis of $T_p(M)$ we denote by e_i a basis of $T_p^c(M)$ such that

$$ e_i = \frac{1}{\sqrt{2}}(E_i - \sqrt{-1}JE_i), \quad \bar{e}_i = e_{\bar{i}} = \frac{1}{\sqrt{2}}(E_i + \sqrt{-1}JE_i), \tag{2.7}$$

where $\bar{e}_i$ is a conjugate vector of e_i. Then we have the following lemma.

Lemma 1.2:

(i) $\tilde{R}(\bar{x},\bar{y}) = \overline{\tilde{R}(x,y)}$,

(ii) $\tilde{R}(x,\bar{y}) = 0$,

(iii) $\tilde{R}(\bar{x},\bar{y})z = \overline{\tilde{R}(x,y)\bar{z}}$,

(iv) $\tilde{R}_{\bar{p}\bar{q}i}{}^{j} = 0$,

(v) $\tilde{R}(e_p, e_q)e_{\bar{i}} = 0$,

for all holomorphic vector fields x, y and an arbitrary complex vector field z.

Proof: If x, y are holomorphic vector fields then we have $x = U - \sqrt{-1}JU$, $y = V - \sqrt{-1}JV$ and consequently $\bar{x} = U + \sqrt{-1}JU$, $\bar{y} = V + \sqrt{-1}JV$. We use now (1.15) to compute

$$ \tilde{R}(\bar{x},\bar{y}) = 2(R(U,V) + \sqrt{-1}R(U,JV)), \tag{2.8}$$

$$ \tilde{R}(x,y) = 2(R(U,V) - \sqrt{-1}R(U,JV)). \tag{2.9}$$

Hence it follows (i). One can check (ii) using (1.11), the relations $x = U - \sqrt{-1}JU$, $\bar{y} = V + \sqrt{-1}JV$ and skew-symmetry of R.

To prove (iii) we use (1.16), (1.17) and an arbitrary complex vector field $z = Z_1 + \sqrt{-1}Z_2$. Therefore one can compute

$$ \tilde{R}(\bar{x},\bar{y})z = 2[(R(U,V)Z_1 - R(U,JV)Z_2) + \sqrt{-1}(R(U,JV)Z_1 + R(U,V)Z_2)], \tag{2.10}$$

$$ \tilde{R}(x,y)\bar{z} = 2[(R(U,V)Z_1 - R(U,JV)Z_2) - \sqrt{-1}(R(U,JV)Z_1 + R(U,V)Z_2)]. \tag{2.11}$$

These relations imply (iii). Similarly one can check (iv) and (v). ■

§3 The Chern characteristic classes of a complex equiaffine surface

Let $E_1, JE_1, \cdots, E_n, JE_n$ be a real base for tangent space T_pM. We denote by $\theta^1, \bar{\theta}^1, \cdots, \theta^n, \bar{\theta}^n$ the corresponding dual base for T_pM^*. We write $E_{n+p} = JE_p = E_{\bar{p}}$ and similarly $\theta^{n+q} = \bar{\theta}^q$, $1 \leq p, q \leq n$. We use also the following ranges of indexes $i, j, p, q = 1, 2, \cdots, n$, and $I, J, P, Q = 1, 2, \cdots, 2n$. We denote

$$R(X,Y)E_P = R_{XYP}{}^Q E_Q \tag{3.1}$$

and for $X = E_I, Y = E_J$ we simplify notation and write $R_{E_I,E_JP}{}^Q = R_{IJP}{}^Q$.

We put

$$\Omega_I^J(X,Y) = R_{XYI}{}^J, \tag{3.2}$$

$$\Xi_i^j(X,Y) = -(\Omega_i^j(X,Y) - \sqrt{-1}\Omega_{\bar{i}}^j(X,Y)) \tag{3.3}$$

and

$$\Xi_i^j = \tilde{R}_{PQi}{}^j \theta^P \wedge \theta^Q, \tag{3.4}$$

for $X, Y \in \mathcal{X}(M^n)$. Then (Ξ_i^j) is a matrix of complex 2-forms and

$$\det\left(\delta_i^j - \frac{1}{2\pi\sqrt{-1}}\Xi_i^j\right) = 1 + \gamma_1 + \cdots + \gamma_n \tag{3.5}$$

is a globaly defined closed form which represents the total Chern class of M via de Rham's theorem (see [6, p.307]). In particular, the Chern forms γ_1 and γ_1^2 and γ_2 are given by

$$\gamma_1 = \frac{\sqrt{-1}}{2\pi}\sum \Xi_i^i = \frac{\sqrt{-1}}{2\pi}(\Omega_i^i - \sqrt{-1}\Omega_{\bar{i}}^i), \tag{3.6}$$

$$\begin{aligned}\gamma_1^2 &= -\frac{1}{4\pi^2}\sum_{1\leq i<j\leq n} \Xi_i^i \wedge \Xi_j^j \\ &= -\frac{1}{4\pi^2}\sum_{1\leq i<j\leq n}\{(\Omega_i^i \wedge \Omega_j^j - \Omega_{\bar{i}}^i \wedge \Omega_{\bar{j}}^j) \\ &\quad - \sqrt{-1}(\Omega_i^i \wedge \Omega_{\bar{j}}^j + \Omega_{\bar{i}}^i \wedge \Omega_j^j)\},\end{aligned} \tag{3.7}$$

$$\begin{aligned}\gamma_2 &= -\frac{1}{8\pi^2}\sum_{1\leq i<j\leq n}(\Xi_i^i \wedge \Xi_j^j - \Xi_i^j \wedge \Xi_j^i) \\ &= -\frac{1}{8\pi^2}\sum_{1\leq i<j\leq n}\{(\Omega_i^i \wedge \Omega_j^j - \Omega_{\bar{i}}^i \wedge \Omega_{\bar{j}}^j - \Omega_i^j \wedge \Omega_j^i - \Omega_{\bar{i}}^j \wedge \Omega_{\bar{j}}^i) \\ &\quad - \sqrt{-1}(\Omega_i^i \wedge \Omega_{\bar{j}}^j + \Omega_{\bar{i}}^i \wedge \Omega_j^j) - \Omega_i^j \wedge \Omega_{\bar{j}}^i - \Omega_{\bar{i}}^j \wedge \Omega_j^i)\}.\end{aligned} \tag{3.8}$$

In the following theorem we consider complex surfaces with all vanishing characteristic classes.

Theorem 3.1: *Let M be a complex surface (dim $M = 2$) endowed with a complex affine connection D satisfying (2.3). Then its Chern characteristic classes c_2 and c_1^2 vanish. Moreover, if M is a complex equiaffine surface then c_1 also vanishes.*

Proof: We introduce the volume form by (1.8). Consequently for our choice of a basis of a complex surface M (dim M=2) we have $\omega = \theta^1 \wedge \theta^{\bar{1}} \wedge \theta^2 \wedge \theta^{\bar{2}}$. Then we use (3.4) and (3.7) to see

$$\begin{aligned}\gamma_2 = -\frac{1}{4\pi^2}\sum_{i<j}(&\tilde{R}_{PQi}{}^i\tilde{R}_{RNj}{}^j \\ &- \tilde{R}_{PQi}{}^j\tilde{R}_{RNj}{}^i)\theta^P\wedge\theta^Q\wedge\theta^R\wedge\theta^N.\end{aligned} \tag{3.9}$$

But, since M is complex surface the indexes P, Q, R, N run over the range $p, q, \bar{p}, \bar{q}$, where $p, q \in \{1, 2\}$. Consequently we use (3.9) to see

$$\begin{aligned}\gamma_2 = -\frac{1}{4\pi^2}\sum_{i<j}(&\tilde{R}_{pqi}{}^i\tilde{R}_{\bar{p}\bar{q}j}{}^j - \tilde{R}_{p\bar{q}i}{}^i\tilde{R}_{q\bar{p}i}{}^i - \tilde{R}_{p\bar{p}i}{}^i\tilde{R}_{q\bar{q}i}{}^i \\ &- \tilde{R}_{pqi}{}^j\tilde{R}_{\bar{p}\bar{q}j}{}^i - \tilde{R}_{p\bar{q}i}{}^j\tilde{R}_{q\bar{p}j}{}^i - \tilde{R}_{p\bar{p}i}{}^j\tilde{R}_{q\bar{q}j}{}^i)\theta^p\wedge\theta^{\bar{p}}\wedge\theta^q\wedge\theta^{\bar{q}}\end{aligned} \tag{3.10}$$

.

Lemma 1.2 (ii), (iv) imply all components of $\tilde{R}$ in (3.10) vanish and consequently γ_2 vanish too. By the same procedure using (3.7) we compute $\gamma_1^2 = 0$. As we know if (M, D, J) is an equiaffine manifold it is equivallent with the condition to have the symmetric Ricci tensor for D. Consequently since $\Xi_i^i = \tilde{R}_{PQi}{}^i\theta^P\wedge\theta^Q = (\tilde{R}_{iPQ}{}^i - \tilde{R}_{iQP}{}^i)\theta^P\wedge\theta^Q$ we use (3.6) and (1.4) to prove $c_1 = 0$ for an equiaffine complex surface. ∎

§4 4-dimensional manifolds endowed with symmetric affine connection

This section is devoted to the construction of nonflat symmetric affine connections satisfying the curvature condition (2.3) on the torus T^4 and the Euclidean space $\mathbf{R}^4$.

Let the almost complex structure J is given by the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tag{4.1}$$

Then the standard basis $\{E_1, E_2, E_3, E_4\}$ in $\mathbf{R}^4$ is maped as follows

$$JE_1 = E_3, \quad JE_2 = E_4, \quad JE_3 = -E_1, \quad JE_4 = -E_2. \tag{4.2}$$

Let D be a symmetric complex affine connection and $\Gamma_{ij}{}^k$ $(i, j, k = 1, 2, 3, 4)$ its

Cristoffel's symbols. We use the condition $DJ = 0$ and (4.1) to see

$$
\begin{aligned}
&{\Gamma_{11}}^1 = {\Gamma_{13}}^3 = {\Gamma_{31}}^3 = -{\Gamma_{33}}^1,\\
&{\Gamma_{12}}^1 = {\Gamma_{21}}^1 = {\Gamma_{14}}^3 = {\Gamma_{41}}^3 = {\Gamma_{23}}^3 = {\Gamma_{32}}^3 = -{\Gamma_{34}}^3 = -{\Gamma_{43}}^3,\\
&{\Gamma_{13}}^1 = {\Gamma_{31}}^1 = -{\Gamma_{11}}^3 = {\Gamma_{33}}^3,\\
&{\Gamma_{14}}^1 = {\Gamma_{41}}^1 = -{\Gamma_{12}}^3 = -{\Gamma_{21}}^3 = {\Gamma_{23}}^1 = {\Gamma_{32}}^1 = {\Gamma_{34}}^3 = {\Gamma_{43}}^3,\\
&{\Gamma_{22}}^1 = -{\Gamma_{44}}^1 = {\Gamma_{24}}^3 = {\Gamma_{42}}^3,\\
&{\Gamma_{24}}^1 = {\Gamma_{42}}^1 = -{\Gamma_{22}}^3 = {\Gamma_{44}}^3,\\
&{\Gamma_{11}}^2 = {\Gamma_{13}}^4 = {\Gamma_{31}}^4 = -{\Gamma_{33}}^2,\\
&{\Gamma_{12}}^2 = {\Gamma_{21}}^2 = {\Gamma_{14}}^4 = {\Gamma_{41}}^4 = {\Gamma_{23}}^4 = {\Gamma_{32}}^4 = -{\Gamma_{34}}^2 = -\Gamma_{43}^2,\\
&{\Gamma_{13}}^2 = {\Gamma_{31}}^2 = -{\Gamma_{11}}^4 = {\Gamma_{33}}^4,\\
&{\Gamma_{14}}^2 = {\Gamma_{41}}^2 = {\Gamma_{23}}^2 = {\Gamma_{32}}^2 = -{\Gamma_{12}}^4 = -{\Gamma_{21}}^4 = {\Gamma_{34}}^4 = {\Gamma_{43}}^4,\\
&{\Gamma_{22}}^2 = -{\Gamma_{44}}^2 = {\Gamma_{24}}^4 = {\Gamma_{42}}^4,\\
&{\Gamma_{24}}^2 = {\Gamma_{42}}^2 = {\Gamma_{44}}^4 = -{\Gamma_{22}}^4.
\end{aligned}
\tag{4.3}
$$

Let us denote

$$
\begin{aligned}
A &= {\Gamma_{11}}^1, & E &= {\Gamma_{22}}^1, & K &= {\Gamma_{13}}^2,\\
B &= {\Gamma_{12}}^1, & F &= {\Gamma_{24}}^1, & L &= {\Gamma_{14}}^2,\\
C &= {\Gamma_{13}}^1, & G &= {\Gamma_{11}}^2, & M &= {\Gamma_{22}}^2,\\
D &= {\Gamma_{14}}^1, & H &= {\Gamma_{12}}^2, & N &= {\Gamma_{24}}^2.
\end{aligned}
\tag{4.4}
$$

We use (2.3) and (4.2) to get

$$
\begin{aligned}
&R(E_1, E_3) = 0,\\
&R(E_2, E_4) = 0,\\
&R(E_1, E_2) = -R(E_3, E_4),\\
&R(E_1, E_4) = -R(E_2, E_3).
\end{aligned}
\tag{4.5}
$$

Let (x^1, x^2, x^3, x^4) be coordinates. The relations (4.5) imply ${R_{132}}^1 = {R_{132}}^3 = {R_{241}}^1 = {R_{241}}^3 = 0$. We substitute (4.3) into these relations and having in mind (1.2) and (4.4) we find

$$
\begin{aligned}
&\frac{\partial B}{\partial x^3} - \frac{\partial D}{\partial x^1} = 0, \quad \frac{\partial B}{\partial x^4} - \frac{\partial D}{\partial x^2} = 0,\\
&\frac{\partial D}{\partial x^3} + \frac{\partial B}{\partial x^1} = 0, \quad \frac{\partial B}{\partial x^2} + \frac{\partial D}{\partial x^4} = 0.
\end{aligned}
\tag{4.6}
$$

Consequently

$$
\frac{\partial^2 B}{(\partial x^1)^2} + \frac{\partial^2 B}{(\partial x^2)^2} + \frac{\partial^2 B}{(\partial x^3)^2} + \frac{\partial^2 B}{(\partial x^4)^2} = 0, \tag{4.7}
$$

and therefore $B = const$ on T^4 since there do not exist harmonic functions on compact manifolds different from constants. We use now (4.6) to see $D = const$.

Similarly, the relations (4.5) imply $R_{131}{}^1 = R_{131}{}^3 = 0$, $R_{123}{}^1 = -R_{343}{}^1$, $R_{121}{}^1 = -R_{341}{}^1$. By the similar procedure it follows $A, C = const$. We use $R_{132}{}^2 = R_{132}{}^4 = R_{241}{}^2 = R_{241}{}^4 = 0$ to prove $H, L = const$. Similarly, $R_{122}{}^2 = -R_{342}{}^2$, $R_{124}{}^2 = -R_{344}{}^2$, $R_{242}{}^2 = R_{244}{}^2 = 0$ give $M, N = const$. Finally, $R_{242}{}^1 = R_{242}{}^3 = 0$, $R_{122}{}^1 = -R_{342}{}^1$, $R_{124}{}^1 = -R_{344}{}^1$ imply $E, F = const$ and $R_{131}{}^3 = R_{131}{}^4 = 0$, $R_{123}{}^4 = -R_{343}{}^4$, $R_{123}{}^2 = -R_{343}{}^2$ give $G, K = const$.

We compute now the components of the curvature tensor

$$
\begin{aligned}
R_{121}{}^1 &= EG + DL - CH - FK, \\
R_{121}{}^2 &= L^2 - H^2 + AH + DK + GM - H^2 - BL - CG - KN, \\
R_{122}{}^1 &= C^2 + BF + HE + DN - D^2 - AE - CM - FL, \\
R_{121}{}^3 &= CL + DH - EK - FG, \\
R_{121}{}^4 &= CK + DG + 2HL - AL - BH - GN - KM, \\
R_{124}{}^1 &= 2CD + EL + FH - AF - CN - BE - DM.
\end{aligned} \tag{4.8}
$$

These components are muttually different. We use Bianchi identity, skewsymmetry of R, (4.3)-(4.5) to see other components of R are equal either to zero or to some of these components given by (4.8), modulo sign.

If we consider the equations (4.6) on $\mathbf{R}^4$ one can see there are many different solutions that give nonflat connections D. To ilustrate it we give only one:

$$
\begin{aligned}
A &= e^{x^1}\sin x^3, & B &= ae^{x^1}\sin x^3, & E &= ce^{x^1}\sin x^3, \\
C &= e^{x^1}\cos x^3, & D &= ae^{x^1}\cos x^3, & F &= ce^{x^1}\cos x^3, \\
G &= e^{x^2}\sin x^4, & H &= be^{x^1}\sin x^3, & M &= de^{x^1}\sin x^4, \\
K &= e^{x^2}\cos x^4, & L &= be^{x^1}\cos x^3, & N &= de^{x^1}\cos x^4,
\end{aligned} \tag{4.9}
$$

where a, b, c, d some constants. By direct computations we see $R \neq 0$ and consequently $(\mathbf{R}^4, J, D)$ is nonflat space.

§5 Complex affine hypersurfaces

In this section we give an example of manifolds endowed with complex affine connection D that its curvature tensor satisfies (2.3). For more details see [4].

Let M^{2n} be *a complex hypersurface* of the complex affine space $\mathbf{R}^{2(n+1)}$, i.e. for each point p of $M : J(T_p(M)) = T_pM$. Then J induces a complex structure on M, which we will also denote by J.

Let $\xi = M \longrightarrow T\mathbf{R}^{2(n+1)} : p \mapsto \xi(p) \in T_p\mathbf{R}^{2(n+1)}$ be a local transversal vector field on M, i.e. for each point p in the domain of ξ, $\xi(p)$ does not belong to T_pM. Then $\xi(p)$, $J\xi(p)$ and T_pM together span $T_p\mathbf{R}^{2(n+1)}$. Consequently, for all tangent vector fields X and Y to M, we can decompose $\nabla_X Y$ and $\nabla_X \xi$ into a component tangent to M and into a component lying in the plane spanned by ξ and $J\xi$:

$$
\begin{aligned}
\nabla_X Y &= D_X Y + h_1(X,Y)\xi + h_2(X,Y)J\xi, \quad \text{(formula of Gauss)}, \\
\nabla_X \xi &= -SX + \tau_1(X)\xi + \tau_2(X)J\xi, \quad \text{(formula of Weingarten)},
\end{aligned} \tag{5.1}
$$

where D is *a torsion-free affine connection* on M, h_1 and h_2 are *symmetric bilinear forms* on M, S is a (1,1)-*tensor field* on M and τ_1 and τ_2 are 1-*forms* on M. Then one can prove

$$\begin{aligned} &DJ = 0, \\ &h_1(X, JY) = h_1(JX, Y) = -h_2(X, Y), \\ &h_2(X, JY) = h_2(JX, Y) = h_1(X, Y). \end{aligned} \tag{5.2}$$

Next, we define the $2n$-*form* θ_ξ on M by

$$\theta_\xi(X_1, \ldots, X_{2n}) = \omega(X_1, \ldots, X_{2n}, \xi, J\xi). \tag{5.3}$$

Then, we have $\theta_\xi J = \theta_\xi$ and moreover

$$D\theta_\xi = 2\tau_1\theta_\xi. \tag{5.4}$$

Further, let $\{X_1, \ldots, X_{2n}\}$ be a local basis in TM such that $\theta_\xi(X_1, \ldots, X_{2n}) = 1$. Then, consider the *function* H_ξ on M defined by

$$H_\xi = \det[h_1(X_i, X_j)], \quad (i, j \in \{1, \ldots, 2n\}); \tag{5.5}$$

(this definition is independent of the choice of basis). Now, we search for the transversal vector fields ξ for which

$$|H_\xi| = 1, \tag{5.6}$$

$$\tau_1 = 0; \tag{5.7}$$

such vector fields will be called *affine normal vector fields*. Condition (5.6) is a kind of normalization, and condition (5.7) implies that $\nabla\theta_\xi = 0$, because of (5.4). We will only consider the hypersurfaces for which h_1 is non-degenerate, i.e. defines a semi-Riemannian metric on M; such hypersurfaces are said to be *non-degenerate*. One can prove that on each non-degenerate complex hypersurface M^{2n} in $\mathbf{R}^{2(n+1)}$, there always exists an affine normal vector field. In terms of such affine normal vector field ξ, a canonically determined complex affine structure (D, J, ξ) can be defined on M.

Let R be the Riemann-Christoffel curvature tensor of the affine connection D on M. Then the equations of Gauss is given by

$$R(X, Y)Z = h_1(Y, Z)SX - h_1(X, Z)SY + h_2(Y, Z)JSX - h_2(X, Z)JSY. \tag{5.8}$$

We have also

$$\begin{aligned} &h_1(X, SY) = h_1(SX, Y), \\ &h_2(X, SY) = h_2(SX, Y) + (d\tau_2)(X, Y). \end{aligned} \tag{5.9}$$

We use $DJ = 0$ in (5.2) to see that M can be endowed with a complex structure such that M^n is a complex hypersurface of $\mathbf{C}^{n+1}$. One can prove if ξ is an affine normal vector field with $\tau_2 = 0$, ξ is also a complex affine normal vector field; i.e. ξ is analytic vector field such that

$$\nabla_X Y = D_X Y + h(X,Y)\xi, \tag{5.10}$$

$$\nabla_X \xi = -SX, \tag{5.11}$$

$$^{\mathbf{C}}H_\xi = |\det h(X_i, X_j)|^2 = 1, \tag{5.12}$$

where D is a torsion-free complex affine connection, $D_X Y$ is analytic for arbitrary tangent analytic vector fields X, Y on M, h is a complex-valued analytic symmetric $\mathbf{C}$-bilinear form, S is a $\mathbf{C}$-linear transformation of TM and $\{X_1, \ldots, X_n\}$ analytic tangent vector fields on M.

Now, using (4.2) and (4.9) one can prove

$$R(JX, Y) = R(X, JY) = R(X,Y) \circ J = J \circ R(X,Y),$$

i.e. D is the complex affine connection satisfying (2.3).

Remark: Let us mention that the relation (2.3) is satisfied also for the Riemann curvature tensor of the Levi-Civita connection in this class of complex hypersurfaces.

§6 Norden manifolds

In the section §5 we gave examples of hypersurfaces in $\mathbf{C}^n$ with the connection D satisfying the condition (2.3) globally. In this section we study some examples of the connection D that exists locally on some compact manifolds.

More precisely, we give in this section a short review of the Norden construction.

Let X^n be a complex manifold with complex coordinates $\omega^\alpha = u^\alpha + iv^\alpha$ ($\alpha = 1, \ldots, n$) and analytic transition functions. Let X^{2n} be a smooth manifold with real coordinates x^i and smooth transition functions. '1-1' correspondence $X^n \longleftrightarrow X^{2n}$ is given by smooth mapping

$$u^\alpha = u^\alpha(x^i), \quad v^\alpha = v^\alpha(x^i), \quad (i = 1, 2, \ldots, 2n). \tag{5.1}$$

Then

$$\omega_i^\alpha = \partial_i \omega^\alpha \tag{5.2}$$

are coordinates of an endomorphism.

The real decomposition

$$-i\omega_j^\alpha = \gamma_j^k \omega_k^\alpha \tag{5.3}$$

define an almost complex structure γ as

$$\gamma_j^k \gamma_k^i = -\delta_j^i. \tag{5.4}$$

Let $\Gamma_{\gamma\beta}^\alpha$ be a coefficients of an affine connection D on X^n. This connection is analytic if $\Gamma_{\gamma\beta}^\alpha$ are analytic functions of ω^α. One can use the functions

$$\Gamma_{i\beta}^\alpha = \Gamma_{\gamma\beta}^\alpha \omega_i^\gamma, \tag{5.5}$$

to obtain its real decomposition

$$\partial_j \omega_i^\alpha + \Gamma_{j\beta}^\alpha \omega_i^\beta = \Gamma_{ji}^k \omega_k^\alpha. \tag{5.6}$$

The real coefficients Γ_{ji}^k define a connection on X^{2n}. We say this connection is *induced* by a connection of X^n or *realizes* this one. It is easy to see a torsion free connection induces also the torsion free connection. We study here only such connections. Using the technic of mixed differentiation in [10] the relation (5.6) can be written as follows

$$D_j \omega_i^\alpha = 0. \tag{5.7}$$

and

$$D_r \gamma_j^\alpha = 0. \tag{5.8}$$

This implies the connection D is complex one.

The equation

$$D_\gamma G_{\alpha\beta} = 2\Omega_\gamma G_{\alpha\beta} \tag{5.9}$$

characterizes a Weyl connection D. It makes sense in a complex space if the field of symmetric nondegenerate fundamental tensor $G_{\alpha\beta}$ is analytic. A Weyl connection is given by the following formula

$$\hat{\Gamma}_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma - (\Omega_\alpha \delta_\beta^\gamma + \Omega_\beta \delta_\alpha^\gamma - \tilde{\Omega}^\gamma G_{\alpha\beta}), \tag{5.10}$$

where $\Omega_\alpha = G_{\alpha\delta}\tilde{\Omega}^\delta$ and $\Gamma_{\alpha\beta}^\gamma$ are Christoffel symbols of Levi-Civita connection corresponding to the $G_{\alpha\beta}$.

If we put

$$G_{ij} = g_{ij} + i\gamma_{ij} = \omega_i^\alpha \omega_j^\beta G_{\alpha\beta}, \tag{5.11}$$

$$\Omega_k = \omega_k + i\tilde{\omega}_k = \omega_k^\gamma \Omega_\gamma, \tag{5.12}$$

we use (5.8) to see

$$D_k G_{ij} = 2\Omega_k G_{ij}. \tag{5.13}$$

Moreover, (5.4) implies $iG_{ij} = ig_{ij} - \gamma_{ij} = -\gamma_i^k((g_{kj} + i\gamma_{kj})$ or

$$\gamma_{ij} - \gamma_i^k g_{kj} \tag{5.14}$$

and

$$g_{ij} = -\gamma_i^k \gamma_{kj} = -\gamma_i^k \gamma_j^t g_{kt}, \tag{5.15}$$

as γ_{kj} is symmetric. Analogously, we have

$$\tilde{\omega}_k = \gamma_k^j \omega_j. \tag{5.16}$$

Consequently, the connection on X^{2n}, which realizes a Weyl connection of complex manifold X^n, is determined by the relations (5.4), (5.13)-(5.16). This connection exists in the line space. We denote this connection by (g, γ, ω) as it is determined by these quantities. Let us remark that the relation between this connection and the corresponding quantities is not unique.

If 1-form Ω is changed and $G_{\alpha\beta}$ is fixed then following Norden terminology we say that the corresponding transformation of a Weyl connection is *conformal* what is in nowdays terminology gauge. We say also that the corresponding transformation of the connection (g,γ,ω) is conformal. Consequently we have the transformation $(g,\gamma,\omega) \leftrightarrow (g,\gamma,\bar{\omega})$, where

$$\omega_k - \bar{\omega}_k = p_k \tag{5.17}$$

is 1-form of this gauge transformation.

One can prove if (g,γ,ω) is equiaffine connection then ω is a gradient. Especially, if both ω and $\tilde{\omega}$ are gradients then it exists an analytical transformation of the complex tensor $G_{\alpha\beta}$ such that the corresponding $\Omega = 0$ and consequently the geometry of X^n is Hermitian. Moreover, the geometry $(g,\gamma,0)$ is also pseudo-Riemannian endowed with two tensors g_{ij} and γ_{ij} such that

$$D_k g_{ij} = \nabla_k g_{ij} = 0; \quad D_k \gamma_{ij} = \nabla_k \gamma_{ij} = 0. \tag{5.18}$$

Consequently, the relations (5.4), (5.13)-(5.16) and (5.28) imply that the manifold X^{2n} $(n=2)$ with geometry $(g,\gamma,0)$ satisfies also the relation (2.3).

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